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A new study on m-quasi-totally-(α, β)-normal operators in relation to polynomials

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Abstract. Creating new operators that act as a superclass to existing ones and studying their spectral and geometrical properties is an interesting area in linear operator theory. From that perspective, the study introduces a new class of operators called polynomially m-quasi-totally-(α, β)-normal. This new class integrates features from (α, β)-normal, quasi-(α, β)-normal and m-quasi-totally-(α, β)-normal operators. This article analyzes several properties of polynomially m-quasi-totally-(α, β)-normal operators.

1. Introduction

Let $\mathscr{H}$ be a non zero complex Hilbert space and let $\mathscr{B}(\mathscr{H})$ denote the algebra of all bounded linear operators on $\mathscr{H}$. Let m be a natural number.

Definition 1.1. *Let $\mathcal{S} \in \mathscr{B}(\mathscr{H})$.*

1. *An operator $\mathcal{S}$ is called (α, β)-normal [9] $(0 \leq \alpha \leq 1 \leq \beta)$ if*

$$\alpha^2 \mathcal{S}^* \mathcal{S} \leq \mathcal{S}\mathcal{S}^* \leq \beta^2 \mathcal{S}^* \mathcal{S}.$$

2. *An operator $\mathcal{S}$ is called quasi-(α, β)-normal [22] $(0 \leq \alpha \leq 1 \leq \beta)$ if*

$$\alpha^2 \mathcal{S}^{*2} \mathcal{S}^2 \leq \mathcal{S}^* \mathcal{S}\mathcal{S}^* \mathcal{S} \leq \beta^2 \mathcal{S}^{*2} \mathcal{S}^2.$$

3. *An operator $\mathcal{S}$ is called m-quasi-(α, β)-normal [22] $(0 \leq \alpha \leq 1 \leq \beta)$ if*

$$\alpha^2 \mathcal{S}^{(m+1)*} \mathcal{S}^{m+1} \leq \mathcal{S}^{m*} (\mathcal{S}\mathcal{S}^*) \mathcal{S}^m \leq \beta^2 \mathcal{S}^{(m+1)*} \mathcal{S}^{m+1}$$

for a natural number m.

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4. *An operator $\mathcal{S}$ is called m-quasi-totally-(α, β)-normal [22]* $(0 \le \alpha \le 1 \le \beta)$ *if*

$$\alpha^2 \mathcal{S}^{m*}(\mathcal{S}-\lambda)^*(\mathcal{S}-\lambda)\mathcal{S}^m \le \mathcal{S}^{m*}(\mathcal{S}-\lambda)(\mathcal{S}-\lambda)^*\mathcal{S}^m$$
$$\le \beta^2 \mathcal{S}^{m*}(\mathcal{S}-\lambda)^*(\mathcal{S}-\lambda)\mathcal{S}^m$$

for a natural number m and for all $\lambda \in \mathbb{C}$.

In general the following implications holds:

$$(\alpha,\beta)-\text{ normal } \subseteq \text{quasi}-(\alpha,\beta)-\text{ normal}$$
$$\subseteq m-\text{quasi}-(\alpha,\beta)-\text{normal} \subseteq m-\text{quasi}-\text{totally}-(\alpha,\beta)-\text{normal}.$$

In the papers [8, 14], the authors have studied the class of polynomially normal operator as follows: An operator $\mathcal{S}$ is said to be polynomially normal if there exists a nontrivial polynomial $\mathcal{P} = \sum_{0\le k\le n} b_k z^k \in \mathbb{C}(z)$ with

$$\mathcal{P}(\mathcal{S})\mathcal{S}^* - \mathcal{S}^*\mathcal{P}(\mathcal{S}) = 0.$$

One of the current trends in operator theory is studying new extension for normal operators. In [21], the authors have introudced polynomially quasi-M-hyponormal operators.
An operator $\mathcal{S}$ is said to be polynomially quasi-M-hyponormal if there exists a nontrivial polynomial $\mathcal{P} \in \mathbb{C}(z)$ and a postive constant M such that

$$\mathcal{P}(\mathcal{S})^*\Big(M^2(\mathcal{S}-\lambda)^*(\mathcal{S}-\lambda) - (\mathcal{S}-\lambda)(\mathcal{S}-\lambda)^*\Big)\mathcal{P}(\mathcal{S}) \ge 0.$$

for all $\lambda \in \mathbb{C}$.
In the following, we introduce a new class of operators called the class of polynomially m-quasi-totally-(α,β)-normal operators as a new extension of m-quasi-totally-(α,β)-normal operators.

An operator $\mathcal{S} \in \mathscr{B}(\mathscr{H})$ is called polynomially m-quasi-totally-(α,β)-normal $(0 \le \alpha \le 1 \le \beta)$ if there exists a nontrivial polynomial $\mathcal{P} \in \mathbb{C}(z)$ such that

$$\alpha^2\mathcal{P}(\mathcal{S}^m)^*(\mathcal{S}-\lambda)^*(\mathcal{S}-\lambda)\mathcal{P}(\mathcal{S}^m) \le \mathcal{P}(\mathcal{S}^m)^*(\mathcal{S}-\lambda)(\mathcal{S}-\lambda)^*\mathcal{P}(\mathcal{S}^m) \le \beta^2\mathcal{P}(\mathcal{S}^m)^*(\mathcal{S}-\lambda)^*(\mathcal{S}-\lambda)\mathcal{P}(\mathcal{S}^m)$$

for all $\lambda \in \mathbb{C}$.

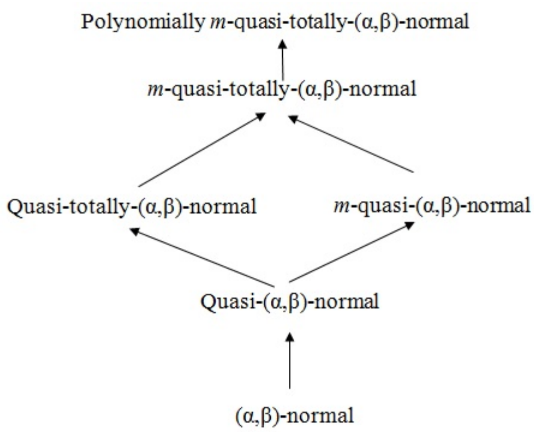

Figure 1: Inclusion relation between generalized (α,β)-normal operator

In 1966, R.G. Douglas [10] proved an equivalence of factorization, range inclusion and majorization of operators, known as Douglas lemma. Note that polynomially m-quasi-totally-(α,β)-normal operator is equivalent to the study of mutual majorization between $(\mathcal{S}-\lambda)\mathcal{P}(\mathcal{S}^m)$ and $(\mathcal{S}-\lambda)^*\mathcal{P}(\mathcal{S}^m)$. It can be said that both $(\mathcal{S}-\lambda)\mathcal{P}(\mathcal{S}^m)$ majorizes $(\mathcal{S}-\lambda)^*\mathcal{P}(\mathcal{S}^m)$ and $(\mathcal{S}-\lambda)^*\mathcal{P}(\mathcal{S}^m)$ majorizes $(\mathcal{S}-\lambda)\mathcal{P}(\mathcal{S}^m)$ for a natural number m. Using Douglas' result, it is observed that $\mathcal{S}$ is polynomially m-quasi-totally-(α,β)-normal if and only if

$$ran\,((\mathcal{S}-\lambda)\mathcal{P}(\mathcal{S}^m)) = ran\,((\mathcal{S}-\lambda)^*\mathcal{P}(\mathcal{S}^m))$$

or equivalently

$$ker\,((\mathcal{S}-\lambda)\mathcal{P}(\mathcal{S}^m)) = ker\,((\mathcal{S}-\lambda)^*\mathcal{P}(\mathcal{S}^m))\,.$$

In particular (choose $\lambda = 0$), an operator $\mathcal{S}$ is called polynomially m-quasi-(α,β)-normal $(0 \le \alpha \le 1 \le \beta)$ if there exists a nontrivial polynomial $\mathcal{P} \in \mathbb{C}(z)$ such that

$$\alpha^2\mathcal{P}(\mathcal{S}^m)^*\mathcal{S}^*\mathcal{S}\mathcal{P}(\mathcal{S}^m) \le \mathcal{P}(\mathcal{S}^m)^*\mathcal{S}\mathcal{S}^*\mathcal{P}(\mathcal{S}^m) \le \beta^2\mathcal{P}(\mathcal{S}^m)^*\mathcal{S}^*\mathcal{S}\mathcal{P}(\mathcal{S}^m).$$

Remark 1.2.
1. *Every m-quasi-totally-(α,β)-normal is polynomially m-quasi-totally-(α,β)-normal with $\mathcal{P}(z) = z$.*
2. *Every m-quasi-(α,β)-normal is polynomially quasi-(α,β)-normal with $\mathcal{P}(z) = z^m$.*

Example 1.3. *The following operator $\mathcal{S}$ in $\mathscr{B}(\mathbb{C}^2)$ is polynomially 2-quasi-(α,β)-normal for $\alpha = 0.04$ and $\beta = 3.8$ with respect to the polynomial $\mathcal{P}(z) = z^2 + 2z$, which is not normal, quasi-normal, hyponormal and quasi-hyponormal.*

$$\mathcal{S} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

2. Main Results

We begin with:

Theorem 2.1. *Let $\mathcal{S} \in \mathscr{B}(\mathscr{H})$ and let $\mathcal{P} \in \mathbb{C}(z)$ be any nontrivial polynomial. $\mathcal{S}$ is an polynomially m-quasi-totally-(α,β)-normal operator iff*

$$\alpha\|(\mathcal{S}-\lambda)\mathcal{P}(\mathcal{S}^m)x\| \le \|(\mathcal{S}-\lambda)^*\mathcal{P}(\mathcal{S}^m)x\| \le \beta\|(\mathcal{S}-\lambda)\mathcal{P}(\mathcal{S}^m)x\|$$

for all $\lambda \in \mathbb{C}$ and for all $x \in \mathscr{H}$.

Proof. Assume that $\mathcal{S}$ is an polynomially m-quasi-totally-(α,β)-normal operator, then there exist $\mathcal{P} \in \mathbb{C}(z)$ for which

$$\begin{aligned}
\alpha^2\|(\mathcal{S}-\lambda)\mathcal{P}(\mathcal{S}^m)x\|^2 &= \alpha^2\,\langle(\mathcal{S}-\lambda)\mathcal{P}(\mathcal{S}^m)x,(\mathcal{S}-\lambda)\mathcal{P}(\mathcal{S}^m)x\rangle\\
&= \alpha^2\,\langle\mathcal{P}(\mathcal{S}^m)^*(\mathcal{S}-\lambda)^*(\mathcal{S}-\lambda)\mathcal{P}(\mathcal{S}^m)x,x\rangle\\
&\le \langle\mathcal{P}(\mathcal{S}^m)^*(\mathcal{S}-\lambda)(\mathcal{S}-\lambda)^*\mathcal{P}(\mathcal{S}^m)x,x\rangle\\
&= \langle(\mathcal{S}-\lambda)^*\mathcal{P}(\mathcal{S}^m)x,(\mathcal{S}-\lambda)^*\mathcal{P}(\mathcal{S}^m)x\rangle\\
&= \|(\mathcal{S}-\lambda)^*\mathcal{P}(\mathcal{S}^m)x\|^2\\
&= \langle(\mathcal{S}-\lambda)^*\mathcal{P}(\mathcal{S}^m)x,(\mathcal{S}-\lambda)^*\mathcal{P}(\mathcal{S}^m)x\rangle\\
&= \langle\mathcal{P}(\mathcal{S}^m)^*(\mathcal{S}-\lambda)(\mathcal{S}-\lambda)^*\mathcal{P}(\mathcal{S}^m)x,x\rangle\\
&\le \beta^2\,\langle\mathcal{P}(\mathcal{S}^m)^*(\mathcal{S}-\lambda)^*(\mathcal{S}-\lambda)\mathcal{P}(\mathcal{S}^m)x,x\rangle\\
&= \beta^2\,\langle(\mathcal{S}-\lambda)\mathcal{P}(\mathcal{S}^m)x,(\mathcal{S}-\lambda)\mathcal{P}(\mathcal{S}^m)x\rangle\\
&= \beta^2\|(\mathcal{S}-\lambda)\mathcal{P}(\mathcal{S}^m)x\|^2.
\end{aligned}$$

Conversely, assume that $\mathcal{S}$ satisfies

$$\alpha\|(\mathcal{S}-\lambda)\mathcal{P}(\mathcal{S}^m)x\| \leq \|(\mathcal{S}-\lambda)^*\mathcal{P}(\mathcal{S}^m)x\| \leq \beta\|(\mathcal{S}-\lambda)\mathcal{P}(\mathcal{S}^m)x\|$$

for all $\lambda \in \mathbb{C}$ and for all $x \in \mathscr{H}$, we have

$$\begin{aligned}\alpha^2 \langle(\mathcal{S}-\lambda)\mathcal{P}(\mathcal{S}^m)x, (\mathcal{S}-\lambda)\mathcal{P}(\mathcal{S}^m)x\rangle &\leq \langle(\mathcal{S}-\lambda)^*\mathcal{P}(\mathcal{S}^m)x, (\mathcal{S}-\lambda)^*\mathcal{P}(\mathcal{S}^m)x\rangle \\ &\leq \beta^2 \langle(\mathcal{S}-\lambda)\mathcal{P}(\mathcal{S}^m)x, (\mathcal{S}-\lambda)\mathcal{P}(\mathcal{S}^m)x\rangle .\end{aligned}$$

So one can obtain that

$$\begin{aligned}\left\langle\alpha^2\mathcal{P}(\mathcal{S}^m)^*(\mathcal{S}-\lambda)^*(\mathcal{S}-\lambda)\mathcal{P}(\mathcal{S}^m)x, x\right\rangle &\leq \langle\mathcal{P}(\mathcal{S}^m)^*(\mathcal{S}-\lambda)(\mathcal{S}-\lambda)^*\mathcal{P}(\mathcal{S}^m)x, x\rangle \\ &\leq \left\langle\beta^2\mathcal{P}(\mathcal{S}^m)^*(\mathcal{S}-\lambda)^*(\mathcal{S}-\lambda)\mathcal{P}(\mathcal{S}^m)x, x\right\rangle .\end{aligned}$$

Therefore

$$\alpha^2\mathcal{P}(\mathcal{S}^m)^*(\mathcal{S}-\lambda)^*(\mathcal{S}-\lambda)\mathcal{P}(\mathcal{S})^m \leq \mathcal{P}(\mathcal{S}^m)^*(\mathcal{S}-\lambda)(\mathcal{S}-\lambda)^*\mathcal{P}(\mathcal{S})^m \leq \beta^2\mathcal{P}(\mathcal{S}^m)^*(\mathcal{S}-\lambda)^*(\mathcal{S}-\lambda)\mathcal{P}(\mathcal{S})^m.$$

Hence $\mathcal{S}$ is polynomially m-quasi-totally-(α,β)-normal operator. □

Proposition 2.2. *Suppose $\mathcal{S} \in \mathscr{B}(\mathscr{H})$ and $\mathcal{P} \in \mathbb{C}(z)$ is any nontrivial polynomial. Then $\mathcal{S}$ is polynomially m-quasi-totally-(α,β)-normal operator if and only if*

$$\begin{aligned}k^2\mathcal{P}(\mathcal{S}^m)^*(\mathcal{S}-\lambda)(\mathcal{S}-\lambda)^*\mathcal{P}(\mathcal{S}^m) + 2k\alpha^2\mathcal{P}(\mathcal{S}^m)^*(\mathcal{S}-\lambda)^*(\mathcal{S}-\lambda)\mathcal{P}(\mathcal{S}^m) \\ + \mathcal{P}(\mathcal{S}^m)^*(\mathcal{S}-\lambda)(\mathcal{S}-\lambda)^*\mathcal{P}(\mathcal{S}^m) \geq 0\end{aligned}$$

and

$$\begin{aligned}k^2\mathcal{P}(\mathcal{S}^m)^*(\mathcal{S}-\lambda)^*(\mathcal{S}-\lambda)\mathcal{P}(\mathcal{S}^m) + 2k\mathcal{P}(\mathcal{S}^m)^*(\mathcal{S}-\lambda)(\mathcal{S}-\lambda)^*\mathcal{P}(\mathcal{S}^m) \\ + \beta^4\mathcal{P}(\mathcal{S}^m)^*(\mathcal{S}-\lambda)^*(\mathcal{S}-\lambda)\mathcal{P}(\mathcal{S}^m) \geq 0\end{aligned}$$

for all $k \in \mathbb{R}$.

Proof. By using elementary properties of real quadratic forms,

$$\begin{aligned}k^2\mathcal{P}(\mathcal{S}^m)^*(\mathcal{S}-\lambda)(\mathcal{S}-\lambda)^*\mathcal{P}(\mathcal{S}^m) + 2k\alpha^2\mathcal{P}(\mathcal{S}^m)^*(\mathcal{S}-\lambda)^*(\mathcal{S}-\lambda)\mathcal{P}(\mathcal{S}^m) \\ +\mathcal{P}(\mathcal{S}^m)^*(\mathcal{S}-\lambda)(\mathcal{S}-\lambda)^*\mathcal{P}(\mathcal{S}^m) \geq 0 \\ \Leftrightarrow k^2\|(\mathcal{S}-\lambda)^*\mathcal{P}(\mathcal{S}^m)x\|^2 + 2k\alpha^2\|(\mathcal{S}-\lambda)\mathcal{P}(\mathcal{S}^m)x\|^2 \\ +\|(\mathcal{S}-\lambda)^*\mathcal{P}(\mathcal{S}^m)x\|^2 \geq 0 \quad \forall\, x \in \mathscr{H} \text{ and } \quad \forall\, k \in \mathbb{R} \\ \Leftrightarrow \alpha\|(\mathcal{S}-\lambda)\mathcal{P}(\mathcal{S}^m)x\| \leq \|(\mathcal{S}-\lambda)^*\mathcal{P}(\mathcal{S}^m)x\| \quad \forall\, x \in \mathscr{H}.\end{aligned}$$

Similarly,

$$\begin{aligned}k^2\mathcal{P}(\mathcal{S}^m)^*(\mathcal{S}-\lambda)^*(\mathcal{S}-\lambda)\mathcal{P}(\mathcal{S}^m) + 2k\mathcal{P}(\mathcal{S}^m)^*(\mathcal{S}-\lambda)(\mathcal{S}-\lambda)^*\mathcal{P}(\mathcal{S}^m) \\ +\beta^4\mathcal{P}(\mathcal{S}^m)^*(\mathcal{S}-\lambda)^*(\mathcal{S}-\lambda)\mathcal{P}(\mathcal{S}^m) \geq 0 \\ \Leftrightarrow k^2\|(\mathcal{S}-\lambda)\mathcal{P}(\mathcal{S}^m)x\|^2 + 2k\|(\mathcal{S}-\lambda)^*\mathcal{P}(\mathcal{S}^m)x\|^2 \\ +\beta^4\|(\mathcal{S}-\lambda)\mathcal{P}(\mathcal{S}^m)x\|^2 \geq 0 \quad \forall\, x \in \mathscr{H} \text{ and } \quad \forall\, k \in \mathbb{R} \\ \Leftrightarrow \beta\|(\mathcal{S}-\lambda)\mathcal{P}(\mathcal{S}^m)x\| \geq \|(\mathcal{S}-\lambda)^*\mathcal{P}(\mathcal{S}^m)x\| \quad \forall\, x \in \mathscr{H}.\end{aligned}$$

Therefore $\mathcal{S}$ is polynomially m-quasi-totally-(α,β)-normal operator. □

Theorem 2.3. *Let $\mathcal{S} \in \mathscr{B}(\mathscr{H})$ such that $\mathcal{P}(\mathcal{S}^m)$ does not have a dense range, then the following are equivalent.*

(1) *$\mathcal{S}$ is a polynomially m-quasi-totally-(α,β)-normal operator.*

(2) $\mathcal{S} = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$ *on $\mathscr{H} = \overline{ran(\mathcal{P}(\mathcal{S}^m))} \oplus ker(\mathcal{P}(\mathcal{S}^m)^*)$, where $A = \mathcal{S}|_{\overline{ran(\mathcal{P}(\mathcal{S}^m))}}$ satisfies*

$$\alpha^2(A-\lambda)^*(A-\lambda) \leq (A-\lambda)(A-\lambda)^* + BB^* \leq \beta^2(A-\lambda)^*(A-\lambda),$$

for all $\lambda \in \mathbb{C}$ and $\mathcal{P}(C^m) = 0$. Furthermore $\sigma(\mathcal{S}) = \sigma(A) \cup \{0\}$.

Proof. (1) $\Rightarrow$ (2). Consider the matrix representation of $\mathcal{S}$ with respect to the decomposition $\mathscr{H} = \overline{ran(\mathcal{P}(\mathcal{S}^m))} \oplus ker(\mathcal{P}(\mathcal{S}^{*m}))$: $\mathcal{S} = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$. Let P be the projection onto $\overline{ran(\mathcal{P}(\mathcal{S}^m))}$. Then $\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} = \mathcal{S}P = P\mathcal{S}P$. Since $\mathcal{S}$ is polynomially m-quasi totally-(α,β)-normal operator, we have then

$$\alpha^2 P\Big(\mathcal{P}(\mathcal{S}^{*m})(\mathcal{S}-\lambda)^*(\mathcal{S}-\lambda)\mathcal{P}(\mathcal{S}^m)\Big)P \leq P\Big(\mathcal{P}(\mathcal{S}^{*m})(\mathcal{S}-\lambda)(\mathcal{S}-\lambda)^*\mathcal{P}(\mathcal{S}^m)\Big)P$$

$$\leq \beta^2 P\Big(\mathcal{P}(\mathcal{S}^{*m})(\mathcal{S}-\lambda)^*(\mathcal{S}-\lambda)\mathcal{P}(\mathcal{S}^m)\Big)P$$

That is

$$\alpha^2(A-\lambda)^*(A-\lambda) \leq (A-\lambda)(A-\lambda)^* + BB^* \leq \beta^2(A-\lambda)^*(A-\lambda),$$

for all $\lambda \in \mathbb{C}$.

On the other hand, let $x = x_1 + x_2 \in \mathscr{H} = \overline{ran(\mathcal{P}(\mathcal{S}^m))} \oplus ker(\mathcal{P}(\mathcal{S}^{*m}))$. A simple computation shows that

$$\begin{aligned} \langle \mathcal{P}(C^m)x_2, x_2\rangle &= \langle \mathcal{P}(\mathcal{S}^m)(I-P)x, (I-P)x\rangle \\ &= \langle (I-P)x, \mathcal{P}(\mathcal{S}^{*m})(I-P)x\rangle = 0. \end{aligned}$$

So, $\mathcal{P}(C^m) = 0$.

Since $\sigma(\mathcal{S}) \cup \mathcal{T} = \sigma(A) \cup \sigma(C)$, where $\mathcal{T}$ is the union of the holes in $\sigma(\mathcal{S})$ which happen to be subset of $\sigma(A) \cap \sigma(C)$ by Corollary 7 of [12], and $\sigma(A) \cap \sigma(C)$ has no interior point and C is nilpotent, we have $\sigma(\mathcal{S}) = \sigma(A) \cup \{0\}$.

(2) $\Rightarrow$ (1) Suppose that $\mathcal{S} = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$ onto $\mathscr{H} = \overline{ran(\mathcal{P}(\mathcal{S}^m))} \oplus ker(\mathcal{P}(\mathcal{S}^{*m}))$, with

$$\alpha^2\Big((A-\lambda)^*(A-\lambda)\Big) \leq (A-\lambda)(A-\lambda)^* + BB^* \leq \beta^2\Big((A-\lambda)^*(A-\lambda)\Big),$$

for all $\lambda \in \mathbb{C}$ and $\mathcal{P}(C^m) = 0$.

Since $\mathcal{S}^m = \begin{pmatrix} A^m & \sum_{j=0}^{m-1} A^j B C^{m-1-j} \\ 0 & 0 \end{pmatrix}$, $\mathcal{P}(\mathcal{S}^m) = \begin{pmatrix} \mathcal{P}(A^m) & Y \\ 0 & 0 \end{pmatrix}$

$$(\mathcal{S}-\lambda)^*(\mathcal{S}-\lambda) = \begin{pmatrix} (A-\lambda)^*(A-\lambda) & (A-\lambda)^*B \\ B^*(A-\lambda) & B^*B + (C-\lambda)^*(C-\lambda) \end{pmatrix}$$

and

$$(\mathcal{S}-\lambda)(\mathcal{S}-\lambda)^* = \begin{pmatrix} (A-\lambda)(A-\lambda)^* + BB^* & B(C-\lambda)^* \\ (C-\lambda)B^* & (C-\lambda)(C-\lambda)^* \end{pmatrix}.$$

Further

$$\mathcal{P}(\mathcal{S}^m)\mathcal{P}(\mathcal{S}^{*m}) = \begin{pmatrix} \mathcal{P}(A^m)\mathcal{P}(A^{*m}) + YY^* & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}.$$

where $D = \mathcal{P}(A^m)\mathcal{P}(A^{*m}) + YY^* = D^*$.
Hence for all $\lambda \in \mathbb{C}$, we have

$$\begin{aligned} &\alpha^2\mathcal{P}(\mathcal{S}^m)\mathcal{P}(\mathcal{S}^{*m})\Big((\mathcal{S}-\lambda)^*(\mathcal{S}-\lambda)\Big)\mathcal{P}(\mathcal{S}^m)\mathcal{P}(\mathcal{S}^{*m}) \\ = &\begin{pmatrix} \alpha^2 D(A-\lambda)^*(A-\lambda)D & 0 \\ 0 & 0 \end{pmatrix} \\ \leq &\begin{pmatrix} D\Big((A-\lambda)(A-\lambda)^* + BB^*\Big)D & 0 \\ 0 & 0 \end{pmatrix} = \mathcal{P}(\mathcal{S}^m)\mathcal{P}(\mathcal{S}^{*m})\Big((\mathcal{S}-\lambda)(\mathcal{S}-\lambda)^*\Big)\mathcal{P}(\mathcal{S}^m)\mathcal{P}(\mathcal{S}^{*m}) \\ \leq &\begin{pmatrix} \beta^2 D(A-\lambda)^*(A-\lambda)D & 0 \\ 0 & 0 \end{pmatrix} = \beta^2\mathcal{P}(\mathcal{S}^m)\mathcal{P}(\mathcal{S}^{*m})\Big((\mathcal{S}-\lambda)^*(\mathcal{S}-\lambda)\Big)\mathcal{P}(\mathcal{S}^m)\mathcal{P}(\mathcal{S}^{*m}). \end{aligned}$$

It follows that

$$\begin{aligned} &\alpha^2\mathcal{P}(\mathcal{S}^m)\mathcal{P}(\mathcal{S}^{*m})\Big((\mathcal{S}-\lambda)^*(\mathcal{S}-\lambda)\Big)\mathcal{P}(\mathcal{S}^m)\mathcal{P}(\mathcal{S}^{*m}) \\ &\leq \mathcal{P}(\mathcal{S}^m)\mathcal{P}(\mathcal{S}^{*m})\Big((\mathcal{S}-\lambda)(\mathcal{S}-\lambda)^*\Big)\mathcal{P}(\mathcal{S}^m)\mathcal{P}(\mathcal{S}^{*m}) \\ &\leq \beta^2\mathcal{P}(\mathcal{S}^m)\mathcal{P}\mathcal{S}^{*m}\Big((\mathcal{S}-\lambda)^*(\mathcal{S}-\lambda)\Big)\mathcal{P}(\mathcal{S}^m)\mathcal{P}(\mathcal{S}^{*m}). \end{aligned}$$

This means that

$$\begin{aligned} \alpha^2\mathcal{P}(\mathcal{S}^{*m})\Big((\mathcal{S}-\lambda)^*(\mathcal{S}-\lambda)\Big)\mathcal{P}(\mathcal{S}^m) &\leq \mathcal{P}(\mathcal{S}^{*m})\Big((\mathcal{S}-\lambda)(\mathcal{S}-\lambda)^*\Big)\mathcal{P}(\mathcal{S}^m) \\ &\leq \beta^2\mathcal{P}(\mathcal{S}^{*m})\Big((\mathcal{S}-\lambda)^*(\mathcal{S}-\lambda)\Big)\mathcal{P}(\mathcal{S}^m), \end{aligned}$$

on $\mathscr{H} = \overline{ran(\mathcal{P}(\mathcal{S}^{*m}))} \oplus ker(\mathcal{P}(\mathcal{S}^m))$. Consequently, $\mathcal{S}$ is polynomially m-quasi-totally-(α,β)-normal. □

Theorem 2.4. *Let $(\alpha,\beta) \in \mathbb{R}^2$ such that $0 < \alpha \leq 1 \leq \beta$ and let $\mathcal{S} \in \mathscr{B}(\mathscr{H})$ such that $ran(\mathcal{P}(\mathcal{S}^m)) = ran(\mathcal{P}(\mathcal{S}^m)^*)$. If $\mathcal{S}$ is polynomially m-quasi-(α,β)-normal, then $\mathcal{S}^*$ is polynomially m-quasi-$\left(\frac{1}{\beta},\frac{1}{\alpha}\right)$-normal.*

Proof. Since $\mathcal{S}$ is polynomially m-quasi-(α,β)-normal, it follows that

$$\alpha\|\mathcal{S}\mathcal{P}(\mathcal{S}^m)x\| \leq \|\mathcal{S}^*\mathcal{P}(\mathcal{S}^m)x\| \leq \beta\|\mathcal{S}\mathcal{P}(\mathcal{S}^m)x\|, \quad \forall\, x \in \mathscr{H}.$$

This means that

$$\alpha\|\mathcal{S}\mathcal{P}(\mathcal{S}^m)^*x\| \leq \|\mathcal{S}^*\mathcal{P}(\mathcal{S}^m)^*x\| \leq \beta\|\mathcal{S}\mathcal{P}(\mathcal{S}^m)^*x\|, \quad \forall\, x \in \mathscr{H}.$$

Combining these inequalities,

$$\frac{1}{\beta}\|\mathcal{S}^*\mathcal{P}(\mathcal{S}^m)^*x\| \leq \|\mathcal{S}\mathcal{P}(\mathcal{S}^m)^*x\| \leq \frac{1}{\alpha}\|\mathcal{S}^*\mathcal{P}(\mathcal{S}^m)^*x\|.$$

So, $\mathcal{S}^*$ is polynomially m-quasi-$(\frac{1}{\beta},\frac{1}{\alpha})$-normal. □

Theorem 2.5. *Let $\mathcal{S}$ be polynomially m-quasi-totally-(α,β)-normal operator. If $\mathcal{P}(\mathcal{S}^m)$ has dense range, then $\mathcal{S}$ is totally-(α,β)-normal.*

Proof. Since $\mathcal{P}(\mathcal{S}^m)$ has a dense range, it follows that $\overline{ran(\mathcal{P}(\mathcal{S}^m))} = \mathscr{H}$. Let $y \in \mathscr{H}$. Then there exists a sequence (x_n) in $\mathscr{H}$ such that $\mathcal{P}(\mathcal{S}^m)(x_n) \to y$ as $n \to \infty$.
Since $\mathcal{S}$ is polynomially m-quasi-totally-(α, β)-normal operator, we have

$$\alpha \| (\mathcal{S} - \lambda) \mathcal{P}(\mathcal{S}^m)x\| \leq \| (\mathcal{S} - \lambda)^* \mathcal{P}(\mathcal{S}^m)x\| \leq \beta \| (\mathcal{S} - \lambda) \mathcal{P}(\mathcal{S}^m)x\|$$

for all $x \in \mathscr{H}$ and for all $\lambda \in \mathbb{C}$.
In particular,

$$\alpha \| (\mathcal{S} - \lambda) \mathcal{P}(\mathcal{S}^m)x_n\| \leq \| (\mathcal{S} - \lambda)^* \mathcal{P}(\mathcal{S}^m)x_n\| \leq \beta \| (\mathcal{S} - \lambda) \mathcal{P}(\mathcal{S}^m)x_n\|$$

for all $x_n \in \mathscr{H}$ and for all $\lambda \in \mathbb{C}$.
It follows that

$$\alpha \| (\mathcal{S} - \lambda) y\| \leq \| (\mathcal{S} - \lambda)^* y\| \leq \beta \| (\mathcal{S} - \lambda) y\|$$

for all $y \in \mathscr{H}$ and for all $\lambda \in \mathbb{C}$. Therefore $\mathcal{S}$ is totally-(α, β) – normal operator. □

Corollary 2.6. *Let $\mathcal{S}$ be polynomially m-quasi-totally-(α, β)-normal operator. If $\mathcal{P}(\mathcal{S}^m) \neq 0$ and if $\mathcal{S}$ has no nontrivial $\mathcal{P}(\mathcal{S}^m)$-invariant closed subspace, then $\mathcal{S}$ is totally-(α, β)-normal.*

Proof. Since $\mathcal{P}(\mathcal{S}^m)$ has no nontrivial invariant closed subspace, it has no nontrivial hyperinvariant subspace. But $ker(\mathcal{P}(\mathcal{S}^m))$ and $\overline{ran(\mathcal{P}(\mathcal{S}^m))}$ are hyperinvariant subspaces, and $\mathcal{P}(\mathcal{S}^m) \neq 0$, hence $ker(\mathcal{P}(\mathcal{S}^m)) = 0$ and $\overline{ran(\mathcal{P}(\mathcal{S}^m))} = \mathscr{H}$. Therefore $\mathcal{S}$ is totally-(α, β)-normal operator. □

Corollary 2.7. *If $\mathcal{S}$ is such that $a_1 + a_2\mathcal{S}$ is polynomially m-quasi-totally-(α, β)-normal operator for all scalars a_1 and a_2, then $\mathcal{S}$ is totally-(α, β)-normal.*

Proof. If $\mathcal{S}$ is polynomially m-quasi-totally-(α, β)-normal operator but not totally-(α, β)-normal operator, then $\mathcal{P}(\mathcal{S}^m)$ is not invertible. It is possible to find scalars a_1 and $a_2 \neq 0$ such that $\mathcal{T} = a_1 + a_2\mathcal{S}$ is invertible polynomially m-quasi-totally-(α, β)-normal operator. Therefore $\mathcal{T}$ is totally-(α, β)-normal operators.

$$\mathcal{T} = a_1 + a_2\mathcal{S} \Rightarrow \mathcal{S} = \frac{1}{a_2}(\mathcal{T} - a_1).$$

Therefore $\mathcal{S}$ is also totally-(α, β)-normal. □

In the following theorem, the stability of the sum of two polynomially m-quasi-totally-(α, β)-normal operators is preserved under the specific conditions.

Theorem 2.8. *Let $\mathcal{S}, \mathcal{T} \in \mathscr{B}(\mathscr{H})$. $\mathcal{S}, \mathcal{T}$ are polynomially m-quasi-totally-(α, β)-normal operator satisfies the following conditions for some $\mathcal{P} \in \mathbb{C}(z)$:*

- $(\mathcal{S} - \lambda)\mathcal{P}(\mathcal{T}) = (\mathcal{T} - \lambda)\mathcal{P}(\mathcal{S}) = 0$
- $\mathcal{P}(\mathcal{T})^*(\mathcal{S} - \lambda) = \mathcal{P}(\mathcal{S})^*(\mathcal{T} - \lambda) = 0$
- $(\mathcal{S} - \lambda)(\mathcal{T} - \lambda)^* = (\mathcal{S} - \lambda)^*(\mathcal{T} - \lambda) = 0$
- $\mathcal{S}\mathcal{T} = \mathcal{T}\mathcal{S} = 0$

Then $\mathcal{S} + \mathcal{T}$ is polynomially m-quasi-totally-(α, β)-normal operator.

Proof. Set $\mathcal{P}(z) = \sum_{0 \leq k \leq n} a_k z^k$.
We have $\mathcal{P}(\mathcal{S} + \mathcal{T})^m = \mathcal{P}(\mathcal{S}^m) + \mathcal{P}(\mathcal{T})^m$ since $\mathcal{S}\mathcal{T} = \mathcal{T}\mathcal{S} = 0$.
Since $\mathcal{S}, \mathcal{T}$ are polynomially m-quasi-totally-(α, β)-normal operator, we have

$$\begin{aligned}\alpha^2\mathcal{P}(\mathcal{S}^m)^*(\mathcal{S} - \lambda)^*(\mathcal{S} - \lambda)\mathcal{P}(\mathcal{S}^m) &\leq \mathcal{P}(\mathcal{S}^m)^*(\mathcal{S} - \lambda)(\mathcal{S} - \lambda)^*\mathcal{P}(\mathcal{S}^m) \\ &\leq \beta^2\mathcal{P}(\mathcal{S}^m)^*(\mathcal{S} - \lambda)^*(\mathcal{S} - \lambda)\mathcal{P}(\mathcal{S}^m),\end{aligned}$$

$$\alpha^2\mathcal{P}(\mathcal{T}^m)^*(\mathcal{T}-\lambda)^*(\mathcal{T}-\lambda)\mathcal{P}(\mathcal{T}^m) \le \mathcal{P}(\mathcal{T}^m)^*(\mathcal{T}-\lambda)(\mathcal{T}-\lambda)^*\mathcal{P}(\mathcal{T}^m)$$
$$\le \beta^2\mathcal{P}(\mathcal{T}^m)^*(\mathcal{T}-\lambda)^*(\mathcal{T}-\lambda)\mathcal{P}(\mathcal{T}^m)$$

for all $\lambda \in \mathbb{C}$.

To show that $\mathcal{S}+\mathcal{T}$ is polynomially m-quasi-totally-(α,β)-normal operator.

First we have,

$$\begin{aligned}
&\mathcal{P}\Big((\mathcal{S}+\mathcal{T})^m\Big)^*\Big[\alpha^2\Big((\mathcal{S}-\lambda)^*+(\mathcal{T}-\lambda)^*\Big)\Big((\mathcal{S}-\lambda)+(\mathcal{T}-\lambda)\Big)\\
&-\Big((\mathcal{S}-\lambda)+(\mathcal{T}-\lambda)\Big)\Big((\mathcal{S}-\lambda)^*+(\mathcal{T}-\lambda)^*\Big)\Big]\mathcal{P}\Big((\mathcal{S}+\mathcal{T})^m\Big)\\
=\;& \mathcal{P}(\mathcal{S}^{m*}+\mathcal{T}^{m*})\Big[\alpha^2\Big((\mathcal{S}-\lambda)^*+(\mathcal{T}-\lambda)^*\Big)\Big((\mathcal{S}-\lambda)+(\mathcal{T}-\lambda)\Big)\\
&-\Big((\mathcal{S}-\lambda)+(\mathcal{T}-\lambda)\Big)\Big((\mathcal{S}-\lambda)^*+(\mathcal{T}-\lambda)^*\Big)\Big]\mathcal{P}(\mathcal{S}^m+\mathcal{T}^m)\\
=\;& \mathcal{P}(\mathcal{S}^{m*})\Big[\alpha^2\Big((\mathcal{S}-\lambda)^*+(\mathcal{T}-\lambda)^*\Big)\Big((\mathcal{S}-\lambda)+(\mathcal{T}-\lambda)\Big)\\
&-\Big((\mathcal{S}-\lambda)+(\mathcal{T}-\lambda)\Big)\Big((\mathcal{S}-\lambda)^*+(\mathcal{T}-\lambda)^*\Big)\Big]\mathcal{P}(\mathcal{S}^m)\\
&+\mathcal{P}(\mathcal{S}^{m*})\Big[\alpha^2\Big((\mathcal{S}-\lambda)^*+(\mathcal{T}-\lambda)^*\Big)\Big((\mathcal{S}-\lambda)+(\mathcal{T}-\lambda)\Big)\\
&-\Big((\mathcal{S}-\lambda)+(\mathcal{T}-\lambda)\Big)\Big((\mathcal{S}-\lambda)^*+(\mathcal{T}-\lambda)^*\Big)\Big]\mathcal{P}(\mathcal{T}^m)\\
&+\mathcal{P}(\mathcal{T}^{m*})\Big[\alpha^2\Big((\mathcal{S}-\lambda)^*+(\mathcal{T}-\lambda)^*\Big)\Big((\mathcal{S}-\lambda)+(\mathcal{T}-\lambda)\Big)\\
&-\Big((\mathcal{S}-\lambda)+(\mathcal{T}-\lambda)\Big)\Big((\mathcal{S}-\lambda)^*+(\mathcal{T}-\lambda)^*\Big)\Big]\mathcal{P}(\mathcal{S}^m)\\
&+\mathcal{P}(\mathcal{T}^{m*})\Big[\alpha^2\Big((\mathcal{S}-\lambda)^*+(\mathcal{T}-\lambda)^*\Big)\Big((\mathcal{S}-\lambda)+(\mathcal{T}-\lambda)\Big)\\
&-\Big((\mathcal{S}-\lambda)+(\mathcal{T}-\lambda)\Big)\Big((\mathcal{S}-\lambda)^*+(\mathcal{T}-\lambda)^*\Big)\Big]\mathcal{P}(\mathcal{T}^m)\\
=\;& \mathcal{P}(\mathcal{S}^{m*})\Big[\alpha^2\Big((\mathcal{S}-\lambda)^*(\mathcal{S}-\lambda)\Big)-\Big((\mathcal{S}-\lambda)(\mathcal{S}-\lambda)^*\Big)\Big]\mathcal{P}(\mathcal{S}^m)\\
&+\mathcal{P}(\mathcal{T}^{m*})\Big[\alpha^2\Big((\mathcal{T}-\lambda)^*(\mathcal{T}-\lambda)\Big)-\Big((\mathcal{T}-\lambda)(\mathcal{T}-\lambda)^*\Big)\Big]\mathcal{P}(\mathcal{T}^m)\\
\le\;& 0.
\end{aligned}$$

Secondly,

$$\begin{aligned}
&\mathcal{P}\Big((\mathcal{S}+\mathcal{T})^m\Big)^*\Big[\beta^2\Big((\mathcal{S}-\lambda)^*+(\mathcal{T}-\lambda)^*\Big)\Big((\mathcal{S}-\lambda)+(\mathcal{T}-\lambda)\Big)\\
&-\Big((\mathcal{S}-\lambda)+(\mathcal{T}-\lambda)\Big)\Big((\mathcal{S}-\lambda)^*+(\mathcal{T}-\lambda)^*\Big)\Big]\mathcal{P}\Big((\mathcal{S}+\mathcal{T})^m\Big)
\end{aligned}$$

$$
\begin{aligned}
&= \mathcal{P}(\mathcal{S}^{m*}+\mathcal{T}^{m*})\Big[\beta^2\Big((\mathcal{S}-\lambda)^*+(\mathcal{T}-\lambda)^*\Big)\Big((\mathcal{S}-\lambda)+(\mathcal{T}-\lambda)\Big)\\
&\quad -\Big((\mathcal{S}-\lambda)+(\mathcal{T}-\lambda)\Big)\Big((\mathcal{S}-\lambda)^*+(\mathcal{T}-\lambda)^*\Big)\Big]\mathcal{P}(\mathcal{S}^m+\mathcal{T}^m)\\
&= \mathcal{P}(\mathcal{S}^{m*})\Big[\beta^2\Big((\mathcal{S}-\lambda)^*+(\mathcal{T}-\lambda)^*\Big)\Big((\mathcal{S}-\lambda)+(\mathcal{T}-\lambda)\Big)\\
&\quad -\Big((\mathcal{S}-\lambda)+(\mathcal{T}-\lambda)\Big)\Big((\mathcal{S}-\lambda)^*+(\mathcal{T}-\lambda)^*\Big)\Big]\mathcal{P}(\mathcal{S}^m)\\
&\quad +\mathcal{P}(\mathcal{S}^{m*})\Big[\beta^2\Big((\mathcal{S}-\lambda)^*+(\mathcal{T}-\lambda)^*\Big)\Big((\mathcal{S}-\lambda)+(\mathcal{T}-\lambda)\Big)\\
&\quad -\Big((\mathcal{S}-\lambda)+(\mathcal{T}-\lambda)\Big)\Big((\mathcal{S}-\lambda)^*+(\mathcal{T}-\lambda)^*\Big)\Big]\mathcal{P}(\mathcal{T}^m)\\
&\quad +\mathcal{P}(\mathcal{T}^{m*})\Big[\beta^2\Big((\mathcal{S}-\lambda)^*+(\mathcal{T}-\lambda)^*\Big)\Big((\mathcal{S}-\lambda)+(\mathcal{T}-\lambda)\Big)\\
&\quad -\Big((\mathcal{S}-\lambda)+(\mathcal{T}-\lambda)\Big)\Big((\mathcal{S}-\lambda)^*+(\mathcal{T}-\lambda)^*\Big)\Big]\mathcal{P}(\mathcal{S}^m)\\
&\quad +\mathcal{P}(\mathcal{T}^{m*})\Big[\beta^2\Big((\mathcal{S}-\lambda)^*+(\mathcal{T}-\lambda)^*\Big)\Big((\mathcal{S}-\lambda)+(\mathcal{T}-\lambda)\Big)\\
&\quad -\Big((\mathcal{S}-\lambda)+(\mathcal{T}-\lambda)\Big)\Big((\mathcal{S}-\lambda)^*+(\mathcal{T}-\lambda)^*\Big)\Big]\mathcal{P}(\mathcal{T}^m)\\
&= \mathcal{P}(\mathcal{S}^{m*})\Big[\beta^2\Big((\mathcal{S}-\lambda)^*(\mathcal{S}-\lambda)\Big)-\Big((\mathcal{S}-\lambda)(\mathcal{S}-\lambda)^*\Big)\Big]\mathcal{P}(\mathcal{S}^m)\\
&\quad +\mathcal{P}(\mathcal{T}^{m*})\Big[\beta^2\Big((\mathcal{T}-\lambda)^*(\mathcal{T}-\lambda)\Big)-\Big((\mathcal{T}-\lambda)(\mathcal{T}-\lambda)^*\Big)\Big]\mathcal{P}(\mathcal{T}^m)\\
&\geq 0.
\end{aligned}
$$

Therefore $\mathcal{S}+\mathcal{T}$ is polynomially m-quasi-totally-(α,β)-normal operator. □

Theorem 2.9. *Let $\mathcal{S}$ is polynomially m-quasi-totally-(α,β)-normal operator with respect to the polynomial $\mathcal{P}\in\mathbb{C}[z]$. Then*

$$ker(\mathcal{S}-k)\subseteq ker(\mathcal{S}-k)^*,$$

for all $k\in\mathbb{C}$ such that $\mathcal{P}(k^m)\neq 0$.

Proof. Let $x\in ker(\mathcal{S}-k)$. Since $\mathcal{S}$ is polynomially m-quasi-totally-(α,β)-normal operator, we have

$$\alpha\|(\mathcal{S}-\lambda)\mathcal{P}(\mathcal{S}^m)x\|\leq\|(\mathcal{S}-\lambda)^*\mathcal{P}(\mathcal{S}^m)x\|\leq\beta\|(\mathcal{S}-\lambda)\mathcal{P}(\mathcal{S}^m)x\|$$

since $\mathcal{S}x=kx$, we get $\mathcal{P}(\mathcal{S}^m)x=\mathcal{P}(k^m)x$, and therefore

$$\alpha\|(\mathcal{S}-\lambda)\mathcal{P}(k^m)x\|\leq\|(\mathcal{S}-\lambda)^*\mathcal{P}(k^m)x\|\leq\beta\|(\mathcal{S}-\lambda)\mathcal{P}(k^m)x\|$$

According to $(\mathcal{S}-k)x=0$ we obtain $\|(\mathcal{S}-\lambda)^*\mathcal{P}(\mathcal{S}^m)x\|=0$. Therefore $x\in ker(\mathcal{S}-k)^*$. □

Proposition 2.10. *Let $\mathcal{S}$ be polynomially m-quasi-totally-(α,β)-normal operator. If a_1,a_2 are non-zero eigenvalues of $\mathcal{S}$ such that $a_1\neq a_2$, then $ker\,(\mathcal{S}-a_1)\perp ker\,(\mathcal{S}-a_2)$.*

Proof. Let $x\in ker(\mathcal{S}-a_1)$ and $y\in ker(\mathcal{S}-a_2)$. Then $\mathcal{S}x=a_1x$ and $\mathcal{S}y=a_2y$. Therefore $a_1<x,y>=a_2<x,y>$, and so $(a_1-a_2)<x,y>=0$. Hence $ker\,(\mathcal{S}-a_1)\perp ker\,(\mathcal{S}-a_2)$. □

Theorem 2.11. *If $\mathcal{S}$ is polynomially m-quasi-(α,β)-normal such that $\alpha\beta=1$, then*

$$\alpha^2\mathcal{P}(\mathcal{S}^m)^*\mathcal{S}\mathcal{S}^*\mathcal{P}(\mathcal{S}^m)\leq\mathcal{P}(\mathcal{S}^m)^*\mathcal{S}^*\mathcal{S}\mathcal{P}(\mathcal{S}^m)\leq\beta^2\mathcal{P}(\mathcal{S}^m)^*\mathcal{S}\mathcal{S}^*\mathcal{P}(\mathcal{S}^m).$$

Proof. $\mathcal{S}$ is polynomially m-quasi-totally-(α,β)-normal if and only if

$$\alpha^2\mathcal{P}(\mathcal{S}^m)^*\mathcal{S}^*\mathcal{S}\mathcal{P}(\mathcal{S}^m)\leq\mathcal{P}(\mathcal{S}^m)^*\mathcal{S}\mathcal{S}^*\mathcal{P}(\mathcal{S}^m)\leq\beta^2\mathcal{P}(\mathcal{S}^m)^*\mathcal{S}^*\mathcal{S}\mathcal{P}(\mathcal{S}^m).$$

Therefore

$$\alpha^4\mathcal{P}(\mathcal{S}^m)^*\mathcal{S}^*\mathcal{S}\mathcal{P}(\mathcal{S}^m) \le \alpha^2\mathcal{P}(\mathcal{S}^m)^*\mathcal{S}\mathcal{S}^*\mathcal{P}(\mathcal{S}^m) \le \alpha^2\beta^2\mathcal{P}(\mathcal{S}^m)^*\mathcal{S}^*\mathcal{S}\mathcal{P}(\mathcal{S}^m)$$

and

$$\alpha^2\beta^2\mathcal{P}(\mathcal{S}^m)^*\mathcal{S}^*\mathcal{S}\mathcal{P}(\mathcal{S}^m) \le \beta^2\mathcal{P}(\mathcal{S}^m)^*\mathcal{S}\mathcal{S}^*\mathcal{P}(\mathcal{S}^m) \le \beta^4\mathcal{P}(\mathcal{S}^m)^*\mathcal{S}^*\mathcal{S}\mathcal{P}(\mathcal{S}^m).$$

Combining these inequalities, $\alpha\beta = 1$, then

$$\alpha^2\mathcal{P}(\mathcal{S}^m)^*\mathcal{S}\mathcal{S}^*\mathcal{P}(\mathcal{S}^m) \le \mathcal{P}(\mathcal{S}^m)^*\mathcal{S}^*\mathcal{S}\mathcal{P}(\mathcal{S}^m) \le \beta^2\mathcal{P}(\mathcal{S}^m)^*\mathcal{S}\mathcal{S}^*\mathcal{P}(\mathcal{S}^m).$$

□

Theorem 2.12. *Let $(\alpha,\beta) \in \mathbb{R}^2$ such that $0 < \alpha \le 1 \le \beta$ and let $\mathcal{S} \in \mathscr{B}\big(\mathscr{H}\big)$ such that $ran\big(\mathcal{P}(\mathcal{S}^m)\big) = ran\big(\mathcal{P}(\mathcal{S}^m)^*\big)$. If $\alpha\beta = 1$ then $\mathcal{S}$ is polynomially m-quasi-(α,β)-normal if and only if $\mathcal{S}^*$ is polynomially m-quasi-(α,β)-normal.*

Proof. Since $\mathcal{S}$ is polynomially m-quasi-(α,β)-normal, it follows that

$$\alpha\|\mathcal{S}\mathcal{P}(\mathcal{S}^m)x\| \le \|\mathcal{S}^*\mathcal{P}(\mathcal{S}^m)x\| \le \beta\|\mathcal{S}\mathcal{P}(\mathcal{S}^m)x\|, \quad \forall\, x \in \mathscr{H}.$$

The condition $ran(\mathcal{P}(\mathcal{S}^m)) = ran(\mathcal{P}(\mathcal{S}^m)^*)$ implies

$$\alpha\|\mathcal{S}\mathcal{P}(\mathcal{S}^m)^*x\| \le \|\mathcal{S}^*\mathcal{P}(\mathcal{S}^m)^*x\| \le \beta\|\mathcal{S}\mathcal{P}(\mathcal{S}^m)^*x\|, \quad \forall\, x \in \mathscr{H}.$$

From the above two inequalities,

$$\frac{1}{\beta}\|\mathcal{S}^*\mathcal{P}(\mathcal{S}^m)^*x\| \le \|\mathcal{S}\mathcal{P}(\mathcal{S}^m)^*x\| \le \frac{1}{\alpha}\|\mathcal{S}^*\mathcal{P}(\mathcal{S}^m)^*x\|.$$

Here $\alpha\beta = 1$, so, $\mathcal{S}^*$ is polynomially m-quasi-(α,β)-normal. □

Theorem 2.13. *Let $\mathcal{S} \in \mathscr{B}(\mathscr{H})$ and $\mathcal{N} \in \mathscr{B}(\mathscr{H})$ be an invertible operator such that $\mathcal{N}^*N$ commutes with S. Then $\mathcal{S}$ is polynomially m-quasi-totally-(α,β)-normal operator if and only if $\mathcal{N}\mathcal{S}\mathcal{N}^{-1}$ is polynomially m-quasi-totally-(α,β)-normal operator.*

Proof. Assume that $\mathcal{S}$ is polynomially m-quasi-totally-(α,β)-normal operator.
Consider,

$$\begin{aligned}
&\alpha^2\mathcal{P}((\mathcal{N}\mathcal{S}\mathcal{N}^{-1})^m)^*(\mathcal{N}\mathcal{S}\mathcal{N}^{-1}-\lambda)^*(\mathcal{N}\mathcal{S}\mathcal{N}^{-1}-\lambda)\mathcal{P}((\mathcal{N}\mathcal{S}\mathcal{N}^{-1})^m)\\
&\le \mathcal{P}((\mathcal{N}\mathcal{S}\mathcal{N}^{-1})^m)^*(\mathcal{N}\mathcal{S}\mathcal{N}^{-1}-\lambda)(\mathcal{N}\mathcal{S}\mathcal{N}^{-1}-\lambda)^*\mathcal{P}((\mathcal{N}\mathcal{S}\mathcal{N}^{-1})^m)\\
&\le \beta^2\mathcal{P}((\mathcal{N}\mathcal{S}\mathcal{N}^{-1})^m)^*(\mathcal{N}\mathcal{S}\mathcal{N}^{-1}-\lambda)^*(\mathcal{N}\mathcal{S}\mathcal{N}^{-1}-\lambda)\mathcal{P}((\mathcal{N}\mathcal{S}\mathcal{N}^{-1})^m).
\end{aligned}$$

We have

$$\begin{aligned}
&\alpha^2(\mathcal{N}^{-1})^*\mathcal{P}(\mathcal{S}^m)^*\mathcal{N}^*((\mathcal{N}^{-1})^*(\mathcal{S}-\lambda)^*\mathcal{N}^*)(\mathcal{N}(\mathcal{S}-\lambda)\mathcal{N}^{-1})\mathcal{N}\mathcal{P}(\mathcal{S}^m)\mathcal{N}^{-1}\\
&\le (\mathcal{N}^{-1})^*\mathcal{P}(\mathcal{S}^m)^*\mathcal{N}^*(\mathcal{N}(\mathcal{S}-\lambda)\mathcal{N}^{-1})((\mathcal{N}^{-1})^*(\mathcal{S}-\lambda)^*\mathcal{N}^*)\mathcal{N}\mathcal{P}(\mathcal{S}^m)\mathcal{N}^{-1}\\
&\le \beta^2(\mathcal{N}^{-1})^*\mathcal{P}(\mathcal{S}^m)^*\mathcal{N}^*((\mathcal{N}^{-1})^*(\mathcal{S}-\lambda)^*\mathcal{N}^*)(\mathcal{N}(\mathcal{S}-\lambda)\mathcal{N}^{-1})\mathcal{N}\mathcal{P}(\mathcal{S}^m)\mathcal{N}^{-1}.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\alpha^2(\mathcal{N}^{-1})^*\mathcal{P}(\mathcal{S}^m)^*(\mathcal{S}-\lambda)^*\mathcal{N}^*\mathcal{N}(\mathcal{S}-\lambda)\mathcal{P}(\mathcal{S}^m)\mathcal{N}^{-1}\\
&\le (\mathcal{N}^{-1})^*\mathcal{P}(\mathcal{S}^m)^*(\mathcal{S}-\lambda)\mathcal{N}^*\mathcal{N}\mathcal{N}^{-1}(\mathcal{N}^{-1})^*(\mathcal{S}-\lambda)^*\mathcal{N}^*\mathcal{N}\mathcal{P}(\mathcal{S}^m)\mathcal{N}^{-1}\\
&\le \beta^2(\mathcal{N}^{-1})^*\mathcal{P}(\mathcal{S}^m)^*(\mathcal{S}-\lambda)^*\mathcal{N}^*\mathcal{N}(\mathcal{S}-\lambda)\mathcal{P}(\mathcal{S}^m)\mathcal{N}^{-1}.
\end{aligned}$$

Hence,

$$\begin{aligned}&\alpha^2\mathcal{N}\mathcal{P}(\mathcal{S}^m)^*(\mathcal{S}-\lambda)^*(\mathcal{S}-\lambda)\mathcal{P}(\mathcal{S}^m)\mathcal{N}^{-1}\\ &\leq \mathcal{N}\mathcal{P}(\mathcal{S}^m)^*(\mathcal{S}-\lambda)(\mathcal{S}-\lambda)^*\mathcal{P}(\mathcal{S}^m)\mathcal{N}^{-1}\\ &\leq \beta^2\mathcal{N}\mathcal{P}(\mathcal{S}^m)^*(\mathcal{S}-\lambda)^*(\mathcal{S}-\lambda)\mathcal{P}(\mathcal{S}^m)\mathcal{N}^{-1}.\end{aligned}$$

We have

$$\mathcal{N}\Big(\alpha^2\mathcal{P}(\mathcal{S}^m)^*(\mathcal{S}-\lambda)^*(\mathcal{S}-\lambda)\mathcal{P}(\mathcal{S}^m) \leq \mathcal{P}(\mathcal{S}^m)^*(\mathcal{S}-\lambda)(\mathcal{S}-\lambda)^*\mathcal{P}(\mathcal{S}^m)$$
$$\leq \beta^2\mathcal{P}(\mathcal{S}^m)^*(\mathcal{S}-\lambda)^*(\mathcal{S}-\lambda)\mathcal{P}(\mathcal{S}^m)\Big)\mathcal{N}^{-1}.$$

Therefore, $\mathcal{N}\mathcal{S}\mathcal{N}^{-1}$ is polynomially m-quasi-totally-(α,β)-normal operator.
Conversely, assume that $N\mathcal{S}N^{-1}$ is polynomially m-quasi-totaly-(α,β)-normal.
Set $\mathcal{T} = \mathcal{N}\mathcal{S}\mathcal{N}^{-1}$. We observe that $\mathcal{T}$ commutes with $(\mathcal{N}^{-1})^*\mathcal{N}^{-1}$ and $\mathcal{N}^{-1}\mathcal{T}\mathcal{N} = \mathcal{S}$. By taking into account the preceding part of the theorem, we have $\mathcal{N}^{-1}\mathcal{T}\mathcal{N}$ is polynomially m-quasi-totaly-(α,β)-normal. □

For $\mathcal{T},\mathcal{S} \in \mathscr{B}(\mathscr{H})$ the operator $\Gamma_{\mathcal{T},\mathcal{S}}$ defined as $\Gamma_{\mathcal{T},\mathcal{S}} : C_2(\mathscr{H})$ such that $X \to \mathcal{T}X\mathcal{S} \in C_2(\mathscr{H})$ has been studied in [6].
The following results extends A. Bachir[3, Theorem 9]

Theorem 2.14. *If $\mathcal{T} \in \mathscr{B}(\mathscr{H})$ is polynomially m-quasi-(α,β)-normal operator with respect to the polynomial $\mathcal{P}(z) = z$ and $\mathcal{S}$ is normal, then $\Gamma_{\mathcal{T},\mathcal{S}}$ is polynomially m-quasi-(α,β)-normal operator with respect to the polynomial $\mathcal{P}(z) = z$.*

Proof. Here,

$$\Gamma_{\mathcal{T},\mathcal{S}}(X) = \mathcal{T}X\mathcal{S},$$

$$\Gamma^*_{\mathcal{T},\mathcal{S}}(X) = \mathcal{T}^*X\mathcal{S}^*,$$

$$\Gamma^m_{\mathcal{T},\mathcal{S}}(X) = \mathcal{T}^mX\mathcal{S}^m,$$

$$\Gamma^{m*}_{\mathcal{T},\mathcal{S}}(X) = \mathcal{T}^{m*}X\mathcal{S}^{m*}.$$

First we have,

$$\begin{aligned}&\left(\Gamma^{m*}_{\mathcal{T},\mathcal{S}}\Gamma_{\mathcal{T},\mathcal{S}}\Gamma^*_{\mathcal{T},\mathcal{S}}\Gamma^m_{\mathcal{T},\mathcal{S}} - \alpha^2\Gamma^{m*}_{\mathcal{T},\mathcal{S}}\Gamma^*_{\mathcal{T},\mathcal{S}}\Gamma_{\mathcal{T},\mathcal{S}}\Gamma^m_{\mathcal{T},\mathcal{S}}\right)(X)\\
&= \mathcal{T}^{m*}\mathcal{T}\mathcal{T}^*\mathcal{T}^mX\mathcal{S}^m\mathcal{S}^*\mathcal{S}\mathcal{S}^{m*} - \alpha^2\mathcal{T}^{m*}\mathcal{T}^*\mathcal{T}\mathcal{T}^mX\mathcal{S}^m\mathcal{S}\mathcal{S}^*\mathcal{S}^{m*}\\
&= \mathcal{T}^{m*}\mathcal{T}\mathcal{T}^*\mathcal{T}^mX\mathcal{S}^m\mathcal{S}^*\mathcal{S}\mathcal{S}^{m*} - \alpha^2\mathcal{T}^{m*}\mathcal{T}^*\mathcal{T}\mathcal{T}^mX\mathcal{S}^m\mathcal{S}^*\mathcal{S}\mathcal{S}^{m*}\\
&\quad+\alpha^2\mathcal{T}^{m*}\mathcal{T}^*\mathcal{T}\mathcal{T}^mX\mathcal{S}^m\mathcal{S}^*\mathcal{S}\mathcal{S}^{m*} - \alpha^2\mathcal{T}^{m*}\mathcal{T}^*\mathcal{T}\mathcal{T}^mX\mathcal{S}^m\mathcal{S}\mathcal{S}^*\mathcal{S}^{m*}\\
&= \left(\mathcal{T}^{m*}\mathcal{T}\mathcal{T}^*\mathcal{T}^m - \alpha^2\mathcal{T}^{m*}\mathcal{T}^*\mathcal{T}\mathcal{T}^m\right)X\mathcal{S}^m\mathcal{S}^*\mathcal{S}\mathcal{S}^{m*}\\
&\quad+\alpha^2\mathcal{T}^{m*}\mathcal{T}^*\mathcal{T}\mathcal{T}^mX\left(\mathcal{S}^m\mathcal{S}^*\mathcal{S}\mathcal{S}^{m*} - \mathcal{S}^m\mathcal{S}\mathcal{S}^*\mathcal{S}^{m*}\right)\\
&= \left(\mathcal{T}^{m*}\mathcal{T}\mathcal{T}^*\mathcal{T}^m - \alpha^2\mathcal{T}^{m*}\mathcal{T}^*\mathcal{T}\mathcal{T}^m\right)X\mathcal{S}^m\mathcal{S}^*\mathcal{S}\mathcal{S}^{m*}\\
&\geq 0\end{aligned}$$

Secondly,

$$
\begin{aligned}
&\left(\beta^2\Gamma^{m*}_{\mathcal{T},\mathcal{S}}\Gamma^{*}_{\mathcal{T},\mathcal{S}}\Gamma_{\mathcal{T},\mathcal{S}}\Gamma^{m}_{\mathcal{T},\mathcal{S}} - \Gamma^{m*}_{\mathcal{T},\mathcal{S}}\Gamma_{\mathcal{T},\mathcal{S}}\Gamma^{*}_{\mathcal{T},\mathcal{S}}\Gamma^{m}_{\mathcal{T},\mathcal{S}}\right)(X)\\
&\quad = \beta^2\mathcal{T}^{m*}\mathcal{T}^{*}\mathcal{T}\mathcal{T}^{m}X\mathcal{S}^{m}\mathcal{S}\mathcal{S}^{*}\mathcal{S}^{m*} - \mathcal{T}^{m*}\mathcal{T}\mathcal{T}^{*}\mathcal{T}^{m}X\mathcal{S}^{m}\mathcal{S}^{*}\mathcal{S}\mathcal{S}^{m*}\\
&\quad = \beta^2\mathcal{T}^{m*}\mathcal{T}^{*}\mathcal{T}\mathcal{T}^{m}X\mathcal{S}^{m}\mathcal{S}\mathcal{S}^{*}\mathcal{S}^{m*} - \mathcal{T}^{m*}\mathcal{T}\mathcal{T}^{*}\mathcal{T}^{m}X\mathcal{S}^{m}\mathcal{S}\mathcal{S}^{*}\mathcal{S}^{m}\\
&\qquad +\mathcal{T}^{m*}\mathcal{T}\mathcal{T}^{*}\mathcal{T}^{m}X\mathcal{S}^{m}\mathcal{S}\mathcal{S}^{*}\mathcal{S}^{m} - \mathcal{T}^{m*}\mathcal{T}\mathcal{T}^{*}\mathcal{T}^{m}X\mathcal{S}^{m*}\mathcal{S}^{*}\mathcal{S}\mathcal{S}^{m}\\
&\quad = \left(\beta^2\mathcal{T}^{m*}\mathcal{T}^{*}\mathcal{T}\mathcal{T}^{m} - \mathcal{T}^{m}\mathcal{T}\mathcal{T}^{*}\mathcal{T}^{m*}\right)X\mathcal{S}^{m*}\mathcal{S}\mathcal{S}^{*}\mathcal{S}^{m}\\
&\qquad +\mathcal{T}^{m*}\mathcal{T}\mathcal{T}^{*}\mathcal{T}^{m}X\left(\mathcal{S}^{m*}\mathcal{S}\mathcal{S}^{*}\mathcal{S}^{m} - \mathcal{S}^{m*}\mathcal{S}^{*}\mathcal{S}\mathcal{S}^{m}\right)\\
&\quad = \left(\beta^2\mathcal{T}^{m*}\mathcal{T}^{*}\mathcal{T}\mathcal{T}^{m} - \mathcal{T}^{m}\mathcal{T}\mathcal{T}^{*}\mathcal{T}^{m*}\right)X\mathcal{S}^{m*}\mathcal{S}\mathcal{S}^{*}\mathcal{S}^{m}\\
&\quad \geq 0
\end{aligned}
$$

Hence $\Gamma_{\mathcal{T},\mathcal{S}}$ is polynomially m-quasi-(α,β)-normal operator with respect to the polynomial $\mathcal{P}(z) = z$ □

Theorem 2.15. *Let $\mathcal{T},\mathcal{S}$ be polynomially m-quasi-(α,β)-normal operator with respect to the polynomial $\mathcal{P}(z) = z$. If $\mathcal{S}$ is invertible and $\mathcal{S}^*\mathcal{S} = \mathcal{S}\mathcal{S}^*$ such that $\mathcal{T}X = X\mathcal{S}$ for some $X \in \mathbb{C}_2(\mathscr{H})$, then $\mathcal{T}^*X = X\mathcal{S}^*$.*

Proof. Let $\Gamma_{\mathcal{T},\mathcal{S}^{-1}}(Y) = \mathcal{T}Y\mathcal{S}^{-1}$. Since $\mathcal{T}$ and $\mathcal{S}$ are m-quasi-(α,β)-normal operator, then $\Gamma_{\mathcal{T},\mathcal{S}^{-1}}$ is also m-quasi-(α,β)-normal operator by Theorem 2.14. Moreover $\Gamma_{\mathcal{T},\mathcal{S}^{-1}}(X) = \mathcal{T}X\mathcal{S}^{-1} = X$ because of $\mathcal{T}X = X\mathcal{S}$. Hence X is an eigenvector of $\Gamma_{\mathcal{T},\mathcal{S}^{-1}}$. By Theorem 2.9, we have $\Gamma^*_{\mathcal{T},\mathcal{S}^{-1}}(X) = \mathcal{T}^*X(\mathcal{S}^{-1})^*$, that is, $\mathcal{T}^*X = X\mathcal{S}^*$ as desired. □

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