Filomat 39:16 (2025), 5661–5673
https://doi.org/10.2298/FIL2516661C

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Slant helices along an isotropic Riemannian maps

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Abstract. This paper aims to introduce the notion of slant helix along an isotropic Riemannian map. The necessary conditions for a curve along an isotropic Riemannian maps to be a slant helix are obtained in terms of differential equations. In addition, certain conditions were found for the slant helix along an isotropic Riemannian map to be a slant helix in the ambient space. The characterizations are obtained for the transportation of slant helices and helices on the total manifold to the target manifold along a Riemannian map (or vice versa).

1. Introduction

Curves are fundamental geometric structures, and analyzing their behavior under specific maps is a crucial technique for reaching geometric conclusions. Nomizu and Yano first utilized this method for circular and isometric immersions in their work cited as reference [9]. They demonstrated that submanifolds are umbilical and have a parallel mean curvature vector field when a circle on the submanifolds is transported along the immersion to the ambient manifold. The concept of isotropic immersions has been studied in various geometric contexts, including Kähler geometry, as shown by O'Neill [2]. In reference [23], Ikawa obtained a similar characterization for helices. The result has been expanded to the semi-Riemannian situation in references [5], [8], [19], [20] and [24]. The papers demonstrate that analyzing how a certain curve behaves under transformation provides valuable insights when comparing the geometry of two manifolds. Tükel et al. explore isotropic Riemannian maps as a generalization of isotropic immersions and helices along Riemannian maps [3]. In recent years, various generalizations of Riemannian maps between almost some manifolds have been introduced and systematically studied. These include pointwise slant, hemi-slant, semi-slant, and bi-slant Riemannian maps [12–15, 25, 26, 28], as well as their conformal counterparts such as conformal anti-invariant, semi-invariant, and slant Riemannian maps to Kaehler manifolds [16–18, 27]. Furthermore, several of these works also focus on associated geometric inequalities, including Casorati-type estimates [14, 28].

Izumiya and Takeuchi have established definitions for slant helices and conical geodesic curves in three-dimensional Euclidean space. Those concepts are abstractions of cylindrical helices. Kula et al. [10] (see also [11]) identified space curves as slant helices by analyzing specific differential equations. Slant helices' geometry has been examined in semi-Riemannian geometry by many researchers cited in references [4], [7], and [22]. Çalışkan and Şahin introduce the notion of slant helix on Riemannian manifolds, obtaining

2020 *Mathematics Subject Classification*. Primary 53A35; Secondary 53B20.
Keywords. Helix, Slant helix, Riemannian maps, Isotropic submanifold, Totally geodesic submanifold.
Received: 05 August 2024; Revised: 31 May 2025; Accepted: 05 July 2025
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necessary conditions for curves and slant helix along immersions, and providing criteria for immersions [1].

The main aim is to provide a precise definition of a horizontal slant helix along an isotropic Riemannian maps and explore its fundamental characteristics. In section 2, the fundamental concepts relevant to the focus of this study are introduced. The third section introduces the definition of a horizontal slant helix along an isotropic Riemannian maps. This term aligns with the horizontal slant helix concept described in ambient spaces. This section provides a characterization for a curve on the manifold to be classified as a horizontal slant helix. The submanifold's characterization is achieved by ensuring that the curve on a certain base manifold is converted into the ambient manifold as a horizontal slant helix. Section 4 discusses the transformation of a horizontal helix and a horizontal slant helix into each other by an isotropic Riemannian map. When a horizontal slant helix is mapped into a horizontal helix using an isometric Riemannian map, the map is proven to be completely geodesic.

2. Preliminaries

Let c be immersed unit speed curve in a n-dimensional Riemannian manifold. We denote the unit tangent vector field, the unit normal vector field, and the binormal vector field of the curve by X, Y, and Z, respectively. $\tau = \langle \nabla_X Z, Y\rangle$ is the torsion of the curve. The curve has also curvatures $\kappa > 0, \tau, k_3, k_4, ..., k_{n-1}$ and Frenet frame $N_0 = X, N_1 = Y, N_2 = Z, N_3, N_4, ..N_{n-1}$. Then, the Frenet equations are given by

$$\nabla_X N_i = -k_i N_{i-1} + k_{i+1} N_{i+1}, \;\; 0 \leq i \leq n-1$$

In this case, c is called a Frenet curve of order n, [6]. An helix satisfies the following equation

$$\nabla^3{}_X X + K \nabla_X X = 0 \tag{1}$$

where K is constant [23].

Definition 2.1. *[21] Let $\alpha(s)$ be a Frenet curve and denote the tangent vector field of $\alpha(s)$ by ξ_s. A regular Frenet curve $\alpha = \alpha(s)$ parameterized by arc length s with $\kappa \neq 0$ is called a slant helix if there are unit vector fields V_2, V_3 along α such that*

$$\begin{aligned} \nabla_{\xi_s} \xi_s &= \kappa V_2, \\ \nabla_{\xi_s} V_2 &= -\kappa \xi_s + \tau V_3, \\ \nabla_{\xi_s} V_3 &= -\tau V_2, \end{aligned} \tag{2}$$

and $\frac{\kappa^2 \left(\frac{\tau}{\kappa}\right)'}{(\kappa^2+\tau^2)^{\frac{3}{2}}}$ is non-zero constant. The numbers κ and τ are called curvature and torsion of the slant helix, respectively.

Definition 2.2. *[1] Let $c(s)$ be a Frenet curve with curvatures $\kappa, \tau \neq 0$ on a Riemannian manifold M ($dimM \geq 3$). If $c(s)$ is a slant helix, then the unit tangent vector field X and the unit vector field Y of the curve satisfy*

$$\nabla^3{}_X X = 2\kappa' \nabla_X Y + \kappa \nabla_X^2 Y + \left(\frac{\tau''}{\tau} - \frac{3}{2} \frac{\kappa}{\tau} \left(\frac{\tau}{\kappa}\right)' \left(\ln(\kappa^2 + \tau^2)\right)' \right) \nabla_X X. \tag{3}$$

$\mathcal{T} : (M_1, g_{M_1}) \rightarrow (M_2, g_{M_2})$ is a map from the Riemannian manifold M_1 with $dimM_1 = m$ to the Riemannian manifold M_2 with $dimM_2 = n$, where $0 < rank\mathcal{T} < min\{m, n\}$. Thus, we represent the kernel space of $\mathcal{T}_*$ by $\ker \mathcal{T}_*$ and $H = (ker\mathcal{T}_*)^\perp$ is orthogonal complementary space to $\ker \mathcal{T}_*$. So, we have

$$TM_1 = ker\mathcal{T}_* \oplus (ker\mathcal{T}_*)^\perp$$

where TM_1 is the tangent bundle of M_1. range$\mathcal{T}_*$ denotes the range of $\mathcal{T}_*$ and $(range\mathcal{T}_*)^\perp$ denotes the orthogonal complementary space to range$\mathcal{T}_*$ in TM_2. The tangent bundle TM_2 of M_2 is given by

$$TM_2 = range\mathcal{T}_* \oplus (range\mathcal{T}_*)^\perp.$$

Now, a smooth map $T : (M_1^m, g_{M_1}) \to (M_2^n, g_{M_2})$ is called Riemannian map if $\mathcal{T}_*$ satisfies

$$g_{M_2}(\mathcal{T}_* X_1, \mathcal{T}_* X_2) = g_{M_1}(X_1, X_2)$$

for X_1, X_2 vector fields tangent to H.

Assume that (M_1, g_{M_1}) and (M_2, g_{M_2}) are Riemannian manifolds, $\mathcal{T} : (M_1, g_{M_1}) \to (M_2, g_{M_2})$ a smooth map between them, and γ a curve on M_1. γ is called a horizontal curve if $\dot{\gamma}(t) \in (\ker \mathcal{T}_*)^\perp$ for any $t \in I$. If γ is an helix with $\dot{\gamma}(t) \in (\ker \mathcal{T}_*)^\perp$ for any $t \in I$, then it is called a horizontal helix.

Let $\mathcal{T}$ be a Riemannian map between the manifolds (M_1, g_{M_1}) and (M_2, g_{M_2}), $p_2 = T(p_1)$ for each $p_1 \in M_1$. Suppose that ∇^{M_2} and ∇^{M_1} represent the connections on (M_2, g_{M_2}) and (M_1, g_{M_1}) respectively. The second fundamental form of $\mathcal{T}$ can be given as follows:

$$(\nabla \mathcal{T}_*)(X_1, X_2) = \overset{M_2}{\nabla^{\mathcal{T}}}_{X_1} \mathcal{T}_*(X_2) - \mathcal{T}_*(\overset{M_1}{\nabla}_{X_1} X_2) \tag{4}$$

for $X_1, X_2 \in \Gamma(TM_1)$, where $\overset{M_2}{\nabla^{\mathcal{T}}}$ is the pullback connection of $\overset{M_2}{\nabla}$. For all $X_1, X_2 \in \Gamma((\ker T_{*p_1})^\perp)$, $(\nabla \mathcal{T}_*)$ is symmetric and has no components in range $\mathcal{T}_*$. So, we can write the following:

$$g_{M_2}((\nabla \mathcal{T}_*)(X_1, X_2), \mathcal{T}_*(X_3)) = 0 \tag{5}$$

for all $X_1, X_2, X_3 \in \Gamma((\ker \mathcal{T}_{*p_1})^\perp)$ [3].

Now, we give some basic formulas for Riemannian maps defined from the total manifold (M_1, g_{M_1}) to the target manifold (M_2, g_{M_2}). For $X_1, X_2 \in (\Gamma(\ker T_{*p_1})^\perp)$ and $U_1 \in \Gamma((\text{range } \mathcal{T}_*)^\perp)$, we have:

$$\overset{M_2}{\nabla^{\mathcal{T}}}_{X_1} U_1 = -S_{U_1} \mathcal{T}_* X_1 + \nabla_{X_1}^{T^\perp} U_1 \tag{6}$$

where $S_{U_1} \mathcal{T}_* X_1$ is the tangential component of $\overset{M_2}{\nabla^{\mathcal{T}}}_{X_1} U_1$, and $\nabla_{X_1}^{\mathcal{T}^\perp}$ is the orthogonal projection of $\overset{M_2}{\nabla^{\mathcal{T}}}_{X_1}$ on $\Gamma((\text{range } \mathcal{T}_*)^\perp)$. Then, we have

$$g_{M_2}(S_{U_1} \mathcal{T}_* X_1, \mathcal{T}_* X_2) = g_{M_2}(U_1, (\nabla \mathcal{T}_*)(X_1, X_2)). \tag{7}$$

Since $(\nabla \mathcal{T}_*)$ is symmetric, S_{U_1} is a symmetric linear transformation of range $\mathcal{T}_*$. On the other hand, we have the following covariant derivatives:

$$(\widetilde{\nabla}_{X_1}(\nabla \mathcal{T}_*))(X_2, X_3) = \nabla_{X_1}^{\mathcal{T}^\perp}(\nabla \mathcal{T}_*)(X_2, X_3) - (\nabla \mathcal{T}_*)(\overset{M_1}{\nabla}_{X_1} X_2, X_3) - (\nabla \mathcal{T}_*)(X_2, \overset{M_1}{\nabla}_{X_1} X_3), \tag{8}$$

$$(\widetilde{\nabla}_{X_1} S)_{U_1} \mathcal{T}_*(X_2) = \mathcal{T}_*(\overset{M_1}{\nabla}_{X_1} {}^*\mathcal{T}_*(S_{U_1} \mathcal{T}_*(X_2))) - S_{\nabla_{X_1}^{\mathcal{T}^\perp} U_1} \mathcal{T}_*(X_2) - S_{U_1} P \overset{M_2}{\nabla^{\mathcal{T}}}_{X_1} \mathcal{T}_*(X_2) \tag{9}$$

where P denotes the projection morphism on range $\mathcal{T}_*$ and ${}^*\mathcal{T}_*$ is the adjoint map of $\mathcal{T}_*$ [3].

In the following lemma, we give a relation obtained from (8) and (9).

Lemma 2.3. *[3] Let (M_1, g_{M_1}), (M_2, g_{M_2}) be Riemannian manifolds, and $\mathcal{T}$ a Riemannian map between them. For all $X_1, X_2 \in \Gamma((ker \mathcal{T}_* p_1)^\perp)$ and $U_1 \in \Gamma((range \mathcal{T}_*)^\perp)$, we have*

$$g_{M_2}\left((\widetilde{\nabla}_{X_1}(\nabla \mathcal{T}_*))(X_2, X_3), U_1\right) = g_{M_2}\left((\widetilde{\nabla}_{X_1} S)_{U_1} \mathcal{T}_*(X_2), \mathcal{T}_*(X_3)\right). \tag{10}$$

A Riemannian map $\mathcal{T} : (M_1, g_{M_1}) \to (M_2, g_{M_2})$ is said to be h-isotropic at $p \in M_1$ if $\lambda(X_1) = \|(\nabla \mathcal{T}_*)(X_1, X_1)\| / \|\mathcal{T}_* X_1\|$ doesn't depend upon the selection of X_1. If the map is h-isotropic at all points, the map is called h-isotropic.

The following lemma gives a criteria for a h-isotropic Riemannian map.

Lemma 2.4. *[3] Let (M_1, g_{M_1}), (M_2, g_{M_2}) be Riemannian manifolds, and $\mathcal{T}$ is h-isotropic at $p \in M_1$ the second fundamental form $\nabla\mathcal{T}_*$ satisfies*

$$g_{M_2}\left((\nabla\mathcal{T}_*)(X_1, X_1), (\nabla\mathcal{T}_*)(X_1, X_2)\right) = 0 \tag{11}$$

for an arbitrary orthogonal couple $X_1, X_2 \in \Gamma((ker\mathcal{T}_* p_1)^\perp)$.

Corollary 2.5. *A totally umbilical Riemannian map $\mathcal{T}$ is a h-isotropic at the point p_1. Conversely a h-isotropic Riemannian map $\mathcal{T}$ is totally umbilical at p_1 if it satisfies $(\nabla\mathcal{T}_*)(X, Y) = 0$ for orthonormal vector fields X and Y at p_1 in $\Gamma((ker\mathcal{T}_* p_1)^\perp)$.*

3. Slant Helices Along An Isotropic Riemannian Maps

Let $\alpha(s)$ be a be a horizontal curve with curvature κ in M_1 and $\gamma = \mathcal{T} \circ \alpha$ a curve with curvature $\bar{\kappa}$ in M_2 along $\gamma = \mathcal{T} \circ \alpha$. We denote a vector field $\mathcal{T}_*\xi$ along $\mathcal{T} \circ \alpha$ by $\mathcal{T}_*\xi(s) = \mathcal{T}_{*\alpha}\xi(s)$, for each vector field ξ_s along α, where ξ_s is the unit tangent vector field along α and s is the arc length parameter. Unless otherwise stated, a unit speed curve α will be considered in this paper.

We first give the necessary definition for a horizontal slant helix curve along an isotropic Riemannian maps.

Definition 3.1. *Let $\alpha(s)$ be horizantal curve with curvatures $\kappa, \tau \neq 0$ on a Riemannian manifold M_1 (dimM≥3). If $\alpha(s)$ is a horizontal slant helix, then the unit tangent vector field ξ_s and the unit vector field V_2 of the curve satisfy*

$$\overset{M_1}{\nabla}{}^3_{\xi_s}\xi_s = 2\kappa' \overset{M_1}{\nabla}_{\xi_s} V_2 + \kappa \overset{M_1}{\nabla}{}^2_{\xi_s} V_2 + \left(\frac{\tau''}{\tau} - \frac{3}{2}\frac{\kappa}{\tau}\left(\frac{\tau}{\kappa}\right)'\left(\ln(\kappa^2+\tau^2)\right)'\right)\overset{M_1}{\nabla}_{\xi_s}\xi_s. \tag{12}$$

We give the following proposition which allows that $\mathcal{T}$ is an isotropic Riemannian map under certain conditions.

Proposition 3.2. *Let $\mathcal{T} : M_1 \to M_2$ be a smooth map between Riemannian manifolds (M_1, g_{M_1}) and (M_1, g_{M_1}). For each pair (u, v) of orthonormal tangent vectors, there is a horizontal slant helix α in M_1 which is not a general horizontal helix and that is a slant helix in M_2 satisfying the following:*

i) $\alpha'(0) = u, (\nabla_{\alpha'}\alpha')(0) = \kappa(0)v,$

ii) $6\kappa(0) \neq \bar{\kappa}(0), \kappa, \bar{\kappa} > 0, \tau, \bar{\tau} \neq 0$

where κ, τ and $\bar{\kappa}$, $\bar{\tau}$ are curvatures of α in M_1 and that in M_2, respectively. $\mathcal{T}$ is h-isotropic Riemannian map.

Proof. We assume that $p \in M_1$ and $\alpha(s)$ is a horizontal slant helix with curvature $\kappa > 0$ and torsion $\tau \neq 0$ on the base manifold M_1, $\mathcal{T} \circ \alpha : I \to M_2$ is the corresponding curve and we can define a vector field $\mathcal{T}_*\xi$ along $\mathcal{T} \circ \alpha$ by $\mathcal{T}_*\xi(s) = \mathcal{T}_{*\alpha(s)}\xi_s$, for each vector field ξ_s along α, where ξ_s is the unit tangent vector field along α and s is the arc length parameter. Now we consider that $\mathcal{T} \circ \alpha$ is a horizontal slant helix with the curvature $\bar{\kappa}$ and $\bar{\tau}$ on M_2:

$$\left(\overset{M_2}{\nabla}{}^{\mathcal{T}}_{\xi_s}\right)^3 \mathcal{T}_*(\xi_s) = 2\bar{\kappa}'\left(\overset{M_2}{\nabla}{}^{\mathcal{T}}_{\xi_s}\right)\mathcal{T}_*(V_2) + \bar{\kappa}\left(\overset{M_2}{\nabla}{}^{\mathcal{T}}_{\xi_s}\right)^2\mathcal{T}_*(V_2) + \bar{K}\left(\overset{M_2}{\nabla}{}^{\mathcal{T}}_{\xi_s}\right)\mathcal{T}_*(\xi_s) \tag{13}$$

where $\bar{K} = \frac{\bar{\tau}''}{\bar{\tau}} - \frac{3}{2}\frac{\bar{\kappa}}{\bar{\tau}}\left(\frac{\bar{\tau}}{\bar{\kappa}}\right)'\left(\ln(\bar{\kappa}^2+\bar{\tau}^2)\right)'$. From (4), the derivatives are given by

$$\begin{aligned}\left(\overset{M_2}{\nabla}{}^{\mathcal{T}}_{\xi_s}\right)^2 \mathcal{T}_*(V_2) &= \mathcal{T}_*\left(\left(\overset{M_1}{\nabla}_{\xi_s}\right)^2 V_2\right) + (\nabla\mathcal{T}_*)(\xi_s, \overset{M_1}{\nabla}_{\xi_s} V_2)\\ &-S_{(\nabla\mathcal{T}_*)(\xi_s, V_2)}\mathcal{T}_*(\xi_s) + \nabla^{\mathcal{T}^\perp}_{\xi_s}(\nabla\mathcal{T}_*)(\xi_s, V_2)\end{aligned} \tag{14}$$

and

$$\begin{aligned}
\left(\overset{M_2}{\mathcal{T}}_{\xi_s}\right)^3 \mathcal{T}_*(\xi_s) \quad = \quad & \mathcal{T}_*((\overset{M_1}{\nabla}_{\xi_s})^3 \xi_s) + (\nabla\mathcal{T}_*)(\xi_s, \left(\overset{M_1}{\nabla}_{\xi_s}\right)^2 \xi_s) - S_{(\nabla\mathcal{T}_*)(\xi_s, \overset{M_1}{\nabla}_{\xi_s}\xi_s)} \mathcal{T}_*(\xi_s) \\
& + \nabla^{\mathcal{T}^\perp}_{\xi_s}(\nabla\mathcal{T}_*)(\xi_s, \overset{M_1}{\nabla}_{\xi_s}\xi_s) - \mathcal{T}_*(\overset{M_1}{\nabla}_{\xi_s} {}^*\mathcal{T}_* S_{(\nabla\mathcal{T}_*)(\xi_s,\xi_s)}\mathcal{T}_*(\xi_s)) \\
& - (\nabla\mathcal{T}_*)(\xi_s, S_{(\nabla\mathcal{T}_*)(\xi_s,\xi_s)}\mathcal{T}_*(\xi_s)) - S_{\nabla^{\mathcal{T}^\perp}_{\xi_s}(\nabla\mathcal{T}_*)(\xi_s,\xi_s)}\mathcal{T}_*(\xi_s) \\
& + \left(\nabla^{\mathcal{T}^\perp}_{\xi_s}\right)^2 (\nabla\mathcal{T}_*)(\xi_s, \xi_s). \qquad (15)
\end{aligned}$$

Substituting (14) and (15) into (13), we obtain

$$\begin{aligned}
& \mathcal{T}_*((\overset{M_1}{\nabla}_{\xi_s})^3 \xi_s) + (\nabla\mathcal{T}_*)(\xi_s, \left(\overset{M_1}{\nabla}_{\xi_s}\right)^2 \xi_s) - S_{(\nabla\mathcal{T}_*)(\xi_s, \overset{M_1}{\nabla}_{\xi_s}\xi_s)} \mathcal{T}_*(\xi_s) + \nabla^{\mathcal{T}^\perp}_{\xi_s}(\nabla\mathcal{T}_*)(\xi_s, \overset{M_1}{\nabla}_{\xi_s}\xi_s) \\
& - \mathcal{T}_*(\overset{M_1}{\nabla}_{\xi_s} {}^*\mathcal{T}_* S_{(\nabla\mathcal{T}_*)(\xi_s,\xi_s)}\mathcal{T}_*(\xi_s)) - (\nabla\mathcal{T}_*)(\xi_s, S_{(\nabla\mathcal{T}_*)(\xi_s,\xi_s)}\mathcal{T}_*(\xi_s)) - S_{\nabla^{\mathcal{T}^\perp}_{\xi_s}(\nabla\mathcal{T}_*)(\xi_s,\xi_s)}\mathcal{T}_*(\xi_s) \\
& + \left(\nabla^{\mathcal{T}^\perp}_{\xi_s}\right)^2 (\nabla\mathcal{T}_*)(\xi_s, \xi_s) = 2\bar{\kappa}'\Big(\mathcal{T}_*(\overset{M_1}{\nabla}_{\xi_s} V_2) + (\nabla\mathcal{T}_*)(\xi_s, V_2)\Big) + \bar{\kappa}\Big[\mathcal{T}_*\Big(\left(\overset{M_1}{\nabla}_{\xi_s}\right)^2 V_2\Big) \\
& + (\nabla\mathcal{T}_*)(\xi_s, \overset{M_1}{\nabla}_{\xi_s} V_2) - S_{(\nabla\mathcal{T}_*)(\xi_s,V_2)}\mathcal{T}_*(\xi_s) + \nabla^{\mathcal{T}^\perp}_{\xi_s}(\nabla\mathcal{T}_*)(\xi_s, V_2)\Big] + \bar{K}\Big(\mathcal{T}_*(\overset{M_1}{\nabla}_{\xi_s}\xi_s) + (\nabla\mathcal{T}_*)(\xi_s, \xi_s)\Big).
\end{aligned} \qquad (16)$$

By looking at *range*$\mathcal{T}_*$ and *range*$\mathcal{T}_*^\perp$ components of (16), we have

$$\begin{aligned}
& \mathcal{T}_*((\overset{M_1}{\nabla}_{\xi_s})^3 \xi_s) - S_{(\nabla\mathcal{T}_*)(\xi_s, \overset{M_1}{\nabla}_{\xi_s}\xi_s)} \mathcal{T}_*(\xi_s) - \mathcal{T}_*(\overset{M_1}{\nabla}_{\xi_s} {}^*\mathcal{T}_* S_{(\nabla\mathcal{T}_*)(\xi_s,\xi_s)}\mathcal{T}_*(\xi_s)) - S_{\nabla^{\mathcal{T}^\perp}_{\xi_s}(\nabla\mathcal{T}_*)(\xi_s,\xi_s)}\mathcal{T}_*(\xi_s) \\
& = 2\bar{\kappa}'\mathcal{T}_*(\overset{M_1}{\nabla}_{\xi_s} V_2) + \bar{\kappa}\Big[\mathcal{T}_*\Big(\left(\overset{M_1}{\nabla}_{\xi_s}\right)^2 V_2\Big) - S_{(\nabla\mathcal{T}_*)(\xi_s,V_2)}\mathcal{T}_*(\xi_s)\Big] + \bar{K}\mathcal{T}_*(\overset{M_1}{\nabla}_{\xi_s}\xi_s)
\end{aligned} \qquad (17)$$

and

$$\begin{aligned}
& (\nabla\mathcal{T}_*)(\xi_s, \left(\overset{M_1}{\nabla}_{\xi_s}\right)^2 \xi_s) + \nabla^{\mathcal{T}^\perp}_{\xi_s}(\nabla\mathcal{T}_*)(\xi_s, \overset{M_1}{\nabla}_{\xi_s}\xi_s) - (\nabla\mathcal{T}_*)(\xi_s, S_{(\nabla\mathcal{T}_*)(\xi_s,\xi_s)}\mathcal{T}_*(\xi_s)) \\
& + \left(\nabla^{\mathcal{T}^\perp}_{\xi_s}\right)^2 (\nabla\mathcal{T}_*)(\xi_s, \xi_s) = 2\bar{\kappa}'(\nabla\mathcal{T}_*)(\xi_s, V_2) + \bar{\kappa}\Big[(\nabla\mathcal{T}_*)(\xi_s, \overset{M_1}{\nabla}_{\xi_s} V_2) + \nabla^{\mathcal{T}^\perp}_{\xi_s}(\nabla\mathcal{T}_*)(\xi_s, V_2)\Big] \\
& + \bar{K}(\nabla\mathcal{T}_*)(\xi_s, \xi_s). \qquad (18)
\end{aligned}$$

Using (9) and Frenet formulas in *range*$\mathcal{T}_*$, we get

$$\begin{aligned}
& (-3\kappa\kappa' + 2\bar{\kappa}'\kappa + \bar{\kappa}\kappa')\mathcal{T}_*(\xi_s) + (-\kappa^3 + \kappa'' - \kappa\tau^2 + \bar{\kappa}(\kappa^2 + \tau^2) - \bar{K}\kappa)\mathcal{T}_*(V_2) \\
& + (2\kappa'\tau + \kappa\tau' - \bar{\kappa}\tau' - 2\bar{\kappa}'\tau)\mathcal{T}_*(V_3) - \kappa S_{(\nabla\mathcal{T}_*)(\xi_s,V_2)}\mathcal{T}_*(\xi_s) - (\widetilde{\nabla}_{\xi_s} S)_{(\nabla\mathcal{T}_*)(\xi_s,\xi_s)}\mathcal{T}_*(\xi_s) \\
& = 2S_{\nabla^{\mathcal{T}^\perp}_{\xi_s}(\nabla\mathcal{T}_*)(\xi_s,\xi_s)}\mathcal{T}_*(\xi_s) + S_{(\nabla\mathcal{T}_*)(\xi_s,\xi_s)} P \overset{M_2}{\nabla^{\mathcal{T}}}_{\xi_s}\mathcal{T}_*(\xi_s) - \bar{\kappa} S_{(\nabla\mathcal{T}_*)(\xi_s,V_2)}\mathcal{T}_*(\xi_s). \qquad (19)
\end{aligned}$$

Putting (8) in (19), we derive

$$\begin{aligned}
& (-3\kappa\kappa' + 2\bar{\kappa}'\kappa + \bar{\kappa}\kappa')\mathcal{T}_*(\xi_s) + (-\kappa^3 + \kappa'' - \kappa\tau^2 + \bar{\kappa}(\kappa^2 + \tau^2) - \bar{K}\kappa)\mathcal{T}_*(V_2) \\
& + (2\kappa'\tau + \kappa\tau' - \bar{\kappa}\tau' - 2\bar{\kappa}'\tau)\mathcal{T}_*(V_3) - (\widetilde{\nabla}_{\xi_s} S)_{(\nabla\mathcal{T}_*)(\xi_s,\xi_s)}\mathcal{T}_*(\xi_s) \\
& = 2S_{(\widetilde{\nabla}_{\xi_s}(\nabla\mathcal{T}_*))(\xi_s,\xi_s)}\mathcal{T}_*(\xi_s) + 5\kappa S_{(\nabla\mathcal{T}_*)(\xi_s,V_2)}\mathcal{T}_*(\xi_s) \\
& + S_{(\nabla\mathcal{T}_*)(\xi_s,\xi_s)} P \overset{M_2}{\nabla^{\mathcal{T}}}_{\xi_s}\mathcal{T}_*(\xi_s) - \bar{\kappa} S_{(\nabla\mathcal{T}_*)(\xi_s,V_2)}\mathcal{T}_*(\xi_s). \qquad (20)
\end{aligned}$$

Changing V_2 into $-V_2$ in (20) and subtracting each other, it follows that

$$(-\kappa^3 + \kappa'' - \kappa\tau^2 + \bar{\kappa}(\kappa^2 + \tau^2) - \bar{K}\kappa)\mathcal{T}_*(V_2)$$
$$= 5\kappa S_{(\nabla\mathcal{T}_*)(\xi_s,V_2)}\mathcal{T}_*(\xi_s) + S_{(\nabla\mathcal{T}_*)(\xi_s,\xi_s)}P\overset{M_2}{\nabla^{\mathcal{T}}}_{\xi_s}\mathcal{T}_*(\xi_s) - \bar{\kappa}S_{(\nabla\mathcal{T}_*)(\xi_s,V_2)}\mathcal{T}_*(\xi_s). \tag{21}$$

Taking inner product with the unit vector field $\mathcal{T}_*(\xi_s)$, we have

$$0 = 5\kappa g_{M_2}(S_{(\nabla\mathcal{T}_*)(\xi_s,V_2)}\mathcal{T}_*(\xi_s), \mathcal{T}_*(\xi_s)) - \bar{\kappa} g_{M_2}(S_{(\nabla\mathcal{T}_*)(\xi_s,V_2)}\mathcal{T}_*(\xi_s), \mathcal{T}_*(\xi_s))$$
$$+ g_{M_2}(S_{(\nabla\mathcal{T}_*)(\xi_s,\xi_s)}P\overset{M_2}{\nabla^{\mathcal{T}}}_{\xi_s}\mathcal{T}_*(\xi_s), \mathcal{T}_*(\xi_s))$$

and

$$0 = (6\kappa - \bar{\kappa})g_{M_2}((\nabla\mathcal{T}_*)(\xi_s, V_2), (\nabla\mathcal{T}_*)(\xi_s, \xi_s)).$$

If we choose $6\kappa \neq \bar{\kappa}$, we get

$$0 = g_{M_2}((\nabla\mathcal{T}_*)(\xi_s, V_2), (\nabla\mathcal{T}_*)(\xi_s, \xi_s)).$$

Then Lemma 3 implies that $\mathcal{T}$ is an h-isotropic Riemannian map. □

In the theory of submanifolds, it is well known that a necessary and sufficient condition for a submanifold to be totally geodesic is that every geodesic on the submanifold is also a geodesic of the ambient Riemannian manifold. The following theorem investigates this condition in the specific context of horizontal slant helices along an isotropic Riemannian maps.

Theorem 3.3. *Let $\mathcal{T} : M_1 \to M_2$ be a smooth map between Riemannian manifolds (M_1, g_{M_1}) and (M_1, g_{M_1}). Let α be a horizontal curve which is not a horizontal general helix. If, for $6\kappa \neq \bar{\kappa}$ and $\kappa'' \neq 0$, a horizontal slant helix with curvatures $\kappa > 0$ and $\tau \neq 0$ in M_1 is a horizontal slant helix with curvatures $\bar{\kappa} > 0$ and $\bar{\tau} \neq 0$ in M_2, then $\mathcal{T}$ is a totally geodesic map.*

Proof. We suppose that $\alpha = \alpha(s)$ is a horizantal slant helix curve with curvatures κ and $\tau \neq 0$. Then, we have (16). Using Frenet formulas in $range\mathcal{T}_*^{\perp}$, we obtain

$$(\nabla\mathcal{T}_*)(\xi_s, -\kappa^2\xi_s + \kappa' V_2 + \kappa\tau V_3) + \nabla^{\mathcal{T}^\perp}_{\xi_s}(\nabla\mathcal{T}_*)(\xi_s, \kappa V_2) - (\nabla\mathcal{T}_*)(\xi_s, S_{(\nabla\mathcal{T}_*)(\xi_s,\xi_s)}\mathcal{T}_*(\xi_s))$$
$$+\left(\nabla^{\mathcal{T}^\perp}_{\xi_s}\right)^2(\nabla\mathcal{T}_*)(\xi_s, \xi_s) = 2\bar{\kappa}'(\nabla\mathcal{T}_*)(\xi_s, V_2) + \bar{\kappa}\Big[(\nabla\mathcal{T}_*)(\xi_s, -\kappa\xi_s + \tau V_3) + \nabla^{\mathcal{T}^\perp}_{\xi_s}(\nabla\mathcal{T}_*)(\xi_s, V_2)\Big]$$
$$+\bar{K}(\nabla\mathcal{T}_*)(\xi_s, \xi_s). \tag{22}$$

From the equation (8), when $\nabla^{\mathcal{T}^\perp}_{\xi_s}(\nabla\mathcal{T}_*)(\xi_s, \xi_s) = (\widetilde{\nabla}_{\xi_s}(\nabla\mathcal{T}_*))(\xi_s, \xi_s) + 2(\nabla\mathcal{T}_*)(\kappa V_2, \xi_s)$, the derivative yields

$$\left(\nabla^{\mathcal{T}^\perp}_{\xi_s}\right)^2(\nabla\mathcal{T}_*)(\xi_s, \xi_s) = (\widetilde{\nabla^2}_{\xi_s}(\nabla\mathcal{T}_*))(\xi_s, \xi_s) + 4(\widetilde{\nabla}_{\xi_s}(\nabla\mathcal{T}_*))(\xi_s, \kappa V_2)$$
$$+2\kappa^2(\nabla\mathcal{T}_*)(V_2, V_2) - 2\kappa^2(\nabla\mathcal{T}_*)(\xi_s, \xi_s) + 2\kappa\tau(\nabla\mathcal{T}_*)(\xi_s, V_3) + 2\kappa'(\nabla\mathcal{T}_*)(\xi_s, V_2). \tag{23}$$

By substituting and rearranging the last equation in (22), we derive the following equation:

$$\begin{aligned}
&(\nabla\mathcal{T}_*)(\xi_s,-\kappa^2\xi_s+\kappa'V_2+\kappa\tau V_3)+(\widetilde{\nabla}_{\xi_s}(\nabla\mathcal{T}_*))(\xi_s,\kappa V_2)\\
&+\kappa^2(\nabla\mathcal{T}_*)(V_2,V_2)-\kappa^2(\nabla\mathcal{T}_*)(\xi_s,\xi_s)+\kappa\tau(\nabla\mathcal{T}_*)(\xi_s,V_3)+\kappa'(\nabla\mathcal{T}_*)(\xi_s,V_2)\\
&-(\nabla\mathcal{T}_*)(\xi_s,S_{(\nabla\mathcal{T}_*)(\xi_s,\xi_s)}\mathcal{T}_*(\xi_s))+(\widetilde{\nabla^2}_{\xi_s}(\nabla\mathcal{T}_*))(\xi_s,\xi_s)+4(\widetilde{\nabla}_{\xi_s}(\nabla\mathcal{T}_*))(\xi_s,\kappa V_2)\\
&+2\kappa^2(\nabla\mathcal{T}_*)(V_2,V_2)-2\kappa^2(\nabla\mathcal{T}_*)(\xi_s,\xi_s)+2\kappa\tau(\nabla\mathcal{T}_*)(\xi_s,V_3)+2\kappa'(\nabla\mathcal{T}_*)(\xi_s,V_2)\\
&=2\bar{\kappa}'(\nabla\mathcal{T}_*)(\xi_s,V_2)+\bar{\kappa}\Big[(\nabla\mathcal{T}_*)(\xi_s,-\kappa\xi_s+\tau V_3)+\nabla^{\mathcal{T}^\perp}_{\xi_s}(\nabla\mathcal{T}_*)(\xi_s,V_2)\Big]\\
&+\bar{K}(\nabla\mathcal{T}_*)(\xi_s,\xi_s)
\end{aligned}\tag{24}$$

or

$$\begin{aligned}
&4\kappa'(\nabla\mathcal{T}_*)(\xi_s,V_2)-4\kappa^2(\nabla\mathcal{T}_*)(\xi_s,\xi_s)+4\kappa\tau(\nabla\mathcal{T}_*)(\xi_s,V_3)\\
&-(\nabla\mathcal{T}_*)(\xi_s,S_{(\nabla\mathcal{T}_*)(\xi_s,\xi_s)}\mathcal{T}_*(\xi_s))+(\widetilde{\nabla^2}_{\xi_s}(\nabla\mathcal{T}_*))(\xi_s,\xi_s)+5(\widetilde{\nabla}_{\xi_s}(\nabla\mathcal{T}_*))(\xi_s,\kappa V_2)\\
&+3\kappa^2(\nabla\mathcal{T}_*)(V_2,V_2)=2\bar{\kappa}'(\nabla\mathcal{T}_*)(\xi_s,V_2)+\bar{\kappa}\Big[(\nabla\mathcal{T}_*)(\xi_s,-\kappa\xi_s+\tau V_3)\\
&+(\widetilde{\nabla}_{\xi_s}(\nabla\mathcal{T}_*))(\xi_s,V_2)+(\nabla\mathcal{T}_*)(\xi_s,-\kappa\xi_s)+(\nabla\mathcal{T}_*)(\xi_s,\tau V_3,)\Big]+\bar{K}(\nabla\mathcal{T}_*)(\xi_s,\xi_s).
\end{aligned}\tag{25}$$

From (8), the equation (25) becomes

$$\begin{aligned}
&4\kappa'(\nabla\mathcal{T}_*)(\xi_s,V_2)-4\kappa^2(\nabla\mathcal{T}_*)(\xi_s,\xi_s)+4\kappa\tau(\nabla\mathcal{T}_*)(\xi_s,V_3)\\
&-(\nabla\mathcal{T}_*)(\xi_s,S_{(\nabla\mathcal{T}_*)(\xi_s,\xi_s)}\mathcal{T}_*(\xi_s))+(\widetilde{\nabla^2}_{\xi_s}(\nabla\mathcal{T}_*))(\xi_s,\xi_s)\\
&+5\kappa\nabla^{\mathcal{T}^\perp}_{\xi_s}(\nabla\mathcal{T}_*)(\xi_s,V_2)-5(\nabla\mathcal{T}_*)(\kappa V_2,\kappa V_2)-5(\nabla\mathcal{T}_*)(\xi_s,\kappa'V_2-\kappa^2\xi_s+\kappa\tau V_3)\\
&+3\kappa^2(\nabla\mathcal{T}_*)(V_2,V_2)=2\bar{\kappa}'(\nabla\mathcal{T}_*)(\xi_s,V_2)+\bar{\kappa}\Big[(\nabla\mathcal{T}_*)(\xi_s,-\kappa\xi_s+\tau V_3)\\
&+(\widetilde{\nabla}_{\xi_s}(\nabla\mathcal{T}_*))(\xi_s,V_2)+(\nabla\mathcal{T}_*)(\xi_s,-\kappa\xi_s)+(\nabla\mathcal{T}_*)(\xi_s,\tau V_3,)\Big]+\bar{K}(\nabla\mathcal{T}_*)(\xi_s,\xi_s).
\end{aligned}\tag{26}$$

Changing V_3 into $-V_3$ and subtracting each other, it follows that

$$-\kappa\tau(\nabla\mathcal{T}_*)(\xi_s,V_3)-2\bar{\kappa}\tau(\nabla\mathcal{T}_*)(\xi_s,V_3)=0\Rightarrow(2\kappa+\bar{\kappa})\tau(\nabla\mathcal{T}_*)(\xi_s,V_3)=0.$$

When $\tau\neq 0$, then either $(\nabla\mathcal{T}_*)(\xi_s,V_3)=0$ or $2\kappa+\bar{\kappa}=0$. If $2\kappa+\bar{\kappa}=0$, then $\bar{\kappa}<0$ since $\bar{\kappa}>0$ (κ must be nonzero as it is a horizontal slant helix with a non-zero).Then $(\nabla\mathcal{T}_*)(\xi_s,V_3)=0$. From Corollary 4, $\mathcal{T}$ becomes umbilical.

Changing V_2 into $-V_2$ and subtracting each other in (24), it follows that

$$5\kappa(\widetilde{\nabla}_{\xi_s}(\nabla\mathcal{T}_*))(\xi_s,V_2)=0.\tag{27}$$

Taking inner product with the unit vector field $\mathcal{T}_*(\xi_s)$ in (21), we have

$$\begin{aligned}
&g_{M_2}((-\kappa^3+\kappa''-\kappa\tau^2+\bar{\kappa}(\kappa^2+\tau^2)-\bar{K}\kappa)\mathcal{T}_*(V_2),\mathcal{T}_*(V_2))=5\kappa g_{M_2}(S_{(\nabla\mathcal{T}_*)(\xi_s,V_2)}\mathcal{T}_*(\xi_s),\mathcal{T}_*(V_2))\\
&+g_{M_2}(S_{(\nabla\mathcal{T}_*)(\xi_s,\xi_s)}P\overset{M_2}{\nabla^{\mathcal{T}}}_{\xi_s}\mathcal{T}_*(\xi_s),\mathcal{T}_*(V_2))-\bar{\kappa}g_{M_2}(S_{(\nabla\mathcal{T}_*)(\xi_s,V_2)}\mathcal{T}_*(\xi_s),\mathcal{T}_*(V_2))
\end{aligned}$$

that is

$$-\kappa^2+\frac{\kappa''}{\kappa}-\tau^2+\frac{\bar{\kappa}}{\kappa}(\kappa^2+\tau^2)-\bar{K}=||H_2||^2.\tag{28}$$

The equtaion (20) is multiplied by $\mathcal{T}_*(\xi_s)$ and $\mathcal{T}_*(V_3)$, respectively and considering that $\mathcal{T}$ is umbilical, then the equations

$$-3\kappa\kappa' + 2\bar{\kappa}'\kappa + \bar{\kappa}\kappa' = 0 \tag{29}$$

and

$$2\kappa'\tau + \kappa\tau' - \bar{\kappa}\tau' - 2\bar{\kappa}'\tau = 0$$

are obtained. When equation (28) is inserted into the final equation, given that $\kappa\tau' \neq \kappa'\tau$ is not constant, then

$$\begin{aligned}
&2\kappa'\tau + \kappa\tau' - \bar{\kappa}\tau' - 2\bar{\kappa}'\tau = 0\\
&2\kappa'\tau + \kappa\tau' - \bar{\kappa}\tau' - 2\left(\frac{3\kappa'}{2} - \frac{\bar{\kappa}\kappa'}{2\kappa}\right)\tau = 0\\
&\kappa\tau' - \kappa'\tau - \bar{\kappa}\tau' + \frac{\bar{\kappa}\kappa'}{\kappa}\tau = 0\\
&\kappa(\kappa\tau' - \kappa'\tau) = \bar{\kappa}(\kappa\tau' - \kappa'\tau)\\
&\kappa = \bar{\kappa}.
\end{aligned}$$

When this expression is inserted into equation (28), it results

$$\frac{\kappa''}{\kappa} - \bar{K} = ||H_2||^2. \tag{30}$$

By considering umbilical and (27), we determine

$$(\kappa\bar{\kappa} - \kappa^2 - \bar{K})H_2 - (\nabla\mathcal{T}_*)(\xi_s, S_{(\nabla\mathcal{T}_*)(\xi_s,\xi_s)}\mathcal{T}_*(\xi_s)) + (\widetilde{\nabla^2}_{\xi_s}(\nabla\mathcal{T}_*))(\xi_s, \xi_s) = 0$$

Changing $T(\xi_s)$ into $-T(\xi_s)$ and subtracting each other in the last equation, it follows that $(\widetilde{\nabla^2}_{\xi_s}(\nabla\mathcal{T}_*))(\xi_s, \xi_s) = 0$ that is

$$(-\bar{K} - ||H_2||^2)H_2 = 0.$$

Using (30), we obtain $\frac{\kappa''}{\kappa}H_2 = 0$ When $\kappa'' \neq 0$, H_2 is zero. Then T is totally geodesic. □

4. Horizantal helices and Horizontal slant helices along an isotropic Riemannian maps

We prove the following theorem which shows the effect of transforming helices and slant helices into the base manifold along Riemannian maps in this section.

Theorem 4.1. *Let $\mathcal{T} : M_1 \to M_2$ be a smooth map between Riemannian manifolds (M_1, g_{M_1}) and (M_1, g_{M_1}) If a horizantal slant helix with curvatures $\kappa > 0$ and $\tau \neq 0$ in M_1 is a horizantal helix with curvatures $\bar{\kappa} > 0$ and $\bar{\tau} \neq 0$ in M_2, then $\mathcal{T}$ is a totally geodesic map.*

Proof. We assume that the horizontal slant helix curve α on M_1 is also an helix curve on M_2 with the map $T \circ \alpha$. The vector field along $T \circ \alpha$ is denoted as $T_\xi(s) = T_{\alpha(s)}\xi(s)$, where ξ_s is the unit tangent vector field along α. From the equation (1), we have following equation

$$\left(\overset{M_2}{\nabla^{\mathcal{T}}}_{\xi_s}\right)^3 \mathcal{T}_*(\xi_s) + \bar{K}\overset{M_2}{\nabla^{\mathcal{T}}}_{\xi_s}\mathcal{T}_*(\xi_s) = 0, \tag{31}$$

where $\bar{K} = \bar{\kappa}^2 + \bar{\tau}^2$. Using (4) and arranging the above equation, we obtain

$$\begin{aligned} &\mathcal{T}_*((\overset{M_1}{\nabla}_{\xi_s})^3\xi_s) + (\nabla\mathcal{T}_*)(\xi_s, \left(\overset{M_1}{\nabla}_{\xi_s}\right)^2\xi_s) \\ &-S_{(\nabla\mathcal{T}_*)(\xi_s, \overset{M_1}{\nabla}_{\xi_s}\xi_s)}\mathcal{T}_*(\xi_s) + \nabla^{\mathcal{T}^\perp}_{\xi_s}(\nabla\mathcal{T}_*)(\xi_s, \overset{M_1}{\nabla}_{\xi_s}\xi_s) \\ &-\mathcal{T}_*(\overset{M_1}{\nabla}_{\xi_s}{}^*\mathcal{T}_*S_{(\nabla\mathcal{T}_*)(\xi_s,\xi_s)}\mathcal{T}_*(\xi_s)) - (\nabla\mathcal{T}_*)(\xi_s, S_{(\nabla\mathcal{T}_*)(\xi_s,\xi_s)}\mathcal{T}_*(\xi_s)) \\ &-S_{\nabla^{\mathcal{T}^\perp}_{\xi_s}(\nabla\mathcal{T}_*)(\xi_s,\xi_s)}\mathcal{T}_*(\xi_s) + \left(\nabla^{\mathcal{T}^\perp}_{\xi_s}\right)^2(\nabla\mathcal{T}_*)(\xi_s,\xi_s) \\ &+\bar{K}\mathcal{T}_*(\overset{M_1}{\nabla}_{\xi_s}\xi_s) + \bar{K}(\nabla\mathcal{T}_*)(\xi_s,\xi_s) = 0. \end{aligned} \tag{32}$$

If $range\mathcal{T}_*$ and $range\mathcal{T}_*^\perp$ are separated, they become

$$\begin{aligned} &\mathcal{T}_*((\overset{M_1}{\nabla}_{\xi_s})^3\xi_s) - S_{(\nabla\mathcal{T}_*)(\xi_s, \overset{M_1}{\nabla}_{\xi_s}\xi_s)}\mathcal{T}_*(\xi_s) - \mathcal{T}_*(\overset{M_1}{\nabla}_{\xi_s}{}^*\mathcal{T}_*S_{(\nabla\mathcal{T}_*)(\xi_s,\xi_s)}\mathcal{T}_*(\xi_s)) \\ &-S_{\nabla^{\mathcal{T}^\perp}_{\xi_s}(\nabla\mathcal{T}_*)(\xi_s,\xi_s)}\mathcal{T}_*(\xi_s) + \bar{K}\mathcal{T}_*(\overset{M_1}{\nabla}_{\xi_s}\xi_s) = 0 \end{aligned} \tag{33}$$

and

$$\begin{aligned} &(\nabla\mathcal{T}_*)(\xi_s, \left(\overset{M_1}{\nabla}_{\xi_s}\right)^2\xi_s) + \nabla^{\mathcal{T}^\perp}_{\xi_s}(\nabla\mathcal{T}_*)(\xi_s, \overset{M_1}{\nabla}_{\xi_s}\xi_s) - (\nabla\mathcal{T}_*)(\xi_s, S_{(\nabla\mathcal{T}_*)(\xi_s,\xi_s)}\mathcal{T}_*(\xi_s)) \\ &+\left(\nabla^{\mathcal{T}^\perp}_{\xi_s}\right)^2(\nabla\mathcal{T}_*)(\xi_s,\xi_s) + \bar{K}(\nabla\mathcal{T}_*)(\xi_s,\xi_s) = 0. \end{aligned} \tag{34}$$

Considering (9) and Frenet formulas, the equation (33) turns into

$$\begin{aligned} &-3\kappa\kappa'\mathcal{T}_*(\xi_s) + (-\kappa^3 - \kappa\tau^2 + \bar{K}\kappa)\mathcal{T}_*(V_2) + (2\kappa'\tau + \kappa\tau')\mathcal{T}_*(V_3) \\ &-\kappa S_{(\nabla\mathcal{T}_*)(\xi_s,V_2)}\mathcal{T}_*(\xi_s) - (\widetilde{\nabla}_{\xi_s}S)_{(\nabla\mathcal{T}_*)(\xi_s,\xi_s)}\mathcal{T}_*(\xi_s) \\ &-2S_{\nabla^{\mathcal{T}^\perp}_{\xi_s}(\nabla\mathcal{T}_*)(\xi_s,\xi_s)}\mathcal{T}_*(\xi_s) - S_{(\nabla\mathcal{T}_*)(\xi_s,\xi_s)}P\overset{M_2}{\nabla^{\mathcal{T}}}_{\xi_s}\mathcal{T}_*(\xi_s) = 0. \end{aligned} \tag{35}$$

By utilizing equation (8) and substituting it into (35), we can compute the component related to $range\mathcal{T}_*$. This computation yields

$$\begin{aligned} &-3\kappa\kappa'\mathcal{T}_*(\xi_s) + (-\kappa^3 - \kappa\tau^2 + \bar{K}\kappa)\mathcal{T}_*(V_2) + (2\kappa'\tau + \kappa\tau')\mathcal{T}_*(V_3) - (\widetilde{\nabla}_{\xi_s}S)_{(\nabla\mathcal{T}_*)(\xi_s,\xi_s)}\mathcal{T}_*(\xi_s) \\ &= 2S_{(\widetilde{\nabla}_{\xi_s}(\nabla\mathcal{T}_*))(\xi_s,\xi_s)}\mathcal{T}_*(\xi_s) + 5\kappa S_{(\nabla\mathcal{T}_*)(\xi_s,V_2)}\mathcal{T}_*(\xi_s) + S_{(\nabla\mathcal{T}_*)(\xi_s,\xi_s)}P\overset{M_2}{\nabla^{\mathcal{T}}}_{\xi_s}\mathcal{T}_*(\xi_s). \end{aligned} \tag{36}$$

Now, if we substitute $-V_2$ for V_2 in this equation, subtract the resulting equation from (36), and then multiply both sides by $\mathcal{T}_*(\xi_s)$, we obtain

$$(-\kappa^3 - \kappa\tau^2 + \bar{K}\kappa)\mathcal{T}_*(V_2) = 5\kappa S_{(\nabla\mathcal{T}_*)(\xi_s,V_2)}\mathcal{T}_*(\xi_s) + S_{(\nabla\mathcal{T}_*)(\xi_s,\xi_s)}P\overset{M_2}{\nabla^{\mathcal{T}}}_{\xi_s}\mathcal{T}_*(\xi_s) \tag{37}$$

and

$$0 = 5\kappa g_{M_2}(S_{(\nabla\mathcal{T}_*)(\xi_s,V_2)}\mathcal{T}_*(\xi_s), \mathcal{T}_*(\xi_s)) + g_{M_2}(S_{(\nabla\mathcal{T}_*)(\xi_s,\xi_s)}P\overset{M_2}{\nabla^{\mathcal{T}}}_{\xi_s}\mathcal{T}_*(\xi_s), \mathcal{T}_*(\xi_s)).$$

Moreover, considering the expression:

$$g_{M_2}(S_{(\nabla\mathcal{T}_*)(\xi_s,\xi_s)}P\overset{M_2}{\nabla^{\mathcal{T}}}_{\xi_s}\mathcal{T}_*(\xi_s),\mathcal{T}_*(\xi_s)) = g_{M_2}((\nabla\mathcal{T}_*)(\xi_s,\overset{M_1}{\nabla}_{\xi_s}\xi_s),(\nabla\mathcal{T}_*)(\xi_s,\xi_s))$$
$$= \kappa g_{M_2}((\nabla\mathcal{T}_*)(\xi_s,V_2),(\nabla\mathcal{T}_*)(\xi_s,\xi_s))$$

and making use of the fact that $\kappa \neq 0$, we can simplify to

$$0 = g_{M_2}((\nabla\mathcal{T}_*)(\xi_s,V_2),(\nabla\mathcal{T}_*)(\xi_s,\xi_s)).$$

This implies that $\mathcal{T}$ is h-isotropic.

On the other hand combining equations (8) and (34), we derive

$$\begin{aligned}
&(\nabla\mathcal{T}_*)(\xi_s,-\kappa^2\xi_s+\kappa'V_2+\kappa\tau V_3)+(\widetilde{\nabla}_{\xi_s}(\nabla\mathcal{T}_*))(\xi_s,\kappa V_2)\\
&+\kappa^2(\nabla\mathcal{T}_*)(V_2,V_2)-\kappa^2(\nabla\mathcal{T}_*)(\xi_s,\xi_s)+\kappa\tau(\nabla\mathcal{T}_*)(\xi_s,V_3)+\kappa'(\nabla\mathcal{T}_*)(\xi_s,V_2)\\
&-(\nabla\mathcal{T}_*)(\xi_s,S_{(\nabla\mathcal{T}_*)(\xi_s,\xi_s)}\mathcal{T}_*(\xi_s))+(\widetilde{\nabla^2}_{\xi_s}(\nabla\mathcal{T}_*))(\xi_s,\xi_s)+4(\widetilde{\nabla}_{\xi_s}(\nabla\mathcal{T}_*))(\xi_s,\kappa V_2)\\
&+2\kappa^2(\nabla\mathcal{T}_*)(V_2,V_2)-2\kappa^2(\nabla\mathcal{T}_*)(\xi_s,\xi_s)+2\kappa\tau(\nabla\mathcal{T}_*)(\xi_s,V_3)+2\kappa'(\nabla\mathcal{T}_*)(\xi_s,V_2)+\bar{K}(\nabla\mathcal{T}_*)(\xi_s,\xi_s)=0
\end{aligned}$$

or equivalently:

$$\begin{aligned}
&4\kappa'(\nabla\mathcal{T}_*)(\xi_s,V_2)-4\kappa^2(\nabla\mathcal{T}_*)(\xi_s,\xi_s)+4\kappa\tau(\nabla\mathcal{T}_*)(\xi_s,V_3)\\
&-(\nabla\mathcal{T}_*)(\xi_s,S_{(\nabla\mathcal{T}_*)(\xi_s,\xi_s)}\mathcal{T}_*(\xi_s))+(\widetilde{\nabla^2}_{\xi_s}(\nabla\mathcal{T}_*))(\xi_s,\xi_s)\\
&+5\kappa\nabla^{\mathcal{T}^\perp}_{\xi_s}(\nabla\mathcal{T}_*)(\xi_s,V_2)-5(\nabla\mathcal{T}_*)(\kappa V_2,\kappa V_2)-5(\nabla\mathcal{T}_*)(\xi_s,\kappa'V_2-\kappa^2\xi_s+\kappa\tau V_3)\\
&+3\kappa^2(\nabla\mathcal{T}_*)(V_2,V_2)+\bar{K}(\nabla\mathcal{T}_*)(\xi_s,\xi_s)=0.
\end{aligned} \tag{38}$$

By substituting V_3 with $-V_3$ in the last equation and taking their difference, we arrive at

$$-\kappa\tau(\nabla\mathcal{T}_*)(\xi_s,V_3)=0.$$

Since κ and τ are both non-zero, we conclude that $(\nabla\mathcal{T}_*)(\xi_s,V_3)=0$. According to Corollary 4, this implies that $\mathcal{T}$ is an umbilical map. Similarly, by replacing V_2 with $-V_2$ in equation (38) and subtracting, it follows that

$$5\kappa(\widetilde{\nabla}_{\xi_s}(\nabla\mathcal{T}_*))(\xi_s,V_2)=0. \tag{39}$$

Taking the inner product with the unit vector field $\mathcal{T}_*(V_2)$ in equation (37), we derive

$$-\kappa^2-\tau^2+\bar{K}=\|H_2\|^2. \tag{40}$$

Given that $\mathcal{T}$ is umbilical and combining equations (39) and (40), equation (38) becomes

$$(-\kappa^2+\|H_2\|^2+\kappa^2+\tau^2-\|H_2\|^2)H_2=0 \Rightarrow H_2=0$$

Thus, we conclude that $\mathcal{T}$ is a totally geodesic map. □

Theorem 4.2. *Let $\mathcal{T}: M_1 \to M_2$ be a smooth map between Riemannian manifolds (M_1, g_{M_1}) and (M_1, g_{M_1}). If, for $6\kappa \neq \bar{\kappa}$ and $\kappa'' \neq 0$, a horizantal helix with curvatures $\kappa > 0$ and $\tau \neq 0$ in M_1 is a horizantal slant helix with curvatures $\bar{\kappa} > 0$ and $\bar{\tau} \neq 0$ in M_2, then $\mathcal{T}$ is a totally geodesic map.*

Proof. We assume that the horizontal helix curve α on M_1 is also a horizontal slant helix curve on M_2 with the map $T \circ \alpha$. From our assumption that a curve $T \circ \alpha$ is a slant helix on M_2, it should be

$$\left(\overset{M_2}{\nabla^{\mathcal{T}}}_{\xi_s}\right)^3\mathcal{T}_*(\xi_s)=2\bar{\kappa}'\left(\overset{M_2}{\nabla^{\mathcal{T}}}_{\xi_s}\right)\mathcal{T}_*(V_2)+\bar{\kappa}\left(\overset{M_2}{\nabla^{\mathcal{T}}}_{\xi_s}\right)^2\mathcal{T}_*(V_2)+\bar{K}\left(\overset{M_2}{\nabla^{\mathcal{T}}}_{\xi_s}\right)\mathcal{T}_*(\xi_s). \tag{41}$$

Where $\bar{K}=\frac{\bar{\tau}''}{\bar{\tau}}-\frac{3}{2}\frac{\bar{\kappa}}{\bar{\tau}}\left(\frac{\bar{\tau}}{\bar{\kappa}}\right)'\left(\ln(\bar{\kappa}^2+\bar{\tau}^2)\right)'$. Using (4) and arranging the above equation, we have

$$\begin{aligned}
&\mathcal{T}_*((\overset{M_1}{\nabla}_{\xi_s})^3\xi_s)+(\nabla\mathcal{T}_*)(\xi_s,\left(\overset{M_1}{\nabla}_{\xi_s}\right)^2\xi_s)-S_{(\nabla\mathcal{T}_*)(\xi_s,\overset{M_1}{\nabla}_{\xi_s}\xi_s)}\mathcal{T}_*(\xi_s)+\nabla^{\mathcal{T}^\perp}_{\xi_s}(\nabla\mathcal{T}_*)(\xi_s,\overset{M_1}{\nabla}_{\xi_s}\xi_s)\\
&-\mathcal{T}_*(\overset{M_1}{\nabla}_{\xi_s}{}^*\mathcal{T}_*S_{(\nabla\mathcal{T}_*)(\xi_s,\xi_s)}\mathcal{T}_*(\xi_s))-(\nabla\mathcal{T}_*)(\xi_s,S_{(\nabla\mathcal{T}_*)(\xi_s,\xi_s)}\mathcal{T}_*(\xi_s))-S_{\nabla^{\mathcal{T}^\perp}_{\xi_s}(\nabla\mathcal{T}_*)(\xi_s,\xi_s)}\mathcal{T}_*(\xi_s)\\
&+\left(\nabla^{\mathcal{T}^\perp}_{\xi_s}\right)^2(\nabla\mathcal{T}_*)(\xi_s,\xi_s)=2\bar{\kappa}'\Big(\mathcal{T}_*(\overset{M_1}{\nabla}_{\xi_s}V_2)+(\nabla\mathcal{T}_*)(\xi_s,V_2)\Big)+\bar{\kappa}\Big[\mathcal{T}_*\Big(\left(\overset{M_1}{\nabla}_{\xi_s}\right)^2V_2\Big)\\
&+(\nabla\mathcal{T}_*)(\xi_s,\overset{M_1}{\nabla}_{\xi_s}V_2)-S_{(\nabla\mathcal{T}_*)(\xi_s,V_2)}\mathcal{T}_*(\xi_s)+\nabla^{\mathcal{T}^\perp}_{\xi_s}(\nabla\mathcal{T}_*)(\xi_s,V_2)\Big]\\
&+\bar{K}\Big(\mathcal{T}_*(\overset{M_1}{\nabla}_{\xi_s}\xi_s)+(\nabla\mathcal{T}_*)(\xi_s,\xi_s)\Big).
\end{aligned}$$

Since α is a horizontal helix, $\mathcal{T}_*((\overset{M_1}{\nabla}_{\xi_s})^3\xi_s)=-K\mathcal{T}_*(\overset{M_1}{\nabla}_{\xi_s}\xi_s)$. if this equation is substituted into the above expression, the following equation is obtained:

$$\begin{aligned}
&-K\mathcal{T}_*(\overset{M_1}{\nabla}_{\xi_s}\xi_s)+(\nabla\mathcal{T}_*)(\xi_s,\left(\overset{M_1}{\nabla}_{\xi_s}\right)^2\xi_s)-S_{(\nabla\mathcal{T}_*)(\xi_s,\overset{M_1}{\nabla}_{\xi_s}\xi_s)}\mathcal{T}_*(\xi_s)+\nabla^{\mathcal{T}^\perp}_{\xi_s}(\nabla\mathcal{T}_*)(\xi_s,\overset{M_1}{\nabla}_{\xi_s}\xi_s)\\
&-\mathcal{T}_*(\overset{M_1}{\nabla}_{\xi_s}{}^*\mathcal{T}_*S_{(\nabla\mathcal{T}_*)(\xi_s,\xi_s)}\mathcal{T}_*(\xi_s))-(\nabla\mathcal{T}_*)(\xi_s,S_{(\nabla\mathcal{T}_*)(\xi_s,\xi_s)}\mathcal{T}_*(\xi_s))-S_{\nabla^{\mathcal{T}^\perp}_{\xi_s}(\nabla\mathcal{T}_*)(\xi_s,\xi_s)}\mathcal{T}_*(\xi_s)\\
&+\left(\nabla^{\mathcal{T}^\perp}_{\xi_s}\right)^2(\nabla\mathcal{T}_*)(\xi_s,\xi_s)=2\bar{\kappa}'\Big(\mathcal{T}_*(\overset{M_1}{\nabla}_{\xi_s}V_2)+(\nabla\mathcal{T}_*)(\xi_s,V_2)\Big)+\bar{\kappa}\Big[\mathcal{T}_*\Big(\left(\overset{M_1}{\nabla}_{\xi_s}\right)^2V_2\Big)\\
&+(\nabla\mathcal{T}_*)(\xi_s,\overset{M_1}{\nabla}_{\xi_s}V_2)-S_{(\nabla\mathcal{T}_*)(\xi_s,V_2)}\mathcal{T}_*(\xi_s)+\nabla^{\mathcal{T}^\perp}_{\xi_s}(\nabla\mathcal{T}_*)(\xi_s,V_2)\Big]\\
&+\bar{K}\Big(\mathcal{T}_*(\overset{M_1}{\nabla}_{\xi_s}\xi_s)+(\nabla\mathcal{T}_*)(\xi_s,\xi_s)\Big) \qquad (42)
\end{aligned}$$

If *range*$\mathcal{T}_*$ and *range*$\mathcal{T}_*^\perp$ are separated, they become

$$\begin{aligned}
&-K\mathcal{T}_*(\overset{M_1}{\nabla}_{\xi_s}\xi_s)-S_{(\nabla\mathcal{T}_*)(\xi_s,\overset{M_1}{\nabla}_{\xi_s}\xi_s)}\mathcal{T}_*(\xi_s)-\mathcal{T}_*(\overset{M_1}{\nabla}_{\xi_s}{}^*\mathcal{T}_*S_{(\nabla\mathcal{T}_*)(\xi_s,\xi_s)}\mathcal{T}_*(\xi_s))\\
&-S_{\nabla^{\mathcal{T}^\perp}_{\xi_s}(\nabla\mathcal{T}_*)(\xi_s,\xi_s)}\mathcal{T}_*(\xi_s)=2\bar{\kappa}'\mathcal{T}_*(\overset{M_1}{\nabla}_{\xi_s}V_2)+\bar{\kappa}\Big[\mathcal{T}_*\Big(\left(\overset{M_1}{\nabla}_{\xi_s}\right)^2V_2\Big)\\
&-S_{(\nabla\mathcal{T}_*)(\xi_s,V_2)}\mathcal{T}_*(\xi_s)\Big]+\bar{K}\mathcal{T}_*(\overset{M_1}{\nabla}_{\xi_s}\xi_s)
\end{aligned} \qquad (43)$$

and

$$\begin{aligned}
&(\nabla\mathcal{T}_*)(\xi_s,\left(\overset{M_1}{\nabla}_{\xi_s}\right)^2\xi_s)+\nabla^{\mathcal{T}^\perp}_{\xi_s}(\nabla\mathcal{T}_*)(\xi_s,\overset{M_1}{\nabla}_{\xi_s}\xi_s)-(\nabla\mathcal{T}_*)(\xi_s,S_{(\nabla\mathcal{T}_*)(\xi_s,\xi_s)}\mathcal{T}_*(\xi_s))\\
&+\left(\nabla^{\mathcal{T}^\perp}_{\xi_s}\right)^2(\nabla\mathcal{T}_*)(\xi_s,\xi_s)=2\bar{\kappa}'(\nabla\mathcal{T}_*)(\xi_s,V_2)+\bar{\kappa}\Big[(\nabla\mathcal{T}_*)(\xi_s,\overset{M_1}{\nabla}_{\xi_s}V_2)\\
&+\nabla^{\mathcal{T}^\perp}_{\xi_s}(\nabla\mathcal{T}_*)(\xi_s,V_2)\Big]+\bar{K}(\nabla\mathcal{T}_*)(\xi_s,\xi_s). \qquad (44)
\end{aligned}$$

Using (9), (44) and Frenet formulas, we get

$$\begin{aligned}
&2\bar{\kappa}'\kappa\mathcal{T}_*(\xi_s) + (-K\kappa - \bar{K}\kappa + \bar{\kappa}(\kappa^2+\tau^2))\mathcal{T}_*(V_2) - 2\bar{\kappa}'\tau\mathcal{T}_*(V_3)\\
&-(\widetilde{\nabla}_{\xi_s}S)_{(\nabla\mathcal{T}_*)(\xi_s,\xi_s)}\mathcal{T}_*(\xi_s) = 2S_{(\widetilde{\nabla}_{\xi_s}(\nabla\mathcal{T}_*))(\xi_s,\xi_s)}\mathcal{T}_*(\xi_s) + 5\kappa S_{(\nabla\mathcal{T}_*)(\xi_s,V_2)}\mathcal{T}_*(\xi_s)\\
&+S_{(\nabla\mathcal{T}_*)(\xi_s,\xi_s)}P\overset{M_2}{\nabla^{\mathcal{T}}}_{\xi_s}\mathcal{T}_*(\xi_s) - \bar{\kappa}S_{(\nabla\mathcal{T}_*)(\xi_s,V_2)}\mathcal{T}_*(\xi_s).
\end{aligned} \tag{45}$$

Changing V_2 into $-V_2$ in the last equation and subtracting each other, it follows that

$$\begin{aligned}
&(-K\kappa - \bar{K}\kappa + \bar{\kappa}(\kappa^2+\tau^2))\mathcal{T}_*(V_2)\\
&= 5\kappa S_{(\nabla\mathcal{T}_*)(\xi_s,V_2)}\mathcal{T}_*(\xi_s) + S_{(\nabla\mathcal{T}_*)(\xi_s,\xi_s)}P\overset{M_2}{\nabla^{\mathcal{T}}}_{\xi_s}\mathcal{T}_*(\xi_s) - \bar{\kappa}S_{(\nabla\mathcal{T}_*)(\xi_s,V_2)}\mathcal{T}_*(\xi_s)
\end{aligned} \tag{46}$$

and taking inner product with the unit vector field $\mathcal{T}_*(\xi_s)$ in last equation, we have

$$0 = (6\kappa - \bar{\kappa})g_{M_2}((\nabla\mathcal{T}_*)(\xi_s,V_2),(\nabla\mathcal{T}_*)(\xi_s,\xi_s)).$$

Since κ is constant and $\bar{\kappa}$ is not constant, it becomes $6\kappa \neq \bar{\kappa}$.

$$0 = g_{M_2}((\nabla\mathcal{T}_*)(\xi_s,V_2),(\nabla\mathcal{T}_*)(\xi_s,\xi_s)).$$

$\mathcal{T}$ is isotropic. On the other hand, using (8) and (44), it can be found that

$$\begin{aligned}
&(\nabla\mathcal{T}_*)(\xi_s,-\kappa^2\xi_s+\kappa\tau V_3) + (\widetilde{\nabla}_{\xi_s}(\nabla\mathcal{T}_*))(\xi_s,\kappa V_2)\\
&+\kappa^2(\nabla\mathcal{T}_*)(V_2,V_2) - \kappa^2(\nabla\mathcal{T}_*)(\xi_s,\xi_s) + \kappa\tau(\nabla\mathcal{T}_*)(\xi_s,V_3)\\
&-(\nabla\mathcal{T}_*)(\xi_s,S_{(\nabla\mathcal{T}_*)(\xi_s,\xi_s)}\mathcal{T}_*(\xi_s)) + (\widetilde{\nabla^2}_{\xi_s}(\nabla\mathcal{T}_*))(\xi_s,\xi_s) + 4(\widetilde{\nabla}_{\xi_s}(\nabla\mathcal{T}_*))(\xi_s,\kappa V_2)\\
&+2\kappa^2(\nabla\mathcal{T}_*)(V_2,V_2) - 2\kappa^2(\nabla\mathcal{T}_*)(\xi_s,\xi_s) + 2\kappa\tau(\nabla\mathcal{T}_*)(\xi_s,V_3)\\
&= 2\bar{\kappa}'(\nabla\mathcal{T}_*)(\xi_s,V_2) + \bar{\kappa}\Big[(\nabla\mathcal{T}_*)(\xi_s,-\kappa\xi_s+\tau V_3) + \nabla^{\mathcal{T}^{\perp}}_{\xi_s}(\nabla\mathcal{T}_*)(\xi_s,V_2)\Big]\\
&+\bar{K}(\nabla\mathcal{T}_*)(\xi_s,\xi_s)
\end{aligned} \tag{47}$$

or

$$\begin{aligned}
&-4\kappa^2(\nabla\mathcal{T}_*)(\xi_s,\xi_s) + 4\kappa\tau(\nabla\mathcal{T}_*)(\xi_s,V_3)\\
&-(\nabla\mathcal{T}_*)(\xi_s,S_{(\nabla\mathcal{T}_*)(\xi_s,\xi_s)}\mathcal{T}_*(\xi_s)) + (\widetilde{\nabla^2}_{\xi_s}(\nabla\mathcal{T}_*))(\xi_s,\xi_s)\\
&+5\kappa\nabla^{\mathcal{T}^{\perp}}_{\xi_s}(\nabla\mathcal{T}_*)(\xi_s,V_2) - 5(\nabla\mathcal{T}_*)(\kappa V_2,\kappa V_2) - 5(\nabla\mathcal{T}_*)(\xi_s,-\kappa^2\xi_s+\kappa\tau V_3)\\
&+3\kappa^2(\nabla\mathcal{T}_*)(V_2,V_2) = 2\bar{\kappa}'(\nabla\mathcal{T}_*)(\xi_s,V_2) + \bar{\kappa}\Big[(\nabla\mathcal{T}_*)(\xi_s,-\kappa\xi_s+\tau V_3)\\
&+(\widetilde{\nabla}_{\xi_s}(\nabla\mathcal{T}_*))(\xi_s,V_2) + (\nabla\mathcal{T}_*)(\xi_s,-\kappa\xi_s) + (\nabla\mathcal{T}_*)(\xi_s,\tau V_3,)\Big] + \bar{K}(\nabla\mathcal{T}_*)(\xi_s,\xi_s).
\end{aligned} \tag{48}$$

Changing V_3 into $-V_3$ in the last equation and subtracting each other, it follows that

$$-\kappa\tau(\nabla\mathcal{T}_*)(\xi_s,V_3) - 2\bar{\kappa}\tau(\nabla\mathcal{T}_*)(\xi_s,V_3) = 0 \Rightarrow (2\kappa+\bar{\kappa})\tau(\nabla\mathcal{T}_*)(\xi_s,V_3) = 0.$$

Where Burada $\tau \neq 0$. Since κ is constant $\bar{\kappa}$ is not constant, we get$(\nabla\mathcal{T}_*)(\xi_s,V_3) = 0$. From the corollary 4, T is umbilical. Also, Changing V_2 into $-V_2$ in the equation (47) and subtracting each other, it follows that

$$5\kappa(\widetilde{\nabla}_{\xi_s}(\nabla\mathcal{T}_*))(\xi_s,V_2) = 0. \tag{49}$$

Taking inner product with the unit vector field $\mathcal{T}_*(V_2)$ in the equation (46), we have

$$-K - \bar{K} + \frac{\bar{\kappa}}{\kappa}(\kappa^2+\tau^2) = \|H_2\|^2. \tag{50}$$

If equation (49), (50) and T umbilical are used in the equation (47), we have

$$\tau^2(\frac{\kappa - \bar{\kappa}}{\kappa})H_2 = 0.$$

Where $\tau \neq 0$ and $\kappa \neq \bar{\kappa}$. Then $H_2 = 0$. Thus T is a totally geodesic map. □

Acknowledgement
The author would like to thank the referee for his/her careful reading and valuable suggestions for improving this paper.

References

[1] A. Çalışkan, B. Şahin, *Slant helices on Riemannian manifolds*, Filomat 38(22) (2024), 7743–7754.
[2] B. O'Neill, *Isotropic and Kähler immersions*, Can. J. Math. 17 (1965), 907–915.
[3] G. Ö. Tükel, B. Şahin, T. Turhan, *Isotropic Riemannian maps and helices along Riemannian maps*, Sci. Bull. Series A 84(4) (2022), 89–100.
[4] F. Ateş, I. Gok, N. F. Ekmekci, *A new kind of slant helix in Lorentzian (n+2)-spaces*, Kyungpook Math. J. 56(3) (2016), 1003–1016.
[5] H. H. Song, *On proper helices in pseudo-Riemannian submanifolds*, J. Geom. 91(1-2) (2009), 150–168.
[6] J. Langer, D. Singer, *The total squared curvature of closed curves*, J. Differ. Geom. 20 (1984), 1–22.
[7] J. E. Lee, *On slant curves in Sasakian Lorentzian 3-manifolds*, Int. Electron. J. Geom. 13(2) (2020), 108–115.
[8] K. İlarslan, *Characterizations of spacelike general helices in Lorentzian manifolds*, Kragujevac J. Math. 25 (2003), 209–218.
[9] K. Yano, K. Nomizu, *On circles and spheres in Riemannian geometry*, Math. Ann. 210(2) (1974), 163–170.
[10] L. Kula, N. F. Ekmekci, Y. Yaylı, K. İlarslan, *Characterizations of slant helices in Euclidean 3-space*, Turk. J. Math. 34(2) (2010), 261–274.
[11] L. Kula, Y. Yaylı, *On slant helix and its spherical indicatrix*, Appl. Math. Comput. 169(1) (2005), 600–607.
[12] M. A. Akyol, Y. Gündüzalp, *Pointwise slant Riemannian maps ($\mathcal{PSRM}$) to almost Hermitian manifolds*, Mediterr. J. Math. 20 (2023), 116.
[13] M. A. Akyol, Y. Gündüzalp, *Pointwise hemi-slant Riemannian maps ($\mathcal{PSRM}$) to almost Hermitian manifolds*, Hacettepe J. Math. 53(5) (2024), 1218–1237.
[14] M. A. Akyol, Y. Gündüzalp, *Pointwise semi-slant Riemannian maps into almost Hermitian manifolds and Casorati inequalities*, Ukr. Math. J. 76 (2025), 1435–1456.
[15] M. A. Akyol, Y. Gündüzalp, *Pointwise bi-slant Riemannian maps ($\mathcal{PBSRM}$) from almost Hermitian manifolds*, Filomat 39(13) (2025).
[16] M. A. Akyol, B. Şahin, *Conformal anti-invariant Riemannian maps to Kähler manifolds*, Sci. Bull. Series A 80(4) (2018), 187–198.
[17] M. A. Akyol, B. Şahin, *Conformal semi-invariant Riemannian maps to Kähler manifolds*, Rev. Union Mat. Argent. 60(2) (2019), 459–468.
[18] M. A. Akyol, B. Şahin, *Conformal slant Riemannian maps to Kähler manifolds*, Tokyo J. Math. 42(1) (2019), 225–237.
[19] M. Barros, A. Ferrandez, P. Lucas, M. A. Merono, *General helices in the three-dimensional Lorentzian space forms*, Rocky Mountain J. Math. 31(2) (2001), 373–388.
[20] N. F. Ekmekci, *On general helices and submanifolds of an indefinite-Riemannian manifold*, An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.) 46 (2001), 263–270.
[21] S. Izumiya, N. Takeuchi, *New special curves and developable surfaces*, Turk. J. Math. 28(2) (2004), 153–164.
[22] S. Uddin, M. S. Stanković, M. Iqbal, S. K. Yadav, M. Aslam, *Slant helices in Minkowski 3-space E_1^3 with Sasai's modified frame fields*, Filomat 36(1) (2022), 151–164.
[23] T. Ikawa, *On some curves in Riemannian geometry*, Soochow J. Math. 7 (1980), 37–44.
[24] T. Ikawa, *On curves and submanifolds in an indefinite Riemannian manifold*, Tsukuba J. Math. 9(2) (1985), 353–371.
[25] Y. Gündüzalp, M. A. Akyol, *Pointwise slant Riemannian maps from Kähler manifolds*, J. Geom. Phys. 179 (2022), 104589.
[26] Y. Gündüzalp, M. A. Akyol, *Pointwise semi-slant Riemannian (PSSR) maps from almost Hermitian manifolds*, Filomat 37(13) (2023), 4271–4286.
[27] Y. Gündüzalp, M. A. Akyol, *Remarks on conformal anti-invariant Riemannian maps to cosymplectic manifolds*, Hacet. J. Math. Stat. 50(4) (2021), 1131–1139.
[28] Y. Gündüzalp, M. A. Akyol, B. Şahin, *Pointwise hemi-slant Riemannian (PSSR) maps into almost Hermitian manifolds and Casorati inequalities*, Int. J. Maps Math. 7(1) (2024), 76–96.