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On direct sums of (L, M)-fuzzy convex spaces

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Abstract. In this paper, the notion of direct sum of a family of convex spaces is generalized to that of a family of (L, M)-fuzzy convex spaces. Firstly, the related properties between the direct sum of a family of (L, M)-convex spaces and its factor spaces are discussed. Secondly, the (L, M)-fuzzy direct sum convex spaces are characterized by means of its some level L-direct sum convex spaces. Finally, the additivity of separability (S_{-1}, sub-S_0, S_0, S_1, S_2) are investigated.

1. Introduction

Convex set is a specific mathematical concept derived from convexity, which generally exists in many mathematical structures such as vector spaces, metric spaces, median algebras, graphs, posets, and topological spaces, etc [2]. Although the definition of convex sets varies in different mathematical structures, they share certain properties in some aspects. Specifically, both empty set and universe set are convex sets; the intersection of any family of non-empty convex sets is a convex set; the union of a family of totally ordered convex sets is a convex set. These characteristics of convex sets inspire people to study the properties of convexity in different mathematical structures from an axiomatic perspective, resulting in the theory of abstract convex structures [24].

With the development of fuzzy mathematics, fuzzy convex theory has attracted the attention of researchers. In 1994, Rosa [14] first proposed the concept of fuzzy convex spaces by combining fuzzy set theory and convex structure theory. In 2009, based on Rosa's work, Marugama [8] further proposed the notion of L-convex spaces, where L is a completely distributive lattice. Recently, many scholars have conducted research on L-convex spaces and obtained rich results. For example, Pang and Shi [9] introduced various types of L-convex spaces and established internal relationships among them and classical convex spaces from a categorical viewpoint. Pang and Xiu [9] established an axiom system for the basis and subbases of L-convex spaces and provided their related applications. Shen and Shi [17] gave some novel characterizations of L-convex structures based on way-below relations in a continuous lattice. Zhou and Shi [35, 36] introuced some separability of L-convex spaces and proposed the concept of the sum of a family

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of L-convex spaces. Specifically, they proved that the separability of L-convex spaces satisfies additivity. There are also some more comprehensive research on L-convex spaces [1, 13, 16, 29, 34].

In the framework of L-convex spaces, the convex structure on a set is a classical subset of its fuzzy power set, that is, the convex structure is distinct. From the viewpoint of degree, in 2014, Shi and Xiu [20] first proposed the concept of M-fuzzifying convex spaces. They discussed the basic properties of M-fuzzifying convex spaces such as convexity preserving mapping, convex to convex mapping, subspaces, product spaces and quotient spaces. After the concept of M-fuzzifying convex spaces was proposed, many related research results have been generated. Xiu and Shi [30] proposed the notion of M-fuzzifying interval operators, and discussed the relationship between the categories of M-fuzzifying convex spaces and M-fuzzifying interval spaces. Liu and Shi [7] provided a method for inducing M-fuzzifying convex structures using M-hazy lattices, and studied their related properties. Dong and Shi [3] studied the properties of disjoint sums of a family of M-fuzzifying convex spaces. Shi [22] established equivalent axioms of M-fuzzifying convex matroids. The latest research progress of M-fuzzifying convex spaces can be found in [19, 23, 27, 28, 31].

Combining the ideas and methods of L-convex spaces and M-fuzzifying convex spaces, Shi and Xiu [21] introduced a new approach to fuzzification of convex spaces, namely an (L, M)-fuzzy convex space. In the framework of (L, M)-fuzzy convex spaces, Liang et.al. [6] used the properties of implication operators to study the separability of (L, M)-fuzzy convex spaces. Subsequently, Zhao et.al. [33] gave some new investigations on separation axioms in (L, M)-fuzzy convex spaces by L-fuzzy hull operators and r-L-fuzzy biconvex. Pang [12] established axioms for bases,subbases,convex hull operators, and interval operators, and used them to study the related properties of (L, M)-fuzzy convex spaces. Zhang and Pang [32] presented the concepts of (L, M)-remotehood spaces and (L, M)-convergence spaces, and studied their related properties. Due to the fact that classical convex spaces, L-convex spaces, and M-fuzzifying convex spaces can all be treated as special cases of (L, M)-fuzzy convex spaces, studying this more general case of fuzzy convex spaces becomes more complex.

The direct sum of a family of convex spaces is a basic and very useful operation [24]. At present, there are no researchers discussing the problem of the direct sum of (L, M)-fuzzy convex spaces. Therefore, studying the direct sum of a family of (L, M)-fuzzy convex spaces has important theoretical value. It can further improve the theory of fuzzy convex spaces. In this paper, on the idea of [36], we shall generalize the direct sum of a family of L-convex spaces to (L, M)-fuzzy setting. This paper is organized as follows. In Section 2, we recall some basic and necessary concepts that are required in subsequence sections. In Section 3, the direct sum of a family of (L, M)-fuzzy convex spaces is introduced and some basic properties are discussed. In Section 4, we characterize an (L, M)-fuzzy direct sum convex space by its level L-direct sum convex space. In Section 5, we examine the additivity of S_{-1}, sub-S_0, S_0, S_1 and S_2 separability for (L, M)-fuzzy convex spaces.

2. Preliminaries

Throughout this paper, both L and M denote completely distributive lattices with order-reversing involution $'$, unless otherwise stated. For a nonempty set X, L^X denotes the family of all L-sets on X. The smallest element and the largest element(called also the zero-element and the unit element)in L are denoted by $\bot_L$ and $\top_L$, respectively. Obviously, L^X is also a completely distributive lattice with an order-reversing involution $'$ under the pointwise order. The two L-sets $\bot_{L^X}$ and $\top_{L^X}$ are smallest and largest elements of L^X, respectively.

An element $a \in L$ is called a prime element if $a \geqslant b \wedge c$ implies $a \geqslant b$ or $a \geqslant c$. a in L is called co-prime element provided that $a \leqslant b \vee c$ implies $a \leqslant b$ or $a \leqslant c$ for all $b, c \in L$. The set of non-unit prime elements in L is denoted by $P(L)$. The set of non-zero co-prime elements in L is denoted by $J(L)$. From [25] we know that in a completely distributive lattice, each element is the sup of co-prime elements and the inf of prime elements.

In the sequel, the binary relation $\prec$ on L can be used to define two new cut sets of an L-set. The binary relation $\prec$ in L is defined as follows: for $a, b \in L, a \prec b$ if and only if for every subset $D \subseteq L$, the relation

$b \leqslant \sup D$ always implies the existence of $d \in D$ with $a \leqslant d$. Moreover, the binary relation $\prec^{op}$ in L is defined as follows: for $a, b \in L, a \prec^{op} b$ if and only if for every subset $D \subseteq L$, the relation $b \geqslant \inf D$ always implies the existence of $d \in D$ with $a \geqslant d$. $\{a \in L : a \prec^{op} b\}$ is called the greatest maximal family of b in sense of [25], denoted by $\alpha(b)$. In a completely distributive lattice L, $b = \wedge\alpha(b)$.

Let $A \in L^X$ and $a \in L$. Define $A_{[a]} = \{x \in X : A(x) \geqslant a\}$, $A^{[a]} = \{x \in X : a \notin \alpha\big(A(x)\big)\}$ and $A^{(a)} = \{x \in X : A(x) \nleqslant a\}$. Some properties of these cut sets can be found in [4].

For each $x \in X$ and $\lambda \in L$, the L-set x_λ, defined by

$$\forall y \in X,\ x_\lambda(y) = \begin{cases} \lambda, & y = x; \\ \bot_L, & y \neq x. \end{cases}$$

is called an L-fuzzy point of X.

Let X and Y be two nonempty sets. For a mapping $f : X \longrightarrow Y$, we define $f_L^{\rightarrow} : L^X \longrightarrow L^Y$ and $f_L^{\leftarrow} : L^Y \longrightarrow L^X$ as follows:

$$\forall A \in L^X, y \in Y, f_L^{\rightarrow}(A)(y) = \bigvee_{f(x)=y} A(x);$$

and

$$\forall B \in L^Y, x \in X, f_L^{\leftarrow}(B)(x) = B(f(x)).$$

Definition 2.1. ([25]) Let $A \in L^X$ and $\emptyset \neq Y \subseteq X$. The L-set $A|_Y \in L^Y$ defined by

$$(A|_Y)(y) = A(y) \text{ for all } y \in Y,$$

is called the restriction of A to Y.

Proposition 2.2. ([25]) *For each $\{A_t\}_{t\in T} \subseteq L^X$, $A \in L^X$, $\emptyset \neq Y \subseteq X$. We have the following results:*

(1) $(\bigvee_{t\in T} A_t)|_Y = \bigvee_{t\in T}(A_t|_Y)$.
(2) $(\bigwedge_{t\in T} A_t)|_Y = \bigwedge_{t\in T}(A_t|_Y)$.
(3) $A'|_Y = (A|_Y)'$.

Definition 2.3. ([25]) Let $\emptyset \neq Y \subseteq X$ and $A \in L^Y$. Define two L-sets $A^*, A^\star \in L^X$ as follows:

$$\forall x \in X,\ A^*(x) = \begin{cases} A(x), & x \in Y; \\ \bot_L, & x \notin Y, \end{cases}$$

$$\forall x \in X,\ A^\star(x) = \begin{cases} A(x), & x \in Y; \\ \top_L, & x \notin Y. \end{cases}$$

Clearly, $x_\lambda \in J(L^Y)$ implies $x_\lambda^* \in J(L^X)$. In particular, $A^*|_Y = A$ and $A^\star|_Y = A$.

In [12, 21], the authors extended the notion of classical convex structures to the notion of (L, M)-fuzzy convex structures as follows.

Definition 2.4. ([12, 21]) A mapping $\mathfrak{C} : L^X \longrightarrow M$ is called an (L, M)-fuzzy convex structure on X if it fulfills the following assertions:

(LMFC1) $\mathfrak{C}(\bot_{L^X}) = \mathfrak{C}(\top_{L^X}) = \top_M$;
(LMFC2) If $\{A_i\}_{i\in I} \subseteq L^X$ is nonempty, then $\mathfrak{C}(\bigwedge\limits_{i\in I} A_i) \geqslant \bigwedge\limits_{i\in I} \mathfrak{C}(A_i)$;
(LMFC3) If $\{A_j\}_{j\in J} \subseteq L^X$ is nonempty and totally ordered by inclusion, then $\mathfrak{C}(\bigvee\limits_{j\in J} A_j) \geqslant \bigwedge\limits_{j\in J} \mathfrak{C}(A_j)$.

If $\mathfrak{C}$ is an (L, M)-fuzzy convex structure on X, then $(X, \mathfrak{C})$ is called an (L, M)-fuzzy convex space. In this case, $\mathfrak{C}(A)$ can be regarded as the degree to which A is an L-convex set.

Remark 2.5. From Definition 2.4, we know that an $(L,\mathbf{2})$-fuzzy convex space is also called an L-convex space. A $(\mathbf{2},M)$-fuzzy convex space is also called an M-fuzzifying convex space which has been considered in [20]. A crisp convex space can be regarded as a $(\mathbf{2},\mathbf{2})$-fuzzy convex space.

Proposition 2.6. ([21]) *Let $(X,\mathfrak{C})$ be an (L,M)-fuzzy convex space and $\emptyset \neq Y \in 2^X$. Then $(Y,\mathfrak{C}|_Y)$ is an (L,M)-fuzzy convex structure on Y, where for each $A \in L^Y$,*

$$(\mathfrak{C}|_Y)(A) = \bigvee\{\mathfrak{C}(B) : B \in L^X, B|_Y = A\}.$$

We call $(Y,\mathfrak{C}|_Y)$ an (L,M)-fuzzy subspace of $(X,\mathfrak{C})$.

Definition 2.7. ([12, 21]) Let $f : (X,\mathfrak{C}) \longrightarrow (Y,\mathfrak{D})$ be a mapping between (L,M)-fuzzy convex spaces. Then

(1) f is called (L,M)-fuzzy convexity preserving if $\mathfrak{C}(f_L^{\leftarrow}(B)) \geqslant \mathfrak{D}(B)$ for all $B \in L^Y$;

(2) f is called (L,M)-fuzzy convex-to-convex if $\mathfrak{C}(A) \leqslant \mathfrak{D}(f_L^{\rightarrow}(A))$ for all $A \in L^X$.

Based on the above definitions, we have the following concepts.

Definition 2.8. Let $f : (X,\mathfrak{C}) \longrightarrow (Y,\mathfrak{D})$ be a mapping between (L,M)-fuzzy convex spaces.

(1) If f is a bijective (L,M)-fuzzy convexity preserving and (L,M)-fuzzy convex-to-convex mapping, then f is (L,M)-fuzzy isomorphic.

(2) If $f|^{f^{\rightarrow}(X)} : (X,\mathfrak{C}) \longrightarrow (f^{\rightarrow}(X),\mathfrak{D}|_{f^{\rightarrow}(X)})$ is an (L,M)-fuzzy isomorphism, then f is called (L,M)-emdedding.

3. Direct sums of (L,M)-fuzzy convex spaces

In this section, we will establish the connections between the sum of a family of (L,M)-fuzzy convex spaces and its factor spaces. These results will be useful in the following sections. First of all, we give a lemma to show that a family of (L,M)-fuzzy convex structures can induce a new (L,M)-fuzzy convex structure with respect to a mapping family.

Lemma 3.1. *Let $\{(X_t,\mathfrak{C}_t)\}_{t\in T}$ be a family of (L,M)-fuzzy convex spaces and X be a nonempty set. If $f_t : X_t \longrightarrow X$ is a mapping for all $t \in T$, then the mapping $\mathfrak{C} : L^X \longrightarrow M$ defined by*

$$\forall B \in L^X, \mathfrak{C}(B) = \bigwedge_{t\in T} \mathfrak{C}_t\Big((f_t)_L^{\leftarrow}(B)\Big)$$

is an (L,M)-fuzzy convex structure on X.

Proof. It suffices to verify that $\mathfrak{C}$ satisfies (LMFC1)-(LMFC3). In fact,

(LMFC1) For each $t \in T$, we can easily obtain that $(f_t)_L^{\leftarrow}(\bot_{L^X}) = \bot_{L^{X_t}}$ and $(f_t)_L^{\leftarrow}(\top_{L^X}) = \top_{L^{X_t}}$.

Since $(X_t,\mathfrak{C}_t)$ is an (L,M)-fuzzy convex space for all $t \in T$, we have

$$\mathfrak{C}(\bot_{L^X}) = \bigwedge_{t\in T} \mathfrak{C}_t\Big((f_t)_L^{\leftarrow}(\bot_{L^X})\Big) = \bigwedge_{t\in T} \mathfrak{C}_t(\bot_{L^X}) = \top_M$$

and

$$\mathfrak{C}(\top_{L^X}) = \bigwedge_{t\in T} \mathfrak{C}_t\Big((f_t)_L^{\leftarrow}(\top_{L^X})\Big) = \bigwedge_{t\in T} \mathfrak{C}_t(\top_{L^X}) = \top_M.$$

(LMFC2) Let $\{A_i\}_{i\in I} \subseteq L^X$ be nonempty. Then we obtain

$$\begin{aligned}
\bigwedge_{i\in I} \mathfrak{C}(A_i) &= \bigwedge_{i\in I}\bigwedge_{t\in T} \mathfrak{C}_t\Big((f_t)_L^{\leftarrow}(A_i)\Big) \\
&= \bigwedge_{t\in T}\bigwedge_{i\in I} \mathfrak{C}_t\Big((f_t)_L^{\leftarrow}(A_i)\Big) \\
&\leqslant \bigwedge_{t\in T} \mathfrak{C}_t\Big(\bigwedge_{i\in I}(f_t)_L^{\leftarrow}(A_i)\Big) \\
&= \bigwedge_{t\in T} \mathfrak{C}_t\Big((f_t)_L^{\leftarrow}(\bigwedge_{i\in I} A_i)\Big) \\
&= \mathfrak{C}(\bigwedge_{i\in I} A_i).
\end{aligned}$$

(LMFC3) Let $\{A_j\}_{j\in J}$ be a totally ordered subset of L^X. Then $\{(f_t)_L^{\leftarrow}(A_j)\}_{j\in J}$ is totally ordered subset of L^{X_t} for all $t \in T$. It follows that

$$\begin{aligned}\bigwedge_{j\in J}\mathfrak{C}(A_j) &= \bigwedge_{j\in J}\bigwedge_{t\in T}\mathfrak{C}_t\Big((f_t)_L^{\leftarrow}(A_j)\Big)\\ &= \bigwedge_{t\in T}\bigwedge_{j\in J}\mathfrak{C}_t\Big((f_t)_L^{\leftarrow}(A_j)\Big)\\ &\leqslant \bigwedge_{t\in T}\mathfrak{C}_t\Big(\bigvee_{j\in J}(f_t)_L^{\leftarrow}(A_j)\Big)\\ &= \bigwedge_{t\in T}\mathfrak{C}_t\Big((f_t)_L^{\leftarrow}(\bigvee_{j\in J}A_j)\Big)\\ &= \mathfrak{C}(\bigvee_{j\in J}A_j).\end{aligned}$$

This shows that $\mathfrak{C}$ is an (L, M)-fuzzy convex structure on X. □

Based on the above lemma, we introduce the definition of the direct sum of a family of (L, M)-fuzzy convex spaces. .

Definition 3.2. Let $\{(X_t, \mathfrak{C}_t)\}_{t\in T}$ be a family of pairwise disjoint (L, M)-fuzzy convex spaces, i.e., $X_{t_1} \cap X_{t_2} = \emptyset$ for $t_1 \neq t_2$. Consider the set $X = \bigcup_{t\in T} X_t$ and $\forall t \in T, j_t : X_t \longrightarrow X$ is the usual inclusion mapping (i.e., $\forall x \in X_t$, $j_t(x) = x$). Define

$$\forall B \in L^X, \mathfrak{C}(B) = \bigwedge_{t\in T}\mathfrak{C}_t\Big((j_t)_L^{\leftarrow}(B)\Big)$$

Then $\mathfrak{C}$ is called the (L, M)-fuzzy sum convex structure of $\{\mathfrak{C}_t\}_{t\in T}$ and denoted by $\sum_{t\in T}\mathfrak{C}_t$, briefly $\sum\mathfrak{C}_t$. The (L, M)-fuzzy convex space $(X, \sum\mathfrak{C}_t)$ is called the (L, M)-fuzzy sum convex space of $\{(X_t, \mathfrak{C}_t)\}_{t\in T}$, written as $\sum_{t\in T}(X_t, \mathfrak{C}_t)$ and briefly $\sum(X_t, \mathfrak{C}_t)$.

Remark 3.3. The preceding definition requires that X_t's ($t \in T$) must be disjoint. In fact, this requirement will not limit us seriously. Let $\{(X_t, \mathfrak{C}_t)\}_{t\in T}$ be a family of (L, M)-fuzzy convex spaces. Let $Y_t = X_t \times \{t\}$ for all $t \in T$. Then $Y_t \cap Y_s = \emptyset$ for $t \neq s$. For each $t \in T$, we know that the usual mapping $f_t : Y_t \longrightarrow X_t$, $(x, t) \mapsto x$ is bijective. Define $\mathfrak{D}_t : L^{Y_t} \longrightarrow M$ such that $\mathfrak{D}_t(A) = \mathfrak{C}_t\Big((f_t)_L^{\rightarrow}(A)\Big)$ for all $A \in L^{Y_t}$. One can readily verify that $\mathfrak{D}_t$ is an (L, M)-fuzzy convex structure on Y_t and $f_t : (Y_t, \mathfrak{D}_t) \longrightarrow (X_t, \mathfrak{C}_t)$ is an (L, M)-fuzzy isomorphism. Therefore, there is no difference between $(Y_t, \mathfrak{D}_t)$ and $(X_t, \mathfrak{C}_t)$ from the point of view of isomorphism, and we can define the sum of any family of (L, M)-fuzzy convex spaces (up to an (L, M)-fuzzy isomorphism). For convenience, we still use Definition 3.2 to study the related problems in later discussions.

The following propositions shows the close relationships between the direct sum of a family of (L, M)-fuzzy convex spaces and its factor spaces.

Proposition 3.4. *Let $(X, \mathfrak{C}) = \sum(X_t, \mathfrak{C}_t)$. Then for any $A \in L^X$, we have*

(1) $\mathfrak{C}(A) = \bigwedge_{t\in T}\mathfrak{C}_t(A|_{X_t})$.

(2) $\mathfrak{C}(A) = \bigvee_{A=\bigvee_{t\in T}A_t^*}\bigwedge_{t\in T}\mathfrak{C}_t(A_t)$.

Proof. (1) It is easy to verify that $(j_t)_L^{\leftarrow}(A) = A|_{X_t}$ for all $t \in T$ and $A \in L^X$. By Definition 3.2, we obtain

$$\mathfrak{C}(A) = \bigwedge_{t\in T}\mathfrak{C}_t\Big((j_t)_L^{\leftarrow}(A)\Big) = \bigwedge_{t\in T}\mathfrak{C}_t(A|_{X_t}).$$

(2) For each $x \in X = \bigcup_{t\in T} X_t$, there exists $s \in T$ such that $x \in X_s$ and $x \notin X_t (t \neq s)$. Let $A_t = (j_t)_L^{\leftarrow}(A)$ for all $t \in T$. Then we have

$$A_t^* = \Big((j_t)_L^{\leftarrow}(A)\Big)^* = (A|_{X_t})^*.$$

It follows that

$$
\begin{aligned}
(\bigvee_{t\in T} A_t^*)(x) &= \Big(\bigvee_{t\in T}\big((j_t)_L^{\leftarrow}(A)\big)^*\Big)(x)\\
&= \Big(\bigvee_{t\in T, t\neq s}\big((j_t)_L^{\leftarrow}(A)\big)^*\Big)(x)\vee\big((j_s)_L^{\leftarrow}(A)\big)^*(x)\\
&= \Big(\bigvee_{t\in T, t\neq s}(A|_{X_t})^*\Big)(x)\vee(A|_{X_s})^*(x)\\
&= (A|_{X_s})^*(x)\\
&= A(x).
\end{aligned}
$$

It implies that $\bigvee_{t\in T} A_t^* = A$. Thus, we obtain that

$$\bigvee_{A=\bigvee_{t\in T} A_t^*}\bigwedge_{t\in T}\mathfrak{C}_t(A_t)\geqslant\bigwedge_{t\in T}\mathfrak{C}_t\Big((j_t)_L^{\leftarrow}(A)\Big)=\mathfrak{C}(A).$$

Conversely, let a be any element in L with the property of

$$a\prec\bigvee_{A=\bigvee_{t\in T} A_t^*}\bigwedge_{t\in T}\mathfrak{C}_t(A_t).$$

Then there exists a family of $\{A_t\}_{t\in T}$ such that $A=\bigvee_{t\in T} A_t^*$ and $\mathfrak{C}_t(A_t)\geqslant a$ for all $t\in T$. Note that $\forall k\in T, (j_k)_L^{\leftarrow}(A)=A_k$, so we obtain that

$$\mathfrak{C}_k(A_k)=\mathfrak{C}_k\Big((j_k)_L^{\leftarrow}(A)\Big)\geqslant a$$

for all $k\in T$. Finally we get $\mathfrak{C}(A)=\bigwedge\limits_{t\in T}\mathfrak{C}_t(A_t)\geq a$. This shows that

$$\mathfrak{C}(A)\geqslant\bigvee_{A=\bigvee_{t\in T} A_t^*}\bigwedge_{t\in T}\mathfrak{C}_t(A_t).$$

Therefore the equality (2) holds. □

Proposition 3.5. *Let $(X,\mathfrak{C})=\sum(X_t,\mathfrak{C}_t)$. Then we have the following assertions:*

(1) $\forall t\in T$, $\mathfrak{C}(\top^*_{L^{X_t}})=\mathfrak{C}(\bot^*_{L^{X_t}})=\top_M$.
(2) $\forall t\in T$, $\mathfrak{C}(\top^\star_{L^{X_t}})=\mathfrak{C}(\bot^\star_{L^{X_t}})=\top_M$.
(3) $\forall t\in T$ *and* $A_t\in L^{X_t}$, $\mathfrak{C}(A_t^*)=\mathfrak{C}_t(A_t)$.
(4) $\mathfrak{C}$ *is the unique (L,M)-fuzzy convex structure on X which possess the following properties:*
 (i) $\forall t\in T$, $(X_t,\mathfrak{C}_t)$ *is a subspace of* $(X,\mathfrak{C})$, *i.e.,* $\mathfrak{C}|_{X_t}=\mathfrak{C}_t$,
 (ii) $\forall t\in T$ *and* $A_t\in L^{X_t}$, $\mathfrak{C}(A_t^\star)=\mathfrak{C}_t(A_t)$.

Proof. (1) Fixing any $k\in T$, then $\mathfrak{C}(\top^*_{L^{X_k}})=\bigwedge\limits_{t\in T}\mathfrak{C}_t\Big(j_t^{\leftarrow}(\top^*_{L^{X_k}})\Big)$. Since $j_k^{\leftarrow}(\top^*_{L^{X_k}})=\top_{L^{X_k}}$ and when $t\in T-\{k\}$, $j_k^{\leftarrow}(\top^*_{L^{X_k}})=\bot_{L^{X_t}}$, we have

$$\mathfrak{C}(\top^*_{L^{X_k}})=\bigwedge_{t\in T}\mathfrak{C}_t\Big(j_t^{\leftarrow}(\top^*_{L^{X_k}})\Big)=\mathfrak{C}_k(\top_{L^{X_k}})=\top_M.$$

Analogously, we obtain $\mathfrak{C}(\bot^*_{L^{X_k}})=\top_M$. This indicates that the conclusion (1) is true.

(2) The proof process is similar to (1).

(3) Fixing any $s\in T$, since $j_s^{\leftarrow}(A_s^*)=A_s$ and when $t\neq t_0$, $j_s^{\leftarrow}(A_s^*)=\bot_{L^{X_t}}$, we obtain

$$\mathfrak{C}(A_s^*)=\bigwedge_{t\in T}\mathfrak{C}_t\Big(j_s^{\leftarrow}(A_s^*)\Big)=\mathfrak{C}_s(A_s).$$

It follows that $\mathfrak{C}(A_t^*)=\mathfrak{C}_t(A_t)$ for all $t\in T$ and $A_t\in L^{X_t}$.

(4) Firstly, we prove that $\mathfrak{C}$ satisfies the properties of (i) and (ii). In fact, let $a \prec (\mathfrak{C}|_{X_t})(A)$, where $A \in L^{X_t}$ and $a \in L$. Since

$$(\mathfrak{C}|_{X_t})(A) = \bigvee_{B|_{X_t}=A} \mathfrak{C}(B),$$

it follows that there exists $B \in L^X$ such that $B|_{X_t} = A$ and $a \leqslant \mathfrak{C}(B)$. Notice that

$$\mathfrak{C}(B) = \bigwedge_{t\in T} \mathfrak{C}_t\Big(j_t^{\leftarrow}(B)\Big) = \bigwedge_{t\in T} \mathfrak{C}_t(B|_{X_t}),$$

so we obtain that $\mathfrak{C}_t(A) = \mathfrak{C}_t(B|_{X_t}) \geqslant a$ for all $t \in T$. It implies that $\mathfrak{C}|_{X_t} \leqslant \mathfrak{C}_t$.

Conversely, let $s \in T$ and $A \in L^{X_s}$. Suppose that $B = \bigvee_{t\in T} A_t^*$, where $\forall t \in T - \{s\}, A_t = \bot_{X_t}$ and $A_s = A$, then

$$\mathfrak{C}(B) = \bigvee_{B=\bigvee_{t\in T} A_t^*} \bigwedge_{t\in T} \mathfrak{C}_t(A_t) \geqslant \mathfrak{C}_s(A)$$

by Proposition 3.4. Notice that $B|_{X_s} = A$, so we obtain that

$$(\mathfrak{C}|_{X_t})(A) = \bigvee_{B|_{X_t}=A} \mathfrak{C}(B) \geqslant \mathfrak{C}(B) \geqslant \mathfrak{C}_s(A)$$

for all $s \in T$, i.e., $\mathfrak{C}|_{X_s} \geqslant \mathfrak{C}_s$. It shows that $\mathfrak{C}$ possesses the property (i).

Furthermore, taking any $k \in T$, we obtain

$$j_k^{\leftarrow}(A_k^{\star}) = \begin{cases} A_k, & t = k; \\ \top_{L^{X_t}}, & t \neq k. \end{cases}$$

Therefore, we have

$$\mathfrak{C}(A_k^{\star}) = \bigwedge_{t\in T} \mathfrak{C}_t\Big(j_k^{\leftarrow}(A_k^{\star})\Big) = \mathfrak{C}_k(A_k).$$

for all $k \in T$. This means that $\mathfrak{C}$ possesses the property (ii).

Secondly, suppose that $\mathfrak{D}$ is an arbitrary (L, M)-fuzzy convex structure on X with properties (i) and (ii). We need to prove that $\mathfrak{D} = \mathfrak{C}$. In fact, let $\mathfrak{D}(A) \geqslant a$ for $a \in L, A \in L^X$. Then we obtain that

$$\mathfrak{C}_t(A|_{X_t}) = (\mathfrak{D}|_{X_t})(A|_{X_t}) = \bigvee_{B\in L^X, B|_{X_t}=A|_{X_t}} \mathfrak{D}(B) \geqslant \mathfrak{D}(A) \geqslant a.$$

It follows that

$$\mathfrak{C}(A) = \bigwedge_{t\in T} \mathfrak{C}_t\Big(j_t^{\leftarrow}(A)\Big) = \bigwedge_{t\in T} \mathfrak{C}_t(A|_{X_t}) \geqslant a$$

It implies that $\mathfrak{D}(A) \leqslant \mathfrak{C}(A)$. Thus, $\mathfrak{D} \leqslant \mathfrak{C}$.

Conversely, let $a \prec \mathfrak{C}(A)$ for $a \in L$, $A \in L^X$. Then

$$\forall t \in T, a \prec \mathfrak{C}_t(A|_{X_t}) = \mathfrak{C}_t\Big(j_t^{\leftarrow}(A)\Big).$$

Moreover, we obtain that

$$a \prec \mathfrak{C}_t(A|_{X_t}) = (\mathfrak{D}|_{X_t})(A|_{X_t}) = \bigvee_{B\in L^X, B|_{X_t}=A|_{X_t}} \mathfrak{D}(B)$$

for all $t \in T$. Hence, there exists $B(t) \in L^X$ such that $B(t)|_{X_t} = A|_{X_t}$ and $a \leqslant \mathfrak{D}\Big(B(t)\Big)$ for all $t \in T$.

For each $x \in X$, there exists $s \in T$ such that $x \in X_s$ and $x \notin X_t (t \neq s)$. Thus, we obtain

$$A(x) = (A|_{X_s})(x) = A_s^{\star}(x) = (\bigwedge_{t\in T} A_t^{\star})(x).$$

This means that $A = \bigwedge_{t\in T} A_t^{\star}$. It follows that

$$\begin{aligned}
\mathfrak{D}(A) &= \mathfrak{D}(\bigwedge_{t\in T} A_t^{\star}) \geqslant \bigwedge_{t\in T} \mathfrak{D}(A_t^{\star}) \\
&= \bigwedge_{t\in T} \mathfrak{C}_t(A_t) = \bigwedge_{t\in T} \mathfrak{C}_t(A|_{X_t}) \\
&= \bigwedge_{t\in T} \mathfrak{C}_t\Big(B(t)|_{X_t}\Big) \\
&= \bigwedge_{t\in T} \bigvee_{B\in L^X, B|_{X_t}=B(t)|_{X_t}} \mathfrak{D}(B) \\
&\geqslant \bigwedge_{t\in T} \mathfrak{D}\Big(B(t)\Big) \\
&\geqslant a.
\end{aligned}$$

This shows that $\mathfrak{C}(A) \leqslant \mathfrak{D}(A)$. Consequently, we obtain $\mathfrak{D} = \mathfrak{C}$. This completes the proof. □

Proposition 3.6. *Let $(X, \mathfrak{C}) = \sum(X_t, \mathfrak{C}_t)$. Then we have the following assertions:*

(1) *$\mathfrak{C}$ is the finest (L, M)-fuzzy convex structure on X such that j_t is an (L, M)-fuzzy convexity preserving mapping for all $t \in T$.*

(2) *For each (L, M)-fuzzy convex space $(Y, \mathfrak{D})$, $g : (X, \mathfrak{C}) \longrightarrow (Y, \mathfrak{D})$ is an (L, M)-fuzzy convexity preserving mapping if and only if $g \circ j_t : (X_t, \mathfrak{C}_t) \longrightarrow (Y, \mathfrak{D})$ is an (L, M)-fuzzy convexity preserving mapping for all $t \in T$.*

(3) *$j_t : (X_t, \mathfrak{C}_t) \longrightarrow (X, \mathfrak{C})$ is an (L, M)-fuzzy embedding mapping for all $t \in T$.*

(4) *Let $(Y, \mathfrak{D})$ be an (L, M)-fuzzy convex space and $\{g_t : (X_t, \mathfrak{C}_t) \longrightarrow (Y, \mathfrak{D})\}$ be a family of (L, M)-fuzzy convexity preserving mappings. Then there exists an (L, M)-fuzzy convexity preserving mapping $h : (X, \mathfrak{C}) \longrightarrow (Y, \mathfrak{D})$ such that $h \circ j_t = g_t$ for all $t \in T$.*

Proof. (1) Let $\mathfrak{D}$ be an (L, M)-fuzzy convex structure on X such that j_t is an (L, M)-fuzzy convexity preserving mapping for all $t \in T$. Then we obtain

$$\forall t \in T, A \in L^X, \mathfrak{D}(A) \leqslant \mathfrak{C}_t\Big((j_t)_L^{\leftarrow}(A)\Big).$$

It follows that $\mathfrak{D}(A) \leqslant \bigwedge_{t\in T} \mathfrak{C}_t\Big((j_t)_L^{\leftarrow}(A)\Big) = \mathfrak{C}(A)$. Hence, $\mathfrak{D} \leqslant \mathfrak{C}$.

(2) Since $g : (X, \mathfrak{C}) \longrightarrow (Y, \mathfrak{D})$ is an (L, M)-fuzzy convexity preserving mapping, it follows that

$$\mathfrak{D}(B) \leqslant \mathfrak{C}\Big(g_L^{\leftarrow}(B)\Big) = \bigwedge_{t\in T} \mathfrak{C}_t\Big((j_t)_L^{\leftarrow}(g_L^{\leftarrow}(B))\Big) \leqslant \mathfrak{C}_t\Big((g \circ j_t)_L^{\leftarrow}(B)\Big)$$

for all $B \in \mathfrak{D}$ and $t \in T$. This means that $g \circ j_t : (X_t, \mathfrak{C}_t) \longrightarrow (Y, \mathfrak{D})$ is an (L, M)-fuzzy convexity preserving mapping for all $t \in T$.

Conversely, notice that $g \circ j_t : (X_t, \mathfrak{C}_t) \longrightarrow (Y, \mathfrak{D})$ is an (L, M)-fuzzy convexity preserving mapping, so we obtain

$$\forall A \in L^Y, \mathfrak{D}(A) \leqslant \mathfrak{C}_t\Big((g \circ j_t)_L^{\leftarrow}(A)\Big) = \mathfrak{C}_t\Big((j_t)_L^{\leftarrow}(g_L^{\leftarrow}(A))\Big)$$

for all $t \in T$. Thus

$$\mathfrak{D}(A) \leqslant \bigwedge_{t\in T} \mathfrak{C}_t\Big((j_t)_L^{\leftarrow}(g_L^{\leftarrow}(A))\Big) = \mathfrak{C}\Big(g_L^{\leftarrow}(A)\Big).$$

It implies that $g : (X, \mathfrak{C}) \longrightarrow (Y, \mathfrak{D})$ is an (L, M)-fuzzy convexity preserving mapping.

(3) We need to show that

$$j_t|^{j_t^{\rightarrow}(X_t)} : (X_t, \mathfrak{C}_t) \longrightarrow (j_t^{\rightarrow}(X_t), \mathfrak{C}|_{j_t^{\rightarrow}(X_t)})$$

is an (L, M)-fuzzy isomorphism. Note that

$$j_t^{\rightarrow}(X_t) = X_t, \mathfrak{C}|_{X_t} = \mathfrak{C}_t$$

and

$$id_{X_t} : (X_t, \mathfrak{C}_t) \longrightarrow (X_t, \mathfrak{C}_t)$$

is an (L, M)-fuzzy isomorphism, so we obtain that $j_t|^{j_t^{\rightarrow}(X_t)} = id_{X_t}$ is an (L, M)-fuzzy isomorphism. Hence $j_t : (X_t, \mathfrak{C}_t) \longrightarrow (X, \mathfrak{C})$ is (L, M)-fuzzy embedding for all $t \in T$.

(4) For each $x \in X = \bigcup_{t\in T} X_t$, there exists $t \in T$ such that $x \in X_t$. We define $h(x) = g_t(x)$. Obviously, $h \circ j_t = g_t$ for all $t \in T$. Since $g_t : (X_t, \mathfrak{C}_t) \longrightarrow (Y, \mathfrak{D})$ is an (L, M)-fuzzy convexity preserving mapping for all $t \in T$, it follows that $\mathfrak{D}(D) \leqslant \mathfrak{C}_t\big((g_t)_L^{\leftarrow}(D)\big)$ for all $D \in L^Y$. Thus, we have

$$\mathfrak{D}(D) \leqslant \mathfrak{C}_t\big((g_t)_L^{\leftarrow}(D)\big) = \mathfrak{C}_t\big((j_t)_L^{\leftarrow}(h_L^{\leftarrow}(D))\big) = \mathfrak{C}_t\big((h \circ j_t)_L^{\leftarrow}(D)\big)$$

for all $t \in T$. Hence

$$\mathfrak{D}(D) \leqslant \bigwedge_{t\in T} \mathfrak{C}_t\big((h \circ j_t)_L^{\leftarrow}(D)\big) = \mathfrak{C}_t\big((j_t)_L^{\leftarrow}(h_L^{\leftarrow}(B))\big).$$

This means that $h : (X, \mathfrak{C}) \longrightarrow (Y, \mathfrak{D})$ is an (L, M)-fuzzy convexity preserving mapping. □

Proposition 3.7. *Let $(X, \mathfrak{C}) = \sum(X_t, \mathfrak{C}_t)$. For each $t \in T$, if $Y_t \subseteq X_t$ and $Y = \bigcup_{t\in T} Y_t$, then $\mathfrak{C}|_Y = \sum_{t\in T}(\mathfrak{C}_t|_{Y_t})$.*

Proof. Let $j_t : X_t \longrightarrow X = \bigcup_{t\in T} X_t$ be the usual inclusion mapping and $j_t|_{Y_t} : Y_t \longrightarrow Y = \bigcup_{t\in T} Y_t$ be the restriction of j_t to Y_t for all $t \in T$. We shall first prove $\mathfrak{C}|_Y \leq \sum_{t\in T}(\mathfrak{C}_t|_{Y_t})$. It is easy to verity that $(j_t|_{Y_t})_L^{\leftarrow}(B) = B|_{Y_t}$ for all $B \in L^Y$.

From Definition 3.2, we obtain

$$\begin{aligned}\textstyle\sum_{t\in T}(\mathfrak{C}_t|_{Y_t})(B) &= \bigwedge_{t\in T}(\mathfrak{C}_t|_{Y_t})\big((j_t|_{Y_t})_L^{\leftarrow}(B)\big)\\ &= \bigwedge_{t\in T}(\mathfrak{C}_t|_{Y_t})(B|_{Y_t})\\ &= \bigwedge_{t\in T}\bigvee\{\mathfrak{C}_t(D) : D \in L^{X_t}, D|_{Y_t} = B|_{Y_t}\}\end{aligned}$$

for all $B \in L^Y$.

Furthermore, for any $B \in L^Y$, we have

$$\begin{aligned}(\mathfrak{C}|_Y)(B) &= \bigvee\{\mathfrak{C}(C) : C \in L^X, C|_Y = B\}\\ &= \bigvee\{\bigwedge_{t\in T}\mathfrak{C}_t\big((j_t)_L^{\leftarrow}(C)\big) : C \in L^X, C|_Y = B\}\\ &= \bigvee\{\bigwedge_{t\in T}\mathfrak{C}_t(C|_{X_t}) : C \in L^X, C|_Y = B\}\end{aligned}$$

Note that for any $C \in L^X$ with $C|_Y = B$ and $Y_t \subseteq X_t$ for all $t \in T$, so we obtain $(C|_{X_t})|_{Y_t} = C|_{Y_t} = B|_{Y_t}$. It follows that

$$\mathfrak{C}_t(C|_{X_t}) \leqslant \bigvee\{\mathfrak{C}_t(D) : D \in L^{X_t}, D|_{Y_t} = B|_{Y_t}\}.$$

Hence

$$\begin{aligned}(\mathfrak{C}|_Y)(B) &= \bigvee\{\bigwedge_{t\in T}\mathfrak{C}_t(C|_{X_t}) : C \in L^X, D|_Y = B)\}\\ &\leqslant \bigwedge_{t\in T}\bigvee\{\mathfrak{C}_t(D) : D \in L^{X_t}, D|_{Y_t} = B|_{Y_t}\}\\ &= \textstyle\sum_{t\in T}(\mathfrak{C}_t|_{Y_t})(B)\end{aligned}$$

for all $B \in L^Y$. It implies that $\mathfrak{C}|_Y \leqslant \sum_{t\in T}(\mathfrak{C}_t|_{Y_t})$.

Next, we need to show that $\sum_{t\in T}(\mathfrak{C}_t|_{Y_t}) \leqslant \mathfrak{C}|_Y$. Suppose that

$$a \prec \textstyle\sum_{t\in T}(\mathfrak{C}_t|_{Y_t})(B) = \bigwedge_{t\in T}\bigvee\{\mathfrak{C}_t(D) : D \in L^{X_t}, D|_{Y_t} = B|_{Y_t}\}$$

for $B \in L^Y$. For any $t \in T$, there exists $D_t \in L^{X_t}$ such that $D_t|_{Y_t} = B|_{Y_t}$ and $a \leqslant \mathfrak{C}_t(D_t)$. Let $C = \bigvee_{t\in T} D_t^* \in L^X$. Then

$$C|_Y = C|_{(\bigcup_{t\in T} Y_t)} = \bigvee_{t\in T}(C|_{Y_t}) = \bigvee_{t\in T}(B|_{Y_t}) = B.$$

Note that $a \leqslant \mathfrak{C}_t(D_t) = \mathfrak{C}_t(C|_{X_t})$ for all $t \in T$, so we obtain $a \leqslant \bigwedge_{t\in T} \mathfrak{C}_t(C|_{X_t})$, where $C|_Y = B$. Thus,

$$(\mathfrak{C}|_Y)(B) = \bigvee\{\bigwedge_{t\in T} \mathfrak{C}_t(C|_{X_t}) : C \in L^X, C|_Y = B\} \geqslant a.$$

This means that $\sum_{t\in T}(\mathfrak{C}_t|_{Y_t})(B) \leq (\mathfrak{C}|_Y)(B)$ for all $B \in L^Y$. Therefore, we obtain $\mathfrak{C}|_Y = \sum_{t\in T}(\mathfrak{C}_t|_{Y_t})$. □

4. Characterizations of direct sums of (L, M)-fuzzy convex spaces

In[36], Zhou and Shi studied the related properties of direct sums of a family of L-convex spaces. In this section, based on the relevant conclusions in[36], we use the cut sets of (L, M)-fuzzy convex spaces to characterize their direct sum properties. For this purpose, we first provide the following lemma.

Lemma 4.1. *Let X be a nonempty set and let $\mathfrak{C} : L^X \to M$ be a mapping. Then the following conditions are equivalent:*

(1) *$(X, \mathfrak{C})$ is an (L, M)-fuzzy convex space;*
(2) *For each $a \in J(M)$, $(X, \mathfrak{C}_{[a]})$ is an L-convex space;*
(3) *For each $a \in \alpha(\bot_M)$, $(X, \mathfrak{C}^{[a]})$ is an L-convex space;*
(4) *For each $a \in P(M)$, $(X, \mathfrak{C}^{(a)})$ is an L-convex space.*
(5) *For each $a \in \beta(\top_M)$, if $\beta(\wedge_{i\in I} a_i) = \cap_{i\in I}\beta(a_i)$ for all $\{a_i\}_{i\in I} \subseteq M$, then $(X, \mathfrak{C}_{(a)}$ is an L-convex space.*

Proof. (1) ⇔ (2) ⇔ (3) have been proved in [21]. Now we prove (1) ⇔ (4) and (1) ⇔ (5).

(1) ⇒ (4). Suppose that $(X, \mathfrak{C})$ is an (L, M)-fuzzy convex space. Then $\mathfrak{C}(\bot_{L^X}) = \mathfrak{C}(\top_{L^X}) = \top_M$, it follows that $\mathfrak{C}(\bot_{L^X}) \nleqslant a$, $\mathfrak{C}(\top_{L^X}) \nleqslant a$ for any $a \in P(M)$. Therefore $\bot_{L^X}, \top_{L^X} \in \mathfrak{C}^{(a)}$ for any $a \in P(M)$.

If $\{A_i\}_{i\in I} \subseteq \mathfrak{C}^{(a)}$, then $\mathfrak{C}(A_i) \nleqslant a$ for all $i \in I$. It implies that $\bigwedge_{i\in I} \mathfrak{C}(A_i) \nleqslant a$. By $\mathfrak{C}(\bigwedge_{i\in I} A_i) \geqslant \bigwedge_{i\in I} \mathfrak{C}(A_i)$, we obtain $\mathfrak{C}(\bigwedge_{i\in I} A_i) \nleqslant a$, i.e.,$\bigwedge_{i\in I} A_i \in \mathfrak{C}^{(a)}$.

If $\{A_j\}_{j\in J} \subseteq \mathfrak{C}^{(a)}$ is totally ordered, then $\mathfrak{C}(A_j) \nleqslant a$ for all $j \in J$. It follows that $\bigvee_{j\in J} \mathfrak{C}(A_j) \nleqslant a$. Thus $\bigvee_{j\in J} A_j \in \mathfrak{C}^{(a)}$.

(4) ⇒ (1). Suppose that $(X, \mathfrak{C}^{(a)})$ is an L-convex space for each $a \in P(M)$. Then we know that $\bot_{L^X} \in \mathfrak{C}^{(a)}$ and $\top_{L^X} \in \mathfrak{C}^{(a)}$ for each $a \in P(M)$. This implies that $\mathfrak{C}(\bot_{L^X}) \nleqslant a$ and $\mathfrak{C}(\top_{L^X}) \nleqslant a$ for each $a \in P(M)$. Therefore $\mathfrak{C}(\bot_{L^X}) = \mathfrak{C}(\top_{L^X}) = \top_M$.

If $\{A_i\}_{i\in I}$ is nonempty, take any $a \in P(M)$ with $\bigwedge_{i\in I} \mathfrak{C}(A_i) \nleqslant a$. Then $\mathfrak{C}(A_i) \nleqslant a$ for any $i \in I$. It follows that $A_i \in \mathfrak{C}^{(a)}$ for all $i \in I$. Thus $\bigwedge_{i\in I} A_i \in \mathfrak{C}^{(a)}$. This means that $\mathfrak{C}(\bigwedge_{i\in I} A_i) \nleqslant a$. By the arbitrariness of a, we obtain $\mathfrak{C}(\bigwedge_{i\in I} A_i) \geqslant \bigwedge_{i\in I} \mathfrak{C}(A_i)$.

Let $\{A_j\}_{j\in J}$ be totally ordered. Take any $a \in P(M)$ with $\bigwedge_{j\in J} \mathfrak{C}(A_j) \nleqslant a$. Then $\mathfrak{C}(A_j) \nleqslant a$ for any $j \in J$. It follows that $A_j \in \mathfrak{C}^{(a)}$ for all $j \in J$. Thus $\bigvee_{j\in J} A_j \in \mathfrak{C}^{(a)}$. This means that $\mathfrak{C}(\bigvee_{j\in J} A_j) \nleqslant a$. By the arbitrariness of a, we obtain $\mathfrak{C}(\bigwedge_{j\in J}) \geqslant \bigwedge_{j\in J} \mathfrak{C}(A_j)$.

(1) ⇒ (5). Suppose that $(X, \mathfrak{C})$ is an (L, M)-fuzzy convex space. Then $\mathfrak{C}(\bot_{L^X}) = \mathfrak{C}(\top_{L^X}) = \top_M$. It follows that $\bot_{L^X}, \top_{L^X} \in \mathfrak{C}_{(a)}$ for all $a \in \beta(\top_M)$.

If $\{A_i\}_{i\in I} \subseteq \mathfrak{C}_{(a)}$, then $a \in \beta\big(\mathfrak{C}(A_i)\big)$ for all $i \in I$. Since $\mathfrak{C}$ is an (L, M)-fuzzy convex space and $\beta(\wedge_{i\in I} a_i) = \cap_{i\in I}\beta(a_i)$, it follows that

$$a \in \cap_{i\in I}\beta\big(\mathfrak{C}(A_i)\big) = \beta\big(\wedge_{i\in I} \mathfrak{C}(A_i)\big) \subseteq \beta\big(\mathfrak{C}(\wedge_{i\in I} A_i)\big).$$

This means that $\wedge_{i\in I} A_i \in \mathfrak{C}_{(a)}$.

If $\{A_j\}_{j\in J} \subseteq \mathfrak{C}_{(a)}$ is totally ordered, then we obtain $a \in \beta\big(\mathfrak{C}(A_j)\big)$ for all $j \in J$. It follows that

$$a \in \cap_{j\in J}\beta\big(\mathfrak{C}(A_j)\big) = \beta\big(\wedge_{j\in J} \mathfrak{C}(A_j)\big) \subseteq \beta\big(\mathfrak{C}(\vee_{j\in I} A_J)\big).$$

It implies that $\vee_{j\in J} A_j \in \mathfrak{C}_{(a)}$.

(5) ⇒ (1). Let $\beta(\wedge_{i\in I}a_i)=\cap_{i\in I}\beta(a_i)$ and $a\in\beta(\top_M)$. If $(X,\mathfrak{C}_{(a)})$ is an L-convex space, then $\perp_{L^{L^X}},\top_{L^{L^X}}\in\mathfrak{C}_{(a)}$ for all $a\in\beta(\top_M)$. It implies that $a\in\beta\Big(\mathfrak{C}(\perp_{L^X})$ and $a\in\beta\Big(\mathfrak{C}(\top_{L^X})$. Therefore $\mathfrak{C}(\perp_{L^X})=\mathfrak{C}(\top_{L^X})=\top_M$.

Let $\{A_i\}_{i\in I}\subseteq L^X$ be nonempty and $\beta(\wedge_{i\in I}a_i)=\cap_{i\in I}\beta(a_i)$, take any $a\in\beta(\top_M)$ with $a\prec\bigwedge_{i\in I}\mathfrak{C}(A_i)$. Then $a\prec\mathfrak{C}(A_i)$ for all $i\in I$, i.e., $a\in\beta\Big((A_i)\Big)$. This means that $A_i\in\mathfrak{C}_{(a)}$ for all $i\in I$. It follows that $\bigwedge_{i\in I}A_i\in\mathfrak{C}_{(a)}$. Therefore $a\in\beta\Big(\mathfrak{C}(\wedge_{i\in I}A_i)\Big)$, i.e., $a\prec\mathfrak{C}(\bigwedge_{i\in I}A_i)$. By the arbitrariness of a, we obtain $\mathfrak{C}(\bigwedge_{i\in I}A_i)\geqslant\bigwedge_{i\in I}\mathfrak{C}(A_i)$.

Let $\{A_j\}_{j\in J}\subseteq L^X$ be totally ordered and $\beta(\wedge_{i\in I}a_i)=\cap_{i\in I}\beta(a_i)$, take any $a\in\beta(\top_M)$ with $a\prec\bigwedge_{j\in J}\mathfrak{C}(A_j)$. Then $a\prec\mathfrak{C}(A_j)$ for all $j\in J$, i.e., $a\in\beta\Big((A_j)\Big)$. This means that $A_j\in\mathfrak{C}_{(a)}$ for all $j\in J$. It follows that $\bigvee_{j\in J}A_j\in\mathfrak{C}_{(a)}$. Therefore $a\in\beta\Big(\mathfrak{C}(\bigvee_{j\in J}A_j)\Big)$, i.e., $a\prec\mathfrak{C}(\bigvee_{j\in J}A_j)\Big)$. By the arbitrariness of a, we obtain $\mathfrak{C}(\bigvee_{i\in I}A_i)\geqslant\bigwedge_{i\in I}\mathfrak{C}(A_i)$. □

An (L,M)-fuzzy direct sum convex space can be characterized by means of its level L-direct sum convex spaces as follows:

Theorem 4.2. *Let $\{(X_t,\mathfrak{C}_t)\}_{t\in T}$ be a family of (L,M)-fuzzy convex spaces and $\mathfrak{C}:L^X\to M$ be a mapping, where X is the disjoint union of $\{X_t\}_{t\in T}$. Then the following conditions are equivalent:*

(1) *$(X,\mathfrak{C})$ is the (L,M)-fuzzy direct sum convex space of $\{(X_t,\mathfrak{C}_t)\}_{t\in T}$;*
(2) *For each $a\in J(M)$, $(X,\mathfrak{C}_{[a]})$ is the L-direct sum convex space of $\{(X_t,(\mathfrak{C}_t)_{[a]})\}_{t\in T}$.*

Proof. (1) ⇒ (2). By Lemma 4.1(2), we know that $\{(X_t,(\mathfrak{C}_t)_{[a]})\}_{t\in T}$ is a family of L-convex spaces for each $a\in J(M)$. Since $(X,\mathfrak{C})$ is the (L,M)-fuzzy direct sum convex space of $\{(X_t,\mathfrak{C}_t)\}_{t\in T}$, $\mathfrak{C}(A)=\bigwedge\limits_{t\in T}\mathfrak{C}_t\Big((f_t)_L^{\leftarrow}(A)\Big)$ for each $A\in L^X$. For each $a\in J(M)$, we obtain

$$\begin{aligned} A\in\mathfrak{C}_{[a]} &\Leftrightarrow \mathfrak{C}(A)=\bigwedge_{t\in T}\mathfrak{C}_t\Big((f_t)_L^{\leftarrow}(A)\Big)\geqslant a\\ &\Leftrightarrow \forall t\in T,\mathfrak{C}_t\Big((f_t)_L^{\leftarrow}(A)\Big)\geqslant a\\ &\Leftrightarrow \forall t\in T,(f_t)_L^{\leftarrow}(A)\in(\mathfrak{C}_t)_{[a]}.\end{aligned}$$

Thus $\mathfrak{C}_{[a]}=\{A\in L^X:\forall t\in T,(f_t)_L^{\leftarrow}(A)\in(\mathfrak{C}_t)_{[a]}\}$ for each $a\in J(M)$. Hence $(X,\mathfrak{C}_{[a]})$ is the L-direct sum convex space of $\{(X_t,(\mathfrak{C}_t)_{[a]})\}_{t\in T}$ for each $a\in J(M)$.

(2) ⇒ (1). Since $(X,\mathfrak{C}_{[a]})$ is the L-direct sum convex space of $\{(X_t,(\mathfrak{C}_t)_{[a]})\}_{t\in T}$ for each $a\in J(M)$, $\mathfrak{C}_{[a]}=\{A\in L^X:\forall t\in T,(f_t)_L^{\leftarrow}(A)\in(\mathfrak{C}_t)_{[a]}\}$ for each $a\in J(M)$. Thus we obtain

$$\mathfrak{C}(A)\geqslant a\Leftrightarrow A\in\mathfrak{C}_{[a]}\Leftrightarrow\forall t\in T,(f_t)_L^{\leftarrow}(A)\in(\mathfrak{C}_t)_{[a]}\Leftrightarrow\bigwedge_{t\in T}\mathfrak{C}_t\Big((f_t)_L^{\leftarrow}(A)\Big)\geqslant a$$

It follows that

$$\begin{aligned}\mathfrak{C}(A) &= \bigvee\{a\in J(M):\mathfrak{C}(A)\geqslant a\}\\ &= \bigvee\{a\in J(M):\bigwedge_{t\in T}\mathfrak{C}_t\Big((f_t)_L^{\leftarrow}(A)\Big)\geqslant a\}\\ &= \bigwedge_{t\in T}\mathfrak{C}_t\Big((f_t)_L^{\leftarrow}(A)\Big).\end{aligned}$$

This implies that $(X,\mathfrak{C})$ is the (L,M)-fuzzy direct sum convex space of $\{(X_t,\mathfrak{C}_t)\}_{t\in T}$. □

Corollary 4.3. *Let $\{(X_t,\mathfrak{C}_t)\}_{t\in T}$ be a family of M-fuzzifying convex spaces and $\mathfrak{C}:2^X\to M$ be a mapping, where X is the disjoint union of $\{X_t\}_{t\in T}$. Then the following conditions are equivalent:*

(1) *$(X,\mathfrak{C})$ is the M-fuzzifying convex space of $\{(X_t,\mathfrak{C}_t)\}_{t\in T}$;*
(2) *For each $a\in J(M)$, $(X,\mathfrak{C}_{[a]})$ is the direct sum convex space of $\{(X_t,(\mathfrak{C}_t)_{[a]})\}_{t\in T}$.*

Theorem 4.4. *Let $\{(X_t,\mathfrak{C}_t)\}_{t\in T}$ be a family of (L,M)-fuzzy convex spaces and $\mathfrak{C}:L^X\to M$ be a mapping, where X is the disjoint union of $\{X_t\}_{t\in T}$. Then the following conditions are equivalent:*

(1) *$(X,\mathfrak{C})$ is the (L,M)-fuzzy direct sum convex space of $\{(X_t,\mathfrak{C}_t)\}_{t\in T}$;*

(2) *For each $a \in \alpha(\perp_M)$, $(X, \mathfrak{C}^{[a]})$ is the L-direct sum convex space of $\{(X_t, (\mathfrak{C}_t)^{[a]})\}_{t\in T}$.*

Proof. (1) ⇒ (2). By Lemma 4.1(3), we know that $\{(X_t, \mathfrak{C}_t{}^{[a]})\}_{t\in T}$ is a family of L-convex spaces for each $a \in \alpha(\perp_M)$. Since $(X, \mathfrak{C})$ is the (L, M)-fuzzy direct sum convex space of $\{(X_t, \mathfrak{C}_t)\}_{t\in T}$, it follows that $\mathfrak{C}(A) = \bigwedge_{t\in T} \mathfrak{C}_t\Big((f_t)_L^{\leftarrow}(A)\Big)$ for each $A \in L^X$. For each $a \in \alpha(\perp_M)$, we obtain

$$\begin{aligned} A \in \mathfrak{C}^{[a]} \quad &\Leftrightarrow a \notin \alpha((A)) = \alpha\Big(\bigwedge_{t\in T} \mathfrak{C}_t\Big((f_t)_L^{\leftarrow}(A)\Big)\Big) \\ &\Leftrightarrow a \notin \bigcup_{t\in T} \alpha\Big(\mathfrak{C}_t((f_t)_L^{\leftarrow}(A))\Big) \\ &\Leftrightarrow \forall t \in T, a \notin \alpha\Big(\mathfrak{C}_t((f_t)_L^{\leftarrow}(A))\Big) \\ &\Leftrightarrow \forall t \in T, (f_t)_L^{\leftarrow}(A) \in (\mathfrak{C}_t)^{[a]}. \end{aligned}$$

Thus $\mathfrak{C}^{[a]} = \{A \in L^X : \forall t \in T, (f_t)_L^{\leftarrow}(A) \in (\mathfrak{C}_t)^{[a]}\}$ for each $a \in \alpha(\perp_M)$. Hence $(X, \mathfrak{C}^{[a]})$ is the L-direct sum convex space of $\{(X_t, (\mathfrak{C}_t)^{[a]})\}_{t\in T}$ for each $\alpha(\perp_M)$.

(2) ⇒ (1). Since $(X, \mathfrak{C}^{[a]})$ is the L-direct sum convex space of $\{(X_t, (\mathfrak{C}_t)^{[a]})\}_{t\in T}$ for each, $\mathfrak{C}^{[a]} = \{A \in L^X : \forall t \in T, (f_t)_L^{\leftarrow}(A) \in (\mathfrak{C}_t)^{[a]}\}$ for each $a \in \alpha(\perp_M)$. Thus we obtain

$$\begin{aligned} a \notin \alpha(\mathfrak{C}(A)) \quad &\Leftrightarrow A \in \mathfrak{C}^{[a]} \\ &\Leftrightarrow \forall t \in T, (f_t)_L^{\leftarrow}(A) \in (\mathfrak{C}_t)^{[a]} \\ &\Leftrightarrow a \notin \bigcup_{t\in T} \alpha\Big(\mathfrak{C}_t((f_t)_L^{\leftarrow}(A))\Big) \\ &\Leftrightarrow a \notin \alpha\Big(\bigwedge_{t\in T} \mathfrak{C}_t((f_t)_L^{\leftarrow}(A))\Big). \end{aligned}$$

Hence $\alpha(\mathfrak{C}(A)) = \alpha\Big(\bigwedge_{t\in T} \mathfrak{C}_t((f_t)_L^{\leftarrow}(A))\Big)$,and thus

$$\mathfrak{C}(A) = \bigwedge \alpha(\mathfrak{C}(A)) = \bigwedge \alpha\Big(\bigwedge_{t\in T} \mathfrak{C}_t((f_t)_L^{\leftarrow}(A))\Big) = \bigwedge_{t\in T} \mathfrak{C}_t((f_t)_L^{\leftarrow}(A))$$

This implies that $(X, \mathfrak{C})$ is the (L, M)-fuzzy direct sum convex space of $\{(X_t, \mathfrak{C}_t)\}_{t\in T}$. □

Corollary 4.5. *Let $\{(X_t, \mathfrak{C}_t)\}_{t\in T}$ be a family of M-fuzzifying convex spaces and $\mathfrak{C} : 2^X \to M$ be a mapping, where X is the disjoint union of $\{X_t\}_{t\in T}$. Then the following conditions are equivalent:*

(1) *$(X, \mathfrak{C})$ is the M-fuzzifying convex space of $\{(X_t, \mathfrak{C}_t)\}_{t\in T}$;*
(2) *For each $a \in \alpha(\perp_M)$, $(X, \mathfrak{C}^{[a]})$ is the direct sum convex space of $\{(X_t, (\mathfrak{C}_t)^{[a]})\}_{t\in T}$.*

Theorem 4.6. *Let $\{(X_t, \mathfrak{C}_t)\}_{t\in T}$ be a family of (L, M)-fuzzy convex spaces and $\mathfrak{C} : L^X \to M$ be a mapping, where X is the disjoint union of $\{X_t\}_{t\in T}$. Then the following conditions are equivalent:*

(1) *$(X, \mathfrak{C})$ is the (L, M)-fuzzy direct sum convex space of $\{(X_t, \mathfrak{C}_t)\}_{t\in T}$;*
(2) *For each $a \in P(M)$, $(X, \mathfrak{C}^{(a)})$ is the L-direct sum convex space $\{(X_t, (\mathfrak{C}_t)^{(a)})\}_{t\in T}$.*

Proof. (1) ⇒ (2). By Lemma 4.1(4), we know that $\{(X_t, (\mathfrak{C}_t)_{(a)})\}_{t\in T}$ is a family of L-convex spaces for each $a \in P(M)$. Since $(X, \mathfrak{C})$ is the (L, M)-fuzzy direct sum convex space of $\{(X_t, \mathfrak{C}_t)\}_{t\in T}$, $\mathfrak{C}(A) = \bigwedge_{t\in T} \mathfrak{C}_t\Big((f_t)_L^{\leftarrow}(A)\Big)$ for each $A \in L^X$. For each $a \in P(M)$, we obtain

$$\begin{aligned} A \in \mathfrak{C}^{(a)} \quad &\Leftrightarrow \mathfrak{C}(A) = \bigwedge_{t\in T} \mathfrak{C}_t\Big((f_t)_L^{\leftarrow}(A)\Big) \nleq a \\ &\Leftrightarrow \forall t \in T, \mathfrak{C}_t\Big((f_t)_L^{\leftarrow}(A)\Big) \nleq a \\ &\Leftrightarrow \forall t \in T, (f_t)_L^{\leftarrow}(A) \in (\mathfrak{C}_t)^{(a)}. \end{aligned}$$

Thus $\mathfrak{C}^{(a)} = \{A \in L^X : \forall t \in T, (f_t)_L^{\leftarrow}(A) \in (\mathfrak{C}_t)^{(a)}\}$ for each $a \in P(M)$. Hence $(X, \mathfrak{C}^{(a)})$ is the L-direct sum convex space of $\{(X_t, (\mathfrak{C}_t)^{(a)})\}_{t\in T}$ for each $a \in P(M)$.

(2) ⇒ (1). Since $(X, \mathfrak{C}^{(a)})$ is the L-direct sum convex space of $\{(X_t, (\mathfrak{C}_t)^{(a)})\}_{t\in T}$ for each $a \in P(M)$, $\mathfrak{C}^{(a)} = \{A \in L^X : \forall t \in T, (f_t)_L^{\leftarrow}(A) \in (\mathfrak{C}_t)^{(a)}\}$ for each $a \in P(M)$. Thus we obtain

$$\mathfrak{C}(A) \nleq a \Leftrightarrow A \in \mathfrak{C}^{(a)} \Leftrightarrow \forall t \in T, (f_t)_L^{\leftarrow}(A) \in (\mathfrak{C}_t)^{(a)} \Leftrightarrow \bigwedge_{t\in T} \mathfrak{C}_t\Big((f_t)_L^{\leftarrow}(A)\Big) \nleq a$$

It follows that

$$\begin{aligned} \mathfrak{C}(A) &= \bigwedge\{a \in P(M) : \mathfrak{C}(A) \nleq a\} \\ &= \bigwedge\{a \in P(M) : \bigwedge_{t\in T} \mathfrak{C}_t\Big((f_t)_L^{\leftarrow}(A)\Big) \nleq a\} \\ &= \bigwedge_{t\in T} \mathfrak{C}_t\Big((f_t)_L^{\leftarrow}(A)\Big). \end{aligned}$$

This implies that $(X, \mathfrak{C})$ is the (L, M)-fuzzy direct sum convex space of $\{(X_t, \mathfrak{C}_t)\}_{t\in T}$. □

Corollary 4.7. *Let $\{(X_t, \mathfrak{C}_t)\}_{t\in T}$ be a family of M-fuzzifying convex spaces and $\mathfrak{C} : 2^X \to M$ be a mapping, where X is the disjoint union of $\{X_t\}_{t\in T}$. Then the following conditions are equivalent:*

(1) *$(X, \mathfrak{C})$ is the M-fuzzifying convex space of $\{(X_t, \mathfrak{C}_t)\}_{t\in T}$;*
(2) *For each $a \in P(M)$, $(X, \mathfrak{C}^{(a)})$ is the direct sum convex space of $\{(X_t, (\mathfrak{C}_t)^{(a)})\}_{t\in T}$.*

Theorem 4.8. *Let $\{(X_t, \mathfrak{C}_t)\}_{t\in T}$ be a family of (L, M)-fuzzy convex spaces and $\mathfrak{C} : L^X \to M$ be a mapping, where X is the disjoint union of $\{X_t\}_{t\in T}$. If $\beta(\wedge_{i\in I} a_i) = \cap_{i\in I}\beta(a_i)$ for all $\{a_i\} \subseteq M$, then the following conditions are equivalent:*

(1) *$(X, \mathfrak{C})$ is the (L, M)-fuzzy direct sum convex space of $\{(X_t, \mathfrak{C}_t)\}_{t\in T}$;*
(2) *For each $a \in \beta(\top_M)$, $(X, \mathfrak{C}_{(a)})$ is the L-direct sum convex space $\{(X_t, (\mathfrak{C}_t)_{(a)})\}_{t\in T}$.*

Proof. (1) ⇒ (2). By Lemma 4.1(5), we know that $\{(X_t, \mathfrak{C}_{t(a)})\}_{t\in T}$ is a family of L-convex spaces for each $a \in \beta(\top_M)$. Since $(X, \mathfrak{C})$ is the (L, M)-fuzzy direct sum convex space of $\{(X_t, \mathfrak{C}_t)\}_{t\in T}$, it follows that $\mathfrak{C}(A) = \bigwedge_{t\in T} \mathfrak{C}_t\Big((f_t)_L^{\leftarrow}(A)\Big)$ for each $A \in L^X$. For each $a \in \beta(\top_M)$, we obtain

$$\begin{aligned} A \in \mathfrak{C}_{(a)} &\Leftrightarrow a \in \beta\Big((A)\Big) = \beta\Big(\bigwedge_{t\in T} \mathfrak{C}_t\Big((f_t)_L^{\leftarrow}(A)\Big)\Big) \\ &\Leftrightarrow a \in \bigcap_{t\in T} \beta\Big(\mathfrak{C}_t((f_t)_L^{\leftarrow}(A))\Big) \\ &\Leftrightarrow \forall t \in T, a \in \beta\Big(\mathfrak{C}_t((f_t)_L^{\leftarrow}(A))\Big) \\ &\Leftrightarrow \forall t \in T, (f_t)_L^{\leftarrow}(A) \in (\mathfrak{C}_t)_{(a)}. \end{aligned}$$

Thus $\mathfrak{C}_{(a)} = \{A \in L^X : \forall t \in T, (f_t)_L^{\leftarrow}(A) \in (\mathfrak{C}_t)_{(a)}\}$ for each $a \in \beta(\top_M)$. Hence $(X, \mathfrak{C}_{(a)})$ is the L-direct sum convex space of $\{(X_t, (\mathfrak{C}_t)_{(a)})\}_{t\in T}$ for each $a \in \beta(\top_M)$.

(2) ⇒ (1). Since $(X, \mathfrak{C}_{(a)})$ is the L-direct sum convex space of $\{(X_t, (\mathfrak{C}_t)_{(a)})\}_{t\in T}$, $\mathfrak{C}_{(a)} = \{A \in L^X : \forall t \in T, (f_t)_L^{\leftarrow}(A) \in (\mathfrak{C}_t)_{(a)}\}$ for each $a \in \beta(\top_M)$. Thus we obtain

$$\begin{aligned} a \in \beta(\mathfrak{C}(A)) &\Leftrightarrow A \in \mathfrak{C}_{(a)} \\ &\Leftrightarrow \forall t \in T, (f_t)_L^{\leftarrow}(A) \in (\mathfrak{C}_t)_{(a)} \\ &\Leftrightarrow a \in \bigcap_{t\in T} \beta\Big(\mathfrak{C}_t((f_t)_L^{\leftarrow}(A))\Big) \\ &\Leftrightarrow a \in \beta\Big(\bigwedge_{t\in T} \mathfrak{C}_t((f_t)_L^{\leftarrow}(A))\Big). \end{aligned}$$

Hence $\beta(\mathfrak{C}(A)) = \beta\Big(\bigwedge_{t\in T} \mathfrak{C}_t((f_t)_L^{\leftarrow}(A))\Big)$, and thus $\mathfrak{C}(A) = \bigwedge_{t\in T} \mathfrak{C}_t((f_t)_L^{\leftarrow}(A))$. This implies that $(X, \mathfrak{C})$ is the (L, M)-fuzzy direct sum convex space of $\{(X_t, \mathfrak{C}_t)\}_{t\in T}$. □

Corollary 4.9. *Let $\{(X_t, \mathfrak{C}_t)\}_{t\in T}$ be a family of M-fuzzifying convex spaces and $\mathfrak{C} : 2^X \to M$ be a mapping, where X is the disjoint union of $\{X_t\}_{t\in T}$. If $\beta(\wedge_{i\in I} a_i) = \cap_{i\in I}\beta(a_i)$ for all $\{a_i\} \subseteq M$, then the following conditions are equivalent:*

(1) *$(X, \mathfrak{C})$ is the M-fuzzifying convex space of $\{(X_t, \mathfrak{C}_t)\}_{t\in T}$;*
(2) *For each $a \in \beta(\top_M)$, $(X, \mathfrak{C}_{(a)})$ is the direct sum convex space of $\{(X_t, (\mathfrak{C}_t)_{(a)})\}_{t\in T}$.*

5. The additivity of separability

In this section, we first introduce the definitions of some low-level separability (S_{-1}, sub-S_0, S_0, S_1 and S_2) of (L, M)-convex spaces, which is different from the concepts in [6, 33]. Then we will discuss the hereditary and additivity of these separability. These results bring convenience for further research on the theory of (L, M)-fuzzy convex spaces in the future.

Definition 5.1. Let $(X, \mathfrak{C})$ be an (L, M)-fuzzy convex space.

(1) The degree to which two distinguished fuzzy points $x_\lambda, x_\mu \in J(L^X)$ are (L, M)-fuzzy S_{-1}, is defined as follows:

$$S_{-1}(x_\lambda, x_\mu) = \bigvee_{x_\mu \not\leqslant A \geqslant x_\lambda} \mathfrak{C}(A)$$

for all $A \in L^X$.

The degree to which $(X, \mathfrak{C})$ is S_{-1}, is defined by

$$S_{-1}(X, \mathfrak{C}) = \bigwedge \{S_{-1}(x_\lambda, x_\mu) : x_\lambda, x_\mu \in J(L^X), \mu \not\leqslant \lambda\}.$$

(2) The degree to which two distinguished fuzzy points $x_\lambda, y_\lambda \in J(L^X)$ are (L, M)-fuzzy sub-S_0, is defined as follows:

$$\text{sub-}S_0(x_\lambda, y_\lambda) = \bigvee_{x_\lambda \not\leqslant A \geqslant y_\lambda} \mathfrak{C}(A) \vee \bigvee_{y_\lambda \not\leqslant B \geqslant x_\lambda} \mathfrak{C}(B)$$

for all $A, B \in L^X$.

The degree to which $(X, \mathfrak{C})$ is sub-S_0, is defined by

$$\text{sub-}S_0(X, \mathfrak{C}) = \bigwedge \{\text{sub-}S_0(x_\lambda, y_\lambda) : x, y \in X, \lambda \in J(L), x \neq y\}.$$

(3) The degree to which two distinguished fuzzy points $x_\lambda, y_\mu \in J(L^X)$ are (L, M)-fuzzy S_0, is defined as follows:

$$S_0(x_\lambda, y_\mu) = \bigvee_{x_\lambda \not\leqslant A \geqslant y_\mu} \mathfrak{C}(A) \vee \bigvee_{y_\mu \not\leqslant B \geqslant x_\lambda} \mathfrak{C}(B)$$

for all $A, B \in L^X$.

The degree to which $(X, \mathfrak{C})$ is S_0, is defined by

$$S_0(X, \mathfrak{C}) = \bigwedge \{S_0(x_\lambda, y_\mu) : x_\lambda, y_\mu \in J(L^X), x_\lambda \neq y_\mu\}.$$

(4) The degree to which two distinguished fuzzy points $x_\lambda, y_\mu \in J(L^X)$ are (L, M)-fuzzy S_1, is defined as follows:

$$S_1(x_\lambda, y_\mu) = \bigvee_{x_\lambda \not\leqslant A \geqslant y_\mu} \mathfrak{C}(A)$$

for all $A \in L^X$.

The degree to which $(X, \mathfrak{C})$ is S_1, is defined by

$$S_1(X, \mathfrak{C}) = \bigwedge \{S_1(x_\lambda, y_\lambda) : x_\lambda, y_\mu \in J(L^X), x_\lambda \not\leqslant y_\mu\}.$$

(5) The degree to which two distinguished fuzzy points $x_\lambda, y_\mu \in J(L^X)$ are (L, M)-fuzzy S_2, is defined as follows:

$$S_2(x_\lambda, y_\mu) = \bigvee_{x_\lambda \leqslant A \leqslant (y_\mu)'} \mathfrak{C}(A) \wedge \mathfrak{C}(A')$$

for all $A \in L^X$.

The degree to which $(X, \mathfrak{C})$ is S_2, is defined by

$$S_2(X, \mathfrak{C}) = \bigwedge \{S_2(x_\lambda, y_\mu) : x_\lambda, y_\mu \in J(L^X), x_\lambda \leqslant (y_\mu)'\}.$$

In the following, we provide an important lemma that is indispensable in subsequent proofs.

Lemma 5.2. *Let* $(X, \mathfrak{C})$ *be an* (L, M)*-fuzzy convex space and* $\emptyset \subseteq Y \subseteq X$*. Then we obtain*
(1) $\forall A \in L^Y$, $x_\lambda, y_\mu \in J(L^Y)$ *implies* $\bigvee\limits_{x_\lambda \nleqslant A \geqslant y_\mu} (\mathfrak{C}|_Y)(A) \leqslant \bigvee\limits_{x^*_\lambda \nleqslant A^\star \geqslant y^*_\mu} \mathfrak{C}(A^\star)$.
(2) $\forall A \in L^X$,$x_\lambda \in J(L^X)$, $y_\mu \in J(L^Y)$ *implies* $\bigvee\limits_{x_\lambda \nleqslant A \geqslant y_\mu} \mathfrak{C}(A) \leqslant \bigvee\limits_{x_\lambda|_Y \nleqslant A|_Y \geqslant y_\mu|_Y} (\mathfrak{C}|_Y)(A|_Y)$.

Proof. (1) Notice that for any $A \in L^Y$, $x_\lambda, y_\mu \in J(L^Y)$ with $x_\lambda \nleqslant A \geqslant y_\mu$,

$$\{C \in L^X : C|_Y = A\} \subseteq \{C \in L^Y : C \leqslant A^\star, x^*_\lambda \nleqslant A^\star \geqslant y^*_\mu\},$$

so we obtain

$$\bigvee_{x_\lambda \nleqslant A \geqslant y_\mu} (\mathfrak{C}|_Y)(A) = \bigvee_{x_\lambda \nleqslant A \geqslant y_\mu} \bigvee_{C|_Y = A} \mathfrak{C}(C) \leqslant \bigvee_{x^*_\lambda \nleqslant A^\star \geqslant y^*_\mu} \mathfrak{C}(A^\star).$$

(2) For any $A \in L^X, x_\lambda \in J(L^X), y_\mu \in J(L^Y)$ with $x_\lambda \nleqslant A \geqslant y_\mu$, it follows that $x_\lambda|_Y \nleqslant A|_Y \geqslant y_\mu|_Y$. Thus we obtain

$$\begin{aligned} \bigvee_{x_\lambda|_Y \nleqslant A|_Y \geqslant y_\mu|_Y} (\mathfrak{C}|_Y)(A|_Y) &\geqslant \bigvee_{x_\lambda \nleqslant A \geqslant y_\mu} (\mathfrak{C}|_Y)(A|_Y) \\ &= \bigvee_{x_\lambda \nleqslant A \geqslant y_\mu} \bigvee_{B|_Y = A|_Y} \mathfrak{C}(B) \\ &= \bigvee_{x_\lambda \nleqslant A \geqslant y_\mu} \mathfrak{C}(A). \end{aligned}$$

□

Proposition 5.3. *Let* $(X, \mathfrak{C})$ *be an* (L, M)*-fuzzy convex space and* $\emptyset \neq Y \subseteq X$*. Then the following inequalities are hold:*

(1) $S^X_{-1}(X, \mathfrak{C}) \leqslant S^Y_{-1}(Y, \mathfrak{C}|_Y)$;
(2) *sub-*$S^X_0(X, \mathfrak{C}) \leqslant$ *sub-*$S^Y_0(Y, \mathfrak{C}|_Y)$;
(3) $S^X_0(X, \mathfrak{C}) \leqslant S^Y_0(Y, \mathfrak{C}|_Y)$;
(4) $S^X_1(X, \mathfrak{C}) \leqslant S^Y_1(Y, \mathfrak{C}|_Y)$.

Proof. We prove only (3) and the others can be proved in a similar way and therefore their proofs are omitted. For any $x_\lambda, y_\mu \in J(L^Y)$ with $x_\lambda \neq y_\mu$, we obtain

$$S^Y_0(x_\lambda, y_\mu) = \bigvee_{x_\lambda \nleqslant A \geqslant y_\mu} (\mathfrak{C}|_Y)(A) \vee \bigvee_{y_\mu \nleqslant B \geqslant x_\lambda} (\mathfrak{C}|_Y)(B)$$

and

$$S^X_0(x^*_\lambda, y^*_\mu) = \bigvee_{x_\lambda * \nleqslant C \geqslant y_\mu *} \mathfrak{C}(C) \vee \bigvee_{y_\mu * \nleqslant D \geqslant x_\lambda *} \mathfrak{C}(D).$$

Let $A \in L^Y$ with $x_\lambda \nleqslant A \geqslant y_\mu$ and $B \in L^Y$ with $y_\mu \nleqslant B \geqslant x_\lambda$. Then $x^*_\lambda \nleqslant A^\star \geqslant y^*_\mu$ and $y^*_\mu \nleqslant B^\star \geqslant x^*_\lambda$ from Definition 2.3. Thus we can obtain that

$$\begin{aligned} S^Y_0(x_\lambda, y_\mu) &= \bigvee_{x_\lambda \nleqslant A \geqslant y_\mu} (\mathfrak{C}|_Y)(A) \vee \bigvee_{y_\mu \nleqslant B \geqslant x_\lambda} (\mathfrak{C}|_Y)(B) \\ &\leqslant \bigvee_{x^*_\lambda \nleqslant A^\star \geqslant y^*_\mu} \mathfrak{C}(A^\star) \vee \bigvee_{y^*_\mu \nleqslant B^\star \geqslant x^*_\lambda} \mathfrak{C}(B^\star) \quad \text{(Lemma 5.2(1))} \\ &\leqslant \bigvee_{x^*_\lambda \nleqslant C \geqslant y^*_\mu} \mathfrak{C}(C) \vee \bigvee_{y^*_\mu \nleqslant D \geqslant x^*_\lambda} \mathfrak{C}(D) \\ &= S^X_0(x^*_\lambda, y^*_\mu). \end{aligned}$$

Moreover, let $C \in L^X$ with $x^*_\lambda \not\leqslant C \geqslant y^*_\mu$ and $D \in L^X$ with $y^*_\mu \not\leqslant D \geqslant x^*_\lambda$. Then we obtain $x^*_\lambda|_Y \not\leqslant C|_Y \geqslant y^*_\mu|_Y$ and $y^*_\mu|_Y \not\leqslant D|_Y \geqslant x^*_\lambda|_Y$. Thus, by Lemma 5.2(2), we obtain

$$\begin{aligned} S^X_0(x^*_\lambda, y^*_\mu) &= \bigvee_{x^*_\lambda \not\leqslant C \geqslant y^*_\mu} \mathfrak{C}(C) \vee \bigvee_{y^*_\mu \not\leqslant D \geqslant x^*_\lambda} \mathfrak{C}(D) \\ &\leqslant \bigvee_{x^*_\lambda|_Y \not\leqslant C|_Y \geqslant y^*_\mu|_Y} (\mathfrak{C}|_Y)(C|_Y) \vee \bigvee_{y^*_\mu|_Y \not\leqslant D|_Y \geqslant x^*_\lambda|_Y} (\mathfrak{C}|_Y)(D|_Y) \\ &= \bigvee_{x_\lambda \not\leqslant C|_Y \geqslant y_\mu} (\mathfrak{C}|_Y)(C|_Y) \vee \bigvee_{y_\mu \not\leqslant D|_Y \geqslant x_\lambda} (\mathfrak{C}|_Y)(D|_Y) \\ &\leqslant \bigvee_{x_\lambda \not\leqslant A \geqslant x_\lambda} (\mathfrak{C}|_Y)(A) \vee \bigvee_{y_\mu \not\leqslant B \geqslant x_\lambda} (\mathfrak{C}|_Y)(B) \\ &= S^Y_0(x_\lambda, y_\mu). \end{aligned}$$

This means that $S^Y_0(x_\lambda, y_\mu) = S^X_0(x^*_\lambda, y^*_\mu)$. Hence we obtain

$$\begin{aligned} S^Y_0(Y, \mathfrak{C}|_Y) &= \bigwedge\{S^Y_0(x_\lambda, y_\mu) : x_\lambda, y_\mu \in J(L^Y), x_\lambda \neq y_\mu\} \\ &= \bigwedge\{S^X_0(x^*_\lambda, y^*_\mu) : x_\lambda, y_\mu \in J(L^Y), x_\lambda \neq y_\mu\} \\ &\geqslant \bigwedge\{S^X_0(p_\lambda, q_\mu) : p_\lambda, q_\mu \in J(L^X), p_\lambda \neq q_\mu\} \\ &= S^X_0(X, \mathfrak{C}). \end{aligned}$$

This proof is completed. □

Proposition 5.4. *Let* $(X, \mathfrak{C}) = \sum(X_t, \mathfrak{C}_t)$. *Then* $S_{-1}(X, \mathfrak{C}) = \bigwedge_{t\in T} S_{-1}(X_t, \mathfrak{C}_t)$.

Proof. By Proposition 3.5 and $\forall t \in T, \mathfrak{C}|_{X_t} = \mathfrak{C}_t$, we always obtain that $S_{-1}(X, \mathfrak{C}) \leqslant \bigwedge_{t\in T} S_{-1}(X_t, \mathfrak{C}_t)$.

Now we need to prove that the converse inequality

$$\bigwedge_{t\in T} S_{-1}(X_t, \mathfrak{C}_t) \leqslant S_{-1}(X, \mathfrak{C})$$

holds. Let $a \in M$ be any element with the property of $a \prec \bigwedge_{t\in T} S_{-1}(X_t, \mathfrak{C}_t)$. Then we obtain

$$a \prec S_{-1}(X_t, \mathfrak{C}_t) = \bigwedge\{S^{X_t}_{-1}(x_\lambda, x_\mu) : x_\lambda, x_\mu \in J(L^{X_t}), \mu \not\leqslant \lambda\}$$

for all $t \in T$.

Let $x_\lambda, x_\mu \in J(L^X)$ with $\mu \not\leqslant \lambda$. Then there exists $r \in T$ such that $x \in X_r$. From $a \prec S^{X_r}_{-1}(X_r, \mathfrak{C}_r)$, we know that there exists $A_r \in L^{X_r}$ such that

$$x_\mu|_{X_r} \not\leqslant A_r \geqslant x_\lambda|_{X_r} \text{ and } \mathfrak{C}_r(A_r) \geqslant a.$$

It implies that $x_\mu = (x_\mu|_{X_r})^* \not\leqslant A^*_r \geqslant (x_\lambda|_{X_r})^* = x_\lambda$. Hence, we obtain

$$\begin{aligned} S^X_{-1}(x_\lambda, x_\mu) &= \textstyle\bigvee_{x_\mu \not\leqslant B \geqslant x_\lambda} \mathfrak{C}(B) \\ &\geqslant \textstyle\bigvee_{x_\mu \not\leqslant A^*_r \geqslant x_\lambda} \mathfrak{C}(A^*_r) \\ &= \textstyle\bigvee_{x_\mu \not\leqslant A^*_r \geqslant x_\lambda} \mathfrak{C}_r(A_r) \ \text{(Proposition3.5(3))} \\ &\geqslant \textstyle\bigvee_{x_\mu|_{X_r} \not\leqslant A_r \geqslant x_\lambda|_{X_r}} \mathfrak{C}_r(A_r) \\ &\geqslant a. \end{aligned}$$

This shows that $S_{-1}(X, \mathfrak{C}) = \bigwedge\{S_{-1}(x_\lambda, x_\mu) : x_\lambda, x_\mu \in J(L^X), \mu \not\leqslant \lambda\} \geqslant a$. Therefore, we obtain that $\bigwedge_{t\in T} S_{-1}(X_t, \mathfrak{C}_t) \leqslant S_{-1}(X, \mathfrak{C})$ for all $t \in T$. □

Proposition 5.5. *Let* $(X, \mathfrak{C}) = \sum(X_t, \mathfrak{C}_t)$. *Then sub-*$S_0(X, \mathfrak{C}) = \bigwedge_{t\in T}$ *sub-*$S_0(X_t, \mathfrak{C}_t)$.

Proof. From Proposition 5.3, we obtain *sub-*$S_0(X, \mathfrak{C}) \leqslant \bigwedge_{t\in T}$ *sub-*$S_0(X_t, \mathfrak{C}_t)$ since $\mathfrak{C}|_{X_t} = \mathfrak{C}_t$ by Proposition 3.5. In the following, we need to prove

$$\bigwedge_{t\in T} sub\text{-}S_0(X_t,\mathfrak{C}_t) \leqslant sub\text{-}S_0(X,\mathfrak{C}).$$

Let $a \in J(M)$ be any element with $a \prec \bigwedge_{t\in T} sub\text{-}S_0(X_t,\mathfrak{C}_t)$. Then we have

$$a \prec sub\text{-}S_0(X_t,\mathfrak{C}_t) = \bigwedge\{sub\text{-}S_0^{X_t}(x_\lambda, y_\lambda) : x, y \in X_t, \lambda \in J(L), x \neq y\}$$

for all $t \in T$. Let $x_\lambda, y_\lambda \in J(L^X)$ with $x \neq y$ and consider two cases below:

Case1: $x, y \in X_r$ for some $r \in T$. Thus we obtain $x_\lambda|_{X_r}, y_\lambda|_{X_r} \in J(L^{X_r})$ with $x \neq y$. From $a \prec sub\text{-}S_0(X_r,\mathfrak{C}_r)$, there exists $A_r \in L^{X_r}$ such that

$$x_\lambda|_{X_r} \not\leqslant A_r \geqslant y_\lambda|_{X_r} \ \text{ and } \ \mathfrak{C}_r(A_r) \geqslant a.$$

or $B_r \in L^{X_r}$ such that

$$y_\lambda|_{X_r} \not\leqslant B_r \geqslant x_\lambda|_{X_r} \ \text{ and } \ \mathfrak{C}_r(B_r) \geqslant a.$$

It implies that

$$x_\lambda = (x_\lambda|_{X_r})^* \not\leqslant A_r^* \geqslant (y_\lambda|_{X_r})^* = y_\lambda, \mathfrak{C}(A_r^*) = \mathfrak{C}_r(A_r) \geqslant a$$

or

$$y_\lambda = (y_\lambda|_{X_r})^* \not\leqslant B_r^* \geqslant (x_\lambda|_{X_r})^* = x_\lambda, \mathfrak{C}(B_r^*) = \mathfrak{C}_r(B_r) \geqslant a.$$

Thus, we obtain

$$\text{sub-}S_0(x_\lambda, y_\lambda) = \bigvee_{x_\lambda \not\leqslant A \geqslant y_\lambda} \mathfrak{C}(A) \vee \bigvee_{y_\lambda \not\leqslant B \geqslant x_\lambda} \mathfrak{C}(B) \geqslant \mathfrak{C}(A_r^*) \vee \mathfrak{C}(B_r^*) \geqslant a.$$

Case 2: If $x \in X_t, y \in X_s$ and $s, t \in T$ with $s \neq t$, then $x_\lambda \not\leqslant \top^*_{L^{X_s}} \geqslant y_\lambda$ and $y_\lambda \not\leqslant \top^*_{L^{X_t}} \geqslant x_\lambda$. From Proposition 5.3, we obtain that

$$\text{sub-}S_0(x_\lambda, y_\lambda) = \bigvee_{x_\lambda \not\leqslant A \geqslant y_\lambda} \mathfrak{C}(A) \vee \bigvee_{y_\lambda \not\leqslant B \geqslant x_\lambda} \mathfrak{C}(B) \geqslant \mathfrak{C}(\top^*_{L^{X_s}}) \vee \mathfrak{C}(\top^*_{L^{X_t}}) \geqslant a.$$

This shows that

$$\text{sub-}S_0(X,\mathfrak{C}) = \bigwedge\{\text{sub-}S_0(x_\lambda, y_\lambda) : x, y \in X, \lambda \in J(L), x \neq y\} \geqslant a.$$

This proof is completed. □

Proposition 5.6. *Let* $(X,\mathfrak{C}) = \sum(X_t,\mathfrak{C}_t)$*. Then* $S_0(X,\mathfrak{C}) = \bigwedge_{t\in T} S_0(X_t,\mathfrak{C}_t)$*.*

Proof. It is similar to Proposition 5.5 and omitted. □

Proposition 5.7. *Let* $(X,\mathfrak{C}) = \sum(X_t,\mathfrak{C}_t)$*. Then* $S_1(X,\mathfrak{C}) = \bigwedge_{t\in T} S_1(X_t,\mathfrak{C}_t)$*.*

Proof. It is similar to Proposition 5.5 and omitted. □

Lemma 5.8. *Let* $(X,\mathfrak{C})$ *be an* (L, M)*-fuzzy convex space and* $\emptyset \subseteq Y \subseteq X$*. Then* $S_2^X(X,\mathfrak{C}) \leqslant S_2^Y(Y,\mathfrak{C}|_Y)$*.*

Proof. For any $x_\lambda, y_\mu \in J(L^Y)$ with $x_\lambda \leqslant (y_\mu)'$, we have

$$S_2^Y(x_\lambda, y_\mu) = \bigvee_{x_\lambda \leqslant A \leqslant (y_\mu)'} (\mathfrak{C}|_Y)(A) \wedge (\mathfrak{C}|_Y)(A')$$

and

$$S_2^X(x_\lambda^*, y_\mu^*) = \bigvee_{x_\lambda^* \leqslant B \leqslant (y_\mu^*)'} \mathfrak{C}(B) \wedge \mathfrak{C}(B')$$

from Definition 5.1.

If $B \in L^X$ with $x_\lambda^* \leqslant B \leqslant (y_\mu^*)'$, letting $A = B|_Y$, then

$$x_\lambda = x_\lambda^*|_Y \leqslant A \leqslant (y_\mu^*)'|_Y = (y_\mu^*|_Y) = (y_\mu)'$$

and

$$(\mathfrak{C}|_Y)(A) = (\mathfrak{C}|_Y)(B|_Y) = \bigvee_{C|_Y = B|_Y} \mathfrak{C}(C) \geqslant \mathfrak{C}(B),$$

$(\mathfrak{C}|_Y)(A') \geqslant \mathfrak{C}(B')$ in a similar way. It follows that

$$\mathfrak{C}(B) \wedge \mathfrak{C}(B') \leqslant (\mathfrak{C}|_Y)(A) \wedge (\mathfrak{C}|_Y)(A').$$

Thus $S_2^X(x_\lambda^*, y_\mu^*) \leqslant S_2^Y(x_\lambda, y_\mu)$ for all $x_\lambda, y_\mu \in J(L^Y)$ with $x_\lambda \leqslant (y_\mu)'$. Moreover it follows that $S_2^X(X, \mathfrak{C}) \leqslant S_2^Y(Y, \mathfrak{C}|_Y)$. □

Proposition 5.9. *Let* $(X, \mathfrak{C}) = \sum(X_t, \mathfrak{C}_t)$. *Then* $S_2(X, \mathfrak{C}) = \bigwedge_{t\in T} S_2(X_t, \mathfrak{C}_t)$.

Proof. From Proposition 5.3, we obtain $S_2(X, \mathfrak{C}) \leqslant \bigwedge_{t\in T} S_2(X_t, \mathfrak{C}_t)$ since $\mathfrak{C}|_{X_t} = \mathfrak{C}_t$ by Proposition 3.5. In the following, we need to prove

$$\bigwedge\nolimits_{t\in T} S_2(X_t, \mathfrak{C}_t) \leqslant S_2(X, \mathfrak{C}).$$

For this purpose, suppose $x_\lambda, y_\mu \in J(L^X)$ with $x_\lambda \leqslant (y_\mu)'$ and $a \in M$ be any element such that $a \prec \bigwedge_{t\in T} S_2(X_t, \mathfrak{C}_t)$. Then $\forall t \in T, a \prec S_2(X_t, \mathfrak{C}_t)$. We are going to show $a \leqslant S_2(X, \mathfrak{C})$. The following two cases must be considered:

Case 1: $\exists r \in T, x, y \in X_r$. Thus we obtain $x_\lambda|_{X_r}, y_\mu|_{X_r} \in J(L^{X_r})$ with $x_\lambda|_{X_r} \leqslant (y_\mu|_{X_r})'$. Since $a \leqslant S_2(X, \mathfrak{C})$, there exists $A_r \in L^{X_r}$ such that

$$x_\lambda|_{X_r} \leqslant A_r \leqslant (y_\mu|_{X_r})' \text{ and } \mathfrak{C}_r(A_r) \wedge \mathfrak{C}_r(A_r') \geqslant a.$$

Note that $\mathfrak{C}_r(A_r) = \mathfrak{C}(A_r^*) \geqslant a$ and $\mathfrak{C}_r(A_r') = \mathfrak{C}((A_r')^\star) = \mathfrak{C}((A_r^*)') \geqslant a$ from Proposition 3.5, we obtain

$$x_\lambda = (x_\lambda|_{X_r})^* \leqslant A_r^* \leqslant ((y_\mu|_{X_r})')^* = (y_\mu)' \text{ and } \mathfrak{C}_r(A_r^*) \wedge \mathfrak{C}((A_r^*)') \geqslant a.$$

It follows that

$$S_2(x_\lambda, y_\mu) = \bigvee_{x_\lambda \leqslant A \leqslant (y_\mu)'} \mathfrak{C}(A) \wedge \mathfrak{C}(A') \geqslant \mathfrak{C}_r(A_r^*) \wedge \mathfrak{C}((A_r^*)') \geqslant a$$

Case 2: If $x \in X_t, y \in X_s$ and $s, t \in T$ with $s \neq t$, then $x_\lambda \leqslant \bot^\star_{L^{X_s}} \geqslant (y_\mu)'$ From Proposition 3.5, we obtain that

$$\mathfrak{C}(\bot^\star_{L^{X_s}}) = \top_M \text{ and } \mathfrak{C}((\bot^\star_{L^{X_s}})') = \mathfrak{C}(\top^*_{L^{X_s}}) = \top_M.$$

It follows that

$$S_2(x_\lambda, y_\mu) = \bigvee_{x_\lambda \leqslant A \leqslant (y_\mu)'} \mathfrak{C}(A) \wedge \mathfrak{C}(A') \geqslant \mathfrak{C}_r(\bot^\star_{L^{X_s}}) \wedge \mathfrak{C}((\bot^\star_{L^{X_s}})') = \top_M \geqslant a$$

This shows that

$$S_2(X, \mathfrak{C}) = \bigwedge\{S_2(x_\lambda, y_\mu) : x_\lambda, y_\mu \in J(L^X), x_\lambda \leqslant (y_\mu)'\} \geqslant a.$$

Finally $a \leqslant S_2(X, \mathfrak{C})$, as desired. □

6. Conclusions

As we all know, the direct sum of a family of convex spaces is a basic and very useful operation [24]. In this paper, we first generalized the direct sum of a family of convex spaces to the (L, M)-fuzzy case. Then we discussed the relationships between the direct sum of a family of (L, M)-convex spaces and its factor spaces. Secondly, we characterize the (L, M)-fuzzy direct sum convex space by means of its level L-direct sum convex spaces. Furthermore, we considered the hereditary and additivity of S_{-1}, sub-S_0, S_0, S_1 and S_2 separability. These research results can further enrich and develop the theory of (L, M)-fuzzy convex spaces. Following this paper, we will consider the following problems in the future.

(1) The additivity of some special L-convex spaces (arity $\leqslant n$, CUP and JHC, respectively.) has been studied in [36]. The notion of (L, M)-fuzzy convex spaces can be seen as a broader form of L-convex spaces, so we will consider the additivity of arity $\leqslant n$, CUP and JHC in the framework of (L, M)-fuzzy convex spaces.

(2) In the framework of M-fuzzifying convex spaces, the additivity of S_3 and S_4 separability have been discussed. The notion of (L, M)-fuzzy convex spaces can be seen as a broader form of M-fuzzifying convex spaces, so we will extend it to the (L, M)-fuzzy case and study the additivity of S_3 and S_4 separability of (L, M)-fuzzy convex spaces.

(3) According to the monograph [24], convex structures exist in many mathematical structures, such as vector spaces, partially ordered sets, and metric spaces, etc. Therefore, we can study the properties of (L, M)-fuzzy convex structures in a specific mathematical structure. This can further enrich and develop the theory of fuzzy convex spaces.

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