Filomat 39:16 (2025), 5601–5615
https://doi.org/10.2298/FIL2516601A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Non-linear mixed Jordan bi-skew Lie-type derivations on ∗-algebras

Asma Ali[a], Mohd Tasleem[a], Abdul Nadim Khan[b,*], Ambreen Bano[a]

[a] *Department of Mathematics, Aligarh Muslim University, Aligarh, India*
[b] *Department of Mathematics, College of Science & Arts- Rabigh, King Abdulaziz University, Saudi Arabia*

Abstract. Let $\mathcal{M}$ be a unital ∗-algebra. For any $M_1, M_2 \in \mathcal{M}$, the Jordan and bi-skew Lie product of M_1 and M_2 are defined as $M_1 \circ M_2 = M_1M_2 + M_2M_1$ and $[M_1, M_2]_\diamond = M_1M_2^* - M_2M_1^*$, respectively. A product defined as $p_n\big(M_1, M_2, \ldots, M_n\big) = [M_1 \circ M_2 \circ \ldots \circ M_{n-1}, M_n]_\diamond$ for all $M_1, M_2, \ldots, M_n \in \mathcal{M}$, is called a mixed Jordan bi-skew Lie n-product of $M_1, M_2, \ldots, M_n$. In this article, we prove that a map $\Psi : \mathcal{M} \to \mathcal{M}$, satisfies $\Psi\Big(p_n\big(M_1, M_2, \ldots, M_n\big)\Big) = \sum_{k=1}^{n} p_n\big(M_1, M_2, \ldots, M_{k-1}, \Psi(M_k), M_{k+1}, \ldots, M_n\big)$ for all $M_1, M_2, \ldots, M_n \in \mathcal{M}$, if and only if Ψ is an additive ∗-derivation. We apply the above result to prime ∗-algebras, factor von Neumann algebras, von Neumann algebras with no central summands of type I_1 and standard operator algebras.

1. Introduction

Let $\mathcal{M}$ be an associative ∗-algebra over $\mathbb{C}$ (the field of complex numbers). The products, $[M_1, M_2] = M_1M_2 - M_2M_1$ and $M_1 \circ M_2 = M_1M_2 + M_2M_1$ are respectively the usual Lie and Jordan product of $M_1, M_2 \in \mathcal{M}$. These products have been extensively studied by many mathematicians (see [1, 5, 6, 17, 18, 23] and the references therein). An involution "∗" over $\mathcal{M}$ is a map $M \to M^*$ satisfies $(\lambda M + N)^* = \bar{\lambda}M^* + N^*$, $(MN)^* = N^*M^*$ and $(M^*)^* = M$ for all $M, N \in \mathcal{M}$ and $\lambda \in \mathbb{C}$, where $\bar{\lambda}$ is the conjugate of λ. An algebra with involution ∗, is called a ∗-algebra. Recall that an additive ∗-derivation is a map $\Psi : \mathcal{M} \to \mathcal{M}$, if it is additive and satisfies $\Psi(M_1M_2) = \Psi(M_1)M_2 + M_1\Psi(M_2)$ and $\Psi(M^*) = \Psi(M)^*$ for all $M, M_1, M_2 \in \mathcal{M}$. Obviously every ∗-derivation is a derivation. A linear map $\Psi : \mathcal{M} \to \mathcal{M}$ is called a Lie (resp. Jordan) derivation if $\Psi([M_1, M_2]) = [\Psi(M_1), M_2] + [M_1, \Psi(M_2)]$ (resp. $\Psi(M_1 \circ M_2) = \Psi(M_1) \circ M_2 + M_1 \circ \Psi(M_2)$) for all $M_1, M_2 \in \mathcal{M}$. If we remove the linearity assumption in the above definitions, then Ψ is said to be a non-linear Lie (resp. non-linear Jordan) derivation on $\mathcal{M}$. Similarly, a map $\Psi : \mathcal{M} \to \mathcal{M}$, is called a non-linear Lie (resp. non-linear Jordan) triple derivation if it only satisfies $\Psi([[M_1, M_2], M_3]) = [[\Psi(M_1), M_2], M_3] + [[M_1, \Psi(M_2)], M_3] + [[M_1, M_2], \Psi(M_3)]$ (resp. $\Psi(M_1 \circ M_2 \circ M_3) = \Psi(M_1) \circ M_2 \circ M_3 + M_1 \circ \Psi(M_2) \circ M_3 + M_1 \circ M_2 \circ \Psi(M_3)$) for all $M_1, M_2, M_3 \in \mathcal{M}$. The new products defined as $[M_1, M_2]_* = M_1M_2 - M_2M_1^*$ and $M_1 * M_2 = M_1M_2 + M_2M_1^*$ are respectively

2020 *Mathematics Subject Classification*. Primary 16W25; Secondary 46L10.
Keywords. Additive ∗-derivation; Mixed Jordan bi-skew Lie-type derivations; ∗-algebras.
Received: 02 August 2024; Revised: 19 April 2025; Accepted: 26 April 2025
Communicated by Dijana Mosić
Research supported by CSIR-UGC Junior Research Fellowship (Ref. No. Nov/06/2020(i)EU-V).
* Corresponding author: Abdul Nadim Khan
Email addresses: asma_ali2@rediffmail.com (Asma Ali), tasleemh59@gmail.com (Mohd Tasleem), abdulnadimkhan@gmail.com (Abdul Nadim Khan), ambreenbn9@gmail.com (Ambreen Bano)
ORCID iDs: https://orcid.org/0000-0001-7602-0268 (Asma Ali), https://orcid.org/0000-0002-2516-0612 (Mohd Tasleem), https://orcid.org/0000-0002-5861-6137 (Abdul Nadim Khan), https://orcid.org/0009-0007-7351-9654 (Ambreen Bano)

called $*$-Lie (or skew Lie) product and $*$-Jordan (or skew Jordan) product of $M_1, M_2 \in \mathcal{M}$. These products are very important as they naturally appear in the problem of representing quadratic functionals by sesquilinear functionals on modules over $*$-algebras. Many mathematicians studied the structure of certain maps (specifically derivations) preserving these products on different rings and operator algebras (see, for example [7, 14, 16, 21, 24]). Recently, a new product called as bi-skew Lie product defined as, for any $M_1, M_2 \in \mathcal{M}$, $[M_1, M_2]_\diamond = M_1 M_2^* - M_2 M_1^*$, has been introduced by Kong and Zhang [10]. They obtained the structure of non-linear bi-skew Lie derivation on factor von Neumann algebra $\mathcal{A}$. In fact, they proved that such a map is an additive $*$-derivation on $\mathcal{A}$. This result was further extended [8] by the third author to the case of non-linear/multiplicative bi-skew Lie triple derivations on $\mathcal{A}$.

In recent years, many mathematicians considered mixed triple products such as $[[M_1, M_2]_*, M_3]$, $[[M_1, M_2], M_3]_*$, $[M_1 * M_2, M_3]_*$, $[M_1, M_2]_* * M_3$, $M_1 * M_2 \circ M_3$ etc. and characterized the structure of derivations preserving these products (see [4, 9, 12, 13, 15, 19, 25, 26]). For instance, Zhou et al. [26] obtained the structure of non-linear mixed Lie triple derivations on prime $*$-algebras. In [15] (resp. [12]) Li and Zhang proved that every non-linear mixed Jordan triple $*$-derivation on factor von Neumann algebras (resp. on $*$-algebras), is an additive $*$-derivation. Kong and Li [9] characterized non-linear mixed Lie triple derivations on prime $*$-rings. Zhao and Fang [25] explored the structure of second non-linear mixed Lie triple derivations on finite von Neumann algebras.

In a recent study, Ferreira and Costa [3] provided a characterization of $*$-Jordan type maps on C^*-algebra $\mathcal{A}$. Their findings revealed that under certain mild conditions imposed on $\mathcal{A}$, every multiplicative $*$-Jordan-type map on $\mathcal{A}$ is, in fact, a $*$-isomorphism. Building on this discovery, Ferreira and Wei [4] extended their investigations to $*$-algebras. Specifically, they demonstrated that on a $*$-algebra $\mathcal{M}$, any non-linear mixed $*$-Jordan-type derivation i.e., the map $\Psi : \mathcal{M} \to \mathcal{M}$, satisfying

$$\Psi(M_1 \circ M_2 \circ \cdots \bullet M_n) = \sum_{k=1}^{n} M_1 \circ M_2 \circ \cdots \circ M_{k-1} \circ \Psi(M_k) \circ M_{k+1} \circ \cdots \bullet M_n$$

for all $M_1, M_2, \ldots, M_n \in \mathcal{M}$ is, indeed, an additive $*$-derivation, where $M_1 \circ M_2 = M_1 M_2 + M_2 M_1$ and $M_1 \bullet M_2 = M_1^* M_2 + M_2^* M_1$.

The above mentioned work motivates us to construct a new type of mixed product called as mixed Jordan bi-skew Lie n-product which we define as

$$p_n\Big(M_1, M_2, \ldots, M_n\Big) = [M_1 \circ M_2 \circ \cdots \circ M_{n-1}, M_n]_\diamond$$

where $M_1 \circ M_2 = M_1 M_2 + M_2 M_1$ and $[M_1, M_2]_\diamond = M_1 M_2^* - M_2 M_1^*$ and we try to give the structure of non-linear mixed Jordan bi-skew Lie-type derivations on $*$-algebras.

Let us first define non-linear mixed Jordan bi-skew Lie triple derivations. A map (not necessarily linear) $\Psi : \mathcal{M} \to \mathcal{M}$, is said to be a non-linear mixed Jordan bi-skew Lie triple derivation if

$$\Psi([M_1 \circ M_2, M_3]_\diamond) = [\Psi(M_1) \circ M_2, M_3]_\diamond + [M_1 \circ \Psi(M_2), M_3]_\diamond + [M_1 \circ M_2, \Psi(M_3)]_\diamond$$

for all $M_1, M_2, M_3 \in \mathcal{M}$, where $M_1 \circ M_2 = M_1 M_2 + M_2 M_1$ and $[M_1, M_2]_\diamond = M_1 M_2^* - M_2 M_1^*$. By considering non-linear mixed Jordan bi-skew Lie triple derivation and the definition of non-linear mixed $*$-Jordan-type derivations in [4], we define non-linear mixed Jordan bi-skew Lie n-derivation as follows: Let $\mathcal{M}$ be a $*$-algebra and $n \geq 3$ be a fixed positive integer. Then a non-linear mixed Jordan bi-skew Lie n-derivation is a map $\Psi : \mathcal{M} \to \mathcal{M}$, which satisfies the following condition

$$\Psi\Big(p_n\Big(M_1, M_2, \ldots, M_n\Big)\Big) = \sum_{k=1}^{n} p_n\Big(M_1, M_2, \ldots, M_{k-1}, \Psi(M_k), M_{k+1}, \ldots, M_n\Big) \tag{1}$$

for all $M_1, M_2, \ldots, M_n \in \mathcal{M}$, where $p_n\Big(M_1, M_2, \ldots, M_n\Big) = [M_1 \circ M_2 \circ \cdots \circ M_{n-1}, M_n]_\diamond$. By the definition, it is evident that every non-linear mixed Jordan bi-skew Lie triple derivation can be categorized as a non-linear mixed Jordan bi-skew Lie 3-derivation. Additionally, it is apparent that any non-linear mixed Jordan bi-skew Lie triple derivation defined on a $*$-algebra is a non-linear mixed Jordan bi-skew Lie n-derivation, although the converse is not be true in general. Non-linear mixed Jordan bi-skew Lie 3-derivation, non-linear mixed Jordan bi-skew Lie 4-derivation and non-linear mixed Jordan bi-skew Lie n-derivations are collectively denoted as non-linear mixed Jordan bi-skew Lie-type derivations.

2. Preliminaries

In the entire text, unless specified otherwise, the symbol $\mathcal{M}$ denotes a $*$-algebra over the field of complex numbers, denoted as $\mathbb{C}$. Let H represent a complex Hilbert space, and $\mathcal{B}(H)$ represent the algebra comprising all bounded linear operators on H. An idempotent operator P belonging to $\mathcal{B}(H)$ is called a projection if it satisfies the condition of being self-adjoint, i.e., $P^2 = P$ and $P^* = P$. Any operator $M \in \mathcal{B}(H)$, can be expressed as $M = RM + iImM$, where $i \in \mathbb{C}$ (i.e., $i^2 = -1$), $RM = \frac{M+M^*}{2}$, and $ImM = \frac{M-M^*}{2i}$. It is noteworthy that both RM and ImM are self-adjoint.

Consider a projection $P = P_1 \in \mathcal{M}$. Define $P_2 = I - P_1$ and $\mathcal{M}_{ij} = P_i\mathcal{M}P_j$. Consequently, $\mathcal{M} = \mathcal{M}_{11} \oplus \mathcal{M}_{12} \oplus \mathcal{M}_{21} \oplus \mathcal{M}_{22}$. Let $\mathcal{R} = \{M \in \mathcal{M} \mid M^* = M\}$ and $\mathcal{S} = \{M \in \mathcal{M} \mid M^* = -M\}$. Additionally, define $\mathcal{S}_{12} = \{P_1SP_2 + P_2SP_1 \mid S \in \mathcal{S}\}$ and $\mathcal{S}_{ii} = P_i\mathcal{S}P_i$ for $i = 1, 2$. Thus, for any $S \in \mathcal{S}$, it can be expressed as $S = S_{11} + S_{12} + S_{22}$, where $S_{12} \in \mathcal{S}_{12}$ and $S_{ii} \in \mathcal{S}_{ii}$ for $i = 1, 2$.

3. Main Result

Theorem 3.1. *Let $\mathcal{M}$ be a unital $*$-algebra containing a nontrivial projection P satisfying*

$$M\mathcal{M}P = (0) \text{ implies } M = 0 \tag{2}$$

and

$$M\mathcal{M}(I - P) = (0) \text{ implies } M = 0. \tag{3}$$

Then, a map $\Psi : \mathcal{M} \to \mathcal{M}$ is a non-linear mixed Jordan bi-skew Lie-type derivation if and only if it is an additive $*$-derivation.

The sufficient part is easy to prove as every additive $*$-derivation satisfies (1). So, we only need to prove the necessary part, which we shall prove in a series of claims that are as follows:

Claim 3.2. $\Psi(0) = 0$.

It follows from the hypothesis that

$$\begin{aligned} \Psi(0) &= \Psi\Big(p_n\Big(0, 0, \ldots, 0\Big)\Big) \\ &= p_n\Big(\Psi(0), 0, \ldots, 0\Big) + p_n\Big(0, \Psi(0), \ldots, 0\Big) + \ldots + p_n\Big(0, 0, \ldots, \Psi(0)\Big) \\ &= 0. \end{aligned}$$

Claim 3.3. $\Psi(S)^* = -\Psi(S)$ *for every* $S \in \mathcal{S}$.

Let $S \in \mathcal{S}$. Then, we can write $S = p_n\Big(S, \frac{I}{2}, \ldots, \frac{I}{2}\Big)$. Now, consider

$$\begin{aligned} \Psi(S) &= \Psi\Big(p_n\Big(S, \frac{I}{2}, \ldots, \frac{I}{2}\Big)\Big) \\ &= p_n\Big(\Psi(S), \frac{I}{2}, \ldots, \frac{I}{2}\Big) + p_n\Big(S, \Psi\Big(\frac{I}{2}\Big), \ldots, \frac{I}{2}\Big) + \cdots + p_n\Big(S, \frac{I}{2}, \ldots, \Psi\Big(\frac{I}{2}\Big)\Big) \\ &= p_{n-1}\Big(\Psi(S), \frac{I}{2}, \ldots, \frac{I}{2}\Big) + p_{n-1}\Big(S\Psi\Big(\frac{I}{2}\Big) + \Psi\Big(\frac{I}{2}\Big)S, \ldots, \frac{I}{2}\Big) + \ldots + p_{n-1}\Big(S, \frac{I}{2}, \ldots, \Psi\Big(\frac{I}{2}\Big)\Big) \end{aligned}$$

$$= \left[\Psi(S), \frac{I}{2}\right]_\diamond + \left[\left(S\Psi\left(\frac{I}{2}\right) + \Psi\left(\frac{I}{2}\right)S\right), \frac{I}{2}\right]_\diamond + \ldots + \left[S, \Psi\left(\frac{I}{2}\right)\right]_\diamond$$
$$= \frac{1}{2}\Big(\Psi(S) - \Psi(S)^*\Big) + \frac{n}{2}\left(S\Psi\left(\frac{I}{2}\right)^* + \Psi\left(\frac{I}{2}\right)S\right) + \frac{(n-2)}{2}\left(S\Psi\left(\frac{I}{2}\right) + \Psi\left(\frac{I}{2}\right)^* S\right).$$

This implies that

$$\Psi(S) = -\Psi(S)^* + n\left(S\Psi\left(\frac{I}{2}\right)^* + \Psi\left(\frac{I}{2}\right)S\right) + (n-2)\left(S\Psi\left(\frac{I}{2}\right) + \Psi\left(\frac{I}{2}\right)^* S\right). \tag{4}$$

It follows that

$$\Psi(S)^* = -\Psi(S) - n\left(\Psi\left(\frac{I}{2}\right)S + S\Psi\left(\frac{I}{2}\right)^*\right) - (n-2)\left(\Psi\left(\frac{I}{2}\right)^* S + S\Psi\left(\frac{I}{2}\right)\right). \tag{5}$$

Combining (4) and (5), we get $\Psi(S)^* = -\Psi(S)$.

Claim 3.4. *For any $S_{11} \in \mathcal{S}_{11}, S_{12} \in \mathcal{S}_{12}$ and $S_{22} \in \mathcal{S}_{22}$, we have*

(*i*) $\Psi(S_{11} + S_{12}) = \Psi(S_{11}) + \Psi(S_{12})$;

(*ii*) $\Psi(S_{12} + S_{22}) = \Psi(S_{12}) + \Psi(S_{22})$.

(*i*) Let $\Delta = \Psi(S_{11} + S_{12}) - \Psi(S_{11}) - \Psi(S_{12})$. It is evident from Claim 3.3 that $\Delta \in \mathcal{S}$, i.e., $\Delta^* = -\Delta$. It would be sufficient to show that $\Delta = \Delta_{11} + \Delta_{12} + \Delta_{22} = 0$. We have

$$\begin{aligned}
&\Psi\Big(p_n\Big(S_{11} + S_{12}, \frac{I}{2}, \ldots, \frac{I}{2}, P_2\Big)\Big)\\
&= \Psi\Big(p_n\Big(S_{11}, \frac{I}{2}, \ldots, \frac{I}{2}, P_2\Big)\Big) + \Psi\Big(p_n\Big(S_{12}, \frac{I}{2}, \ldots, \frac{I}{2}, P_2\Big)\Big)\\
&= p_n\Big(\Psi(S_{11}), \frac{I}{2}, \ldots, \frac{I}{2}, P_2\Big) + p_n\Big(S_{11}, \Psi\left(\frac{I}{2}\right), \ldots, \frac{I}{2}, P_2\Big) + \ldots + p_n\Big(S_{11}, \frac{I}{2}, \ldots, \Psi\left(\frac{I}{2}\right), P_2\Big)\\
&+ p_n\Big(S_{11}, \frac{I}{2}, \ldots, \frac{I}{2}, \Psi(P_2)\Big) + p_n\Big(\Psi(S_{12}), \frac{I}{2}, \ldots, \frac{I}{2}, P_2\Big) + p_n\Big(S_{12}, \Psi\left(\frac{I}{2}\right), \ldots, \frac{I}{2}, P_2\Big)\\
&+ \ldots + p_n\Big(S_{12}, \frac{I}{2}, \ldots, \Psi\left(\frac{I}{2}\right), P_2\Big) + p_n\Big(S_{12}, \frac{I}{2}, \ldots, \frac{I}{2}, \Psi(P_2)\Big)\\
&= p_n\Big(\Psi(S_{11}) + \Psi(S_{12}), \frac{I}{2}, \ldots, \frac{I}{2}, P_2\Big) + p_n\Big(S_{11} + S_{12}, \Psi\left(\frac{I}{2}\right), \ldots, \frac{I}{2}, P_2\Big)\\
&+ \ldots + p_n\Big(S_{11} + S_{12}, \frac{I}{2}, \ldots \Psi\left(\frac{I}{2}\right), P_2\Big) + p_n\Big(S_{11} + S_{12}, \frac{I}{2}, \ldots, \frac{I}{2}, \Psi(P_2)\Big).
\end{aligned}$$

Also, we have

$$\begin{aligned}
&\Psi\Big(p_n\Big(S_{11} + S_{12}, \frac{I}{2}, \ldots, \frac{I}{2}, P_2\Big)\Big)\\
&= p_n\Big(\Psi(S_{11} + S_{12}), \frac{I}{2}, \ldots, \frac{I}{2}, P_2\Big) + p_n\Big(S_{11} + S_{12}, \Psi\left(\frac{I}{2}\right), \ldots, \frac{I}{2}, P_2\Big)\\
&+ \ldots + p_n\Big(S_{11} + S_{12}, \frac{I}{2}, \ldots, \Psi\left(\frac{I}{2}\right), P_2\Big) + p_n\Big(S_{11} + S_{12}, \frac{I}{2}, \ldots, \frac{I}{2}, \Psi(P_2)\Big).
\end{aligned}$$

We obtain from the above two expressions that $p_n\big(\Delta, \frac{I}{2}, \ldots, \frac{I}{2}, P_2\big) = 0$. Which gives $\Delta_{12} = \Delta_{22} = 0$. Now, since $p_n\big(S_{12}, \frac{I}{2}, \ldots, \frac{I}{2}, P_2 - P_1\big) = 0$, then we can write

$$\begin{aligned}
&p_n\Big(\Psi(S_{11} + S_{12}), \frac{I}{2}, \ldots, \frac{I}{2}, P_2 - P_1\Big) + p_n\Big(S_{11} + S_{12}, \Psi\left(\frac{I}{2}\right), \ldots, \frac{I}{2}, P_2 - P_1\Big)\\
&+ \ldots + p_n\Big(S_{11} + S_{12}, \frac{I}{2}, \ldots, \Psi\left(\frac{I}{2}\right), P_2 - P_1\Big) + p_n\Big(S_{11} + S_{12}, \frac{I}{2}, \ldots, \frac{I}{2}, \Psi(P_2 - P_1)\Big)
\end{aligned}$$

$$
\begin{aligned}
&= \Psi\Big(p_n\Big(S_{11} + S_{12}, \frac{I}{2}, \ldots, \frac{I}{2}, P_2 - P_1\Big)\Big)\\
&= \Psi\Big(p_n\Big(S_{11}, \frac{I}{2}, \ldots, \frac{I}{2}, P_2 - P_1\Big)\Big) + \Psi\Big(p_n\Big(S_{12}, \frac{I}{2}, \ldots, \frac{I}{2}, P_2 - P_1\Big)\Big)\\
&= p_n\Big(\Psi(S_{11}), \frac{I}{2}, \ldots, \frac{I}{2}, P_2 - P_1\Big) + p_n\Big(S_{11}, \Psi\left(\frac{I}{2}\right), \ldots, \frac{I}{2}, P_2 - P_1\Big)\\
&+ \ldots + p_n\Big(S_{11}, \frac{I}{2}, \ldots, \Psi\left(\frac{I}{2}\right), P_2 - P_1\Big) + p_n\Big(S_{11}, \frac{I}{2}, \ldots, \frac{I}{2}, \Psi(P_2 - P_1)\Big)\\
&+ p_n\Big(\Psi(S_{12}), \frac{I}{2}, \ldots, \frac{I}{2}, P_2 - P_1\Big) + p_n\Big(S_{12}, \Psi\left(\frac{I}{2}\right), \ldots, \frac{I}{2}, P_2 - P_1\Big)\\
&+ \ldots + p_n\Big(S_{12}, \frac{I}{2}, \ldots, \Psi\left(\frac{I}{2}\right), P_2 - P_1\Big) + p_n\Big(S_{12}, \frac{I}{2}, \ldots, \frac{I}{2}, \Psi(P_2 - P_1)\Big)\\
&= p_n\Big(\Psi(S_{11}) + \Psi(S_{12}), \frac{I}{2}, \ldots, \frac{I}{2}, P_2 - P_1\Big) + p_n\Big(S_{11} + S_{12}, \Psi\left(\frac{I}{2}\right), \ldots, \frac{I}{2}, P_2 - P_1\Big)\\
&+ \ldots + p_n\Big(S_{11} + S_{12}, \frac{I}{2}, \ldots, \Psi\left(\frac{I}{2}\right), P_2 - P_1\Big) + p_n\Big(S_{11} + S_{12}, \frac{I}{2}, \ldots, \frac{I}{2}, \Psi(P_2 - P_1)\Big).
\end{aligned}
$$

The above expression yields that $p_n(\Delta, \frac{I}{2}, \ldots, \frac{I}{2}, P_2 - P_1) = 0$. Using Claim 3.3, we obtain $\Delta_{11} = 0$. Therefore $\Delta = 0$, i.e.,

$$\Psi(S_{11} + S_{12}) = \Psi(S_{11}) + \Psi(S_{12}).$$

Following the similar procedure, one can establish (*ii*). This proves the claim.

Claim 3.5. *For any $S_{11} \in \mathcal{S}_{11}, S_{12} \in \mathcal{S}_{12}$ and $S_{22} \in \mathcal{S}_{22}$, we have*

$$\Psi(S_{11} + S_{12} + S_{22}) = \Psi(S_{11}) + \Psi(S_{12}) + \Psi(S_{22}).$$

Let $\Delta = \Psi(S_{11}+S_{12}+S_{22})-\Psi(S_{11})-\Psi(S_{12})-\Psi(S_{22})$. It follows from Claim 3.4 and $p_n(\frac{I}{2}, \frac{I}{2}, \ldots, \frac{I}{2}, S_{22}, P_1) = 0$ that

$$
\begin{aligned}
&\Psi\Big(p_n\Big(\frac{I}{2}, \frac{I}{2}, \ldots, \frac{I}{2}, S_{11} + S_{12} + S_{22}, P_1\Big)\Big)\\
&= \Psi\Big(p_n\Big(\frac{I}{2}, \frac{I}{2}, \ldots, \frac{I}{2}, S_{11} + S_{12}, P_1\Big)\Big) + \Psi\Big(p_n\Big(\frac{I}{2}, \frac{I}{2}, \ldots, \frac{I}{2}, S_{22}, P_1\Big)\Big)\\
&= p_n\Big(\Psi\left(\frac{I}{2}\right), \frac{I}{2}, \ldots, \frac{I}{2}, S_{11} + S_{12}, P_1\Big) + p_n\Big(\frac{I}{2}, \Psi\left(\frac{I}{2}\right), \ldots, \frac{I}{2}, S_{11} + S_{12}, P_1\Big)\\
&+ \ldots + p_n\Big(\frac{I}{2}, \frac{I}{2}, \ldots, \Psi\left(\frac{I}{2}\right), S_{11} + S_{12}, P_1\Big) + p_n\Big(\frac{I}{2}, \frac{I}{2}, \ldots, \frac{I}{2}, \Psi(S_{11} + S_{12}), P_1\Big)\\
&+ p_n\Big(\frac{I}{2}, \frac{I}{2}, \ldots, \frac{I}{2}, S_{11} + S_{12}, \Psi(P_1)\Big) + p_n\Big(\Psi\left(\frac{I}{2}\right), \frac{I}{2}, \ldots, \frac{I}{2}, S_{22}, P_1\Big)\\
&+ p_n\Big(\frac{I}{2}, \Psi\left(\frac{I}{2}\right), \ldots, \frac{I}{2}, S_{22}, P_1\Big) + \ldots + p_n\Big(\frac{I}{2}, \frac{I}{2}, \ldots, \Psi\left(\frac{I}{2}\right), S_{22}, P_1\Big)\\
&+ p_n\Big(\frac{I}{2}, \frac{I}{2}, \ldots, \frac{I}{2}, \Psi(S_{22}), P_1\Big) + p_n\Big(\frac{I}{2}, \frac{I}{2}, \ldots, \frac{I}{2}, S_{22}, \Psi(P_1)\Big)\\
&= p_n\Big(\Psi\left(\frac{I}{2}\right), \frac{I}{2}, \ldots, \frac{I}{2}, S_{11} + S_{12}, P_1\Big) + p_n\Big(\frac{I}{2}, \Psi\left(\frac{I}{2}\right), \ldots, \frac{I}{2}, S_{11} + S_{12}, P_1\Big)\\
&+ \ldots + p_n\Big(\frac{I}{2}, \frac{I}{2}, \ldots, \Psi\left(\frac{I}{2}\right), S_{11} + S_{12}, P_1\Big) + p_n\Big(\frac{I}{2}, \frac{I}{2}, \ldots, \frac{I}{2}, \Psi(S_{11}) + \Psi(S_{12}), P_1\Big)\\
&+ p_n\Big(\frac{I}{2}, \frac{I}{2}, \ldots, \frac{I}{2}, S_{11} + S_{12}, \Psi(P_1)\Big) + p_n\Big(\Psi\left(\frac{I}{2}\right), \frac{I}{2}, \ldots, \frac{I}{2}, S_{22}, P_1\Big)\\
&+ p_n\Big(\frac{I}{2}, \Psi\left(\frac{I}{2}\right), \ldots, \frac{I}{2}, S_{22}, P_1\Big) + \ldots + p_n\Big(\frac{I}{2}, \frac{I}{2}, \ldots, \Psi\left(\frac{I}{2}\right), S_{22}, P_1\Big)\\
&+ p_n\Big(\frac{I}{2}, \frac{I}{2}, \ldots, \frac{I}{2}, \Psi(S_{22}), P_1\Big) + p_n\Big(\frac{I}{2}, \frac{I}{2}, \ldots, \frac{I}{2}, S_{22}, \Psi(P_1)\Big)
\end{aligned}
$$

$$= p_n\Big(\Psi\left(\frac{I}{2}\right), \frac{I}{2}, \ldots, \frac{I}{2}, S_{11} + S_{12} + S_{22}, P_1\Big) + p_n\Big(\frac{I}{2}, \Psi\left(\frac{I}{2}\right), \ldots, \frac{I}{2}, S_{11} + S_{12} + S_{22}, P_1\Big)$$
$$+ \ldots + p_n\Big(\frac{I}{2}, \frac{I}{2}, \ldots, \Psi\left(\frac{I}{2}\right), S_{11} + S_{12} + S_{22}, P_1\Big) + p_n\Big(\frac{I}{2}, \frac{I}{2}, \ldots, \frac{I}{2}, \Psi(S_{11}) + \Psi(S_{12}) + \Psi(S_{22}), P_1\Big)$$
$$+ p_n\Big(\frac{I}{2}, \frac{I}{2}, \ldots, \frac{I}{2}, S_{11} + S_{12} + S_{22}, \Psi(P_1)\Big).$$

Apparently

$$\Psi\Big(p_n\Big(\frac{I}{2}, \frac{I}{2}, \ldots, \frac{I}{2}, S_{11} + S_{12} + S_{22}, P_1\Big)\Big)$$
$$= p_n\Big(\Psi\left(\frac{I}{2}\right), \frac{I}{2}, \ldots, \frac{I}{2}, S_{11} + S_{12} + S_{22}, P_1\Big) + p_n\Big(\frac{I}{2}, \Psi\left(\frac{I}{2}\right), \ldots, \frac{I}{2}, S_{11} + S_{12} + S_{22}, P_1\Big)$$
$$+ \ldots + p_n\Big(\frac{I}{2}, \frac{I}{2}, \ldots, \Psi\left(\frac{I}{2}\right), S_{11} + S_{12} + S_{22}, P_1\Big) + p_n\Big(\frac{I}{2}, \frac{I}{2}, \ldots, \frac{I}{2}, \Psi(S_{11} + S_{12} + S_{22}), P_1\Big)$$
$$+ p_n\Big(\frac{I}{2}, \frac{I}{2}, \ldots, \frac{I}{2}, S_{11} + S_{12} + S_{22}, \Psi(P_1)\Big).$$

We can conclude from the above two relations that $p_n(\frac{I}{2}, \frac{I}{2}, \ldots, \frac{I}{2}, \Delta, P_1) = 0$. Using the fact that $\Delta^* = -\Delta$, we obtain $\Delta_{11} = \Delta_{12} = 0$. It remains to show that $\Delta_{22} = 0$. Observe that $p_n(\frac{I}{2}, \frac{I}{2}, \ldots, S_{11}, P_2) = 0$. Following the similar technique as above, one can obtain $\Delta_{22} = 0$, and thus $\Delta = 0$, i.e.,

$$\Psi(S_{11} + S_{12} + S_{22}) = \Psi(S_{11}) + \Psi(S_{12}) + \Psi(S_{22}).$$

Claim 3.6. *For any $S_{12}, N_{12} \in \mathcal{S}_{12}$, we have*

$$\Psi(S_{12} + N_{12}) = \Psi(S_{12}) + \Psi(N_{12}).$$

Let $X_{12}, Y_{12} \in \mathcal{M}_{12}$. Assume that $S_{12} = X_{12} - X_{12}^* \in \mathcal{S}_{12}$ and $N_{12} = Y_{12} - Y_{12}^* \in \mathcal{S}_{12}$. Thus,

$$p_n\Big(\frac{I}{2}, \frac{I}{2}, \ldots, \frac{I}{2}, (iP_1 + iX_{12} + iX_{12}^*), (iP_2 + iY_{12} + iY_{12}^*)\Big)$$
$$= (X_{12} - X_{12}^*) + (Y_{12} - Y_{12}^*) + (X_{12}Y_{12}^* + X_{12}^*Y_{12} - Y_{12}X_{12}^* - Y_{12}^*X_{12})$$
$$= S_{12} + N_{12} + S_{12}N_{12}^* - N_{12}S_{12}^*.$$

Note that $S_{12}N_{12}^* - N_{12}S_{12}^* = X_{12}Y_{12}^* - Y_{12}X_{12}^* + X_{12}^*Y_{12} - Y_{12}^*X_{12} = S_{11} + S_{22}$, where $S_{11} = X_{12}Y_{12}^* - Y_{12}X_{12}^* \in \mathcal{S}_{11}$ and $S_{22} = X_{12}^*Y_{12} - Y_{12}^*X_{12} \in \mathcal{S}_{22}$. Since $iX_{12} + iX_{12}^*, iY_{12} + iY_{12}^* \in \mathcal{S}_{12}$, then from Claims 3.4 and 3.5, we have

$$\Psi(S_{12} + N_{12}) + \Psi(S_{11}) + \Psi(S_{22})$$
$$= \Psi(S_{12} + N_{12} + S_{11} + S_{22}) = \Psi(S_{12} + N_{12} + S_{12}N_{12}^* - N_{12}S_{12}^*)$$
$$= \Psi\Big(p_n\Big(\frac{I}{2}, \frac{I}{2}, \ldots, \frac{I}{2}, iP_1 + iX_{12} + iX_{12}^*, iP_2 + iY_{12} + iY_{12}^*\Big)\Big)$$
$$= p_n\Big(\Psi\left(\frac{I}{2}\right), \frac{I}{2}, \ldots, \frac{I}{2}, iP_1 + iX_{12} + iX_{12}^*, iP_2 + iY_{12} + iY_{12}^*\Big)$$
$$+ p_n\Big(\frac{I}{2}, \Psi\left(\frac{I}{2}\right), \ldots, \frac{I}{2}, iP_1 + iX_{12} + iX_{12}^*, iP_2 + iY_{12} + iY_{12}^*\Big)$$
$$+ \ldots + p_n\Big(\frac{I}{2}, \frac{I}{2}, \ldots, \Psi\left(\frac{I}{2}\right), iP_1 + iX_{12} + iX_{12}^*, iP_2 + iY_{12} + iY_{12}^*\Big)$$
$$+ p_n\Big(\frac{I}{2}, \frac{I}{2}, \ldots, \frac{I}{2}, \Psi(iP_1) + \Psi(iX_{12} + iX_{12}^*), iP_2 + iY_{12} + iY_{12}^*\Big)$$
$$+ p_n\Big(\frac{I}{2}, \frac{I}{2}, \ldots, \frac{I}{2}, iP_1 + iX_{12} + iX_{12}^*, \Psi(iP_2) + \Psi(iY_{12} + iY_{12}^*)\Big)$$
$$= \Psi\Big(p_n\Big(\frac{I}{2}, \frac{I}{2}, \ldots, \frac{I}{2}, iP_1, iP_2\Big)\Big) + \Psi\Big(p_n\Big(\frac{I}{2}, \frac{I}{2}, \ldots, \frac{I}{2}, iP_1, iY_{12} + iY_{12}^*\Big)\Big)$$

$$+\Psi\Big(p_n\Big(\frac{I}{2},\frac{I}{2},\dots,\frac{I}{2},iX_{12}+iX_{12}^*,iP_2\Big)\Big)+\Psi\Big(p_n\Big(\frac{I}{2},\frac{I}{2},\dots,\frac{I}{2},iX_{12}+iX_{12}^*,iY_{12}+iY_{12}^*\Big)\Big)$$
$$=\Psi(S_{12})+\Psi(N_{12})+\Psi(S_{12}N_{12}^*-N_{12}S_{12}^*)=\Psi(S_{12})+\Psi(N_{12})+\Psi(S_{11}+S_{22})$$
$$=\Psi(S_{12})+\Psi(N_{12})+\Psi(S_{11})+\Psi(S_{22}).$$

Thus, we obtain

$$\Psi(S_{12}+N_{12})=\Psi(S_{12})+\Psi(N_{12}).$$

Claim 3.7. *For every $S_{ii},N_{ii}\in\mathcal{S}_{ii}$ $(i=1,2)$, we have*

(i) $\Psi(S_{11}+N_{11})=\Psi(S_{11})+\Psi(N_{11})$;

(ii) $\Psi(S_{22}+N_{22})=\Psi(S_{22})+\Psi(N_{22})$.

(i) Let $\Delta=\Psi(S_{11}+N_{11})-\Psi(S_{11})-\Psi(N_{11})$. In order to prove the claim, we show that $\Delta=0$. We have

$$\Psi\Big(p_n\Big(\frac{I}{2},\frac{I}{2},\dots,\frac{I}{2},S_{11}+N_{11},P_2\Big)\Big)$$
$$=\Psi\Big(p_n\Big(\frac{I}{2},\frac{I}{2},\dots,\frac{I}{2},S_{11},P_2\Big)\Big)+\Psi\Big(p_n\Big(\frac{I}{2},\frac{I}{2},\dots,\frac{I}{2},N_{11},P_2\Big)\Big)$$
$$=p_n\Big(\Psi\Big(\frac{I}{2}\Big),\frac{I}{2},\dots,\frac{I}{2},S_{11},P_2\Big)+p_n\Big(\frac{I}{2},\Psi\Big(\frac{I}{2}\Big),\dots,\frac{I}{2},S_{11},P_2\Big)$$
$$+\dots+p_n\Big(\frac{I}{2},\frac{I}{2},\dots,\Psi\Big(\frac{I}{2}\Big),S_{11},P_2\Big)+p_n\Big(\frac{I}{2},\frac{I}{2},\dots,\frac{I}{2},\Psi(S_{11}),P_2\Big)$$
$$+p_n\Big(\frac{I}{2},\frac{I}{2},\dots,\frac{I}{2},S_{11},\Psi(P_2)\Big)+p_n\Big(\Psi\Big(\frac{I}{2}\Big),\frac{I}{2},\dots,\frac{I}{2},N_{11},P_2\Big)$$
$$+p_n\Big(\frac{I}{2},\Psi\Big(\frac{I}{2}\Big),\dots,\frac{I}{2},N_{11},P_2\Big)+\dots+p_n\Big(\frac{I}{2},\frac{I}{2},\dots,\Psi\Big(\frac{I}{2}\Big),N_{11},P_2\Big)$$
$$+p_n\Big(\frac{I}{2},\frac{I}{2},\dots,\frac{I}{2},\Psi(N_{11}),P_2\Big)+p_n\Big(\frac{I}{2},\frac{I}{2},\dots,\frac{I}{2},N_{11},\Psi(P_2)\Big)$$
$$=p_n\Big(\Psi\Big(\frac{I}{2}\Big),\frac{I}{2},\dots,\frac{I}{2},S_{11}+N_{11},P_2\Big)+p_n\Big(\frac{I}{2},\Psi\Big(\frac{I}{2}\Big),\dots,\frac{I}{2},S_{11}+N_{11},P_2\Big)$$
$$+\dots+p_n\Big(\frac{I}{2},\frac{I}{2},\dots,\Psi\Big(\frac{I}{2}\Big),S_{11}+N_{11},P_2\Big)+p_n\Big(\frac{I}{2},\frac{I}{2},\dots,\frac{I}{2},\Psi(S_{11})+\Psi(N_{11}),P_2\Big)$$
$$+p_n\Big(\frac{I}{2},\frac{I}{2},\dots,\frac{I}{2},S_{11}+N_{11},\Psi(P_2)\Big).$$

On the other hand, we have

$$\Psi\Big(p_n\Big(\frac{I}{2},\frac{I}{2},\dots,\frac{I}{2},S_{11}+N_{11},P_2\Big)\Big)$$
$$=p_n\Big(\Psi\Big(\frac{I}{2}\Big),\frac{I}{2},\dots,\frac{I}{2},S_{11}+N_{11},P_2\Big)+p_n\Big(\frac{I}{2},\Psi\Big(\frac{I}{2}\Big),\dots,\frac{I}{2},S_{11}+N_{11},P_2\Big)$$
$$+\dots+p_n\Big(\frac{I}{2},\frac{I}{2},\dots,\Psi\Big(\frac{I}{2}\Big),S_{11}+N_{11},P_2\Big)+p_n\Big(\frac{I}{2},\frac{I}{2},\dots,\frac{I}{2},\Psi(S_{11}+N_{11}),P_2\Big)$$
$$+p_n\Big(\frac{I}{2},\frac{I}{2},\dots,\frac{I}{2},S_{11}+N_{11},\Psi(P_2)\Big).$$

Equating the above two relations, we get $p_n(\frac{I}{2},\frac{I}{2},\dots,\frac{I}{2},\Delta,P_2)=0$, and using the fact that $\Delta^*=-\Delta$, we obtain $\Delta_{12}=\Delta_{22}=0$. Now, for any $X_{12}\in\mathcal{M}_{12}$, we can assume that $W_{12}=X_{12}-X_{12}^*\in\mathcal{S}_{12}$. Then $p_n\Big(\frac{I}{2},\frac{I}{2},\dots,\frac{I}{2},W_{12},S_{11}\Big)$, $p_n\Big(\frac{I}{2},\frac{I}{2},\dots,\frac{I}{2},W_{12},N_{11}\Big)\in\mathcal{S}_{12}$. Therefore, using Claim 3.6, we write

$$p_n\Big(\Psi\Big(\frac{I}{2}\Big),\frac{I}{2},\dots,\frac{I}{2},W_{12},S_{11}+N_{11}\Big)+p_n\Big(\frac{I}{2},\Psi\Big(\frac{I}{2}\Big),\dots,\frac{I}{2},W_{12},S_{11}+N_{11}\Big)$$

$$+\dots+p_n\Big(\frac{I}{2},\frac{I}{2},\dots,\Psi\Big(\frac{I}{2}\Big),W_{12},S_{11}+N_{11}\Big)+p_n\Big(\frac{I}{2},\frac{I}{2},\dots,\frac{I}{2},\Psi(W_{12}),S_{11}+N_{11}\Big)$$
$$+p_n\Big(\frac{I}{2},\frac{I}{2},\dots,\frac{I}{2},W_{12},\Psi(S_{11}+N_{11})\Big)$$
$$=\Psi\Big(p_n\Big(\frac{I}{2},\frac{I}{2},\dots,\frac{I}{2},W_{12},S_{11}+N_{11}\Big)\Big)$$
$$=\Psi\Big(p_n\Big(\frac{I}{2},\frac{I}{2},\dots,\frac{I}{2},W_{12},S_{11}\Big)\Big)+\Psi\Big(p_n\Big(\frac{I}{2},\frac{I}{2},\dots,\frac{I}{2},W_{12},N_{11}\Big)\Big)$$
$$=p_n\Big(\Psi\Big(\frac{I}{2}\Big),\frac{I}{2},\dots,\frac{I}{2},W_{12},S_{11}+N_{11}\Big)+p_n\Big(\frac{I}{2},\Psi\Big(\frac{I}{2}\Big),\dots,\frac{I}{2},W_{12},S_{11}+N_{11}\Big)$$
$$+\dots+p_n\Big(\frac{I}{2},\frac{I}{2},\dots,\Psi\Big(\frac{I}{2}\Big),W_{12},S_{11}+N_{11}\Big)+p_n\Big(\frac{I}{2},\frac{I}{2},\dots,\frac{I}{2},\Psi(W_{12}),S_{11}+N_{11}\Big)$$
$$+p_n\Big(\frac{I}{2},\frac{I}{2},\dots,\frac{I}{2},W_{12},\Psi(S_{11})+\Psi(N_{11})\Big).$$

The above expression yields that $p_n\Big(\frac{I}{2},\frac{I}{2},\dots,\frac{I}{2},W_{12},\Delta\Big)=0$. This implies that $\Delta_{11}=0$. Using the similar procedure one can easily obtain (*ii*). Therefore, the proof is completed.

Remark 3.8. *Claims 3.5–3.7 assert the additivity of* Ψ *on* $\mathcal{S}$.

Claim 3.9. $\Psi(I)=0$.

Let $S\in\mathcal{S}$. Then, using Claim 3.3 and Remark 3.8, we have

$$\begin{aligned}2^{n-1}\Psi(S) &= \Psi(2^{n-1}S)=\Psi\Big(p_n\Big(S,I,\dots,I\Big)\Big)\\ &= p_n\Big(\Psi(S),I,\dots,I\Big)+p_n\Big(S,\Psi(I),\dots,I\Big)+\dots+p_n\Big(S,I,\dots,\Psi(I)\Big)\\ &= 2^{n-1}\Psi(S)+n2^{n-3}\Big(\Psi(I)S+S\Psi(I)^*\Big)+(n-2)2^{n-3}\Big(S\Psi(I)+\Psi(I)^*S\Big).\end{aligned}$$

This implies that

$$n\Big(\Psi(I)S+S\Psi(I)^*\Big)+(n-2)\Big(S\Psi(I)+\Psi(I)^*S\Big)=0. \tag{6}$$

Putting $S=iI$ in (6), we obtain $(2n-2)i(\Psi(I)+\Psi(I)^*)=0$. Thus, we get

$$\Psi(I)^*=-\Psi(I). \tag{7}$$

It follows from (6) and (7) that $\Psi(I)S=S\Psi(I)$ for any $S\in\mathcal{S}$. Now, since for any $M\in\mathcal{M}$, $M=S_1+iS_2$ with $S_1=\frac{M-M^*}{2}\in\mathcal{S}$ and $S_2=\frac{M+M^*}{2i}\in\mathcal{S}$. Therefore, we have

$$\Psi(I)M=M\Psi(I) \tag{8}$$

for all $M\in\mathcal{M}$. Next, since $p_n(I,I,\dots,I)=0$, then we have

$$\begin{aligned}0 &= \Psi(p_n(I,I,\dots,I))\\ &= p_n(\Psi(I),I,\dots,I)+p_n(I,\Psi(I),\dots,I)+\dots+p_n(I,I,\dots,\Psi(I))\\ &= (n-1)2^{n-2}(\Psi(I)-\Psi(I)^*)+2^{n-2}(\Psi(I)^*-\Psi(I))\\ &= (n-2)2^{n-2}(\Psi(I)-\Psi(I)^*).\end{aligned}$$

It follows from (7) that $(n-2)2^{n-1}\Psi(I)=0$. Since (by the hypothesis) $n\geq 3$, then we have $\Psi(I)=0$.

Claim 3.10. *For any* $R\in\mathcal{R}$, $\Psi(R)^*=\Psi(R)$.

Let $R \in \mathcal{R}$. Then, $p_n\Big(R, I, \ldots, I\Big) = 0$, so by the hypothesis and Claim 3.9, we have

$$\begin{aligned} 0 &= \Psi\Big(p_n\Big(R, I, \ldots, I\Big)\Big) \\ &= p_n(\Psi(R), I, \ldots, I) = 2^{n-2}\Big(\Psi(R) - \Psi(R)^*\Big). \end{aligned} \tag{9}$$

This gives $\Psi(R)^* = \Psi(R)$ for all $R \in \mathcal{R}$. Hence the claim.

Claim 3.11. $\Psi(iI) \in \mathcal{Z}(\mathcal{M})$.

It follows from Claims 3.2, 3.3, 3.9, 3.10 and $p_n(I, I, \ldots, I, R, iI, iI) = 0$, that

$$\begin{aligned} 0 &= \Psi\Big(p_n\Big(I, I, \ldots, I, R, iI, iI\Big)\Big) \\ &= p_n\Big(I, I, \ldots, I, \Psi(R), iI, iI\Big) + p_n\Big(I, I, \ldots, I, R, \Psi(iI), iI\Big) + p_n\Big(I, I, \ldots, I, R, iI, \Psi(iI)\Big) \\ &= 2^{n-2} i\Big(\Psi(iI)R - R\Psi(iI)\Big). \end{aligned}$$

This implies that $\Psi(iI)R = R\Psi(iI)$ for all $R \in \mathcal{R}$. Since for any $M \in \mathcal{M}$, $M = R_1 + iR_2$ with $R_1 = \frac{M+M^*}{2} \in \mathcal{R}$ and $R_2 = \frac{M-M^*}{2i} \in \mathcal{R}$. Thus, $\Psi(iI)M = M\Psi(iI)$ for all $M \in \mathcal{M}$, and hence $\Psi(iI) \in \mathcal{Z}(\mathcal{M})$.

Claim 3.12. *For any* $R \in \mathcal{R}$, $\Psi(iR) = i\Psi(R) + \Psi(iI)R$.

In view of Claims 3.3, 3.9, 3.11 and Remark 3.8, we have

$$\begin{aligned} 2^{n-1}\Psi(iR) &= \Psi\Big(p_n\Big(I, I, \ldots, I, iI, R\Big)\Big) \\ &= p_n\Big(I, I, \ldots, I, \Psi(iI), R\Big) + p_n\Big(I, I, \ldots, I, iI, \Psi(R)\Big) \\ &= 2^{n-1}\Big(i\Psi(R) + \Psi(iI)R\Big). \end{aligned}$$

Therefore, we have

$$\Psi(iR) = i\Psi(R) + \Psi(iI)R.$$

Claim 3.13. Ψ *is additive on* $\mathcal{R}$.

Let $R, R' \in \mathcal{R}$. Then, using Remark 3.8 and Claim 3.12, we can write

$$\begin{aligned} i\Psi(R + R') + \Psi(iI)(R + R') &= \Psi(i(R + R')) \\ &= \Psi(iR) + \Psi(iR') \\ &= i(\Psi(R) + \Psi(R')) + \Psi(iI)(R + R'). \end{aligned}$$

This implies that

$$\Psi(R + R') = \Psi(R) + \Psi(R').$$

Claim 3.14. *For any* $R_1, R_2 \in \mathcal{R}$ *and* $M \in \mathcal{M}$, *we have*

(*i*) $\Psi(R_1 + iR_2) = \Psi(R_1) + i\Psi(R_2) + \Psi(iI)R_2$;

(*ii*) $\Psi(M^*) = \Psi(M)^*$.

(*i*) Let $R_1, R_2 \in \mathcal{R}$. Then, in view of Claims 3.10, 3.13 and $p_n(R_1, I, I, \ldots, I) = 0$, we have

$$\begin{aligned} &\Psi\Big(p_n\Big(R_1 + iR_2, I, I, \ldots, I\Big)\Big) = \Psi\Big(p_n\Big(R_1, I, I, \ldots, I\Big)\Big) + \Psi\Big(p_n\Big(iR_2, I, I, \ldots, I\Big)\Big) \\ &= \Psi\Big(p_n\Big(iR_2, I, I, \ldots, I\Big)\Big) = p_n\Big(\Psi(iR_2), I, I, \ldots, I\Big) = 2^{n-1}\Psi(iR_2) \\ &= 2^{n-1}\Big(i\Psi(R_2) + \Psi(iI)R_2\Big). \end{aligned} \tag{10}$$

On the other hand, we have

$$\Psi\Big(p_n\big((R_1+iR_2), I, \ldots, I\big)\Big) = p_n\Big(\Psi(R_1+iR_2), I, \ldots, I\Big)$$
$$= 2^{n-2}\Big(\Psi(R_1+iR_2) - \Psi(R_1+iR_2)^*\Big). \tag{11}$$

From (10) and (11), we have

$$2^{n-1}\Big(i\Psi(R_2) + \Psi(iI)R_2\Big) = 2^{n-2}\Big(\Psi(R_1+iR_2) - \Psi(R_1+iR_2)^*\Big). \tag{12}$$

Since $p_n(iR_2, iI, I, \ldots, I) = 0$, then we have

$$\Psi\Big(p_n\Big(R_1+iR_2, iI, I, \ldots, I\Big)\Big) = \Psi\Big(p_n(R_1, iI, I, \ldots, I)\Big) + \Psi\Big(p_n(iR_2, iI, I, \ldots, I)\Big)$$
$$= \Psi\Big(p_n(R_1, iI, I, \ldots, I)\Big) = p_n(\Psi(R_1), iI, I, \ldots, I) + p_n(R_1, \Psi(iI), I, \ldots, I)$$
$$= 2^{n-1}\Big(i\Psi(R_1) + \Psi(iI)R_1\Big). \tag{13}$$

Apparently, we can write

$$\Psi\Big(p_n\Big(R_1+iR_2, iI, I, \ldots, I\Big)\Big) = p_n\Big(\Psi(R_1+iR_2), iI, I, \ldots, I\Big) + p_n\Big(R_1+iR_2, \Psi(iI), I, \ldots, I\Big)$$
$$= 2^{n-2}i\Big(\Psi(R_1+iR_2) + \Psi(R_1+iR_2)^*\Big) + 2^{n-1}\Psi(iI)R_1. \tag{14}$$

From (13) and (14), we get

$$2^{n-1}\Big(i\Psi(R_1) + \Psi(iI)R_1\Big) = 2^{n-1}\Psi(iI)R_1 + 2^{n-2}i\Big(\Psi(R_1+iR_2) + \Psi(R_1+iR_2)^*\Big).$$

It follows that

$$2^{n-1}\Big(\Psi(R_1) - i\Psi(iI)R_1\Big) = -2^{n-1}i\Psi(iI)R_1 + 2^{n-2}\Big(\Psi(R_1+iR_2) + \Psi(R_1+iR_2)^*\Big). \tag{15}$$

On adding (12) and (15), we obtain

$$\Psi(R_1+iR_2) = \Psi(R_1) + i\Psi(R_2) + \Psi(iI)R_2.$$

(*ii*) Let $M \in \mathcal{M}$. Then $M = R_1 + iR_2$ for some $R_1, R_2 \in \mathcal{R}$. In view of Claims 3.3, 3.10, 3.11, 3.13 and 3.14 (*i*), we have

$$\begin{aligned}\Psi(M)^* &= \Psi(R_1+iR_2)^* = \Big(\Psi(R_1) + i\Psi(R_2) + \Psi(iI)R_2\Big)^* \\ &= \Psi(R_1) - i\Psi(R_2) - \Psi(iI)R_2 = \Psi(R_1 - iR_2) \\ &= \Psi(M^*).\end{aligned}$$

This gives the assertion.

Claim 3.15. *Ψ is additive on $\mathcal{M}$.*

Let $M, M' \in \mathcal{M}$ such that $M = R_1 + iR_2$ and $M' = R_1' + iR_2'$ for $R_1, R_2, R_1', R_2' \in \mathcal{R}$. Observe, from Claims 3.13 and 3.14 (*i*), that

$$\begin{aligned}\Psi(M+M') &= \Psi\Big((R_1+R_1') + i(R_2+R_2')\Big) \\ &= \Psi(R_1+R_1') + i\Psi(R_2+R_2') + \Psi(iI)(R_2+R_2') \\ &= \Psi(R_1) + i\Psi(R_2) + \Psi(iI)R_2 + \Psi(R_1') + i\Psi(R_2') + \Psi(iI)R_2' \\ &= \Psi(R_1+iR_2) + \Psi(R_1'+iR_2') \\ &= \Psi(M) + \Psi(M').\end{aligned}$$

Hence the result.

Claim 3.16. $\Psi(iI) = 0$.

Since $\Psi(I) = 0$, $\Psi(R)^* = \Psi(R)$, for all $R \in \mathcal{R}$, $\Psi(M^*) = \Psi(M)^*$ for all $M \in \mathcal{M}$ and $\Psi(R_1 + iR_2) = \Psi(R_1) + i\Psi(R_2) + \Psi(iI)R_2$, then let us assume that

$$\Psi(P_1) = R \tag{16}$$

for some $R \in \mathcal{R}$ and

$$\Psi(iP_1) = i\Psi(P_1) + \Psi(iI)P_1 = iR + \Psi(iI)P_1. \tag{17}$$

Therefore, we have

$$\begin{aligned} 2^{n-1}\Psi(iP_1) &= \Psi\Big(p_n\Big(iP_1, P_1, I, \dots, I\Big)\Big) \\ &= p_n\Big(\Psi(iP_1), P_1, I, \dots, I\Big) + p_n\Big(iP_1, \Psi(P_1), I, \dots, I\Big) \\ &= p_n\Big((iR + \Psi(iI)P_1), P_1, I, \dots, I\Big) + p_n\Big(iP_1, R, I, \dots, I\Big) \\ &= 2^{n-1}\Big(i(P_1R + RP_1) + \Psi(iI)P_1\Big). \end{aligned}$$

This implies that

$$\Psi(iP_1) = \Psi(iI)P_1 + i(P_1R + RP_1). \tag{18}$$

From (17) and (18), we get

$$R = P_1R + RP_1.$$

This gives

$$P_1RP_1 = P_2RP_2 = 0$$

and hence

$$\Psi(iP_1) = \Psi(iI)P_1 + i(P_1RP_2 + P_2RP_1). \tag{19}$$

Observe, for any $M_{12} \in \mathcal{M}_{12}$, that

$$\Psi\Big(p_n\Big(I, I, \dots, I, iP_1, (M_{12} - M_{12}^*)\Big)\Big) = -2^{n-2}\Psi\Big(i(M_{12} + M_{12}^*)\Big).$$

In view of Claims 3.12 and 3.14 (*ii*), we have

$$-2^{n-2}\Psi\Big(i(M_{12} + M_{12}^*)\Big) = -2^{n-2}\Big(i\Psi(M_{12}) + i\Psi(M_{12})^* + \Psi(iI)(M_{12} + M_{12}^*)\Big).$$

Thus

$$\Psi\Big(p_n\Big(I, I, \dots, I, iP_1, (M_{12} - M_{12}^*)\Big)\Big) = -2^{n-2}\Big(i\Psi(M_{12}) + i\Psi(M_{12})^* + \Psi(iI)(M_{12} + M_{12}^*)\Big). \tag{20}$$

Alternatively, from (19), Claims 3.9 and 3.15, we have

$$\begin{aligned} &\Psi\Big(p_n\Big(I, I, \dots, I, iP_1, (M_{12} - M_{12}^*)\Big)\Big) \\ &= p_n\Big(I, I, \dots, I, \Psi(iP_1), (M_{12} - M_{12}^*)\Big) + p_n\Big(I, I, \dots, I, iP_1, \Psi(M_{12} - M_{12}^*)\Big) \\ &= p_n\Big(I, I, \dots, I, (\Psi(iI)P_1 + iP_1RP_2 + iP_2RP_1), (M_{12} - M_{12}^*)\Big) \\ &+ p_n\Big(I, I, \dots, I, iP_1, (\Psi(M_{12}) - \Psi(M_{12}^*))\Big) \\ &= 2^{n-2}\bigg\{\Big(\Psi(iI)P_1 + i(P_1RP_2 + P_2RP_1)\Big)(M_{12}^* - M_{12}) + (M_{12} - M_{12}^*) \end{aligned}$$

$$\Big(\Psi(iI)P_1 + i(P_1RP_2 + P_2RP_1)\Big) + iP_1\Big(\Psi(M_{12})^* - \Psi(M_{12})\Big) + i\Big(\Psi(M_{12}) - \Psi(M_{12})^*\Big)P_1\Big\}.$$

This implies that

$$\begin{aligned}
&\Psi\Big(p_n\Big(I, I, \ldots, I, iP_1, (M_{12} - M_{12}^*)\Big)\Big) \\
&= 2^{n-2}\Big\{\Big(\Psi(iI)P_1 + i(P_1RP_2 + P_2RP_1)\Big)(M_{12}^* - M_{12}) + (M_{12} - M_{12}^*) \\
&\Big(\Psi(iI)P_1 + i(P_1RP_2 + P_2RP_1)\Big) + iP_1\Big(\Psi(M_{12})^* - \Psi(M_{12})\Big) + i\Big(\Psi(M_{12}) - \Psi(M_{12})^*\Big)P_1\Big\}. \qquad (21)
\end{aligned}$$

From (20) and (21), we get

$$\begin{aligned}
&-i\Psi(M_{12}) - i\Psi(M_{12})^* - \Psi(iI)(M_{12} + M_{12}^*) \\
&= \Big(\Psi(iI)P_1 + i(P_1RP_2 + P_2RP_1)\Big)(M_{12}^* - M_{12}) + (M_{12} - M_{12}^*)\Big(\Psi(iI)P_1 + i(P_1RP_2 + P_2RP_1)\Big) \\
&+iP_1\Big(\Psi(M_{12})^* - \Psi(M_{12})\Big) + i\Big(\Psi(M_{12}) - \Psi(M_{12})^*\Big)P_1. \qquad (22)
\end{aligned}$$

Multiplying (22) by P_1 from left and by P_2 from right, we get

$$P_1\Psi(M_{12})^*P_2 = 0.$$

Next, consider

$$\begin{aligned}
&2^{n-2}\Big(\Psi(M_{12}) - \Psi(M_{12})^*\Big) = \Psi\Big(p_n\Big(I, I, \ldots, I, iP_1, i(M_{12} + M_{12}^*)\Big)\Big) \\
&= p_n\Big(I, I, \ldots, I, \Psi(iP_1), i(M_{12} + M_{12}^*)\Big) + p_n\Big(I, I, \ldots, I, iP_1, \Psi(i(M_{12} + M_{12}^*))\Big) \\
&= p_n\Big(I, I, \ldots, I, (\Psi(iI)P_1 + iP_1RP_2 + iP_2RP_1), i(M_{12} + M_{12}^*)\Big) \\
&+p_n\Big(I, I, \ldots, I, iP_1, (i\Psi(M_{12}) + i\Psi(M_{12}^*) + \Psi(iI)(M_{12} + M_{12}^*))\Big) \\
&= -2^{n-2}\Big\{\Big(i\Psi(iI)P_1 - P_1RP_2 - P_2RP_1\Big)(M_{12} + M_{12}^*) - (M_{12} + M_{12}^*)\Big(i\Psi(iI)P_1 - P_1RP_2 - P_2RP_1\Big) \\
&-P_1\Big(\Psi(M_{12}) + \Psi(M_{12})^*\Big) + i\Psi(iI)M_{12} + \Big(\Psi(M_{12}) + \Psi(M_{12})^* - i\Psi(iI)(M_{12} + M_{12}^*)\Big)P_1\Big\}.
\end{aligned}$$

Multiplying above relation by P_1 from left and by P_2 from right, we obtain $\Psi(iI)M_{12} = 0$ and so by (3), we get $\Psi(iI)P_1 = 0$. Also, by Claim 3.11, we get $\Psi(iI)M_{12}^* = 0$ and thus, by (2), we obtain $\Psi(iI)P_2 = 0$. Hence, $\Psi(iI) = \Psi(iI)P_1 + \Psi(iI)P_2 = 0$. This completes the proof.

Claim 3.17. $\Psi(iM) = i\Psi(M)$ *for all* $M \in \mathcal{M}$.

In light of Claims 3.12 and 3.16, we get $\Psi(iR) = i\Psi(R)$ for all $R \in \mathcal{R}$. Therefore, for any $M \in \mathcal{M}$, assume that $M = R_1 + iR_2$ for some $R_1, R_2 \in \mathcal{R}$. In view of Claim 3.15, we have

$$\Psi(iM) = \Psi(i(R_1 + iR_2)) = i\Big(\Psi(R_1) + i\Psi(R_2)\Big) = i\Psi(M).$$

Hence the result.

Proof of Theorem 3.1: We have shown that Ψ is additive on $\mathcal{M}$ (Claim 3.15) with $\Psi(M^*) = \Psi(M)^*$ for all $M \in \mathcal{M}$ (Claim 3.14 (*ii*)). The final task is to prove that Ψ satisfies the Leibniz rule on $\mathcal{M}$. Now, let $R_1, R_2 \in \mathcal{R}$. Then

$$2^{n-2}\Psi(R_1R_2 - R_2R_1) \quad = \quad \Psi\Big(p_n\Big(I, I, \ldots, I, R_1, R_2\Big)\Big)$$

$$
\begin{aligned}
&= p_n\Big(I, I, \ldots, I, \Psi(R_1), R_2\Big) + p_n\Big(I, I, \ldots, I, R_1, \Psi(R_2)\Big) \\
&= 2^{n-2}\Big(\Psi(R_1)R_2 - R_2\Psi(R_1) + R_1\Psi(R_2) - \Psi(R_2)R_1\Big). \qquad (23)
\end{aligned}
$$

Also

$$
\begin{aligned}
2^{n-2}i\Psi(R_1R_2 + R_2R_1) &= \Psi\Big(p_n\Big(I, I, \ldots, I, iR_1, R_2\Big)\Big) \\
&= p_n\Big(I, I, \ldots, I, \Psi(iR_1), R_2\Big) + p_n\Big(I, I, \ldots, I, iR_1, \Psi(R_2)\Big) \\
&= 2^{n-2}i\Big(\Psi(R_1)R_2 + R_2\Psi(R_1) + R_1\Psi(R_2) + \Psi(R_2)R_1\Big). \qquad (24)
\end{aligned}
$$

Addition of (23) and (24) gives $\Psi(R_1R_2) = \Psi(R_1)R_2 + R_1\Psi(R_2)$ for all $R_1, R_2 \in \mathcal{R}$. Further, for any $M, M' \in \mathcal{M}$ assume that $M = R_1 + iR_2$ and $M' = R_1' + iR_2'$ for some $R_1, R_2, R_1', R_2' \in \mathcal{R}$. Then

$$
\begin{aligned}
\Psi(MM') &= \Psi((R_1 + iR_2)(R_1' + iR_2')) = \Psi(R_1R_1' + iR_1R_2' + iR_2R_1' - R_2R_2') \\
&= \Psi(R_1)R_1' + R_1\Psi(R_1') + i\Psi(R_1)R_2' + iR_1\Psi(R_2') + i\Psi(R_2)R_1' + iR_2\Psi(R_1') \\
&- \Psi(R_2)R_2' - R_2\Psi(R_2'). \qquad (25)
\end{aligned}
$$

On the other hand

$$
\begin{aligned}
\Psi(M)M' + M\Psi(M') &= \Psi(R_1 + iR_2)(R_1' + iR_2') + (R_1 + iR_2)\Psi(R_1' + iR_2') \\
&= (\Psi(R_1) + i\Psi(R_2))(R_1' + iR_2') + (R_1 + iR_2)(\Psi(R_1') + i\Psi(R_2')) \\
&= \Psi(R_1)R_1' + R_1\Psi(R_1') + i\Psi(R_1)R_2' + iR_1\Psi(R_2') + i\Psi(R_2)R_1' + iR_2\Psi(R_1') \\
&- \Psi(R_2)R_2' - R_2\Psi(R_2'). \qquad (26)
\end{aligned}
$$

From (25) and (26), we conclude that Ψ satisfies the Leibniz rule on $\mathcal{M}$, i.e., $\Psi(MM') = \Psi(M)M' + M\psi(M')$ holds for all $M, M' \in \mathcal{M}$. Therefore, the proof of the main theorem is completed.

4. Corollaries

The following corollaries are immediate from our main result.
Let $\mathcal{M}$ be a $*$-algebra. An algebra $\mathcal{M}$ is called prime if for any two non-zero ideals $I, J \subseteq \mathcal{M}$, $IJ \neq (0)$. Alternatively, an algebra $\mathcal{M}$ is said to be prime if for any $X, Y \in \mathcal{M}$, $X\mathcal{M}Y = (0)$ implies that either $X = 0$ or $Y = 0$. Given that prime $*$-algebras satisfy conditions (2) and (3), the subsequent corollary can be deduced.

Corollary 4.1. *Consider a unital prime $*$-algebra $\mathcal{M}$ containing a nontrivial projection P. A mapping Ψ is a non-linear mixed Jordan bi-skew Lie-type derivation on $\mathcal{M}$ if and only if Ψ is an additive $*$-derivation on $\mathcal{M}$.*

A von Neumann algebra $\mathcal{M}$ is defined as a weakly closed self-adjoint algebra of operators on a complex Hilbert space H that includes the identity operator I. The algebra $\mathcal{M}$ is classified as a factor if its centre is trivial. Given that a factor von Neumann algebra is a prime $*$-algebra, the subsequent corollary follows.

Corollary 4.2. *For a factor von Neumann algebra $\mathcal{M}$ with $dim(\mathcal{M}) \geq 2$, a mapping $\Psi : \mathcal{M} \to \mathcal{M}$ is a non-linear mixed Jordan bi-skew Lie-type derivation if and only if Ψ is an additive $*$-derivation.*

It follows from [2] and [11] that every von Neumann algebra having no central summands of type I_1 satisfies (2) and (3). Therefore, we have the following corollary:

Corollary 4.3. *Let $\mathcal{M}$ be a von Neumann algebra with no central summands of type I_1. A mapping $\Psi : \mathcal{M} \to \mathcal{M}$ is a non-linear mixed Jordan bi-skew Lie-type derivation if and only if Ψ is an additive $*$-derivation.*

Consider the algebra of all bounded linear operators on a complex Hilbert space H, denoted as $\mathcal{B}(H)$. A subalgebra $\mathcal{M}$ of $\mathcal{B}(H)$ is termed a standard operator algebra if it contains the subalgebra $\mathcal{F}(H)$, comprising all finite-rank operators on H. As a standard operator algebra is inherently a prime $*$-algebra, the following corollary is derived.

Corollary 4.4. *For an infinite-dimensional complex Hilbert space H and a standard operator algebra $\mathcal{M}$ on H containing the identity operator I, closed under the adjoint operation, a mapping $\Psi : \mathcal{M} \to \mathcal{M}$ is a non-linear mixed Jordan bi-skew Lie-type derivation if and only if Ψ is an additive $*$-derivation. Additionally, there exists an operator $X \in \mathcal{B}(H)$ such that $X + X^* = 0$, and $\Psi(M) = MX - XM$ for all $M \in \mathcal{M}$, indicating that Ψ is inner.*

Proof. As Ψ is an additive $*$-derivation on standard operator algebra $\mathcal{M}$, so from [20] we deduce that Ψ is inner, i.e., there exists $Y \in \mathcal{B}(H)$ such that $\Psi(M) = MY - YM$ for all $M \in \mathcal{M}$. Since $\Psi(M^*) = \Psi(M)^*$ for all $M \in \mathcal{M}$, then we have

$$M^*Y - YM^* = \Psi(M^*) = \Psi(M)^* = Y^*M^* - M^*Y^*$$

for all $M \in \mathcal{M}$. This implies that $M^*(Y + Y^*) = (Y + Y^*)M^*$. Thus, $Y + Y^* = \alpha I$ for some $\alpha \in \mathbb{R}$. Let us set $X = Y - \frac{1}{2}\alpha I$. One can check that $X + X^* = 0$ such that $\Psi(M) = MX - XM$. □

Acknowledgement : The second author was supported by CSIR-UGC Junior Research Fellowship, India, Human Resource Development Group (Ref. No. 09/0112(13455)/2022-EMR-I).
The authors extend their sincere gratitude to the reviewer(s) for their insightful and constructive comments, which has substantially enhanced the quality of this manuscript.

Authors' contribution : Each author has made an equal contribution.

Conflict of interest : The authors of the manuscript declare that they do not have any competing interests.

References

[1] Z. F. Bai, S. P. Du, *The structure of nonlinear Lie derivation on von Neumann algebras*, Linear Algebra Appl. **436** (7) (2012), 2701–2708. https://doi.org/10.1016/j.laa.2011.11.009
[2] L. Dai, F. Lu, *Nonlinear maps preserving Jordan $*$-products*, J. Math. Anal. Appl. **409** (1) (2014), 180–188. https://doi.org/10.1016/j.jmaa.2013.07.019
[3] B. L. M. Ferreira, B. T. Costa, *$*$-Jordan-type maps on C^*-algebras*, Comm. Algebra **49** (12) (2021), 5073–5082. DOI: 10.1080/00927872.2021.1937636
[4] B. L. M. Ferreira, F. Wei, *Mixed $*$-Jordan-type derivations on $*$-algebras*, J. Algebra Appl. **22** (5) (2023), 2350100 https://doi.org/10.1142/S0219498823501001
[5] A. Fošner, F. Wei, Z. Xiao, *Nonlinear Lie-type derivations of von Neumann algebras and related topics*, Colloq. Math. **132** (1) (2013), 53–71.
[6] P. Ji, R. Liu, Y. Zhao, *Nonlinear Lie triple derivations of triangular algebras*, Linear Multilinear Algebra **60** (10) (2012), 1155–1164. https://doi.org/10.1080/03081087.2011.652109
[7] W. Jing, *Nonlinear $*$-Lie derivations of standard operator algebras*, Quaest. Math. **39** (8) (2016), 1037–1046. https://doi.org/10.2989/16073606.2016.1247119
[8] A. N. Khan, *Multiplicative biskew Lie triple derivations on factor von Neumann algebras*, Rocky Mountain J. Math. **51** (6) (2021), 2103–2114. DOI: 10.1216/rmj.2021.51.2103
[9] L. Kong, C. Li, *Nonlinear mixed Lie triple derivations on prime $*$-rings*, ScienceAsia **48** (3) (2022), 335–339. doi: 10.2306/scienceasia1513-1874.2022.034
[10] L. Kong, J. Zhang, *Nonlinear bi-skew Lie derivations on factor von Neumann algebras*, Bull. Iranian Math. Soc. **47** (2021), 1097–1106. https://doi.org/10.1007/s41980-020-00430-5
[11] C. Li, F. Lu, X. Fang, *Non-linear ξ-Jordan $*$-derivations on von Neumann algebras*, Linear Multilinear Algebra **62** (4) (2013), 466-–473. https://doi.org/10.1080/03081087.2013.780603
[12] C. Li, D. Zhang, *Nonlinear mixed Jordan triple $*$-derivations on factor von Neumann algebras*, Filomat **36** (8) (2022), 2637–2644. https://doi.org/10.2298/FIL2208637L
[13] C. Li, D. Zhang, *Nonlinear Mixed Jordan Triple $*$-Derivations on $*$-Algebras*, Sib. Math. J. **63** (2022), 735––742. https://doi.org/10.1134/S0037446622040140
[14] C. Li, Y. Zhao, F. Zhao, *Nonlinear $*$-Jordan-type derivations on $*$-algebras*, Rocky Mountain J. Math. **51** (2) (2021), 601–612. DOI: 10.1216/rmj.2021.51.601
[15] Y. Liang, J. Zhang, *Nonlinear mixed Lie triple derivations on factor von Neumann algebras*, Acta Math. Sin. Chinese Ser. **62** (2019), 1–13.
[16] W. H. Lin, *Nonlinear $*$-Lie-type derivations on von Neumann algebras*, Acta Math. Hungar. **156** (2018), 112–131. https://doi.org/10.1007/s10474-018-0803-1
[17] W. S. Martindale, *Lie isomorphisms of operator algebras*, Pacific J. Math. **38** (1971), 717–735.
[18] C. R. Miers, *Lie triple derivations of von Neumann algebras*, Proc. Amer. Math. Soc. **71** (1) (1978), 57–61. https://doi.org/10.1090/S0002-9939-1978-0487480-9
[19] T. C. Pierin, R. N. Ferreira, F. Borges, B. L. M. Ferreira, *Nonlinear mixed $*$-Jordan type derivations on alternative $*$-algebras*, J. Algebra Appl. **24** (2) (2025), 2550028. https://doi.org/10.1142/S0219498825500288

[20] P. Šemrl, *Additive derivations of some operator algebras*, Illinois J. Math. **35** (2) (1991), 234–240. DOI: 10.1215/ijm/1255987893
[21] A. Taghavi, H. Rohi, V. Darvish, *Non-linear ∗-Jordan derivations on von Neumann algebras*, Linear Multilinear Algebra, **64** (3) (2016), 426–439. https://doi.org/10.1080/03081087.2015.1043855
[22] Z. Yang, J. Zhang, *Nonlinear maps preserving mixed Lie triple products on factor von Neumann algebras*, Ann. Funct. Anal. **10** (3) (2019), 325–336. DOI: 10.1215/20088752-2018-0032
[23] W. Yu, J. Zhang, *Nonlinear Lie derivations of triangular algebras*, Linear Algebra Appl. **432** (11) (2010), 2953–2960. https://doi.org/10.1016/j.laa.2009.12.042
[24] W. Yu, J. Zhang, *Nonlinear ∗-Lie derivations on factor von Neumann algebras*, Linear Algebra Appl. **437** (8) (2012), 1979–1991. http://dx.doi.org/10.1016/j.laa.2012.05.032
[25] X. Zhao, X. Fang, *The Second Nonlinear Mixed Lie Triple Derivations on Finite Von Neumann Algebras*, Bull. Iranian Math. Soc. **47** (1) (2021), 237–254. https://doi.org/10.1007/s41980-020-00380-y
[26] Y. Zhou, Z. Yang, J. Zhang, *Nonlinear mixed Lie triple derivations on prime ∗-algebras*, Comm. Algebra **47** (11) (2019), 4791–4796. https://doi.org/10.1080/00927872.2019.1596277