Filomat 39:16 (2025), 5617–5628
https://doi.org/10.2298/FIL2516617A

Published by Faculty of Sciences and Mathematics, University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

A note on nearly Alster spaces and its interrelations with some covering properties

Necati Can Açikgöz[a,*], Ceren Sultan Elmali[a]

[a] *Department of Mathematics, Erzurum Technical University, Erzurum, Turkey*

Abstract. We introduce a new form of Alster property and investigate the relationships between other weaker forms of Alster spaces. We also scrutinize its relations with some selective covering properties. By giving counter-examples, we show the differences between nearly Alster property and those properties. We also consider the productively nearly Lindelöf spaces. We present some topological properties of nearly Alster spaces and characterize nearly Alster property in terms of some selection principles.

1. Introduction

As having an important role in mathematics, Lindelöfness of a topological space is an interest of many mathematicians. A topological space (X, τ) is called a Lindelöf space if every open cover of X admits a countable subfamily which covers X. However, with the fact that the product of two Lindelöf spaces need not be Lindelöf, many mathematicians tried to characterize productively Lindelöf spaces. A topological space is said to be productively Lindelöf if its product with every Lindelöf space is Lindelöf.

In 1988, K. Alster defined a property known as $(*)$ property (also the Alster property) for characterizing productively Lindelöf topological spaces. He proved that the spaces having Alster property are productively Lindelöf and moreover, assuming Continuum Hypothesis, he proved that every productively Lindelöf topological space of weight at most $\aleph_1$ is Alster [2].

On the other hand, weaker forms of the Lindelöf property such as nearly Lindelöf, almost Lindelöf and weakly Lindelöf properties were introduced, see [6, 14, 36]. The productivity of some of weaker forms of the Lindelöf property were investigated. In [5], authors considered weakly Lindelöf property for characterizing the productivity of such spaces and obtained some results. They also considered weaker versions of the Alster property and investigated its relations with some other weaker forms of the Menger-type covering properties. They characterized Alster property and its weaker form known as the weakly Alster property in terms of selection principles. Kocev in [18] defined a new form of Alster-type covering property called

2020 *Mathematics Subject Classification*. Primary 54D20; Secondary 54G10, 54D10

Keywords. Alster spaces, nearly Alster spaces, Menger spaces, nearly Menger spaces, almost Menger spaces, weakly Alster, almost Alster, nearly Lindelöf, Lindelöf, selection principles

Received: 06 July 2024; Revised: 25 March 2025; Accepted: 14 April 2025

Communicated by Ljubiša D. R. Kočinac

* Corresponding author: Necati Can Açikgöz

Email addresses: necatican.acikgoz@erzurum.edu.tr (Necati Can Açikgöz), ceren.elmali@erzurum.edu.tr (Ceren Sultan Elmali)

ORCID iDs: https://orcid.org/0000-0002-4299-3472 (Necati Can Açikgöz), https://orcid.org/0000-0002-2553-4733 (Ceren Sultan Elmali)

almost Alster and gave some results related to producitivity of almost Alster and almost Menger spaces. (See also [17]).

In this study, we introduce a novel form of the Alster property called nearly Alster. In Section 2, we recall some definitions, notations and concepts which are related to study. In Section 3, we introduce nearly Alster space and investigate the relationships between the nearly Alster and Menger-type covering properties. By giving some counterexamples, we show the differences between nearly Alster spaces and the corresponding covering properties. On the other hand, we investigate under what conditions those mentioned properties are equivalent. We give an example for a problem posed in [18]. We also consider the productively nearly Lindelöf spaces. In Section 4, we give some topological properties of nearly Alster spaces and we give a couple of characterization of it in terms of selection principles. In conclusion section, we pose new types of the Alster covering property.

2. Preliminaries

Throughout the paper, (X,τ) (Sometimes X) will denote a topological space on which no separation axiom will be assumed unless explicitly stated. $Int(A)$ and $\overline{A}$ will denote the interior and closure of a subset A of X. Our terminology will follow [12].

Definition 2.1. A topological space (X,τ) is said to be *nearly Lindelöf* [6] (resp., *almost Lindelöf* [36], *weakly Lindelöf* [14]) if every open cover $\mathcal{O}$ of X admits a countable subfamily $\mathcal{U} \subset \mathcal{O}$ such that $X = \bigcup_{U\in\mathcal{U}} Int(\overline{U})$ $\left(\text{resp., } X = \bigcup_{U\in\mathcal{U}} \overline{U}, X = \overline{\bigcup_{U\in\mathcal{U}} U}\right)$.

Definition 2.2. A topological space (X,τ) is called *Menger* [26] (resp., *nearly Menger* [23], *almost Menger* [19], *weakly Menger* [28]) if each sequence of open covers $(\mathcal{O}_n)_{n\in\mathbb{N}}$ of X admits a sequence $(\mathcal{V}_n)_{n\in\mathbb{N}}$ of finite families, where $\mathcal{V}_n \subset \mathcal{O}_n$ for every $n \in \mathbb{N}$, and $X = \bigcup_{n\in\mathbb{N}} \bigcup \mathcal{V}_n$ $\left(\text{resp., } X = \bigcup_{n\in\mathbb{N}} \bigcup_{V\in\mathcal{V}_n} Int(\overline{V}), X = \bigcup_{n\in\mathbb{N}} \bigcup_{V\in\mathcal{V}_n} \overline{V}, X = \overline{\bigcup_{n\in\mathbb{N}} \bigcup_{V\in\mathcal{V}_n} V}\right)$.

Above-mentioned Menger-type covering properties and many topological concepts can be defined and characterized by selection principles. Let X be an infinite set, $\mathcal{A}$ and $\mathcal{B}$ be the sets that consist of families of subsets of X.

$S_{fin}(\mathcal{A},\mathcal{B})$ denotes the selection principle: For every sequence $(A_n)_{n\in\mathbb{N}}$, where $A_n \in \mathcal{A}$ for each n, there exists a sequence $(B_n)_{n\in\mathbb{N}}$ of finite sets such that for every $n \in \mathbb{N}$, $B_n \subset A_n$, and $\bigcup_{n\in\mathbb{N}} B_n \in \mathcal{B}$.

$S_1(\mathcal{A},\mathcal{B})$ denotes the selection principle: For every sequence $(A_n)_{n\in\mathbb{N}}$, where $A_n \in \mathcal{A}$ for each n, there exists a sequence $(b_n)_{n\in\mathbb{N}}$ such that for every $n \in \mathbb{N}$, $b_n \in A_n$, and $\{b_n : n \in \mathbb{N}\} \in \mathcal{B}$, see [33].

Let $\mathcal{O}$ be the family of all open covers of a topological space (X,τ). The selection principle $S_{fin}(\mathcal{O},\mathcal{O})$ denotes the Menger property. Let $\overline{\mathcal{O}}$ and $\mathcal{D}$ be the collections of families $\mathcal{U}$ of open sets such that $X = \bigcup_{U\in\mathcal{U}} \overline{U}$ and $X = \overline{\bigcup_{U\in\mathcal{U}} U}$, respectively. Then the selection principles $S_{fin}(\mathcal{O},\overline{\mathcal{O}})$ and $S_{fin}(\mathcal{O},\mathcal{D})$ denotes the almost Menger property and weakly Menger property, respectively. We can say that $S_{fin}(\mathcal{O},\mathring{\overline{\mathcal{O}}})$ denotes the nearly Menger property, where $\mathring{\overline{\mathcal{O}}}$ is the collection of families $\mathcal{U}$ of open subsets of X such that $\{Int(\overline{U}) : U \in \mathcal{U}\}$ covers X.

For more extensive and detailed information about the selection principles and covering properties, we refer reader to [16, 20, 21, 24, 34].

In [7], authors introduced *amply Lindelöf spaces* that is in fact the definition of Alster spaces. In [5], authors used the notion of G_δ compact cover for defining Alster spaces. A family $\mathcal{F}$ of G_δ subsets of a topological space X is called G_δ *compact cover* if there exists a $F \in \mathcal{F}$ for each compact $K \subset X$ such that $K \subset F$. A topological space is said to be an *Alster space* if for every G_δ compact cover of the space has a countable subfamily which covers the space. They stated that the definition above is not identical to the definition in [7], but it is equivalent to it and to (∗) property defined by Alster in [2]. For Alster spaces, we will use the definition given below;

Definition 2.3. ([3]) Let (X, τ) be a topological space. A cover $\mathcal{A}$ by G_δ subsets of X is said to be an *Alster cover* if each compact subset of X is included in some element of $\mathcal{A}$. (X, τ) is called an *Alster Space* if every Alster cover of X admits a countable subcover.

Definition 2.4. ([5]) A topological space (X, τ) is called *weakly Alster* if for every Alster cover $\mathcal{A}$ such that $X \notin \mathcal{A}$ of X, there exists a countable subfamily $\mathcal{U}$ of $\mathcal{A}$ such that $\bigcup \mathcal{U}$ dense in X, i.e. $X = \overline{\bigcup_{U \in \mathcal{U}} U}$.

Definition 2.5. ([18]) A topological space is called *almost Alster* if every Alster cover $\mathcal{A}$ of X has a countable subfamily $\mathcal{U}$ such that $X = \bigcup_{U \in \mathcal{U}} \overline{U}$.

For more informations about the Alster spaces, we refer reader to [2–5, 7, 8, 18, 27].

3. Nearly Alster spaces

In this section, we will define a form of Alster space called nearly Alster space and explore relation with some other known covering properties. We will also consider the nearly productively Lindelöf spaces.

Definition 3.1. A topological space (X, τ) is called *nearly Alster* if for all Alster cover $\mathcal{O}$ of X, there exists a countable subfamily $\mathcal{U}$ of $\mathcal{O}$ such that $X = \bigcup\{Int(\overline{U}) : U \in \mathcal{U}\}$

We note that a topological space (X, τ) is *nearly σ-compact* if it can be represented as $X = \bigcup_{n \in \mathbb{N}} Int(\overline{C_n})$, where each C_n is a compact subset of X.

Proposition 3.2. *Nearly σ-compact spaces are nearly Alster.*

Proof. Let (X, τ) be a nearly σ-compact topological space and $\mathcal{A}$ be any Alster cover of X. Since X is nearly σ-compact, then there is a sequence $(C_n)_{n \in \mathbb{N}}$ of compact subsets of X such that

$$X = \bigcup_{n \in \mathbb{N}} Int(\overline{C_n}).$$

For each $n \in \mathbb{N}$, there exists a $U_n \in \mathcal{A}$ such that $C_n \subset U_n$, since $\mathcal{A}$ is an Alster cover of X. Thus

$$X = \bigcup_{n \in \mathbb{N}} Int(\overline{C_n}) \subset \bigcup_{n \in \mathbb{N}} Int(\overline{U_n})$$

is obtained. Hence (X, τ) is nearly Alster. □

In general, the inverse of above proposition does not hold, see Example 3.12. We have the following theorem for the equivalency of the nearly Alster and nearly σ-compact properties.

Theorem 3.3. *Nearly Alster and nearly σ-compact properties are equivalent in metrizable spaces.*

Proof. Let (X, τ) be a metrizable Alster space and $\mathcal{K}$ be the family of all compact subsets of X. Since every metrizable space is Hausdorff, so is (X, τ). Thus each member of $\mathcal{K}$ is closed. Moreover, since every closed subset of metrizable space is a G_δ set, $\mathcal{K}$ is an Alster cover of X. Then there exists a countable subfamily $\mathcal{C}$ of $\mathcal{K}$ such that $X = \bigcup_{C \in \mathcal{C}} Int(\overline{C})$, since (X, τ) is nearly Alster. Thus the family $\mathcal{C}$ witnesses for (X, τ) is nearly σ-compact. □

Kočinac in [23] gave the definition of nearly Menger spaces. Later on Parvez and Khan in [29] defined the nearly Menger property in a different notion and observed that both definitions of nearly Menger property are coincide.

Proposition 3.4. *Every nearly Alster space is nearly Menger.*

Proof. Let $(O_n)_{n\in\mathbb{N}}$ be any sequence of open covers of X. For each $n \in \mathbb{N}$, we may assume that $\mathcal{O}_n$ is closed under finite union. Let

$$\mathcal{U} = \{ \bigcap_{n\in\mathbb{N}} O_n : (\forall n)(O_n \in \mathcal{O}_n)\}.$$

Clearly every member of $\mathcal{U}$ is a G_δ subset of X. Let C be a compact subset of X. Then there exists a $U_C \in \mathcal{U}$ such that $C \subset U_C$. Hence $\mathcal{U}$ is an Alster cover of X.As (X, τ) is nearly Alster, there exists a countable subfamily$\mathcal{V} = \{V_n : n \in \mathbb{N}\}$ of $\mathcal{U}$ such that

$$X = \bigcup_{n\in\mathbb{N}} Int(\overline{V_n}).$$

For each $n \in \mathbb{N}$, let $V_n = \bigcap_{m\in\mathbb{N}} O^n_m$, where $O^n_m \in \mathcal{O}_m$ for each $m \in \mathbb{N}$. Then $V_n \subset O^n_n$ and $O^n_n \in \mathcal{O}_n$ for all $n \in \mathbb{N}$. So $X = \bigcup_{n\in\mathbb{N}} Int(\overline{O^n_n})$, hence (X, τ) is nearly Menger. □

The following example illustrates that a nearly Menger space need not be nearly Alster.

Example 3.5. Let X be the set of all real numbers. Consider X with the countable complement extension topology τ, that is, the smallest topology generated by $\tau_{coc} \cup \sigma$, where τ_{coc} is the countable complement topology and σ is the usual topology. A subset G of X is in τ if and only if $G = U \setminus C$, where $U \in \sigma$ and C is a countable subset of X. (See [37]). Since (X, τ) is Menger (See [22]), it is, therefore, nearly Menger. We show that (X, τ) fails to be nearly Alster. Let $\mathcal{K}$ be the family of all compact subsets of X and $K \in \mathcal{K}$. Since K is finite, it is compact and thus it is closed with respect to σ. On the other hand, since (X, σ) is metrizable and $\sigma \subset \tau$, K is a G_δ subset of X with respect to τ. Then $\mathcal{K}$ is an Alster cover of X. But there is no subfamily of $\mathcal{K}$ whose interiors of closures of its members covers X.

Clearly every nearly Menger space is nearly Lindelöf. Thus by Proposition 3.4, every nearly Alster space is nearly Lindelöf. But every nearly Lindelöf space need not be nearly Alster as:

Example 3.6. Let X be the set of all real numbers and $\mathcal{B}$ be a base for usual topology. Then $\mathcal{R} = \mathcal{B} \cup \{\{q\} : q \in \mathbb{Q}\}$ is a base for a topology τ on X called the discrete rational extension of the usual topology.([37]). (X, τ) is a metrizable non-nearly σ-compact Lindelöf space and thus it is not nearly Alster.

Also, (X, τ) in Example 3.5 is a non-regular Lindelöf space and thus it is nearly Lindelöf. But it is not nearly Alster.

Recall that a topological space is called a P-space if the intersection of every countably many open sets is open.

Theorem 3.7. *If a topological space (X, τ) is nearly Lindelöf and a P-space, then it is nearly Alster.*

Proof. If $\mathcal{A}$ is an Alster cover of X, then each member U of $\mathcal{A}$ is a G_δ subset of X. Since (X, τ) is P-space, then each $U \in \mathcal{A}$ is open in X and thus $\mathcal{A}$ is an open cover of X. It follows that (X, τ) is nearly Lindelöf. □

Corollary 3.8. *Let (X, τ) be a P-space, then the following statements are equivalent:*

1. (X, τ) *is nearly Alster,*
2. (X, τ) *is nearly Menger,*
3. (X, τ) *is nearly Lindelöf.*

With a similar argument in Theorem 3.7., the following theorem can easily be obtained.

Theorem 3.9. *Almost Lindelöf P-spaces are almost Alster.*

Considering the corresponding definitions, every nearly Alster space is almost Alster and every almost Alster space is weakly Alster. The following examples show that the reverse implications are not true in general.

Example 3.10. A Tychonoff weakly Alster space which fails to be almost Alster.

Let D be a discrete space of cardinality ω_1 and let

$$X = (\beta D \times (\omega + 1)) \setminus ((\beta D \setminus D) \times \{\omega\})$$

be the subspace of the product space $\beta D \times (\omega + 1)$, where βD is Čech-Stone compactification of D.

It is clear that $\beta D \times \{n\}$ is a compact subset of X for each $n \in \omega$. So $\beta D \times \omega$ is a σ-compact subset of X. Thus $\beta D \times \omega$ is weakly Alster. Moreover, since $\beta D \times \omega$ is a dense subset of X, hence X is weakly Alster by Lemma 43 in [5]. On the other hand, X is not almost Alster. In [35], Song showed that X is not almost Lindelöf. Since every almost Alster space is almost Lindelöf, X cannot be an almost Alster space.

Example 3.11. An almost Alster space which is not nearly Alster.

Let $A = \{a, b\}$, $B = \{c_i : i < \omega_1\}$, and $C = \{a_{ij} : i < \omega_1, j \in \mathbb{N}\} \cup \{b_{ij} : i < \omega_1, j \in \mathbb{N}\}$. Consider $X = A \cup B \cup C$ with the topology such that the points $\{a_{ij}\}$ and $\{b_{ij}\}$ are isolated and $U^n_{c_i} = \{c_i\} \cup \{a_{ij} : j \geq n\} \cup \{b_{ij} : j \geq n\}$, $U^\alpha_a = \{a\} \cup \{a_{ij} : i \geq \alpha, j \in \mathbb{N}\}$, $U^\alpha_b = \{b\} \cup \{b_{ij} : i \geq \alpha, j \in \mathbb{N}\}$ are the fundamental system of neighborhoods of the points $\{c_i\}$, a, and b, respectively. Since X is not nearly Lindelöf (see [9]), it is not nearly Alster. On the other hand, X is almost Lindelöf P-space (See also [29]) and hence it is almost Alster by Theorem 3.9.

Example 3.12. A nearly Alster space which is not nearly σ-compact

Let X be an uncountable set and $p \in X$ be fixed. Consider X with the topology $\tau = \{U \subset X : p \notin U$ or if $p \in U$ then $X \setminus U$ is countable $\}$ [37]. Then (X, τ) is a nearly Menger space. Indeed, since (X, τ) is Lindelöf, it is nearly Lindelöf and it can easily be seen that the space is a P-space, and thus by Corollary 3.8, it is nearly Alster. But it fails to be nearly σ-compact. Every compact subset C of the space is finite and since (X, τ) is Hausdorff, C is closed. So X cannot be written as the union of interiors of closures of countably many compact subsets.

Since X in Example 3.11 is not Lindelöf, X is an almost Alster space which is not Alster. Thus it gives a positive answer for the Problem 3.8. posed by Kocev in [18].

Recall [30] that a topological space is extremally disconnected if the closure of every open set is open.

Theorem 3.13. *If (X, τ) is an extremally disconnected P-space, then the following statements are equivalent:*

1. (X, τ) *is nearly Alster,*
2. (X, τ) *is almost Alster,*
3. (X, τ) *is weakly Alster.*

Proof. For (3) $\Rightarrow$ (1) Let $\mathcal{A}$ be any Alster cover of X. Since (X, τ) is weakly Alster, there exists a countable subfamily $\mathcal{B} \subset \mathcal{A}$ such that $\cup\mathcal{B}$ dense in X, i.e. $X = \overline{\bigcup_{U \in \mathcal{B}} U}$. Since (X, τ) is a P-space, $\bigcup_{U \in \mathcal{B}} \overline{U}$ is a closed subset of X as the union of countably many closed sets. Since $\overline{\bigcup_{U \in \mathcal{B}} U}$ is the smallest closed subset containing $\bigcup_{U \in \mathcal{B}} U$, then $X = \overline{\bigcup_{U \in \mathcal{B}} U} \subset \bigcup_{U \in \mathcal{B}} \overline{U}$. On the other hand, each member of $\mathcal{A}$ is a G_δ subset of X and as (X, τ) is a P-space, U is open for every $U \in \mathcal{B}$. Since (X, τ) is extremally disconnected, then $\overline{U} = Int(\overline{U})$ for every $U \in \mathcal{B}$ which completes the proof. $\square$

In [5], authors showed that every weakly Lindelöf P-space is weakly Alster. By considering all above-mentioned equivalences, we have the following:

Corollary 3.14. *Let (X, τ) be an extremally disconnected regular P-space. Then the following statements are equivalent:*

1. (X, τ) *is Alster,*
2. (X, τ) *is nearly Alster,*
3. (X, τ) *is almost Alster,*
4. (X, τ) *is weakly Alster,*
5. (X, τ) *is Menger,*
6. (X, τ) *is nearly Menger,*
7. (X, τ) *is almost Menger,*
8. (X, τ) *is weakly Menger,*
9. (X, τ) *is Lindelöf,*
10. (X, τ) *is nearly Lindelöf,*
11. (X, τ) *is almost Lindelöf,*
12. (X, τ) *is weakly Lindelöf.*

We end this section by dealing with the productivity of nearly Lindelöf spaces. As known, the product of two nearly Lindelöf space need not be nearly Lindelöf as the following example shows:

Example 3.15. Let X be the set of all real numbers and τ be the Sorgenfrey topology. (X, τ) is a regular Lindelöf space, hence it is nearly Lindelöf. But (X^2, τ^2) is a regular space which fails to be Lindelöf. Therefore, it is not nearly Lindelöf. (See [9, 37]).

The following theorem shows that a relation with the nearly Alster property and productively nearly Lindelöf spaces.

Theorem 3.16. *If (X, τ) is nearly Alster and (Y, σ) is nearly Lindelöf, then $X \times Y$ is nearly Lindelöf.*

Proof. Let $\mathcal{O}$ be an open cover of the product space $X \times Y$. Without loss of generality, we may assume that $\mathcal{O}$ is closed under finite union. Let C be any compact subset of X. Since $C \times \{y\}$ is a compact subset of $X \times Y$ for every $y \in Y$, we can find an $O_C^y \in \mathcal{O}$ such that $C \times \{y\} \subset O_C^y$. For every $y \in Y$, there exist open sets $U_C^y \subset X$ and $V_C^y \subset Y$ such that $C \subset U_C^y$, $y \in V_C^y$, and $C \times \{y\} \subset U_C^y \times V_C^y \subset O_C^y$ by 3.2.10 in [12]. Then we obtain an open cover $\mathcal{U} = \{V_C^y : y \in Y\}$ of Y and thus by the nearly Lindelófness of Y, there exists a countable subset $Y_C \subset Y$ such that

$$Y = \bigcup\{Int(\overline{V_C^y}) : y \in Y_C\}.$$

Now, let $U_C = \bigcap\{U_C^y : y \in Y_C\}$. Clearly U_C is a G_δ subset of X containing C. Hence

$$\mathcal{A} = \{U_C : C \subset X \text{ is compact }\}$$

is an Alster cover of X. Then there is a countable subfamily $\{C_n : n \in \mathbb{N}\}$ of compact subsets of X such that

$X = \bigcup\{Int(\overline{U_{C_n}}) : n \in \mathbb{N}\}$. Since $C_n \times \{y\} \subset U_{C_n} \times V^y_{C_n} \subset O^y_{C_n}$, we have

$$
\begin{aligned}
X \times Y &= \bigcup_{n\in\mathbb{N}} \left(Int(\overline{U_{C_n}}) \times \bigcup_{y\in Y_{C_n}} Int(\overline{V^y_{C_n}}) \right) \\
&= \bigcup_{n\in\mathbb{N}} \left(\bigcup_{y\in Y_{C_n}} Int(\overline{U_{C_n}}) \times Int(\overline{V^y_{C_n}}) \right) \\
&= \bigcup_{n\in\mathbb{N}} \left(\bigcup_{y\in Y_{C_n}} Int(\overline{U_{C_n} \times V^y_{C_n}}) \right) \\
&\subset \bigcup_{n\in\mathbb{N}} \bigcup_{y\in Y_{C_n}} Int(\overline{O^y_{C_n}})
\end{aligned}
$$

Hence the family $\{O^y_{C_n} : y \in Y_{C_n}, n \in \mathbb{N}\} \subset \mathcal{O}$ witnesses for $X \times Y$ is nearly Lindelöf. □

Corollary 3.17. *Nearly Alster spaces are productively nearly Lindelöf.*

4. Preservation properties of nearly Alster spaces and its some characterization in terms of selection principles

In this section, we will present some topological properties of Alster spaces and characterize the nearly Alster property in terms of some selection principles. We will start with considering the subspaces of nearly Alster spaces.

We note that a subset A of a topological space (X, τ) is nearly Alster if (A, τ_A) is nearly Alster, where τ_A is the induced topology on X.

First, we state that a subset of a nearly Alster space need not be nearly Alster. For example, if X is the set of all real numbers, the usual topological space (X, τ) is nearly Alster. Indeed, the family $\{[-n, n] : n \in \mathbb{N}\}$ is a countable subfamily of compact subsets of X and (X, τ) is metrizable. However, since the interiors of compact subsets of the set of all irrationals $\mathbb{P}$ have an empty interior, $\mathbb{P}$ is a metrizable non-nearly Alster subset of X with the induced topology.

Moreover, a closed subset of a nearly Alster space need not be nearly Alster. If X is an uncountable set and $p \in X$, X with uncountable particular point topology $\tau = \{U \subset X : p \in U\} \cup \{\emptyset\}$ is a nearly Alster space, since it is a nearly Lindelóf P-space. However, $X \setminus \{p\}$ with the induced topology is a closed discrete subspace of (X, τ), hence it can not be nearly Alster.

Theorem 4.1. *Closed and open subsets of nearly Alster spaces are nearly Alster.*

Proof. Let (X, τ) be a nearly Alster space and A be open and closed subset of X. If $\mathcal{A}$ is a τ_A-Alster cover of A, each member U of $\mathcal{A}$ can be written as

$$U = \bigcap\{O_n \cap A : n \in \mathbb{N}\},$$

where each O_n is open in X. Since $X \setminus A$ is open, then $U \cup (X \setminus A) = \bigcap\{O_n \cup (X \setminus A)\}$ is a G_δ subset of X. Hence the family $\mathcal{B} = \{U \cup (X \setminus A) : U \in \mathcal{A}\}$ is an Alster cover of X. Indeed, if C is any compact subset of X, $C \cap A$ is a compact subset of A and thus there exists a $U_C \in \mathcal{A}$ such that $C \cap A \subset U_C$ and hence $C = (C \cap A) \cup (C \setminus A) \subset U_C \cup (X \setminus A)$. Then there exists a countable subfamily $\mathcal{U}$ of $\mathcal{A}$ such that

$$X = \bigcup\{Int(\overline{U \cup (X \setminus A)}) : U \in \mathcal{U}\}.$$

By taking the intersection of each side by A and since A is open, (A, τ_A) is nearly Alster. □

Theorem 4.2. *Nearly Alster property is invariant under open and continuous surjections.*

Proof. Let (X, τ) be a nearly Alster space, (Y, σ) be a topological space, and $f : (X, \tau) \to (Y, \sigma)$ be an open continuous surjection. Let $\mathcal{A}$ be an Alster cover of Y. Since f is continuous, $\mathcal{B} = \{f^{-1}(U) : U \in \mathcal{A}\}$ is an Alster cover of X. Thus there exists a countable subfamily $\{V_n : n \in \mathbb{N}\}$ of $\mathcal{B}$ such that $X = \bigcup_{n\in\mathbb{N}} Int(\overline{V_n})$. On the other hand, for each V_n, there is a $U_n \in \mathcal{A}$ such that $V_n = f^{-1}(U_n)$. Since f is open and continuous surjection, we have

$$\begin{aligned} f(X) = Y &= f\Big(\bigcup_{n\in\mathbb{N}} Int(\overline{V_n})\Big) \\ &= \bigcup_{n\in\mathbb{N}} f\Big(Int(\overline{V_n})\Big) \\ &\subset \bigcup_{n\in\mathbb{N}} Int(\overline{f(V_n)}) \\ &\subset \bigcup_{n\in\mathbb{N}} Int(\overline{U_n}). \end{aligned}$$

Hence the subfamily $\{U_n : n \in \mathbb{N}\}$ of $\mathcal{A}$ witnesses for (Y, σ) is nearly Alster. □

Theorem 4.3. *Product of two nearly Alster space is nearly Alster.*

Proof. Let (X, τ) and (Y, σ) be Alster spaces and $\mathcal{A}$ be an Alster cover of the product space $X \times Y$. For every compact $C \subset X$ and compact $K \subset Y$, we can find an $A_{(C,K)} \in \mathcal{A}$ such that $C \times K \subset A_{(C,K)}$. Since $A_{(C,K)}$ is a G_δ subset of $X \times Y$, we can write it as $A_{(C,K)} = \bigcap\{O_{n,(C,K)} : n \in \mathbb{N}\}$, where for each $n \in \mathbb{N}$, $O_{n,(C,K)}$ is an open subset of $X \times Y$. Since $C \times K$ is compact and $C \times K \subset O_{n,(C,K)}$ for each $n \in \mathbb{N}$, we can find an open subset $U_{n,(C,K)} \subset X$ and $V_{n,(C,K)} \subset Y$ such that

$$C \times K \subset U_{n,(C,K)} \times V_{n,(C,K)} \subset O_{n,(C,K)}.$$

Let $U_{(C,K)} = \bigcap\{U_{n,(C,K)} : n \in \mathbb{N}\}$ and Let $V_{(C,K)} = \bigcap\{V_{n,(C,K)} : n \in \mathbb{N}\}$. Then $U_{(C,K)}$ is a G_δ subset of X containing C and $V_{(C,K)}$ is a G_δ subset of Y containing K. Then $\mathcal{A} = \{V_{(C,K)} : K \subset Y \text{ is compact }\}$ is an Alster cover of Y for each compact subset C of X. Since (Y, σ) is nearly Alster, there exists a countable family $\mathcal{K}$ which consists of compact subsets of Y such that

$$Y = \bigcup\{Int(\overline{V_{(C,K)}}) : K \in \mathcal{K}\}.$$

Now, let $U_C = \bigcap\{U_{(C,K)} : K \in \mathcal{K}\}$. Then $\mathcal{B} = \{U_C : C \subset X \text{ is compact }\}$ is an Alster cover of X. Since (X, τ) is nearly Alster, there exists a countable set $\{C_n : n \in \mathbb{N}\}$, where $C_n \subset X$ is compact for each $n \in \mathbb{N}$ such that

$$X = \bigcup\{Int(\overline{U_{C_n}}) : n \in \mathbb{N}\}.$$

With the fact that $C \times K \subset U_C \times V_{(C,K)} \subset A_{(C,K)}$, we have

$$\begin{aligned} X \times Y &= \bigcup_{n\in\mathbb{N}} \Big(Int(\overline{U_{C_n}}) \times \bigcup_{K\in\mathcal{K}_n} Int(\overline{V_{(C_n,K)}}) \Big) \\ &= \bigcup_{n\in\mathbb{N}} \Big(\bigcup_{K\in\mathcal{K}_n} Int(\overline{U_{C_n}}) \times Int(\overline{V_{(C_n,K)}}) \Big) \\ &= \bigcup_{n\in\mathbb{N}} \Big(\bigcup_{K\in\mathcal{K}_n} Int(\overline{U_{C_n} \times V_{(C_n,K)}}) \Big) \\ &\subset \bigcup_{n\in\mathbb{N}} \bigcup_{K\in\mathcal{K}_n} Int(\overline{A_{(C_n,K)}}). \end{aligned}$$

Hence $X \times Y$ is nearly Alster. □

Corollary 4.4. *If (X, τ) is nearly Alster, then X^n is nearly Alster for each $n \in \mathbb{N}$.*

There exists a nearly Menger space X such that X^2 is not nearly Menger, see Example 2.15 in [29]. From Proposition 3.4, we have:

Corollary 4.5. *If (X, τ) is nearly Alster, then X^n is nearly Menger for each $n \in \mathbb{N}$.*

Alster property and its some weaker forms were characterized in terms of selection principles in [5]. We will give some characterizations of nearly Alster property in terms of selection principles and a result for the productivity of nearly Menger spaces. We need the following notations for a topological space (X, τ):
$\mathcal{G} = \{\mathcal{U} \subset \mathcal{P}(X) : \mathcal{U}$ is a cover of X and each $U \in \mathcal{U}$ is a G_δ subset of $X\}$
$\mathcal{G}_{\mathcal{A}} : \{\mathcal{U} \subset \mathcal{P}(X) : \mathcal{U}$ is an Alster cover of $X\}$.
$\mathcal{G}_\Omega = \{\mathcal{U} \in \mathcal{G} :$ there exists a $U_F \in \mathcal{U}$ for each finite $F \subset X$ such that $F \subset U_F\}$
We also define the following notations as:
$\overline{\mathring{\mathcal{G}}} = \{\mathcal{U} \subset \mathcal{P}(X) :$ each $U \in \mathcal{U}$ is G_δ and $\{Int(\overline{U}) : U \in \mathcal{U}\}$ is a cover of $X\}$
$\overline{\mathring{\mathcal{G}}_\Omega} = \{\mathcal{U} \in \overline{\mathring{\mathcal{G}}} :$ for every finite $F \subset X$ there exists a $U_F \in \mathcal{U}$ such that $F \subset Int(\overline{U_F})\}$

Theorem 4.6. *Let (X, τ) be a topological space. The following statements are equivalent:*

1. *X is nearly Alster,*
2. *X satisfies the selection principle $S_1\big(\mathcal{G}_{\mathcal{A}}, \overline{\mathring{\mathcal{G}}}\big)$,*
3. *X satisfies the selection principle $S_1\big(\mathcal{G}_{\mathcal{A}}, \overline{\mathring{\mathcal{G}}_\Omega}\big)$.*

Proof. (1) ⇒ (2) Let a sequence $(\mathcal{A}_n)_{n\in\mathbb{N}}$ of Alster covers of X be given. Let $\mathcal{A} = \{\bigcap_{n\in\mathbb{N}} A_n : (\forall n)(A_n \in \mathcal{A}_n)\}$. Clearly $\mathcal{A}$ is an Alster cover of X. Since X is nearly Alster, we can find a countable subfamily $\mathcal{B} = \{U_n : n \in \mathbb{N}\}$ of $\mathcal{A}$ such that

$$X = \bigcup_{n\in\mathbb{N}} Int(\overline{U_n}).$$

For each $n \in \mathbb{N}$, let $U_n = \bigcap_{m\in\mathbb{N}} A^n_m$, where $A^n_m \in \mathcal{A}_m$ for each $m \in \mathbb{N}$. Since $U_n \subset A^n_n$ and $A^n_n \in \mathcal{A}_n$ for each $n \in \mathbb{N}$, we have

$$X = \bigcup_{n\in\mathbb{N}} Int(\overline{U_n}) \subset \bigcup_{n\in\mathbb{N}} Int(\overline{A^n_n}).$$

Hence $\{A^n_n : n \in \mathbb{N}\}$ is the desired family.

(2) ⇒ (1) Let $\mathcal{A}$ be an Alster cover of X. Put $\mathcal{A}_n = \mathcal{A}$ for every $n \in \mathbb{N}$. So $(\mathcal{A}_n)_{n\in\mathbb{N}}$ is a sequence of Alster covers of X. Then there exists an $A_n \in \mathcal{A}_n$ for every $n \in \mathbb{N}$ such that $X = \bigcup_{n\in\mathbb{N}} Int(\overline{A_n})$. Hence X is nearly Alster.

(2) ⇒ (3) Let X satisfy the selection principle $S_1\big(\mathcal{G}_{\mathcal{A}}, \overline{\mathring{\mathcal{G}}}\big)$. Then by Corollary 4.4., X^n satisfies $S_1\big(\mathcal{G}_{\mathcal{A}}, \overline{\mathring{\mathcal{G}}}\big)$ for each $n \in \mathbb{N}$. Now, let $(\mathcal{A}_n)_{n\in\mathbb{N}}$ be a sequence of Alster covers of X. Take a partition $\{N_n : n \in \mathbb{N}\}$ of $\mathbb{N}$, where N_n is infinite for each $n \in \mathbb{N}$. Let $\mathcal{U}_m = \{(A)^n : A \in \mathcal{A}_m\}$ for each $n \in \mathbb{N}$ and $m \in N_n$. Then for every $n \in \mathbb{N}$, $(\mathcal{U}_m)_{m\in N_n}$ is a sequence of Alster covers of X^n. Hence, by the assumption, there exists a $U_m \in \mathcal{U}_m$ for every $m \in N_n$ such that

$$X^n = \bigcup_{m\in N_n} Int(\overline{U_m}).$$

For each $n \in \mathbb{N}$ and $m \in N_n$, find an $A_m \in \mathcal{A}_m$ such that $U_m = (A_m)^n$. Then $\mathcal{V} = \{A_n : n \in \mathbb{N}\} \in \overline{\mathring{\mathcal{G}}_\Omega}$. To see this, consider a finite subset $F = \{x_1, x_2, ..., x_k\}$ of X. Since $x = (x_1, x_2, ..., x_k) \in X^k$, we can find an $m \in N_k$ such that $x \in Int(\overline{U_m})$. Then $x_i \in Int(\overline{A_m})$ for each $i = 1, 2, ..., k$ and hence $F \subset Int(\overline{A_m})$ which completes the proof.

(3) ⇒ (2) It is clear with the fact that $\overline{\mathring{\mathcal{G}}_\Omega} \subset \overline{\mathring{\mathcal{G}}}$. □

In [29], It was shown that the product of a nearly compact space and a nearly Menger space is nearly Menger. Now, by using the previous theorem, we have the following:

Theorem 4.7. *If* (X, τ) *is nearly Alster and* (Y, σ) *is nearly Menger, then* $X \times Y$ *is nearly Menger.*

Proof. Let $(\mathcal{O}_n)_{n\in\mathbb{N}}$ be a sequence of open covers of $X \times Y$. Without loss of generality, we may assume that each $\mathcal{O}_n$ is closed under finite union. Since $C \times \{y\}$ is a compact subset of $X \times Y$ for every compact $C \subset X$ and $y \in Y$, we can find an $O^y_{n,C} \in \mathcal{O}_n$ for each $n \in \mathbb{N}$ such that $C \times \{y\} \subset O^y_{n,C}$. On the other hand, there exists an open set $U^y_{n,C} \subset X$ and $V^y_{n,C} \subset Y$ such that $C \times \{y\} \subset U^y_{n,C} \times V^y_{n,C} \subset O^y_{n,C}$. Then for each $n \in \mathbb{N}$ and each compact subset C of X, we obtain an open cover $\mathcal{V}^C_n = \{V^y_{n,C} : y \in Y\}$ of Y. Now let $\{N_m : m \in \mathbb{N}\}$ be a partition of $\mathbb{N}$, where each N_m is infinite. Then $(\mathcal{V}^C_n)_{n\in N_m}$ is a sequence of open covers of Y for each $m \in \mathbb{N}$. Since Y is nearly Menger, for each $m \in \mathbb{N}$ and each $n \in N_m$, there exists a finite $F_{n,C} \subset Y$ such that

$$Y = \bigcup_{n\in N_m} \bigcup_{y\in F_{n,C}} Int(\overline{V^y_{n,C}}).$$

Define

$$U^C_m = \bigcap_{n\in N_m} \bigcap_{y\in F_{n,C}} U^y_{n,C}$$

for each $m \in \mathbb{N}$. Then each U^C_m is a G_δ subset of X containing the compact subset C. Let $\mathcal{U}_m = \{U^C_m : C \subset X \text{ is compact }\}$ for each $m \in \mathbb{N}$. We obtain a sequence $(\mathcal{U}_m)_{m\in\mathbb{N}}$ of Alster covers of X. Since X is nearly Alster, X satisfies the selection principles $S_1\big(\mathcal{G}_{\mathcal{A}}, \overset{\circ}{\overline{\mathcal{G}}}\big)$, hence there exists a $U^{C_m}_m \in \mathcal{U}_m$ for every $m \in \mathbb{N}$ such that

$$X = \bigcup_{m\in\mathbb{N}} Int(\overline{U^{C_m}_m}).$$

Now put $\mathcal{W}_n = \{U^y_{n,C_m} \times V^y_{n,C_m} : y \in F_{n,C_m}\}$ for each $m \in \mathbb{N}$ and $n \in N_m$. So, each $\mathcal{W}_n$ is finite. Now, we will show that

$$X \times Y = \bigcup_{m\in\mathbb{N}} \bigcup_{n\in N_m} \bigcup_{W\in\mathcal{W}_n} Int(\overline{W})$$

Let $(x_0, y_0) \in X \times Y$. We can find an $m \in \mathbb{N}$ such that $x_0 \in Int(\overline{U^{C_m}_m})$. On the other hand, for an $n \in N_m$ and $y \in F_{n,C_m}$, we have $y_0 \in Int(\overline{V^y_{n,C_m}})$. Hence,

$$\begin{aligned}(x_0, y_0) &\in Int(\overline{U^{C_m}_m}) \times Int(\overline{V^y_{n,C_m}}) \\ &\subset Int(\overline{U^y_{n,C_m}}) \times Int(\overline{V^y_{n,C_m}}) \\ &\subset Int(\overline{U^y_{n,C_m} \times V^y_{n,C_m}}).\end{aligned}$$

holds. On the other hand, for every $n \in \mathbb{N}$ and $W \in \mathcal{W}_n$, there exists an $O_W \in \mathcal{O}_n$ such that $W \subset O_W$. Put $\mathcal{S}_n = \{O_W : W \in \mathcal{W}_n\} \subset \mathcal{O}_n$ for every $n \in \mathbb{N}$. Then the sequence $(\mathcal{S}_n)_{n\in\mathbb{N}}$ witnesses for $X \times Y$ is nearly Menger. □

5. Conclusion

We studied a new class of Alster spaces. We established a few properties of this class and investigated the relationships between it and some of ones in earlier works in the literature. The study qualifies as a complement and continuation of selective covering properties as well as Alster-type covering properties. Weaker forms of the Alster covering property in a bitopological setting was considered in [1, 13]. The corresponding properties may be investigated in (a)-topological spaces and a large frame can be obtained. We also believe in that the paper can be a nice initiation for the generalized type of Alster spaces. As known, there are various types of the Menger-type covering properties in terms of generalized notions, see [23, 25, 31]. So it would be interesting to investigate the properties and set up the relations between the existing Alster covering properties and the following:

Call a cover $\mathcal{A}$ of a topological space (X, τ) *semi Alster cover (*resp., *θ-Alster cover*) if each member U of $\mathcal{A}$ is a semi G_δ set [32] (resp., θ-G_δ set [38], i.e. every intersection of countably many θ-open set is θ-open) and for each semi-compact [11] (resp., θ-compact [15]) subset C of X, there exists a $U \in \mathcal{A}$ such that $C \subset U$ (resp., $C \subset U$). (X, τ) is called *semi Alster (*resp., *θ-Alster)* if every semi Alster (resp., θ-Alster) cover of X has a countable subcover. We also define a weaker form of the corresponding properties in a following way;

A topological space is said to be *almost semi Alster*(resp., *almost θ-Alster*) if every semi Alster (resp., θ-Alster) cover $\mathcal{A}$ of X has a countable subfamily $\mathcal{U}$ such that $X = \bigcup_{U \in \mathcal{U}} scl(U)$ (resp., $X = \bigcup_{U \in \mathcal{U}} Cl_\theta(\overline{U})$), where $scl(U)$ and $Cl_\theta(\overline{U})$ are semi-closure of U and θ-closure of $\overline{U}$, respectively. (See [10, 38]).

References

[1] N. C. Açikgöz, C. S. Elmali, *A note on a novel type of Alster covering property in bitopological setting*, Turk. J. Math. **49** (2025), 185–200.
[2] K. Alster, *On the class of all spaces of weight not greater than ω_1 whose Cartesian product with every Lindelöf space is Lindelöf*, Fund. Math. **129** (1988), 133–140.
[3] L. F. Aurichi, R. R. Dias, *Topological games and Alster spaces*, Canad. Math. Bull. **57** (2014), 683–696.
[4] L. F. Aurichi, F. Tall, *Lindelöf spaces which are indestructible, productive, or D*, Topol. Appl. **159** (2012), 331-340.
[5] L. Babinkostova, B. A. Pansera, M. Scheepers, *Weak covering properties and selection principles*, Topol. Appl. **160** (2013), 2251–2271.
[6] G. Balasubramanian, *On some generalizations of compact spaces*, Glas. Mat. **17** (1982), 367–380.
[7] M. Barr, J. F. Kennison, R. Raphael, *Searching for absolute $\mathcal{CR}$-epic spaces*, Canad. J. Math. **59** (2007), 465–487.
[8] M. Barr, J. F. Kennison, R. Raphael, *On productively Lindelöf spaces*, Sci. Math. Jpn. **65** (2007), 310–332.
[9] F. Commaroto, G. Santoro, *Some counterexamples and properties on generalizations of Lindelöf spaces*, Internat. J. Math. & Math. Sci. **19** (1996), 737–746.
[10] S. G. Crossley, S. K. Hildebrand, *Semi-closure*, Texas J. Sci. **22** (1971), 99–112.
[11] C. Dorsett, *Semi compactness, semi separation axioms, and product spaces*, Bull. Malays. Math. Sci. Soc. **4** (1981), 21–28.
[12] R. Engelking, *General Topology*, Heldermann-Verlag, Berlin, 1989.
[13] A.E. Eysen, S. Ozcag, *Investigations on weak versions of the Alster property in bitopological spaces and selection principles*, Filomat **33** (2019), 4561–4571.
[14] Z. Frolik, *Generalizations of compact and Lindelöf spaces*, Czech. Math. **9** (1959), 172–217.
[15] S. Jafari, *Some properties of quasi θ-continuous functions*, Far East J. Math. Sci. **6** (1998), 689–696.
[16] W. Just, A. W. Miller, M. Scheepers, P. J. Szeptycki, *The combinatorics of open covers II*, Topol. Appl. **73** (1996), 241–266.
[17] A. Kilicman, A.J. Fawakhreh, *Product property on generalized Lindelöf spaces*, Int. Sch. Res. Notices **2011** (2011), 7 pages.
[18] D. Kocev, *Menger-type covering properties of topological spaces*, Filomat **29** (2015), 99–106.
[19] Lj. D. R. Kočinac, *Star-Menger and related spaces, II*, Filomat **13** (1999), 129–140.
[20] Lj. D. R. Kočinac, *Selected results on selection principles*, In: Proc. Third Seminar on Geometry and Topology, July 15–17, 2004, Tabriz, Iran, pp. 71–104.
[21] Lj. D. R. Kočinac, *Some covering properties in topological and uniform spaces*, Proc. Steklov Inst. Math. **252** (2006), 122–137.
[22] Lj. D. R. Kočinac, S. Ozcag, *Bitopological spaces and selection principles*, Proc. ICTA 2011, July 4-10, 2011, Islamabad, Pakistan, pp. 243–255.
[23] Lj. D. R. Kočinac, *Generalized open sets and selection properties*, Filomat **33** (2019), 1485–1493.
[24] Lj. D. R. Kočinac, *Variations of classical selection principles: An overview*, Quaest. Math. **43** (2020), 1121–1153.
[25] G. Kumar, B. K. Tyagi, *On variants of θ-Menger spaces*, Filomat **37** (2023), 5075–5085.
[26] K. Menger, *Einige Überdeckungssätze der Punktmengenlehre*, Sitzungsberichte Abt. 2a, Mathematik, Astronomie, Physik, Meteorologie und Mechanik (Wiener Akademie, Wien) **133** (1924), 421–444.
[27] A. V. Osipov, S. Ozcag, *On the Alster, Menger and D-type covering properties*, Quaest. Math. **44** (2021), 369–377.
[28] B. A. Pansera, *Weaker forms of the Menger property*, Quaest. Math. **35** (2012), 161–169.
[29] A. Parvez, M. D. Khan, *On nearly Menger and nearly star-Menger spaces*, Filomat **33** (2019), 6219–6227.
[30] J. R. Porter, G. Woods, *Extensions and Absolutes of Hausdorff Spaces*, Springer Science & Business Media, New York, (2012).
[31] A. Sabah, M. D. Khan, Lj. D. R. Kočinac, *Covering properties defined by semi-open sets*, J. Nonlinear Sci. Appl. **9** (2016), 4388–4398.

[32] A. Sabah, M. D. Khan, *Semi Rothberger and related spaces*, Bull. Iran Math. Soc. **43** (2017), 1969–1987.
[33] M. Scheepers, *Combinatorics of open covers I: Ramsey Theory*, Topol. Appl. **69** (1996), 31–62.
[34] M. Scheepers, *Selection principles and covering properties in topology*, Note Mat. **22** (2003), 3–41.
[35] Y. K. Song, *Some remarks on almost Menger spaces and weakly Menger spaces*, Publ. lnst. Math. **98** (2015), 193–198.
[36] P. Staynova, *A comparison of Lindelöf type covering properties of topological spaces*, Rose-Hulman Undergrad. Math. J. **12** (2011), 163–204.
[37] L. A. Steen, J. A. Seebach, *Counterexamples in Topology*, Dover Publ., New York, (1995).
[38] N. V. Ve1icko, *H-closed topological spaces*, Amer. Math. Soc. Transl. **78** (1968), 103–118.