Filomat 39:16 (2025), 5675–5684
https://doi.org/10.2298/FIL2516675E

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Applications of the measure of noncompactness for solving pantograph differential equations with ψ-Caputo fractional derivatives

Ali El Mfadel[a,b,*], **Said Melliani**[a]

[a] *Laboratory of Applied Mathematics and Scientific Computing, Sultan Moulay Slimane University, Beni Mellal, Morocco*
[b] *Superior School of Technology, Sultan Moulay Slimane University, Khénifra, Morocco*

Abstract. The aim of this manuscript is to investigate the existence and uniqueness of solutions for a class of nonlinear ψ-Caputo fractional pantograph differential equations with damping and nonlocal conditions. The proofs are based on results from topological degree theory for condensing maps, combined with the technique of measures of noncompactness. As an application, a nontrivial example is presented to illustrate the theoretical results.

1. Introduction

Fractional calculus, which extends integration and differentiation to non-integer orders, has become one of the most rapidly expanding mathematical fields following the recognition of its utility in mathematical modeling [11, 21, 22, 33, 35]. Fractional differential equations, which can be used to model and describe non-homogeneous physical events, have recently attracted a lot of attention, particularly initial and boundary value problems for nonlinear fractional differential equations. Different researchers have found some interesting solutions to initial and boundary value problems for fractional differential equations involving various fractional derivatives, including their existence and uniqueness, such as Riemann-Liouville [28], Caputo [3], Hilfer [27], Erdelyi-Kober [30] and Hadamard [2]. Since all these definitions incorporate kernel-dependent formulations, researchers introduced the ψ-Caputo derivative a fractional derivative with respect to another function to provide a unified framework for studying fractional differential equations. This generalized approach encompasses several well-known fractional derivatives as special cases: by selecting specific functions ψ, one can recover the classical Caputo, Caputo-Hadamard, or Caputo-Erdelyi-Kober fractional derivatives. From an applications perspective, this framework offers significant advantages, as the strategic selection of the "trial" function ψ enables researchers to fine-tune the ψ-Caputo derivative for modeling specific phenomena [24, 25]. Almeida et al. [6] established existence and uniqueness results for nonlinear fractional differential equations involving ψ-Caputo-type derivatives using fixed point theorems and the Picard iteration method. Additional details can be found in [8, 18–20, 26, 34] and the references cited therein. In particular, the pantograph equation was employed as a useful tool to shed light on some of the modern problems originating from several scientific disciplines, including electrodynamics, probability,

2020 *Mathematics Subject Classification*. Primary 26A33; Secondary 34A08, 47H08.
Keywords. $\psi-$Caputo fractional derivative Kuratowski measure topological degree theory.
Received: 17 December 2023; Revised: 21 May 2025; Accepted: 30 May 2025
Communicated by Maria Alessandra Ragusa
* Corresponding author: Ali El Mfadel
Email address: a.elmfadel@usms.ma (Ali El Mfadel)

quantum mechanics, and number theory. However, a substantial investigation on the characteristics of this type of fractional differential equation, both analytical and numerical, has been done, and intriguing findings have been published in [1, 4, 7, 9, 10, 12, 15–17, 29, 37].
Motivated by recent advances, this study explores the existence of solutions for a nonlinear pantograph differential equation governed by the ψ-Caputo fractional derivative.

$$\begin{cases} {}^{C}D_{0^+}^{\beta,\psi}x(t) + \lambda x'(t) = g(t, x(t), x(\varepsilon t)), & t \in J = [0, T], \\ x'(0) = 0, \quad x(0) + \omega(x) = x_0. \end{cases} \tag{1}$$

Where ${}^{C}D_{0^+}^{\beta,\psi}$ is the $\psi-$Caputo fractional derivative of u at order $\beta \in (1, 2)$,
$\varepsilon \in (0, 1)$, $T > 0$, $x_0 \in \mathbb{R}$, $g \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, ω is the nonlocal term satisfies some given conditions, and the damping coefficient $|\lambda| \in \left(0, \frac{1}{Q_{\beta,\psi}}\right)$ where $Q_{\beta,\psi} = \frac{\|\psi'\|(\psi(T)-\psi(0))^{\beta-1}}{\Gamma(\beta)}$.
To the best of the authors' knowledge, topological degree theory for condensing maps has not yet been applied to this class of nonlinear pantograph differential equations with damping and the ψ-Caputo fractional derivative.
The structure of the paper is as follows. In Section 2, we present some basic definitions and preliminary results that will be used in establishing our main findings. Section 3 is devoted to proving the existence of solutions for the problem (1). In Section 4, we provide a concrete example to illustrate the applicability of the main results. Finally, Section 5 presents conclusions derived from the study's results.

2. Basic concepts

This section introduces the preliminaries and notations employed throughout the paper. For further details, the reader is referred to [5].

Definition 2.1. *[6] Let $q > 0$, $h \in L^1(J, \mathbb{R})$ and $\psi \in C^n(J, \mathbb{R})$ such that $\psi'(t) > 0$ for all $t \in J$. The ψ-Riemann-Liouville fractional integral at order q of the function h is given by*

$$I_{0^+}^{q,\psi}h(t) = \frac{1}{\Gamma(q)}\int_0^t \psi'(s)(\psi(t) - \psi(s))^{q-1}h(s)ds. \tag{2}$$

where $\Gamma(q) = \int_0^\infty e^{-t}t^{q-1}\,dt$.

Definition 2.2. *[6] Let $q > 0$, $h \in C^{n-1}(J, \mathbb{R})$ and $\psi \in C^n(J, \mathbb{R})$ such that $\psi'(t) > 0$ for all $t \in J$. The $\psi-$Caputo fractional derivative at order q of the function h is given by*

$${}^{C}D_{0^+}^{q,\psi}h(t) = \frac{1}{\Gamma(n-q)}\int_0^t \psi'(s)(\psi(t) - \psi(s))^{n-q-1}h_\psi^{[n]}(s)ds, \tag{3}$$

where

$$h_\psi^{[n]}(s) = \left(\frac{1}{\psi'(s)}\frac{d}{ds}\right)^n h(s) \quad and \quad n = [q] + 1,$$

and $[q]$ denotes the integer part of the real number q.

Remark 2.3. *In particular, note that if $\psi(t) = t$ and $\psi(t) = log(t)$, then the equation (3) is reduced to the the Caputo fractional derivative and Caputo-Hadamard fractional derivative respectively.*

Proposition 2.4. *[6] Let $q > 0$, if $h \in C^{n-1}(J, \mathbb{R})$, then we have*

1) ${}^{C}D_{0^+}^{q,\psi}I_{0^+}^{q,\psi}h(t) = h(t)$.

2) $I_{0^+}^{q,\psi}\, {}^{C}D_{0^+}^{q,\psi} h(t) = h(t) - \sum_{k=0}^{n-1} \frac{h_{\psi}^{[k]}(0)}{k!}(\psi(t)-\psi(0))^k.$

Proposition 2.5. *[6] Let $q > p > 0$ and $t \in J$, then*

1) $I_{0^+}^{q,\psi}(\psi(t)-\psi(0))^{p-1} = \frac{\Gamma(p)}{\Gamma(q+p)}(\psi(t)-\psi(0))^{q+p-1}.$

2) $D_{0^+}^{q,\psi}(\psi(t)-\psi(0))^{p-1} = \frac{\Gamma(p)}{\Gamma(p-q)}(\psi(t)-\psi(0))^{\nu-\mu-1}.$

3) $D_{0^+}^{q,\psi}(\psi(t)-\psi(0))^{k} = 0, \quad \forall k < n \in \mathbb{N}.$

Definition 2.6. *[14] Let X a Banach space and $\mathfrak{B}_X$ be the family of all non-empty and bounded subsets of X. The Kuratowski measure of non-compactness is the mapping $\rho : \mathfrak{B}_X \to [0, +\infty[$ defined by*

$\rho(A) = \inf\{\, r > 0$*: A admits a finite cover by sets of diameter* $\leq r\}$.

Proposition 2.7. *[14] The Kuratowski measure of noncompactness ρ satisfies the following assertions.*

1. $\rho(A) = 0$ *if and only if A is retativety compact.*
2. $\rho(kA) = |k|\rho(A), \quad k \in \mathbb{R}$.
3. $\rho(A_1 + A_2) \leq \rho(A_1) + \rho(A_2)$.
4. *If $A_1 \subset A_2$ then $\rho(A_1) \leq \rho(A_2)$.*
5. $\rho(A_1 \cup A_2) = \max\{\rho(A_1),\ \rho(A_2)\}$.
6. $\rho(A) = \rho(\overline{A}) = \rho(convA)$ *where $\overline{A}$ and $convA$ denote the closure and the convex hull of A respectively.*

Definition 2.8. *[14] Let and $\mathcal{F} : \Omega \subset X \to X$ be a continuous bounded map. We say that $\mathcal{F}$ is ρ-Lipschitz if there exists $k \geq 0$ such that*

$$\rho(\mathcal{F}(A)) \leq k\rho(A), \quad for \ \ every \quad A \subset \Omega.$$

Moreover, if $k < 1$ then we say that $\mathcal{F}$ is a strict ρ-contraction.

Definition 2.9. *[14] We say that the function $\mathcal{F}$ is ρ-condensing if*

$$\rho(\mathcal{F}(A)) < \rho(A),$$

for every bounded subset B of Ω with $\rho(A) > 0$.

In other words,

$$\rho(\mathcal{F}(A)) \geq \rho(A) \Rightarrow \rho(A) = 0.$$

Definition 2.10. *[14] We say that the function $\mathcal{F} : \Omega \to X$ is Lipschitz if there exists $l > 0$ such that*

$$\| \mathcal{F}(x) - \mathcal{F}(y) \| \leq k \| x - y \|, \quad for \ \ all \quad x, y \in \Omega.$$

Moreover, if $k < 1$ then we say that $\mathcal{F}$ is a strict contraction.

Lemma 2.11. *[14] If $\mathcal{L}, \mathcal{F} : \Omega \to X$ are ρ-Lipschitz mappings with constants c_1 respectively c_2, then the mapping $\mathcal{F} + \mathcal{L} : \Omega \to X$ is ρ- Lipschitz with constants $c_1 + c_2$.*

Lemma 2.12. *[14] If $\mathcal{F} : \Omega \to X$ is compact, then $\mathcal{F}$ is $\rho-$ Lipschitz with constant $c = 0$.*

Lemma 2.13. *[14] If $\mathcal{F} : \Omega \to X$ is Lipschitz with constant k, then $\mathcal{F}$ is $\rho-$Lipschitz with the same constant k.*

Theorem 2.14. *(See Isaia [31]). Let $\mathcal{F} : X \to X$ be ρ-condensing and*

$$\mathcal{E}_\gamma = \{x \in X : x = \gamma\mathcal{F}x \quad for \quad some \quad 0 \leq \gamma \leq 1\}.$$

If $\mathcal{E}_\gamma$ is a bounded set in X, then there exists $r > 0$ such that $\mathcal{S}_\gamma \subset B_r$ and we have

$$deg(I - \delta\mathcal{F}, B_r, 0) = 1, \quad for\ all\ \delta \in [0, 1].$$

Consequently, $\mathcal{F}$ has at least one fixed point and the set of the fixed points of $\mathcal{H}$ lies in B_r .

3. Main results

We start this section by introducing necessary notations and hypotheses which we will need in the sequel.

• We denote by $C := C(J, \mathbb{R})$ the space of continuous real-valued functions defined on J provided with supremum norm

$$\| x \| = \sup_{t\in J} | x(t) | .$$

• We denote by B_η the closed ball centered at 0 with radius $\eta > 0$.

Lemma 3.1. *A function $x \in C^1 := C^1(J, \mathbb{R})$ is a solution of (1) if and only if x satisfies the following fractional integral equation*

$$x(t) = x_0 - \omega(x) + \frac{1}{\Gamma(\beta)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\beta-1}(g(s, x(s), x(\varepsilon s)) - \lambda x'(s))ds. \tag{4}$$

Proof. Let x be a solution of (1), then by applying $\psi-$fractional integral $I_{0^+}^{\beta,\psi}$ on both sides of (1) we obtain

$$I_{0^+}^{\beta,\psi}\, {}^C D_{0^+}^{\beta,\psi} x(t) - \lambda I_{0^+}^{\beta,\psi}\, x'(t) = I_{0^+}^{\beta,\psi} g(t, x(t), x(\varepsilon t)),$$

and by using Proposition 2.4 we get

$$x(t) = c_0 + c_1(\psi(t) - \psi(0)) + I_{0^+}^{\beta,\psi} g(t, x(t), x(\varepsilon t)) + \lambda I_{0^+}^{\beta,\psi}\, x'(t),$$

where $c_0, c_1 \in \mathbb{R}$,
hence

$$x'(t) = c_1\Psi'(t) + \frac{1}{\Gamma(\beta)} \int_0^t \Big(\psi'(s)(\psi(t) - \psi(s))^{\beta-1}\Big(g(s, x(s), x(\varepsilon s)) - \lambda x'(s)\Big)\Big)' ds,$$

given that $x(0) + \omega(x) = x_0$ and $x'(0) = 0,$ it follows $c_0 = x_0 - \omega(x)$ and $c_1 = 0$.
Therefore (4) holds.
Conversely, suppose x is a solution to (4). Then, it follows that $x(0) + \omega(x) = x_0$ and $x'(0) = 0$. Moreover, by applying the ψ-Caputo fractional derivative ${}^C D_{0^+}^{\beta,\psi}$ to both sides of (4) and utilizing Proposition 2.4, we obtain

$${}^C D_{0^+}^{\beta,\psi} x(t) - \lambda I_{0^+}^{\beta,\psi} x'(t) = I_{0^+}^{\beta,\psi} g(t, x(t), x(\varepsilon t)).$$

Thus, equation (1) holds. □

In order to establish the existence of a solution for our main problem (1), it is necessary to first state the following hypotheses.

(H_1) There exists a constant $L_\omega \in [0, 1)$ such that

$$|\omega(x) - \omega(y)| \le L_\omega \mid x - y \mid, \quad for\ \ all\ \ x, y \in C.$$

(H_2) There exist two constants $K_\omega, M_\omega > 0$ and $q \in (0, 1)$ such that

$$|\omega(x)| \le K_\omega \parallel x \parallel^q + M_\omega \quad for\ \ all\ \ x \in C.$$

(H_3) There exist two constants $K_g, M_g > 0$ and $p \in (0, 1)$ such that

$$|g(t, x(t), x(\varepsilon t))| \le K_g \parallel x \parallel^p + M_g \quad for\ \ all\ \ x \in C.$$

Lemma 3.2. *If a function $x \in C^1$ is a solution to (1), then*

$$\|x'\| \leq \frac{Q_{\beta,\psi}}{1-|\lambda|Q_{\beta,\psi}}(K_g\|x\|^p + M_g).$$

Proof. Let $x \in C^1$ be a solution of (1), then

$$\begin{aligned}|x'(t)| &= \left|\frac{(\beta-1)}{\Gamma(\beta)}\int_0^t (\psi'(s)\psi'(t)(\psi(t)-\psi(s))^{\beta-2}(g(s,x(s),x(\varepsilon s)) - \lambda x'(s))ds\right| \\ &\leq \frac{(\beta-1)\psi'(t)}{\Gamma(\beta)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\beta-2}\left|(g(s,x(s),x(\varepsilon s)) - \lambda x'(s))\right| ds.\end{aligned}$$

By (H_3), we get

$$|x'(t)| \leq Q_{\beta,\psi}(K_g\|x\|^p + M_g + |\lambda|\|x'\|).$$

It follows

$$\|x'\|(1-|\lambda|)Q_{\psi,\beta} \leq Q_{\beta,\psi}(K_g\|x\|^p + M_g).$$

Thus

$$\|x'\| \leq \frac{Q_{\beta,\psi}}{1-|\lambda|Q_{\beta,\psi}}(K_g\|x\|^p + M_g).$$

□

To prove the (4) has at least one solution $x \in C^1$, we consider the following two operators $\mathcal{B}_1, \mathcal{B}_2 : C^1 \to C^1$ defined by

$$\mathcal{B}_1 x(t) = x_0 - \omega(x), \quad t \in J, \tag{5}$$

and

$$\mathcal{B}_2 x(t) = \frac{1}{\Gamma(\beta)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\beta-1}\Big(g(s,x(s),x(\varepsilon s)) - \lambda x'(s)\Big)ds, \quad t \in J, \tag{6}$$

thus, (4) can be formulated as follows

$$\mathcal{B}x(t) = \mathcal{B}_1 x(t) + \mathcal{B}_2 x(t), \quad t \in J. \tag{7}$$

Theorem 3.3. *Assume that the hypotheses $(H_1)-(H_3)$ are satisfied, then fractional pantograph differential equation (1) has at least one solution $x \in C^1$. Moreover, the set of all solutions for (1) is bounded in C^1.*

In order to prove the Theorem 3.3, we will need to show some lemmas and preliminary results.

Lemma 3.4. *The operator $\mathcal{B}_1$ is ρ- Lipschitz with the constant L_ω. Moreover, $\mathcal{B}_1$ satisfies the following growth condition*

$$\|\mathcal{B}_1 x\| \leq |x_0| + K_\omega\|x\|^q + M_\omega, \quad for \ every \ x \in C^1. \tag{8}$$

Proof. To prove that the operator $\mathcal{B}_1$ is Lipschitz with constant $L\omega$.
Let $x, y \in C^1$, then we have

$$\left|\mathcal{B}_1 x(t) - \mathcal{B}_1 y(t)\right| \leq |\omega(x) - \omega(y)|,$$

by using (H_1) we get

$$\left|\mathcal{B}_1 x(t) - \mathcal{B}_1 y(t)\right| \leq L_\omega\|x-y\|,$$

Taking supremum over t, we obtain

$$\left\|\mathcal{B}_1 x - \mathcal{B}_1 y\right\| \leq L_\omega \|x - y\|,$$

hence $\mathcal{B}_1$ is is Lipschitz with L_ω. By Lemma 2.13, we conclude that $\mathcal{B}_1$ is ρ-Lipschitz with the constant L_ω. To show the growth condition (8), let $x \in C^1$, then we have

$$|\mathcal{B}_1 x(t)| = |x_0 - \omega(x)| \leq |x_0| + |\omega(x)|,$$

by (H_2), we get

$$\|\mathcal{B}_1 x\| \leq |x_0| + K_\omega \|x\|^q + M_\omega.$$

□

Lemma 3.5. *The operator $\mathcal{B}_2$ is continuous. Additionally $\mathcal{B}_2$ satisfies the following growth condition*

$$\|\mathcal{B}_2 x\| \leq \Pi_{\beta,\lambda,\psi}(K_\omega \|x\|^p + M_\omega), \quad \textit{for each } x \in C^1. \tag{9}$$

where $\Pi_{\beta,\lambda,\psi} = \frac{(1-|\lambda|)Q^2_{\beta,\psi} + Q_{\beta,\psi}}{\|\psi'\|\beta(1-|\lambda|Q_{\beta,\psi})}$.

Proof. To prove that the operator $\mathcal{B}_2$ is continuous, let $x_n \in C^1$ converging to x in C^1, it follows that there exists $\delta > 0$ such that $\|x_n\| \leq \delta$ and $\|x\| \leq \delta$. Now let $t \in J$, then we have

$$\begin{aligned} |\mathcal{B}_2 x_n(t) - \mathcal{B}_2 x(t)| \leq & \frac{1}{\Gamma(\beta)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\beta-1} \left|g(s, x_n(s), x_n(\varepsilon s)) - g(s, x(s), x(\varepsilon s))\right| ds \\ & + \frac{\lambda}{\Gamma(\beta)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\beta-1} \left|x'_n(s)) - x'(s)\right| ds. \end{aligned}$$

Given that g is continuous, then we have

$$\lim_{n\to\infty} g(s, x_n(s), x_n(\varepsilon s)) = g(s, x(s), x(\varepsilon s)).$$

Conversely, utilizing Lemma 3.2 and (H_3), we find

$$\begin{aligned} & \frac{1}{\Gamma(\beta)}(\psi'(s)(\psi(t) - \psi(s))^{\beta-1} \left\|(g(s, x_n(s), x_n(\varepsilon s)) - \lambda x'_n(s)) - (g(s, x(s), x(\varepsilon s)) - \lambda x'(s))\right\| \\ & \qquad \leq 2\left(K_g \delta^p + M_g\right)(1 + \frac{Q_{\beta,\psi}}{1 - |\lambda| Q_{\beta,\psi}}) \frac{1}{\Gamma(\beta)}(\psi'(s)(\psi(t) - \psi(s))^{\beta-1}, \end{aligned}$$

since $s \mapsto \frac{1}{\Gamma(\beta)}(\psi'(s)(\psi(t) - \psi(s))^{\beta-1}$ is an integrable function on $[0, t]$, then by Lebesgue dominated convergence theorem, we can conclude that

$$\lim_{n\mapsto+\infty} \frac{1}{\Gamma(\beta)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\beta-1} \left\|(g(s, x_n(s), x_n(\varepsilon s)) - \lambda x'_n(s)) - (g(s, x(s), x(\varepsilon s)) - \lambda x'(s))\right\| ds = 0.$$

Therefore,

$$\lim_{n\mapsto+\infty} \| \mathcal{B}_2 x_n - \mathcal{B}_2 x \| = 0,$$

it follows that $\mathcal{B}_2$ is continuous .
To demonstrate (9), let $x \in C^1$, then we obtain

$$|\mathcal{B}_2 x(t)| \leq \frac{1}{\Gamma(\beta)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\beta-1} \left|(g(s, x(s), x(\varepsilon s)) - \lambda x'(s))\right| ds,$$

by Lemma 3.2 and (H_3) we obtain

$$|\mathcal{B}_2x(t)| \leq \frac{(1+\frac{Q_{\beta,\psi}}{1-|\lambda|Q_{\beta,\psi}})(K_g\|x\|^p+M_g)}{\Gamma(\beta)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\beta-1}ds,$$

it follows

$$\| \mathcal{B}_2x \| \leq \frac{(1+\frac{Q_{\beta,\psi}}{1-|\lambda|Q_{\beta,\psi}})(K_g\|x\|^p+M_g)(\psi(T)-\psi(0))^{\beta}}{\Gamma(\beta+1)}.$$

Thus,

$$\| \mathcal{B}_2x \| \leq \frac{Q_{\beta,\psi}}{\| \psi' \| \beta}(1+\frac{Q_{\beta,\psi}}{1-|\lambda|Q_{\beta,\psi}})(K_g\|x\|^p+M_g).$$

Finally, we get

$$\| \mathcal{B}_2x \| \leq \Pi_{\beta,\lambda,\psi}(K_g\|x\|^p+M_g).$$

□

Lemma 3.6. *$\mathcal{B}_2 : C^1 \to C^1$ is a compact operator.*

Proof. Let us first prove that $\mathcal{B}_2B_\eta$ is bounded. To do so, let $x \in B_\eta$. Then by (9) we have

$$\| \mathcal{B}x \| \leq \Pi_{\beta,\lambda,\psi}(K_g\eta^p+M_g) := \zeta.$$

It follows that $\mathcal{B}_2B_\eta \subset B_\zeta$. Thus $\mathcal{B}_2B_\eta$ is bounded.
Now let us show that $\mathcal{B}B_\eta$ is equicontinuous. Let $x \in \mathcal{B}_2B_\eta$ and $t_1, t_2 \in J$ such that $t_1 < t_2$, then we have

$$\begin{aligned}\left|\mathcal{B}_2x(t_2)-\mathcal{B}_2x(t_1)\right| \leq & \frac{(1+\frac{Q_{\beta,\psi}}{1-|\lambda|Q_{\beta,\psi}}))(K_g\eta^p+M_g)}{\Gamma(\beta)}\int_{t_1}^{t_2} \psi'(s)(\psi(t_2)-\psi(s))^{\beta-1}ds,\\ &+\frac{(1+\frac{Q_{\beta,\psi}}{1-|\lambda|Q_{\beta,\psi}})(K_g\eta^p+M_g)}{\Gamma(\beta)}\int_0^{t_1} \psi'(s)((\psi(t_2)-\psi(s))^{\beta-1}-(\psi(t_1)-\psi(s))^{\beta-1})ds\\ &\leq \frac{(1+\frac{Q_{\beta,\psi}}{1-|\lambda|Q_{\beta,\psi}})(K_g\eta^p+M_g)}{\Gamma(\beta+1)}((\psi(t_2)-\psi(0))^{\beta}+2(\psi(t_2)-\psi(t_1))^{\beta}+(\psi(t_1)-\psi(0))^{\beta}).\end{aligned}$$

Since ψ is a continuous function, then we obtain

$$\lim_{t_1\to t_2}|\mathcal{B}_2x(t_1)-\mathcal{B}_2x(t_2)| = 0.$$

thus $\mathcal{B}_2B_\eta$ is equicontinuous.
Given that $\mathcal{B}_2B_\eta$ is uniformly bounded and equicontinuous. Then by Arzelà–Ascoli Theorem [23] we conclude that $\mathcal{B}_2B_\eta$ is relatively compact, hence $\mathcal{B}_2$ is compact. □

Corollary 3.7. *$\mathcal{B}_2 : C^1 \to C^1$ is ρ-Lipschitz with zero constant.*

Proof. Since the operator $\mathcal{B}_2$ is compact and from Lemma 2.12 it follows that $\mathcal{B}_2$ is ρ-Lipschitz with zero constant. □

Now, we have all tools to establish the proof of Theorem 3.3.
Proof of Theorem 3.3.
Let $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}$: $C^1 \to C^1$ be the operators given by the equations (5),(6) and (7) respectively.
$\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}$ are continuous and bounded. Moreover, by using Lemma 3.4 we have $\mathcal{B}_1$ is ρ-Lipschitz with constant L_ω and by using Corollary 3.7 we have $\mathcal{B}_1$ is ρ-Lipschitz with zero constant. It follows from

Lemma 2.11 that $\mathcal{B}$ is a strict ρ-contraction with constant L_ω.
We consider the following set

$$\mathcal{S}_\gamma = \{x \in C^1 : x = \gamma \mathcal{B} x \ \ for \ \ some \ \ \gamma \in [0,1]\}.$$

Let us show that $\mathcal{S}_\gamma$ is bounded in C^1. For this purpose let $x \in \mathcal{S}_\gamma$, then $x = \gamma \mathcal{B} x = \gamma(\mathcal{A}x + \mathcal{B}_2 x)$. It follows that

$$\|x\| = \gamma \|\mathcal{B}x\| \leq \gamma(\|\mathcal{B}_1 x\| + \|\mathcal{B}_2 x\|),$$

by using Lemmas 3.4 and 3.5 we get

$$\|x\| \leq \Big(|x_0| + K_\omega \|x\|^q + M_\omega + \Pi_{\beta,\lambda,\psi}(K_g\|x\|^p + M_g)\Big). \tag{10}$$

From the inequality (10) we deduce that $\mathcal{S}_\gamma$ is bounded in C^1 with $p < 1$ and $q < 1$.
if it's not the case, we suppose that $\xi := \|x\| \longrightarrow \infty$. Dividing both sides of (10) by ξ, and taking $\xi \to \infty$, it follows that

$$1 \leq \lim_{\xi \to \infty} \frac{\Big(|x_0| + K_\omega \xi^q + M_\omega + \Pi_{\beta,\lambda,\psi}(K_g \xi^p + M_g)\Big)}{\xi} = 0,$$

which is a contradiction.
Finally, by Theorem 2.14, we conclude that $\mathcal{B}$ has at least one fixed point, which serves as a solution to (1).
Moreover, the set of fixed points of $\mathcal{B}$ is bounded in C^1. □

Remark 3.8. *Note that if the assumptions* (H_2) *and* (H_3) *are formulated for* $q = 1$ *and* $p = 1$*, then the conclusions of Theorem 3.3 remain valid provided that*

$$K_\omega + K_g \Pi_{\beta,\lambda,\psi} < 1.$$

4. An illustrative example

In this section, we give an example to illustrate the usefulness of our main result.
Consider the following problem:

$$\begin{cases} {}^C D_{0^+}^{\frac{3}{2},(e^t} x(t) + \dfrac{1}{8} x'(t) = g(t, x(t), x(\varepsilon t)), \ \ t \in J = [0,1], \\ \\ x'(0) = 0, \ \ x(0) = \displaystyle\sum_{j=1}^{20} \theta_j |x(t_j)|, \ \ \theta_j > 0, \ \ 0 < t_j < 1, \ \ j = 1, 2, .., 20. \end{cases} \tag{11}$$

where $g(t, x(t), x(\varepsilon t)) = \frac{sin\left(x\left(\frac{t}{\sqrt{2}}\right)\right)}{(9+e^t)\sqrt{2}} \left(\frac{|x(t)|}{1+\left|x\left(\frac{t}{\sqrt{2}}\right)\right|} \right)$

Here $\varepsilon = \frac{1}{\sqrt{2}}$, $\beta = \frac{3}{2}$, $T = 1$, $\psi(t) = e^t$, $\lambda = \frac{1}{8}$ and $\omega(x) = \displaystyle\sum_{j=1}^{20} \theta_j |x(t_j)|$ with $\sum_{j=1}^{20} \theta_j < 1$.

Clearly (H_1), (H_2) hold with $K_\omega = L_\omega = \sum_{i=j}^{20} \theta_j$, $M_\omega = 0$ and $q = 1$.

Indeed, we can write

$$|\omega(x(t))| = \left| \sum_{j=1}^{20} \theta_j |x(t_j)| \right|,$$

hence

$$|\omega(x)| \leq \sum_{j=1}^{20} \theta_j \, \|x\| \, ,$$

thus $K_\omega = \sum_{j=1}^{20} \theta_j, \quad M_\omega = 0$ and $q = 1$.
Alternatively, we have

$$|\omega(x(t)) - \omega(y(t))| = \left| \sum_{j=1}^{20} \theta_j |x(t_j) - \sum_{j=1}^{20} \theta_j |y(t_j)| \right|,$$

hence

$$|\omega(x) - \omega(y)| \leq \sum_{j=1}^{20} \theta_j \left| x - y \right|,$$

thus $L_\omega = \sum_{j=1}^{20} \theta_j$.
To prove (H_3), let $t \in J$ and $x \in \mathbb{R}$, then we have

$$\left| g(t, x(t), x(\varepsilon t)) \right| = \left| \frac{\cos\left(x\left(\frac{t}{\sqrt{2}}\right)\right)}{(9+e^t)\sqrt{2}} \left(\frac{|x(t)|}{1 + \left| x\left(\frac{t}{\sqrt{2}}\right) \right|} \right) \right| \leq \frac{1}{10\sqrt{2}} \left(\|x\| + 1 \right).$$

Thus, (H_3) holds with $K_g = M_g = \dfrac{1}{10\sqrt{2}}$ and $p = 1$.
Consequently, Theorem 3.3 implies that problem (11) has at least one solution.
Moreover, from the inequality (10) we have

$$\|x\| \leq \frac{\left(1 + e\frac{(e-1)^{(1/2)}}{\Gamma(3/2) - \frac{1}{8}e(e-1)^{(1/2)}}\right)(e-1)^{(3/2)}}{10\sqrt{2}\Gamma(8/3) - 1} = 0.0598798162,$$

thus the set of solutions for (11) is bounded.

5. Conclusion

In this paper, we studied the existence of solutions for a class of nonlocal pantograph differential equations with damping and a ψ-Caputo type fractional derivative. The existence results were established using topological degree theory for condensing maps. Finally, an illustrative example was presented to support the theoretical findings.

Acknowledgements

We are very grateful to the anonymous referee, who helped signifcantly improve the paper.

Conflict of interest

The authors declare that they have no conflict of interest.

Data Availability

No data were used to support this study.

References

[1] M.S. Abdo, T. Abdeljawad, K.D. Kucche, M.A. Alqudah, S.M. Ali and M.B. Jeelani, On nonlinear pantograph fractional differential equations with Atangana-Baleanu-Caputo derivative, *Adv. Differ. Equ.* **2021**(65) (2021), 1–17.

[2] R.P. Agarwal, S.K. Ntouyas, B. Ahmad and A.K. Alzahrani, Hadamard-type fractional functional differential equations and inclusions with retarded and advanced arguments, *Adv. Differ. Equ.* **2016** (2016), 1–15.

[3] R.P. Agarwal, S. Hristova and D. O'Regan, A survey of Lyapunov functions, stability and impulsive Caputo fractional differential equations, *Fract. Calc. Appl. Anal.* **19**(2) (2016), 290–318.
[4] K.S. Akiladevi and K. Balachandran, Existence results for fractional integrodifferential equations with infinite delay and fractional integral boundary conditions, *Filomat* **38**(24) (2024), 8391–8409.
[5] R. Almeida, Caputo fractional derivative of a function with respect to another function, *Commun. Nonlinear Sci. Numer. Simul.* **44** (2017), 460–481.
[6] R. Almeida, A.B. Malinowska and M.T.T. Monteiro, Fractional differential equations with a Caputo derivative with respect to a kernel function and their applications, *Math. Methods Appl. Sci.* **41**(1) (2018), 336–352.
[7] K. Balachandran, S. Kiruthika and J.J. Trujillo, Existence of solutions of nonlinear fractional pantograph equations, *Acta Math. Sci.* **33**(3) (2013), 712–720.
[8] D. Baleanu, H. Jafari, H. Khan and S.J. Johnston, Results for mild solution of fractional coupled hybrid boundary value problems, *Open Math.* **13** (2015), 601–608.
[9] S. Bhalekar and J. Patade, Series solution of the pantograph equation and its properties, *Fractal Fract.* **1**(1) (2017), 1–6.
[10] S. Bouriah, D. Foukrach, M. Benchohra and J. Graef, Existence and uniqueness of periodic solutions for some nonlinear fractional pantograph differential equations with ψ-Caputo derivative, *Arabian J. Math.* **10**(3) (2021), 575–587.
[11] M. Caputo, Linear models of dissipation whose Q is almost frequency independent, *Int. J. Geogr. Inf. Sci.* **13**(5) (1967), 529–539.
[12] N. Chems Eddine, M.A. Ragusa and D.D. Repovs, On the concentration-compactness principle for anisotropic variable exponent Sobolev spaces and its applications, *Fract. Calc. Appl. Anal.* **27**(2) (2024), 725–756.
[13] B.C. Dhage, On β-condensing mappings in Banach algebras, *Math. Student* **63** (1994), 146–152.
[14] K. Deimling, *Nonlinear Functional Analysis*, Springer-Verlag, New York, 1985.
[15] A. El Mfadel, F.E. Bourhim and M. Elomari, Existence of mild solutions for semilinear ψ-Caputo-type fractional evolution equations with nonlocal conditions in Banach spaces, *Results Nonlinear Anal.* **5**(4) (2022), 459–472.
[16] A. El Mfadel, S. Melliani and M. Elomari, On the initial value problem for fuzzy nonlinear fractional differential equations, *Filomat* **48**(4) (2024), 547–554.
[17] A. El Mfadel, S. Melliani and M. Elomari, Existence of solutions for nonlinear ψ-Caputo-type fractional hybrid differential equations with periodic boundary conditions, *Asia Pac. J. Math.* **9** (2022), 1–11.
[18] A. El Mfadel, S. Melliani and M. Elomari, Existence and uniqueness results for Caputo fractional boundary value problems involving the p-Laplacian operator, *U.P.B. Sci. Bull. Ser. A* **84**(1) (2022), 37–46.
[19] A. El Mfadel, S. Melliani and M. Elomari, New existence results for nonlinear functional hybrid differential equations involving the ψ-Caputo fractional derivative, *Results Nonlinear Anal.* **5**(1) (2022), 78–86.
[20] A. El Mfadel, S. Melliani and M. Elomari, Existence results for nonlocal Cauchy problem of nonlinear ψ-Caputo type fractional differential equations via topological degree methods, *Adv. Theory Nonlinear Anal. Appl.* **6**(2) (2022), 270–279.
[21] A. El Mfadel, S. Melliani and M. Elomari, On the existence and uniqueness results for fuzzy linear and semilinear fractional evolution equations involving Caputo fractional derivative, *J. Funct. Spaces* **2021** (2021), 1–9.
[22] A. El Mfadel, S. Melliani and M. Elomari, A note on the stability analysis of fuzzy nonlinear fractional differential equations involving the Caputo fractional derivative, *Int. J. Math. Math. Sci.* **2021** (2021), 1–10.
[23] J.W. Green and F.A. Valentine, On the Arzelà-Ascoli theorem, *Math. Mag.* **34** (1961), 199–202.
[24] N. Hatime, S. Melliani, A. El Mfadel and M. Elomari, Numerical analysis of generalized fractional form of Newton's cooling law under a variable environment temperature, *Int. J. Appl. Comput. Math.* **10**(2) (2024), 61.
[25] N. Hatime, S. Melliani, A. El Mfadel and M. Elomari, Existence, uniqueness, and finite-time stability of solutions for ψ-Caputo fractional differential equations with time delay, *Comput. Methods Differ. Equ.* **11**(4) (2023), 785–802.
[26] N. Hatime, A. El Mfadel, M. Elomari and S. Melliani, Existence and stability of solutions for nonlinear fractional differential equations involving the Gronwall-Fredholm-type inequality, *J. Math. Sci.* **2024** (2024), 1–20.
[27] R. Hilfer, *Applications of Fractional Calculus in Physics*, World Scientific, Singapore, 2000.
[28] A.A. Kilbas, H.M. Srivastava and J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematical Studies 204, Elsevier Science B.V., Amsterdam, 2006.
[29] M.I. Liaqat, A hybrid approach to approximate and exact solutions for linear and nonlinear fractional-order Schrödinger equations with conformable fractional derivatives, *Electron. J. Appl. Math.* **2**(3) (2024), doi: https://doi.org/10.61383/ejam.20242371.
[30] Y. Luchko and J.J. Trujillo, Caputo-type modification of the Erdélyi-Kober fractional derivative, *Fract. Calc. Appl. Anal.* **10**(3) (2007), 249–267.
[31] F. Isaia, On a nonlinear integral equation without compactness, *Acta Math. Univ. Comenianae* **75**(2) (2006), 233–240.
[32] M. Liu and D. Li, Properties of analytic solution and numerical solution of multi-pantograph equation, *Appl. Math. Comput.* **155**(3) (2004), 853–871.
[33] F. Mainardi, *Fractals and Fractional Calculus in Continuum Mechanics*, Springer-Verlag, Wien, 1997.
[34] S.K. Ntouyas, Boundary value problems for nonlinear fractional differential equations and inclusions with nonlocal and fractional integral boundary conditions, *Opuscula Math.* **33**(1) (2013), 117–138.
[35] I. Podlubny, *Fractional Differential Equations*, Mathematics in Science and Engineering, Academic Press, New York, 1999.
[36] J. Schauder, Der Fixpunktsatz in Funktionalräumen, *Studia Math.* **2** (1930), 171–180.
[37] G.B. Saluja, H.K. Nashine, R. Jain, R.W. Ibrahim and H.A. Nabwey, Common fixed point theorems on S-metric spaces for integral type contractions involving rational terms and application to fractional integral equation, *J. Funct. Spaces* **2024** (2024), 1–18.