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A study of reversible DNA cyclic codes over a non-chain ring based on the deletion distance

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Abstract. Let $\mathcal{R} = \mathbb{Z}_8[u]/\langle u^2 - 4, 2u\rangle$ be a non-chain ring of characteristic 8. In this article, DNA codes of odd lengths over the ring $\mathcal{R}$ based on the deletion distance are discussed. For this purpose, we study cyclic codes of any odd length over the ring $\mathcal{R}$ satisfying the reversible and the reversible complement constraints. Also, a bijection ϑ between the elements of the ring $\mathcal{R}$ and $S_{D_{16}}$ is constructed in such a way that the reversibility problem is solved. Moreover, we introduce a homogeneous weight w_{hom} over the ring $\mathcal{R}$ and by utilizing w_{hom}, a new Gray map $\theta_{\text{hom}} : \mathcal{R}^n \to \mathbb{F}_2^{8n}$ is obtained. Furthermore, we study the GC-content of DNA codes and provide some examples of DNA codes with their respective deletion distance.

1. Introduction

Deoxyribonucleic acid or DNA for short, is a molecule that contains the genetic instructions necessary for the development and functioning of all known living organisms. DNA is a double-stranded molecule made up of nucleotides. Each nucleotide consists of a sugar (deoxyribose), a phosphate group, and one of four nitrogenous bases: adenine (A), thymine (T), cytosine (C), or guanine (G). The DNA molecules form a twin-stranded double helix as two sugar-phosphate chains are connected through hydrogen bonds, specifically between G and C as well as between A and T, and linked according to the Watson-Crick complement (WCC) rule. We call a given sequence $\alpha = \alpha_0\alpha_1 \cdots \alpha_{n-1}$, a quaternary n-sequence or a DNA sequence of length n if $\alpha_i \in \mathbf{Y}$, where $\mathbf{Y} = \{A, G, T, C\}$. Moreover, DNA k-bases is defined as the collection of all DNA sequences of length k. The Watson-Crick complement rule states that when a DNA sequence combines with its reversible complement, it forms a helix. Specifically, T pairs with A (and vice versa), while G pairs with C (and vice versa) We denote by $C^c = G$, $G^c = C$, $A^c = T$, $T^c = A$, where C^c, G^c, A^c, and T^c are the complements of C, G, A and T respectively. For example, the Watson-Crick complement of $(ATTGACC)^c = A^cT^cT^cG^cA^cC^cC^c = TAACTGG$. A DNA code $\mathcal{C}$ of length n is the collection of DNA sequences of length n where each sequence occurs with its reverse complement sequence i.e., $\alpha^{rc} \in \mathcal{C}$, where

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$\alpha^{rc} = \alpha^c_{n-1}\alpha^c_{n-2}\cdots\alpha^c_0$ is the reverse complement of $\alpha = \alpha_0\alpha_1\cdots\alpha_{n-1}$.

The ability to construct DNA codes that fulfill particular constraints is of utmost importance in various domains, including biotechnology and security. These applications encompass a range of areas, such as DNA computing, DNA cryptography, and DNA steganography [5], [6]. A cyclic DNA code C always satisfies the Hamming distance constraint and it may satisfy the other three constraints. Following are the constraints:

1. **The Hamming constraint :** If $H(c_1, c_2) \geq d$, where $c_1, c_2 \in C$ and $c_1 \neq c_2$ for some Hamming distance d.

2. **The Reverse constraint :** If $H(c_1, c_2^r) \geq d$, where $c_1, c_2 \in C$ for some Hamming distance d, and $c_2^r = (\alpha_{n-1}, \alpha_{n-2}, \ldots, \alpha_0)$ is the reverse of $c_2 = (\alpha_0, \alpha_1, \ldots, \alpha_{n-1})$

3. **The Reverse-complement constraint :** If $H(c_1, c_2^{rc}) \geq d$, where $c_1, c_2 \in C$ for some Hamming distance d, and $c_2^{rc} = (\alpha^c_{n-1}, \alpha^c_{n-2}, \ldots, \alpha^c_0)$ is the reverse complement of $c_2 = (\alpha_0, \alpha_1, \ldots, \alpha_{n-1})$

4. **The Fixed *GC*-content constraint :** If any codeword contains the same number of G and C.

The first three constraints aim to minimize the likelihood of non-specific hybridization, while the fixed *GC*-content constraint is employed to achieve comparable melting temperatures.

Adleman [4] initiated the exploration of DNA's structural role in computations by solving a well-known NP-hard problem through the application of DNA molecules. He used an approach that was based on the Watson-Crick complement (WCC) property of DNA sequence. Moreover, Adleman et al. [5] formulated a molecular program for breaking the symmetric cryptographic algorithm namely the Data Encryption Standard (DES). Subsequently, Mansuripur et al. [24] demonstrated that DNA molecules can serve as a storage medium. This advancement necessitates the development of various theories for constructing DNA sequences that meet specific constraints. Algebraic coding theory plays a crucial role in creating DNA codes with constraints (refer to [21] for DNA codes over different finite rings). DNA codes, rooted in error-correcting codes, have proven effective in DNA-based computation and storage. For instance, Milenkovic and Kashyap [23] elucidated the design of codes for DNA computing by considering avoidance of formation of secondary structures in single-stranded DNA molecules and non-selective cross-hybridization.

There are some known methods for designing DNA codes that satisfy certain constraints including the study of reversible codes. In 1964 Massey [25] studied the reversible codes over finite fields. Later on, Tzeng and Hartmann[31] obtained the bounds of the minimum distance for certain reversible cyclic codes. Moreover, Srinivasulu and Bhaintwal[29] studied reversible cyclic codes over the finite ring $\mathbb{F}_4 + u\mathbb{F}_4$, $u^2 = 0$ and constructed certain DNA codes. Further, Dinh et al [13] constructed reversible complement codes and obtained cyclic DNA codes over the ring $\mathbb{F}_2[u, v]/\langle u^2 - 1, v^3 - v, uv - vu\rangle$. For the intensive study of reversible codes over finite rings, we refer the readers [1–3, 11, 14, 18–20, 26–28]. D'yachkov et al. [9, 10] identified a similarity distance which is more worthy than the Hamming distance. Recently, Martinez-Moro and Szabo [22] discussed the structure of the local Frobenius non-chain ring of order 16. Later on, Dougherty et al. [15] constructed cyclic codes over a local Frobenius non-chain ring of order 16. Siap et al. [30] studied cyclic DNA codes over the ring $\mathbb{F}_2[u]/\langle u^2 - 1\rangle$ based on the deletion distance. Further, Dinh et al. [12] discussed cyclic DNA codes over the ring $\mathbb{Z}_4[u]/\langle u^2 - 1\rangle$ based on the deletion distance.

Motivated by these studies, we study cyclic DNA codes of odd length over the ring $\mathcal{R} = \mathbb{Z}_8[u]/\langle u^2 - 4, 2u\rangle$ based on the deletion distance. We discuss the reversibility of cyclic codes over the ring $\mathcal{R}$. We have also constructed a bijection ϑ (see Table 1) between the elements of the ring $\mathcal{R}$ and $S_{D_{16}}$, where $S_{D_{16}} = \{x_1x_2 : x_1, x_2 \in \mathbf{Y}\}$. The novelty of this article is that by utilizing the bijection ϑ, we have solved the reversibility problem. Moreover, Dougherty et al. [15] studied binary cyclic codes as the images of the Gray map with respect to the

Lee weight. In this article, we obtain cyclic codes with respect to the homogeneous weight. Also, we study the *GC*-content of cyclic DNA codes and find their deletion distance.

The rest of the article is organized as follows: Section 2 is devoted to familiarize the readers with basic terminology. In Section 3, we discuss the structure of cyclic codes over the ring $\mathcal{R}$. We obtain a new Gray map $\theta_{\text{hom}} : \mathcal{R}^n \to \mathbb{F}_2^{8n}$ with respect to the homogeneous weight w_{hom} in Section 4. In Section 5, we provide some necessary and sufficient conditions for: (*i*) a given cyclic code of odd length over the ring $\mathcal{R}$ to be a reversible cyclic code, and (*ii*) a given cyclic code of odd length over the ring $\mathcal{R}$ to be reversible complement cyclic code. The study of the *GC*-content of cyclic DNA codes of odd length and their deletion distance is included in Section 6. Section 7 consists of examples of DNA codes with respect to their deletion distance.

2. Preliminaries

We begin this section by characterizing the ring $\mathcal{R}$. In [22], the authors obtained all local Frobenius non-chain rings of order 16. Suppose $\mathbb{Z}_8$ is a ring of integers modulo 8. Throughout the article $\mathcal{R} := \mathbb{Z}_8[u]/\langle u^2 - 4, 2u\rangle$. Therefore $\mathcal{R}$ is a finite commutative ring, which is an extension of $\mathbb{Z}_8$. Moreover, $\mathcal{R}$ is a vector space over $\mathbb{F}_2$ with basis $\{1, 2, u, 4\}$. Thus, any element $x \in \mathcal{R}$ can be uniquely expressed as $x = \xi_1 + 2\xi_2 + u\xi_3 + 4\xi_4$, where $\xi_i \in \mathbb{F}_2$. The cardinalty of $\mathcal{R}$ is 16 and characteristic of $\mathcal{R}$ is 8. Then all elements of the ring $\mathcal{R}$ are as follows: $\mathcal{R} = \{0, 1, 2, 3, 4, 5, 6, 7, u, 1 + u, 2 + u, 3 + u, 4 + u, 5 + u, 6 + u, 7 + u\}$. Moreover, the non-trivial ideals of $\mathcal{R}$ are as follows:

$$\langle 4\rangle = \{0, 4\},$$

$$\langle 2\rangle = \{0, 2, 4, 6\},$$

$$\langle u\rangle = \{0, u, 4, u + 4\},$$

$$\langle 2 + u\rangle = \{0, 4, 2 + u, 6 + u\},$$

$$\langle 2, u\rangle = \{0, 2, 4, 6, u, 2 + u, 4 + u, 6 + u\}.$$

The ideal lattice of the ring $\mathcal{R}$ is given in Figure 1. It is worth to notice that ideals of $\mathcal{R}$ do not form a chain under the set theoretical inclusion relation. Therefore, $\mathcal{R}$ is a non-chain ring and the ideal $\langle 2, u\rangle$ is the unique maximal ideal of $\mathcal{R}$. Hence, $\mathcal{R}$ is a local Frobenius non-chain ring of order 16. Recall that a linear code C of length n over $\mathcal{R}$ is an $\mathcal{R}$-submodule of $\mathcal{R}^n$ and members of C are known as codewords. A linear code C of length n over $\mathcal{R}$ is called cyclic if for each codeword $a = (a_0, a_1, \ldots, a_{n-1}) \in C$, the n-tuple $(a_{n-1}, a_0, \ldots, a_{n-2})$ obtained by the cyclic shift of coordinates i to $i + 1 \mod n$ is also in C.

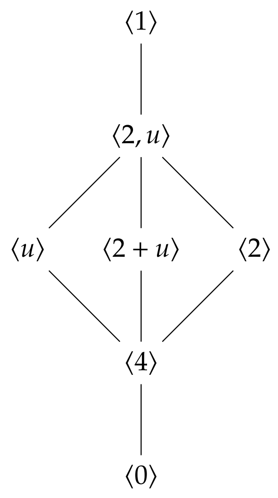

Figure 1: Ideal lattice of the ring $\mathcal{R}$

The inner-product of two given n-tuples $a = (a_0, a_1, \ldots, a_{n-1})$, $b = (b_0, b_1, \ldots, b_{n-1})$ is defined as $a \cdot b = \sum_{i=0}^{n-1} a_i b_i$, a and b are said to be orthogonal if $a \cdot b = 0$.
For a given linear code C, we define the dual code $C^{\perp}$ over the ring $\mathcal{R}$ in the following manner

$$C^{\perp} = \{a \in \mathcal{R}^n \mid a \cdot b = 0 \text{ for all } b \in C\}.$$

One can verify that $C^{\perp}$ is a linear code over the ring $\mathcal{R}$ of same length as C. In addition, If $C \subseteq C^{\perp}$ then a linear code C is said to be self-orthogonal and if $C = C^{\perp}$ then we call a linear code C is self-dual. The reciprocal polynomial $p^*(x)$ of a given polynomial $p(x) = p_0 + p_1 x + \cdots + p_{n-1}x^{n-1} \in \mathcal{R}[x]$, is defined as $p^*(x) = x^{\deg(p(x))} p(\frac{1}{x}) = p_{n-1} + p_{n-2}x + \cdots + p_0 x^{n-1}$. It is worth to mention that $\deg(p(x)) \geq \deg(p^*(x))$ and if $p_0 \neq 0$, then $\deg(p(x)) = \deg(p^*(x))$. A given polynomial $p(x)$ is said to be self-reciprocal if $p^*(x) = p(x)$.

We need the following result which provides the criteria for obtaining the reciprocal of sum and product of polynomials.

Lemma 2.1. *[3] Let $f_1(x)$ and $f_1(x)$ be any two polynomials in $\mathcal{R}[x]$ with $\deg(f_1(x)) \geq \deg(f_2(x))$. Then the following statements hold:*

1. $(f_1(x) \cdot f_1(x))^* = f_1^*(x) \cdot f_2^*(x)$,
2. $(f_1(x) + f_2(x))^* = f_1^*(x) + x^i f_2^*(x)$, *where* $i = \deg(f_1(x)) - \deg(f_2(x))$.

Let $A = \alpha_0\alpha_1 \cdots \alpha_{n-1}$ and $B = \beta_0\beta_1 \cdots \beta_{n-1}$ be any two quaternary n-sequences. Suppose $\ell \in \{1, 2, \ldots, n\}$ and $1 \leq t \leq \ell$. We define by $C = \gamma_0\gamma_1 \cdots \gamma_{\ell-1}$, a common subsequence of length ℓ between A and B if $\gamma_t = \alpha_{p_t} = \beta_{q_t}$, where $1 \leq p_1 < p_2 < \cdots < p_\ell \leq n$, $1 \leq q_1 < q_2 < \cdots < q_\ell \leq n$. The energy of DNA hybridization $E(A, B)$ is measured by the longest common subsequence (not necessarily contiguous) of either strand or the reverse complement of the other strand. The deletion similarity is define as the length of the longest common subsequence for A and B and is denoted by $S(A, B)$. If A and B are any strands of length n, then we have $S(A, B) = n$ and $S(A, B) = S(B, A)$. Moreover, the deletion similarity $S(A, B)$ can be identified by the number of base pair bonds between A and B^{rc} i.e.,

$$S(A, B) = S(B, A) = E(A, B^{rc}) = E(A^{rc}, B) \tag{1}$$

Example 2.2. *Suppose $A = TAGATT$ and $B = TCGATT$, are two DNA sequences of length 6. Clearly, GAT is a largest common subsequence of length 3, TAT is also a common subsequence of length 3. There is no common subsequence of length $\ell > 3$. Therefore, the deletion similarity between the DNA sequences A and B is given by $S(A, B) = 3$.*

Definition 2.3. *[9, 10] Let C be a DNA code of length n. Then C is called a DNA code of distance D based on the deletion similarity or equivalently an (n, D)-code if there exists a smallest positive integer D such that $S(X, Y) \leq n - D - 1$ for all $X, Y \in C$, $X \neq Y$.*

Example 2.4. *Let $B = \{GAGC, GCTC, GCGA, TCGC\}$ be the collection of DNA sequences of length 4. Clearly, $(GAGC)^{rc} = GCTC$, $(GCTC)^{rc} = GAGC$, $(GCGA)^{rc} = TCGC$, $(TCGC)^{rc} = GCGA$. Therefore, we can conclude that B is a DNA code of length 4. Notice that GC is a common subsequence of length 2. There is no subsequence of length $\ell > 2$. Hence, the deletion similarity $S(X, Y) = 2$, for all $X, Y \in B$. By using Definition 2.1, we must have $4 - D - 1 \geq 2$, which further yields $D \leq 1$. Thus, B is a (4,1)-code.*

3. Structure of cyclic codes over $\mathcal{R}$

Dougherty et al. [15] have obtained structure of cyclic codes over a local Frobenius non-chain ring of order 16. In this article, the aforementioned ring $\mathcal{R}$ is also a local Frobenius non-chain ring of order 16, and hence we obtain the structure of cyclic codes over $\mathcal{R}$ using results given in [15]. Throughout this article, we use a symbol N, where N is the ideal generated by $\{2, u\}$, i.e., $N = \langle 2, u\rangle$. For example, $Nf(x) = \langle 2f(x), uf(x)\rangle$, where $f(x) \in \mathcal{R}_n$.

The following theorem provides the structure of cyclic codes of any odd length over the ring $\mathcal{R}$.

Theorem 3.1. *[15] Let C be a cyclic code of odd length n over the ring $\mathcal{R}$. Then C is generated by the following polynomials $\{\widehat{H_1}(x), N\widehat{H_2}(x), 2\widehat{H_3}(x), u\widehat{H_4}(x), (2+u)\widehat{H_5}(x), 4\widehat{H_6}(x)\}$, where $x^n-1 = H_0(x)H_1(x)H_2(x)H_3(x)H_4(x)H_5(x)H_6(x)$; $H_i(x)$ are coprime monic polynomials and $\widehat{H_i}(x) = \dfrac{x^n-1}{H_i(x)}$, $0 \le i \le 6$. Moreover, C is of the type*

$$16^{\deg(H_1(x))}8^{\deg(H_2(x))}4^{(\deg(H_3(x))+\deg(H_4(x))+\deg(H_5(x)))}2^{\deg(H_6(x))}.$$

Observe that there is no relationship between the generating polynomials of the cyclic codes over the ring $\mathcal{R}$ in the aforementioned theorem. The following equivalent structure of cyclic codes over the ring $\mathcal{R}$, in which generating polynomials are dividing some other generating polynomials, is obtained using the above structure of cyclic codes over the ring $\mathcal{R}$.

Theorem 3.2. *Let C be a cyclic code of odd length n over the ring $\mathcal{R}$. Then C is given by*

$$C = \langle g_0(x), Ng_1(x), 2g_2(x), ug_3(x), (2+u)g_4(x), 4g_5(x)\rangle$$

such that $g_j(x) \in \mathcal{R}[x]$ for $0 \le j \le 5$ and $g_5(x)|g_k(x)|g_1(x)|g_0(x)$, where $k = 2, 3, 4$.

Proof. By using Theorem 3.1, we have $C = \langle\widehat{H_1}(x), N\widehat{H_2}(x), 2\widehat{H_3}(x), u\widehat{H_4}(x), (2+u)\widehat{H_5}(x), 4\widehat{H_6}(x)\rangle$, where $x^n-1 = H_0(x)H_1(x)H_2(x)H_3(x)H_4(x)H_5(x)H_6(x)$; $H_i(x)$ are coprime monic polynomials and $\widehat{H_i}(x) = \dfrac{x^n-1}{H_i(x)}$, $0 \le i \le 6$. Now construct the polynomials $g_j(x)$, where $0 \le j \le 5$ using the polynomials $H_i(x)$ as follows:

$$\begin{aligned}
g_0(x) &= H_0(x)H_2(x)H_3(x)H_4(x)H_5(x)H_6(x),\\
g_1(x) &= H_0(x)H_3(x)H_4(x)H_5(x)H_6(x),\\
g_2(x) &= H_0(x)H_4(x)H_5(x)H_6(x),\\
g_3(x) &= H_0(x)H_3(x)H_5(x)H_6(x),\\
g_4(x) &= H_0(x)H_3(x)H_4(x)H_6(x),\\
g_5(x) &= H_0(x).
\end{aligned}$$

Let $C' = \langle g_0(x), Ng_1(x), 2g_2(x), ug_3(x), (2+u)g_4(x), 4g_5(x)\rangle$, where $g_j(x)$ are polynomials as defined above. Our claim is that $C' = C$. First we prove $C \subseteq C'$. To do so, notice that $\widehat{H_1}(x) = g_0(x) \in C'$. Now consider the polynomial

$$N\widehat{H_2}(x) = NH_0(x)H_1(x)H_3(x)H_4(x)H_5(x)H_6(x),$$

and substitute the value of $g_1(x)$ in the above expression we obtain

$$N\widehat{H_2}(x) = Ng_1(x)H_1(x).$$

Therefore, $N\widehat{H_2}(x) \in C'$. Similarly, the polynomials $2\widehat{H_3}(x)=2g_2(x)H_1(x)H_2(x) \in C'$, $u\widehat{H_4}(x)=ug_3(x)H_1(x)H_2(x) \in C'$ and $(2+u)\widehat{H_5}(x)=(2+u)g_4(x)H_1(x)H_2(x) \in C'$. Also, the polynomial $4\widehat{H_6}(x) = 4g_5(x)H_1(x)H_2(x)H_3(x)H_4(x)H_5(x) \in C'$. Hence, we obtain that $C \subseteq C'$. To prove $C' \subseteq C$, we proceed as follows. Notice that $g_0(x) = \widehat{H_1}(x)$, then $g_0(x) \in C$. First, we check if $Ng_1(x) \in C$. Since $H_1(x)$ and $H_2(x)$ are coprime polynomials, there exist polynomials $p_1(x), p_2(x) \in \mathcal{R}[x]$ such that

$$p_1(x)H_1(x) + p_2(x)H_2(x) = 1. \tag{2}$$

Now multiplying by $g_1(x)$ in both sides of (2), we obtain

$$\begin{aligned} g_1(x) &= p_1(x)H_1(x)g_1(x) + p_2(x)H_2(x)g_1(x), \\ g_1(x) &= p_1(x)\widehat{H_2}(x) + p_2(x)g_0(x) \end{aligned}$$

and $Ng_1(x) = Np_1(x)\widehat{H_2}(x) + Np_2(x)g_0(x) \in C$. Next, we check if $2g_2(x) \in C$. Since the polynomials $H_1(x)H_2(x)$ and $H_3(x)$ are coprime, there exist polynomials $q_1(x), q_2(x) \in \mathcal{R}[x]$ such that

$$q_1(x)H_1(x)H_2(x) + q_2(x)H_3(x) = 1. \tag{3}$$

Now multiplying by $g_2(x)$ in both sides of (3), we obtain

$$\begin{aligned} g_2(x) &= q_1(x)H_1(x)H_2(x)g_2(x) + q_2(x)H_3(x)g_2(x), \\ g_2(x) &= p_1(x)\widehat{H_3}(x) + p_2(x)g_1(x). \end{aligned}$$

This implies that $2g_2(x) \in C$. Similarly, we can also verify that the polynomials $ug_3(x), (2+u)g_4(x)$ and $4g_5(x) \in C$. Therefore, we can conclude that $C' \subseteq C$ and hence $C' = C$. □

Now, using the above structure of cyclic codes we define the following particular cases of cyclic codes. These codes are helpful in determining the conditions for reversibility over the ring $\mathcal{R}$.

Definition 3.3. *Let C_i be a cyclic code of odd length n over the ring $\mathcal{R}$, where $i \in \{1, 2, 3\}$. Then*

$$C_i = \langle g_0(x), Ng_1(x), \xi_i g_2(x), 4g_3(x)\rangle,$$

where $g_3(x)|g_2(x)|g_1(x)|g_0(x)$, $\xi_1 = 2$, $\xi_2 = u$ and $\xi_3 = 2 + u$.

The annihilator $A(\mathcal{I})$ for any ideal $\mathcal{I}$ of $\mathcal{R}_n$, is given by the following expression

$$A(\mathcal{I}) = \{g(x) \in \mathcal{R}_n \mid f(x) \cdot g(x) = 0, \forall\, f(x) \in \mathcal{I}\}.$$

Since $A(\mathcal{I})$ is an ideal of the ring $\mathcal{R}_n$, $A(\mathcal{I})$ a cyclic code over the ring $\mathcal{R}$. Moreover, if a cyclic code C of length n over $\mathcal{R}$ is given by $C = \langle \mathcal{I} \rangle$ then $C^{\perp} = \langle A^*(\mathcal{I})\rangle$, where $A^*(\mathcal{I}) = \{a^*(x) \mid a(x) \in A(\mathcal{I})\}$.
The following theorem is useful to obtain the structure of annihilator $A(C_i)$ of cyclic code C_i, where $i \in \{1, 2, 3\}$ over the ring $\mathcal{R}$.

Theorem 3.4. *Let C_i be a cyclic code of odd length n over the ring $\mathcal{R}$, where $i \in \{1, 2, 3\}$. Then annihilator $A(C_i)$ of C_i is given by*

$$A(C_i) = \left\langle \left(\frac{x^n-1}{g_3(x)}\right), N\left(\frac{x^n-1}{g_2(x)}\right), \xi_i\left(\frac{x^n-1}{g_1(x)}\right), 4\left(\frac{x^n-1}{g_0(x)}\right)\right\rangle,$$

where $C_i = \langle g_0(x), Ng_1(x), \xi_i g_2(x), 4g_3(x)\rangle$ such that $g_3(x)|g_2(x)|g_1(x)|g_0(x)$, $\xi_1 = 2$, $\xi_2 = u$ and $\xi_3 = 2 + u$.

Proof. Suppose that

$$K' = \left\langle \left(\frac{x^n-1}{g_3(x)}\right), N\left(\frac{x^n-1}{g_2(x)}\right), \xi_i\left(\frac{x^n-1}{g_1(x)}\right), 4\left(\frac{x^n-1}{g_0(x)}\right)\right\rangle.$$

Given that C_i is a cyclic code of odd length n over the ring $\mathcal{R}$. By using the definition of annihilator, $A(C_i)$ becomes a cyclic code over the ring $\mathcal{R}$. Let

$$A(C_i) = \langle p_1(x), Np_2(x), \xi_i' p_3(x), 4p_4(x)\rangle,$$

where $p_4(x)|p_3(x)|p_2(x)|p_1(x)$. As $g_3(x) \in C_i$ and $p_1(x) \in A(C_i)$, we get

$$g_3(x) \cdot p_1(x) = 0 \mod (x^n - 1).$$

We have $p_1(x) = \frac{x^n-1}{g_3(x)}\alpha(x)$, for some $\alpha(x) \in \mathcal{R}[x]$, $p_1(x) \in \langle\frac{x^n-1}{g_3(x)}\rangle$. Similarly, we also have $p_2(x) \in \langle\frac{x^n-1}{g_2(x)}\rangle$, $p_3(x) \in \langle\frac{x^n-1}{g_1(x)}\rangle$ and $p_4(x) \in \langle\frac{x^n-1}{g_0(x)}\rangle$. This implies that $A(C_i) \subseteq K'$. Similarly, we must have $K' \subseteq A(C_i)$. Thus,

$$A(C_i) = \left\langle \left(\frac{x^n-1}{g_3(x)}\right), N\left(\frac{x^n-1}{g_2(x)}\right), \xi_i\left(\frac{x^n-1}{g_1(x)}\right), 4\left(\frac{x^n-1}{g_0(x)}\right) \right\rangle.$$

□

Now by using the above theorem, we obtain the structure of the dual of a cyclic code C_i, where $i \in \{1, 2, 3\}$ over the ring $\mathcal{R}$.

Theorem 3.5. *Let C_i be a cyclic code of odd length n over the ring $\mathcal{R}$ given by*

$$C_i = \langle g_0(x), Ng_1(x), \xi_i g_2(x), 4g_3(x)\rangle,$$

where $g_3(x)|g_2(x)|g_1(x)|g_0(x)$ and $\xi_1 = 2$, $\xi_2 = u$ and $\xi_3 = 2 + u$. Then dual of the cyclic code C_i is given by $C_i^{\perp} = \left\langle \left(\frac{x^n-1}{g_3(x)}\right)^*, N\left(\frac{x^n-1}{g_2(x)}\right)^*, \xi_i\left(\frac{x^n-1}{g_1(x)}\right)^*, 4\left(\frac{x^n-1}{g_0(x)}\right)^* \right\rangle.$

Proof. The result directly follows from the Theorem 3.4. □

4. Homogeneous weight of cyclic codes over $\mathcal{R}$

The idea of homogeneous weight over the residue class rings of integers was introduced by Constanttinescu and Heise [8]. Subsequently, Greferath and Schmidt [17] generalized this notion to arbitrary finite rings and they have proved the existence of such a weight function on arbitrary finite rings. Dougherty et al.[15] constructed cyclic codes over the ring $\mathcal{R}$ with respect to the Lee weight. In this article, we obtain cyclic codes over the ring $\mathcal{R}$ with respect to the homogeneous weight on $\mathcal{R}$.

Definition 4.1. *Suppose R is a finite ring. A real valued function w on R is called a homogeneous (left) weight, if $w(0) = 0$ and the following are true:*

(*i*) *If $Rx = Ry$, then $w(x) = w(y)$ for all $x, y \in R$.*
(*ii*) *There exists a real number $\alpha \geq 0$ such that*

$$\sum_{t \in Rx} w(t) = \alpha |Rx|$$

for all non-zero $x \in R$.

Yildiz and Karadeniz [32] studied cyclic codes over the non-chain ring $R = \mathbb{F}_2 + u\mathbb{F}_2 + v\mathbb{F}_2 + uv\mathbb{F}_2$, where $u^2 = 0, v^2 = 0$ and $uv = vu$ with respect to the homogeneous weight. Later on, Yildiz and Kelebek [33] studied homogeneous weight for an infinite family of rings $R_k = \mathbb{F}_2[u_1, u_2, \cdots, u_k]/\langle u_i^2, u_iu_j = u_ju_i\rangle$. Motivated by this work, in this article we introduced homogeneous weight over the ring $\mathcal{R}$ as follows:
Let $x \in \mathcal{R}$ be an arbitrary element. Then we define,

$$w_{\text{hom}}(x) = \begin{cases} 0 & \text{if } x = 0, \\ 8 & \text{if } x = 4, \\ 4 & \text{otherwise.} \end{cases}$$

In order to obtain a distance preserving map θ from $\mathcal{R}$ to $\mathbb{F}_2^8$, we assume the following: $\theta(0) = (0, 0, 0, 0, 0, 0, 0, 0)$, $\theta(1) = (1, 0, 1, 0, 1, 0, 1, 0)$, $\theta(2) = (1, 1, 1, 1, 0, 0, 0, 0)$, $\theta(u) = (1, 1, 0, 0, 1, 1, 0, 0)$ and $\theta(4) = (1, 1, 1, 1, 1, 1, 1, 1)$. Now, we extend this map to the whole ring $\mathcal{R}$ in the following manner. Notice that any $x \in \mathcal{R}$ can be expressed uniquely $x = \alpha_1 + 2\alpha_2 + u\alpha_3 + 4\alpha_4$, where $\alpha_i \in \mathbb{F}_2$. Then, we define

$$\theta(\alpha_1 + 2\alpha_2 + u\alpha_3 + 4\alpha_4) = \alpha_1\theta(1) + \alpha_2\theta(2) + \alpha_3\theta(u) + \alpha_4\theta(4).$$

We can verify that the map $\theta : \mathcal{R} \to \mathbb{F}_2^8$ defined above is a distance preserving map (Gray map). Notice that $\theta(6 + 2) \neq \theta(6) + \theta(2)$, therefore we can conclude that the Gray map θ is not linear. Moreover, θ can be extended to the map $\theta_{\text{hom}} : \mathcal{R}^n \to \mathbb{F}_2^{8n}$ such that

$$\theta_{\text{hom}}(\beta_0, \beta_1, \beta_2, \ldots, \beta_{n-1}) = (\theta(\beta_0), \theta(\beta_1), \theta(\beta_2), \ldots, \theta(\beta_{n-1})),$$

where $(\beta_0, \beta_1, \beta_2, \dots, \beta_{n-1}) \in \mathcal{R}^n$. Recall that a cyclic shift on $\mathcal{R}^n$ is the permutation σ given by

$$\sigma(c_0, c_1, \dots, c_{n-1}) = (c_{n-1}, c_0, \dots, c_{n-2}).$$

A code C is said to be an ℓ-quasi-cyclic code if it is invariant under σ^ℓ and we call ℓ the index of the quasi-cyclic code.

Lemma 4.2. *Let σ be the cyclic shift. Then $\theta_{\text{hom}} \circ \sigma = \sigma^8 \circ \theta_{\text{hom}}$.*

Proof. Suppose an element $c = (c_0, c_1, \dots, c_{n-1}) \in \mathcal{R}^n$, then

$$\theta_{\text{hom}} \circ \sigma\Big((c_0, c_1, \dots, c_{n-1})\Big) = \Big(\theta(c_{n-1}), \theta(c_0), \theta(c_1), \dots, \theta(c_{n-2})\Big). \tag{4}$$

Also, we can write $\theta_{\text{hom}}(c_0, c_1, \dots, c_{n-1}) = (\theta(c_0), \theta(c_1), \theta(c_2), \dots, \theta(c_{n-1}))$, where each $\theta(c_i)$ is of length 8. Therefore, if we apply the cyclic shift eight times then we must have

$$\sigma^8 \circ \theta_{\text{hom}}\Big((c_0, c_1, \dots, c_{n-1})\Big) = \Big(\theta(c_{n-1}), \theta(c_0), \theta(c_1), \dots, \theta(c_{n-2})\Big). \tag{5}$$

Hence, by using (4) and (5) we get the desired result. □

Based on the above lemma we have the following result.

Theorem 4.3. *Let C be a cyclic code of length n and minimum homogeneous weight d over the ring $\mathcal{R}$. Then $\theta_{\text{hom}}(C)$ is an 8-quasi-cyclic binary code of length $8n$ and minimum distance d.*

Proof. Given that C is a cyclic code of length n over the ring $\mathcal{R}$. Then, we have $\sigma(C) = C$, where σ is the cyclic shift. Now, apply θ_{hom} to both sides we obtain

$$\theta_{\text{hom}}(\sigma(C)) = \theta_{\text{hom}}(C).$$

Consider the Gray image of a given cyclic code C as

$$\begin{aligned} \theta_{\text{hom}}(C) &= \theta_{\text{hom}}(\sigma(C)) \\ &= (\theta_{\text{hom}} \circ \sigma)(C). \end{aligned}$$

By making use of Lemma 4.2 we obtain

$$\theta_{\text{hom}}(C) = \sigma^8(\theta_{\text{hom}}(C)).$$

This implies that $\theta_{\text{hom}}(C)$ is invariant under σ^8. Therefore, $\theta_{\text{hom}}(C)$ is an 8-quasi cyclic binary code. □

5. The Reversible DNA codes over $\mathcal{R}$

In this section, we discuss the reversibility of cyclic codes of odd length over the ring $\mathcal{R}$. We provide necessary and sufficient condition for a given cyclic code to be reversible. Moreover, we study the reversible complement cyclic codes of odd length over the ring $\mathcal{R}$ and construct a bijection ϑ in such a manner that the reversibility problem is solved.

Definition 5.1. *A cyclic code C of length n over the ring $\mathcal{R}$ is called reversible if $\alpha^r \in C$ whenever $\alpha \in C$, where reverse of a codeword $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_{n-1})$ is given by $\alpha^r = (\alpha_{n-1}, \alpha_{n-2}, \dots, \alpha_0)$.*

The following theorem provides a necessary and sufficient condition for cyclic codes of odd lengths over the ring $\mathcal{R}$ to be reversible.

Theorem 5.2. *Let C be a cyclic code of odd length n over the ring $\mathcal{R}$ given by*

$$C = \langle \widehat{H_1}(x), N\widehat{H_2}(x), 2\widehat{H_3}(x), u\widehat{H_4}(x), (2+u)\widehat{H_5}(x), 4\widehat{H_6}(x)\rangle,$$

where $x^n - 1 = H_0(x)H_1(x)H_2(x)H_3(x)H_4(x)H_5(x)H_6(x)$; $H_i(x)$ are coprime monic polynomials and $\widehat{H_i}(x) = \dfrac{x^n-1}{H_i(x)}$, $0 \le i \le 6$. Then C is a reversible cyclic code if and only if each $\widehat{H_i}(x)$ for $1 \le i \le 6$, is a self-reciprocal polynomial.

Proof. Suppose that C is a reversible cyclic code of odd length n over the ring $\mathcal{R}$. We have to show that each $\widehat{H_i}(x)$ is self-reciprocal polynomial. To do so, let on contrary $\widehat{H_i}(x)$ is not a self-reciprocal polynomial i.e., $\widehat{H_i}(x) \neq \widehat{H_i}^*(x)$, where $i \in \{1,2,3,4,5,6\}$. Now, consider a polynomial $H(x)$ given by $H(x) = \gcd(\widehat{H_i}(x), \widehat{H_i}^*(x))$. There exist $\alpha_1(x)$ and $\alpha_2(x)$ such that

$$H(x) = \alpha_1(x)\widehat{H_i}(x) + \alpha_2\widehat{H_i}^*(x).$$

Let $P = \{N, 2, u, 2+u, 4\}$ be a set. Now, multiplying by $p \in P$ on both sides in the above expression we get

$$pH(x) = \alpha_1(x)p\widehat{H_i}(x) + \alpha_2(x)p\widehat{H_i}^*(x).$$

Since C is a reversible cyclic code, $pH(x) \in C$ for all $p \in P$. Notice that $\deg(pH(x)) < \deg(p\widehat{H_i}(x))$ for all $p \in P$ which contradicts the fact that $\widehat{H_i}(x)$ is the minimal degree polynomial such that $p\widehat{H_i}(x) \in C$. Hence, each $\widehat{H_i}(x)$ is a self-reciprocal polynomial. For the converse part assume that each $\widehat{H_i}(x)$, where $i \in \{1,2,3,4,5,6\}$ is a self-reciprocal polynomial over the ring $\mathcal{R}$. Let $c(x) \in C$ be an arbitrary polynomial. Then there exist $\lambda_i(x) \in \mathcal{R}[x]$, where $1 \le i \le 6$ such that

$$\begin{aligned} c(x) &= \lambda_1(x)\widehat{H_1}(x) + \lambda_2(x)N\widehat{H_2}(x) + \lambda_3(x)2\widehat{H_3}(x) + \lambda_4(x)u\widehat{H_4}(x) \\ &\quad + \lambda_5(x)(2+u)\widehat{H_5}(x) + \lambda_6(x)4\widehat{H_6}(x). \end{aligned}$$

By using Lemma 2.1, the reciprocal polynomial $c^*(x)$ is given by

$$\begin{aligned} c^*(x) &= \lambda_1^*(x)\widehat{H_1}^*(x) + x^{i_1}\lambda_2^*(x)N\widehat{H_2}^*(x) + x^{i_2}\lambda_3^*(x)2\widehat{H_3}^*(x) + x^{i_3}\lambda_4^*(x)u\widehat{H_4}^*(x) \\ &\quad + x^{i_4}\lambda_5^*(x)(2+u)\widehat{H_5}^*(x) + x^{i_5}\lambda_6^*(x)4\widehat{H_6}^*(x). \end{aligned}$$

Since each $\widehat{H_i}(x)$ is self-reciprocal i.e., $\widehat{H_i}(x) = \widehat{H_i}^*(x)$ and $1 \le i \le 6$, we obtain

$$\begin{aligned} c^*(x) &= \lambda_1^*(x)\widehat{H_1}(x) + x^{i_1}\lambda_2^*(x)N\widehat{H_2}(x) + x^{i_2}\lambda_3^*(x)2\widehat{H_3}(x) + x^{i_3}\lambda_4^*(x)u\widehat{H_4}(x) \\ &\quad + x^{i_4}\lambda_5^*(x)(2+u)\widehat{H_5}(x) + x^{i_5}\lambda_6^*(x)4\widehat{H_6}(x). \end{aligned}$$

This implies that $c^*(x) \in C$ for $\lambda_i^*(x) \in \mathcal{R}[x]$. Hence, C is a reversible cyclic code over the ring $\mathcal{R}$. □

The following result provides the reversibility conditions for cyclic codes C_i, where $i \in \{1,2,3\}$.

Theorem 5.3. *Let C_i be a cyclic code of odd length n over the ring $\mathcal{R}$ given by*

$$C_i = \langle g_0(x), Ng_1(x), \xi_i g_2(x), 4g_3(x)\rangle,$$

where $g_3(x)|g_2(x)|g_1(x)|g_0(x)$, $\xi_1 = 2$, $\xi_2 = u$ and $\xi_3 = 2+u$. Then C_i is a reversible cyclic code over the ring $\mathcal{R}$ if and only if $g_0(x), g_1(x), g_2(x)$ and $g_3(x)$ are self-reciprocal polynomials.

Proof. The proof is similar to the Theorem 5.2. □

The next result is useful to obtain the reversibility of the dual of cyclic codes C_i, where $i \in \{1,2,3\}$.

Theorem 5.4. *Let C_i be a reversible cyclic code over the ring $\mathcal{R}$ given by*

$$C_i = \langle g_0(x), Ng_1(x), \xi_i g_2(x), 4g_3(x)\rangle,$$

where $g_3(x)|g_2(x)|g_1(x)|g_0(x)$, $\xi_1 = 2$, $\xi_2 = u$ and $\xi_3 = 2+u$. Then $C_i^{\perp}$ is also a reversible cyclic code over the ring $\mathcal{R}$.

Proof. By making use of Theorem 5.3, we must have $g_0(x), g_1(x), g_2(x)$ and $g_3(x)$ are self-reciprocal polynomials. Suppose that $\frac{x^n-1}{g_3(x)} = s_1(x)$, $\frac{x^n-1}{g_2(x)} = s_2(x)$, $\frac{x^n-1}{g_1(x)} = s_3(x)$, and $\frac{x^n-1}{g_0(x)} = s_4(x)$. Notice that $\left(\frac{x^n-1}{g_3(x)}\right)^* = s_1^*(x)$, which further yields $-\frac{(x^n-1)}{g_3^*(x)} = s_1^*(x)$. Since $g_3(x)$ is self-reciprocal polynomial, we obtain $-\frac{(x^n-1)}{g_3(x)} = s_1^*(x)$ and $-s_1(x) = s_1^*(x)$. Similarly, we obtain $-s_2(x) = s_2^*(x)$, $-s_3(x) = s_3^*(x)$ and $-s_4(x) = s_4^*(x)$. Let $\bar{c}(x) \in C_i^{\perp}$ be an arbitrary codeword. By using Theorem 3.5 there exist polynomials $P_1(x), P_2(x), P_3(x)$ and $P_4(x)$ in $\mathcal{R}[x]$ such that

$$\bar{c}(x) = \left[P_1(x)\left(\frac{x^n-1}{g_3(x)}\right)^* + P_2(x)N\left(\frac{x^n-1}{g_2(x)}\right)^* + P_3(x)\xi_i\left(\frac{x^n-1}{g_1(x)}\right)^* + P_4(x)4\left(\frac{x^n-1}{g_0(x)}\right)^*\right].$$

Which on simplifying yields,

$$\bar{c}(x) = [P_1(x)s_1^*(x) + P_2(x)Ns_2^*(x) + P_3(x)\xi_i s_3^*(x) + P_4(x)4s_4^*(x)].$$

Now, consider the reciprocal polynomial of $\bar{c}(x)$

$$\bar{c}^*(x) = [P_1(x)s_1^*(x) + P_2(x)Ns_2^*(x) + P_3(x)\xi_i s_3^*(x) + P_4(x)4s_4^*(x)]^*.$$

Replace $s_i^*(x)$ by $-s_i(x)$ in above expression we obtain the following

$$\bar{c}^*(x) = [P_1(x)(-s_1(x)) + P_2(x)N(-s_2(x)) + P_3(x)\xi_i(-s_3(x)) + P_4(x)4(-s_4(x))]^*.$$

By using Lemma 2.1 we obtain

$$\bar{c}^*(x) = [-P_1^*(x)s_1^*(x) - x^{\alpha_1}P_2^*(x)Ns_2^*(x) - x^{\alpha_2}P_3^*(x)\xi_i s_3^*(x) - x^{\alpha_3}P_4^*(x)4s_4^*(x)].$$

Equivalently we can write

$$\bar{c}^*(x) = [s_1^*(x)q_1(x) + Ns_2^*(x)q_2(x) + \xi_i s_3^*(x)q_3(x) + 4s_4^*(x)q_4(x)],$$

where $q_1(x) = -P_1^*(x)$, $q_2(x) = -x^{\alpha_1}P_2^*(x)$, $q_3(x) = -x^{\alpha_2}P_3^*(x)$ and $q_4(x) = -x^{\alpha_3}P_4(x)$. Hence, $\bar{c}^*(x) \in C_i^{\perp}$ and which implies that $C_i^{\perp}$ is also a reversible cyclic code over the ring $\mathcal{R}$. □

Our current focus is to characterize the reversible-complement cyclic codes of odd lengths. We will establish a condition that is both necessary and sufficient for a cyclic code to be categorized as a reversible-complement cyclic code over the ring $\mathcal{R}$. To accomplish this, we discuss the foundational concepts of complement over the ring $\mathcal{R}$. We denote the complement of an element $z \in \mathcal{R}$ by $\bar{z}$ and define as: $z^c = z + 4$. For example, the complement of $z = 3+u$ is $z^c = 3+u+4 = 7+u$. Let $p(x) \in \mathcal{R}[x]$, where $p(x) = p_0 + p_1x + p_2x^2 + \cdots + p_{n-1}x^{n-1}$. Then the complement of $p(x)$ is defined as follows: $p^c(x) = p_0^c + p_1^cx + p_2^cx^2 + \cdots + p_{n-1}^cx^{n-1}$, where p_i^c denotes the complement of p_i for $p_i \in \mathcal{R}$, $0 \le i \le n-1$. The reverse-complement of $p(x) \in \mathcal{R}[x]$ is denoted by $p^{rc}(x)$ and is defined as: $p^{rc}(x) = p_{n-1}^c + p_{n-2}^cx + \cdots + p_0^cx^{n-1}$.

Table 1: DNA correspondence ϑ between $\mathcal{R}$ and $S_{D_{16}}$

Elements of the ring $\mathcal{R}$	$\alpha_1\alpha_2 \in S_{D_{16}}$
0	AA
2	AT
$3+u$	GT
3	CT
u	GG
$2+u$	GC
$1+u$	AC
1	AG
4	TT
6	TA
$7+u$	CA
7	GA
$4+u$	CC
$6+u$	CG
$5+u$	TG
5	TC

In Table 1, we establish a bijection ϑ between the elements of the ring $\mathcal{R}$ and $S_{D_{16}}$ to address the reversibility problem. To illustrate the reversibility problem, consider any codeword $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ over the ring $\mathcal{R}$. Let $ATCGGATG$ be the DNA sequence corresponding to $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$, where λ_1 corresponds to AT, λ_2 corresponds to CG, λ_3 corresponds to GA, and λ_4 corresponds to TG. The reverse of λ is denoted as $\lambda^r = (\lambda_4, \lambda_3, \lambda_2, \lambda_1)$, with the corresponding DNA sequence being $TGGACGAT$. It is noteworthy that $TGGACGAT$ is not the reverse of the DNA sequence $ATCGGATG$; rather, the reverse of $ATCGGATG$ is $GTAGGCTA$. We resolve this reversibility issue by employing the correspondence outlined in Table 1. Consequently, we present the following lemma, which plays a pivotal role in solving the reversibility problem.

Lemma 5.5. *Let $x = (x_1, x_2, \ldots, x_n)$ be any codeword over the ring $\mathcal{R}$. Also, suppose that $X = d_1d_2\cdots d_{2n-1}d_{2n}$ is the DNA sequence corresponding to the codeword x. Then, DNA sequence corresponding to the codeword $7x^r$ is the reverse of X i.e., $d_{2n}d_{2n-1}\cdots d_2d_1$.*

Proof. The above result can be easily verified by using Table 1. For example, suppose that $\lambda = (4+u, 6+u, 5+u, 5)$ be a codeword over the ring $\mathcal{R}$. According to Table 1, the DNA sequence corresponding to λ is $CCCGTGTC$. Now consider the codeword $7\lambda^r = (3, 3+u, 2+u, 4+u)$. By using Table 1 corresponding DNA sequence to the codeword $(3, 3+u, 2+u, 4+u)$ is $CTGTGCCC$. We can verify that $CTGTGCCC$ is the reverse of $CCCGTGTC$. □

Lemma 5.6. *If $x, y \in \mathcal{R}$, then the $\overline{x+y} = \bar{x} + \bar{y} + 4$.*

Proof. The above lemma can easily be verified by observing Table 1. □

Now, we provide a necessary and sufficient condition under which a given cyclic code C_i, where $i \in \{1, 2, 3\}$ over $\mathcal{R}$ of any odd length is a reversible complement cyclic code.

Theorem 5.7. *Let C_i be a cyclic code of odd length n over the ring $\mathcal{R}$ given by*

$$C_i = \langle g_0(x), Ng_1(x), \xi_i g_2(x), 4g_3(x)\rangle,$$

where $g_3(x)|g_2(x)|g_1(x)|g_0(x)$, $\xi_1 = 2$, $\xi_2 = u$ and $\xi_3 = 2+u$. Then C_i is a reversible complement cyclic code if and only if (*i*) *C_i is a reversible code;*

(*ii*) *the element* $\dfrac{4(x^n-1)}{x-1} \in C_i$.

Proof. First we prove the second condition. Assume that C_i is a reversible complement cyclic code over the ring $\mathcal{R}$. Since C_i is a reversible complement cyclic code, $\mathbf{0}^{rc} \in C_i$, where $\mathbf{0} = (0,0,0,\dots,0,0) \in C_i$. Now consider the codeword,

$$\begin{aligned}\mathbf{0}^{rc} = (\bar{0},\bar{0},\dots,\bar{0}) &= (4,4,\dots,4) \in C_i\\ &= 4(1,1,\dots,1,1) \in C_i\\ &= \frac{4(x^n-1)}{x-1} \in C_i,\end{aligned}$$

where $(1,1,\dots,1,1) \in C_i$ can be identified by $\frac{x^n-1}{x-1}$ in C_i.
For the first condition take any $c(x) \in C_i$. Then there exist polynomials $q_i(x) \in \mathcal{R}[x]$, $0 \le i \le 3$ such that

$$c(x) = [q_0(x)g_0(x) + Nq_1(x)g_1(x) + \xi_i q_2(x)g_2(x) + 4q_3(x)g_3(x)].$$

Then by using Lemma 2.1, the reciprocal polynomial $c^*(x)$ is given by

$$\begin{aligned}c^*(x) &= [q_0(x)g_0(x) + Nq_1(x)g_1(x) + \xi_i q_2(x)g_2(x) + 4q_3(x)g_3(x)]^*,\\ c^*(x) &= q_0^*(x)g_0^*(x) + x^{i_1}Nq_1^*(x)g_1^*(x) + x^{i_2}\xi_i q_2^*(x)g_2^*(x) + 4x^{i_3}q_3^*(x)g_3^*(x) \qquad (6)\end{aligned}$$

Let

$$g_0(x) = \beta_0' + \beta_1' x + \dots + \beta_{r-1}' x^{r-1} + \beta_r' x^r,$$

be a polynomial over $\mathcal{R}$. Then the reverse of $g_0(x)$ is given by

$$g_0^r(x) = \beta_r' + \beta_{r-1}' x + \dots + \beta_1' x^{r-1} + \beta_0' x^r,$$

and the reverse complement of $g_0(x)$ is given by

$$g_0^{rc}(x) = \bar{\beta}_r' + \bar{\beta}_{r-1}' x + \dots + \bar{\beta}_1' x^{r-1} + \bar{\beta}_0' x^r.$$

Now multiplying both sides of the above expression by x^{n-r-1} we obtain

$$\begin{aligned}(x^{n-r-1})g_0^{rc}(x) &= \bar{\beta}_r' x^{n-r-1} + \bar{\beta}_{r-1}' x^{n-r} + \dots + \bar{\beta}_1' x^{n-2} + \bar{\beta}_0' x^{n-1}\\ &= \bar{0} + \bar{0}x + \dots + \bar{0}x^{n-r-2} + \bar{\beta}_r' x^{n-r-1}\\ &\quad + \bar{\beta}_{r-1}' x^{n-r} + \dots + \bar{\beta}_1' x^{n-2} + \bar{\beta}_0' x^{n-1}\\ (x^{n-r-1})g_0^{rc}(x) &= 4 + 4x + \dots + 4x^{n-r-2} + \bar{\beta}_r' x^{n-r-1} + \bar{\beta}_{r-1}' x^{n-r}\\ &\quad + \dots + \bar{\beta}_1' x^{n-2} + \bar{\beta}_0' x^{n-1}. \qquad (7)\end{aligned}$$

On adding $\dfrac{4(x^n-1)}{x-1}$ in both sides of (7), we obtain

$$\begin{aligned}(x^{n-r-1})g_0^{rc}(x) + \frac{4(x^n-1)}{x-1} &= 0 + 0x + 0x^2 + \dots + 0x^{n-r-2} + \beta_r' x^{n-r-1}\\ &\quad + \beta_{r-1}' x^{n-r} + \dots + \beta_1' x^{n-2} + \beta_0' x^{n-1},\\ &= x^{n-r-1}[\beta_r' + \beta_{r-1}' x + \dots + \beta_0' x^r],\\ (x^{n-r-1})g_0^{rc}(x) + \frac{4(x^n-1)}{x-1} &= x^{n-r-1}g_0^*(x). \qquad (8)\end{aligned}$$

Therefore, using (8) we can conclude $g_0^*(x) \in C_i$. Similarly, $g_1^*(x), g_2^*(x)$ and $g_3^*(x) \in C_i$ and by using (6), we obtain $c^*(x) \in C_i$ for $q_i^*(x) \in \mathcal{R}[x]$. Thus, C_i is a reversible cyclic code over the ring $\mathcal{R}$.

For converse part, let $B(x) = b_0 + b_1x + \cdots + b_m x^m \in C$ be an arbitrary codeword. Then the reverse-complement of the polynomial $B(x)$ is given by

$$B^{rc}(x) = \bar{b}_m + \bar{b}_{m-1}x + \cdots + \bar{b}_1 x^{m-1} + \bar{b}_0 x^m,$$

multiplying by x^{n-m-1} on both sides in the above equation, we obtain

$$\begin{aligned}(x^{n-m-1})B^{rc}(x) &= \bar{b}_m x^{n-m-1} + \bar{b}_{m-1}x^{n-m} + \cdots + \\ &\quad \bar{b}_1 x^{n-2} + \bar{b}_0 x^{n-1}.\end{aligned}$$

On adding $\frac{4(x^n-1)}{x-1}$ in both sides of the above expression, we obtain

$$\begin{aligned}(x^{n-m-1})B^{rc} + \frac{4(x^n-1)}{x-1} &= x^{n-m-1}[b_m + b_{m-1}x + \cdots + b_0 x^m] \\ &= x^{n-m-1}B^*(x).\end{aligned}$$

Now, on simplifying the above expression we find that

$$(x^{n-m-1})B^{rc}(x) = x^{n-m-1}B^*(x) - \frac{4(x^n-1)}{x-1}. \tag{9}$$

Utilizing conditions (*i*) and (*ii*) in theorem with equation (9) leads to the conclusion that $B^{rc}(x)$ belongs to C. Consequently, we must have C is a reversible-complement cyclic code over the ring $\mathcal{R}$. □

6. The *GC*-content and the Deletion distance *D*

The DNA is a long chain of nucleotides. Each base binds with another complementary base with hydrogen bonds. For example, the A forms two hydrogen bonds with the T, vice versa and G forms three hydrogen bonds with the C, vice versa. The percentage of the Guanine (G) and the Cytosine (C), nitrogenous bases present in a DNA, nucleic acid sequence is known as the GC-content. Moreover, in DNA codes the same GC-content in every codeword ensures that the codewords have similar hybridization energy and melting temperature. This motivates to consider such DNA codes in which all codewords have the same GC-content. In this section, we study the GC-content of DNA cyclic codes over the ring $\mathcal{R}$ and deletion distance D of such DNA codes.

Definition 6.1. *Let C be a cyclic code over the ring $\mathcal{R}$. Then Hamming weight enumerator of C is denoted by $W_C(x)$ and is define as follows:*

$$W_C(x) = \sum_k B_k x^k,$$

where $B_k = |\{c : w(c) = k\}|$ i.e., the number of codewords in C whose Hamming weights equal to k. By making use of the Hamming weight enumerator, we can identify the minimum distance of the code as the smallest non-zero exponent of x with a non-zero coefficient in $W_C(x)$.

Let $C_i = \langle g_0(x), Ng_1(x), \xi_i g_2(x), 4g_3(x)\rangle$ be a cyclic code of odd length n over the ring $\mathcal{R}$. Then we define a subcode $C_i^{(4)}$ of C_i such that $C_i^{(4)}$ consist of all codewords in C_i which are multiples of 4. The next result provides the structure of $C_i^{(4)}$.

Theorem 6.2. *Let C_i be a cyclic code of odd length n over the ring $\mathcal{R}$ given by*

$$C_i = \langle g_0(x), Ng_1(x), \xi_i g_2(x), 4g_3(x)\rangle,$$

where $g_3(x)|g_2(x)|g_1(x)|g_0(x)$, $\xi_1 = 2$, $\xi_2 = u$ and $\xi_3 = 2+u$. Then $C_i^{(4)} = \langle 4g_3(x)\rangle$.

Proof. Notice that $\langle 4g_3(x)\rangle \subseteq C_i^{(4)}$. For the reverse inclusion, $C_i^{(4)} \subseteq \langle 4g_3(x)\rangle$ we proceed as follows. Let $c(x) \in C_i$ be any codeword. Then there exist polynomials $a_i(x), b_i(x) \in \mathcal{R}[x]$ such that

$$\begin{aligned} c(x) &= g_0(x)a_1(x) + Ng_1(x)a_2(x) + \xi_i g_2(x)a_3(x) + 4g_0(x)b_1(x) \\ &\quad +4Ng_1(x)b_2(x) + 4\xi_i g_2(x)b_3(x) + 4g_3(x)a_4(x). \end{aligned}$$

If $c(x)$ is a multiple of 4, then we must have $x^n - 1 | g_0(x)a_1(x)$, $x^n - 1|Ng_1(x)a_2(x)$ and $x^n - 1|\xi_i g_2(x)a_3(x)$ and hence we obtain

$$c(x) = 4g_0(x)b_1(x) + 4Ng_1(x)b_2(x) + 4\xi_i g_2(x)b_3(x) + 4g_3(x)a_4(x).$$

Since $g_3(x)|g_2(x)|g_1(x)|g_0(x)$, $c(x) \in \langle 4g_3(x)\rangle$. Therefore, we can conclude $C_i^{(4)} \subseteq \langle 4g_3(x)\rangle$ and hence $C_i^{(4)} = \langle 4g_3(x)\rangle$. □

The next theorem provides the spectra of the GC-content of cyclic codes C_i, where $i \in \{1,2,3\}$ of any odd length over the ring $\mathcal{R}$.

Theorem 6.3. *Let C_i be a cyclic code of odd length n over the ring $\mathcal{R}$ given by*

$$C_i = \langle g_0(x), Ng_1(x), \xi_i g_2(x), 4g_3(x)\rangle,$$

where $g_3(x)|g_2(x)|g_1(x)|g_0(x)$, $\xi_1 = 2$, $\xi_2 = u$ and $\xi_3 = 2+u$. Let $c \in C_i$ be any codeword such that $w(c) = W_H(\bar{c})$, $\bar{c} \in \mathbb{Z}_8[x]$. Then all possible spectra of the GC-content of C_i are determined by the Hamming weight enumerator of the code generated by $g_3(x)$.

Proof. Notice that GC-contents of $\mathfrak{C}_i$ can be obtained by multiplying the elements of $\mathfrak{C}_i$ by 4. By using the structure of $\mathfrak{C}_i^{(4)}$, we can conclude that the Hamming weight enumerator generated by the polynomial $g_3(x)$ provides the spectra of the GC-content. □

By using the following theorem one can obtain the deletion distance of the subcode $C_i^{(4)}$.

Theorem 6.4. *Let $C_i = \langle g_0(x), Ng_1(x), \xi_i g_2(x), 4g_3(x)\rangle$ be an (n, D) cyclic code over the ring $\mathcal{R}$, where n is any odd integer and D_4 be the deletion distance of the code $C_i^{(4)}$. Then $D = D_4$.*

Proof. Observe that $C_i^{(4)} \subseteq C_i$, then for any $A, B \in C_i$ we obtain $S(A, B) \leq n - D - 1$ and hence $D_4 \geq D$. Let $A_1, A_2 \in C_i$ be two codewords such that $S(A_1, A_2) > n - D_4 - 1$. Since $A_1, A_2 \in C_i$ by making use of Theorem 6.2, we obtain $4A_1$ and $4A_2$ are two codewords in $C_i^{(4)}$. Therefore, we can conclude that $S(4A_1, 4A_2) \geq S(A_1, A_2) > n - D_4 - 1$, a contradiction. Hence $D = D_4$. □

7. examples

In this section, we present some examples of reversible cyclic codes over the ring $\mathcal{R}$ and their ϑ images are DNA codes which satisfying reversible and reversible complement constraint (see Tables 2, 3 & 4). In Table 5, we provide some examples of reversible cyclic codes and their Gray images.

Example 7.1. *Consider the factorization of the polynomial $x^3 - 1$ over the ring $\mathbb{Z}_8$ as follows:*

$$\begin{aligned} x^3 - 1 &= (x-1)(x^2+x+1) \\ &= a_1(x)a_2(x). \end{aligned}$$

Let C be a cyclic code of length 3 over the ring $\mathcal{R}$ given by $C = \langle a_2(x)\rangle$. Notice that C satisfies all conditions of Theorem 5.3. Hence, C is a reversible cyclic code of length 3 over the ring $\mathcal{R}$. Since $\frac{4(x^3-1)}{x-1} = 4g_0(x) \in C$, C is a reversible complement code over the ring $\mathcal{R}$. Observe that in Table 2, we have $A^{rc} \in C$ for all $A \in C$. Also, $S(A, B) \in \{0, 3\}$ for all $A, B \in C$ and $A \neq B$. Moreover, $\vartheta(C)$ is a DNA code of length 6 and $S(A, B) \leq 3$ and we obtain $D = 2$. Hence, $\vartheta(C)$ is a $(6, 2)$ DNA cyclic code.

Example 7.2. *Consider the factorization of the polynomial $x^{13}-1$ over the ring $\mathbb{Z}_8$ as follows:*

$$x^{13}-1 \quad = \quad (x-1)(\sum_{i=0}^{i=12} x^i).$$

Let C be a cyclic code of length 13 over the ring $\mathcal{R}$ given by $C = \langle g_0(x)\rangle$, where $g_0(x) = \sum_{i=0}^{i=12} x^i$. Clearly, $g_0(x)$ is a self-reciprocal polynomial. Since C satisfies the Theorem 5.3, C is reversible cyclic code of length 13 over the ring $\mathcal{R}$. Moreover, $\frac{4(x^{13}-1)}{x-1} = 4g_0(x) \in C$, then by using Theorem 5.7, C is a reversible complement code over the ring $\mathcal{R}$. Observe that in Table 3, we have $A^{rc} \in C$ for all $A \in C$. Notice that $S(A,B) \in \{0,13\}$ for all $A,B \in C$ and $A \neq B$. Moreover, $\vartheta(C)$ is a DNA code of length 26 and $S(A,B) \leq 13$ and we obtain $D = 12$. Hence, $\vartheta(C)$ is a $(26,12)$ DNA cyclic code.

Example 7.3. *Consider the factorization of the polynomial $x^{15}-1$ over the ring $\mathbb{Z}_8$ as follows:*

$$\begin{aligned} x^{15}-1 &= (x-1)(x^2+x+1)(x^4+4x^3+6x^2+3x+1)(x^4+3x^3+6x^2+4x+1) \\ &\quad (x^4+x^3+x^2+x+1) \\ &= a_1(x)a_2(x)a_3(x)a_4(x)a_5(x). \end{aligned}$$

Let C be a cyclic code of length 15 over the ring $\mathcal{R}$ given by $C = \langle 4g_0(x)\rangle$, where $g_0(x) = a_2(x)a_5(x)$. Since $a_2(x)$ and $a_5(x)$ both are self-reciprocal polynomial, $g_0(x)$ is also a self-reciprocal polynomial. Hence, C is a reversible cyclic code of length 15 over the ring $\mathcal{R}$.

Example 7.4. *Consider the factorization of the polynomial $x^{17}-1$ over the ring $\mathbb{Z}_8$ as follows:*

$$\begin{aligned} x^{17}-1 &= (x-1)(x^8+4x^7+6x^6+7x^5+x^4+7x^3+6x^2+4x+1) \\ &\quad (x^8+5x^7+7x^6+3x^4+7x^2+5x+1) \\ &= a_1(x)a_2(x)a_3(x). \end{aligned}$$

Let $g_0(x) = a_2(x)a_3(x)$, $g_2(x) = a_3(x)$ be two polynomials. Since $a_2(x)$ and $a_3(x)$ are self-reciprocal polynomials, $g_0(x)$ and $g_2(x)$ are self-reciprocal polynomials.

- (*i*) *Let C be a cyclic code of length 17 over the ring $\mathcal{R}$ given by $C = \langle 4g_2(x)\rangle$. Notice that C satisfying all conditions of Theorem 5.3. Hence, C is a reversible cyclic code of length 17 over the ring $\mathcal{R}$.*
- (*ii*) *Let $C = \langle g_0(x)\rangle$ be a cyclic code of length 17. Notice that C satisfying all conditions of Theorem 5.3. Hence, C is a reversible cyclic code of length 17 over the ring $\mathcal{R}$. Since $\frac{4(x^{17}-1)}{x-1} = 4g_0(x) \in C$, C is a reversible complement code over the ring $\mathcal{R}$. Thus $\vartheta(C)$ is a DNA code of length 34 shown in Table 4.*

Table 2: DNA code of length 6 corresponding to Example 7.1

GAGAGA	*GGGGGG*
GTGTGT	*GCGCGC*
CACACA	*CGCGCG*
CTCTCT	*CCCCCC*
AAAAAA	*AGAGAG*
TGTGTG	*TCTCTC*
ATATAT	*TTTTTT*
ACACAC	*TATATA*

Table 3: DNA code of length 26 corresponding to Example 7.2

TGTGTGTGTGTGTGTGTGTGTGTGTG
CGCGCGCGCGCGCGCGCGCGCGCGCG
ATATATATATATATATATATATATAT
AGAGAGAGAGAGAGAGAGAGAGAGAG
CTCTCTCTCTCTCTCTCTCTCTCTCT
CCCCCCCCCCCCCCCCCCCCCCCCCC
AAAAAAAAAAAAAAAAAAAAAAAAAA
TTTTTTTTTTTTTTTTTTTTTTTTTT
GAGAGAGAGAGAGAGAGAGAGAGAGA
GGGGGGGGGGGGGGGGGGGGGGGGGG
GTGTGTGTGTGTGTGTGTGTGTGTGT
GCGCGCGCGCGCGCGCGCGCGCGCGC
CACACACACACACACACACACACACA
TCTCTCTCTCTCTCTCTCTCTCTCTC
ACACACACACACACACACACACACAC
TATATATATATATATATATATATATA

Table 4: DNA code of length 34 corresponding to Example 7.4 (*ii*)

GGGGGGGGGGGGGGGGGGGGGGGGGGGGGGGGGG
GCGCGCGCGCGCGCGCGCGCGCGCGCGCGCGCGC
TCTCTCTCTCTCTCTCTCTCTCTCTCTCTCTCTC
ACACACACACACACACACACACACACACACACAC
CGCGCGCGCGCGCGCGCGCGCGCGCGCGCGCGCG
AGAGAGAGAGAGAGAGAGAGAGAGAGAGAGAGAG
TGTGTGTGTGTGTGTGTGTGTGTGTGTGTGTGTG
ATATATATATATATATATATATATATATATATAT
CTCTCTCTCTCTCTCTCTCTCTCTCTCTCTCTCT
AAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAA
GAGAGAGAGAGAGAGAGAGAGAGAGAGAGAGAGA
GTGTGTGTGTGTGTGTGTGTGTGTGTGTGTGTGT
CACACACACACACACACACACACACACACACACA
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
TTTTTTTTTTTTTTTTTTTTTTTTTTTTTTTTTT
TATATATATATATATATATATATATATATATATA

Table 5: Gray images of reversible cyclic codes

Length	Generator(s) of cyclic code	Gray Images
3	$\langle x^2+x+1\rangle$	$(24, 2^4, 12)_2^*$
5	$\langle x^4+x^3+x^2+x+1\rangle$	$(40, 2^4, 20)_2^*$
9	$\langle x^2+x+1\rangle$	$(72, 2^{28}, 16)_2$
9	$\langle u(x^2+x+1)(x^6+x^3+1)\rangle$	$(72, 2^4, 36)_2$
15	$\langle 4(x^2+x+1)(x^4+x^3+x^2+x+1)\rangle$	$(120, 2^{36}, 32)_2^*$
17	$\langle 4(x^8+5x^7+7x^6+3x^4+7x^2+5x+1)\rangle$	$(136, 2^{36}, 40)_2$
17	$\langle (x^8+4x^7+6x^6+7x^5+x^4+7x^3+6x^2+4x+1)$ $(x^8+5x^7+7x^6+3x^4+7x^2+5x+1)\rangle$	$(136, 2^4, 68)_2$

REMARK: All codes in the above examples are calculated using Magma software [7] and * delineates that the corresponding binary linear code is optimal with respect to the online data available in [16].

8. Conclusion

Establishing a bijection between the components of an algebraic structure, such as fields or rings, and DNA k-bases poses a substantial challenge. This complexity is particularly giving rise to the reversibility problem when the algebraic structures consist of exactly 4^k elements (where $k > 1$), In this paper, we tackle this challenge by introducing a bijection denoted as ϑ, identifying the elements of the ring $\mathcal{R}$ to $S_{D_{16}}$ (see Table 1). This construction adeptly resolves the reversibility problem occurred in DNA 2-bases. Also, we have discussed DNA codes of odd length based on the deletion distance. Cyclic codes over the ring $\mathcal{R}$ such that they satisfy reversible and reversible complement constraint are studied. We have also established a relation between the GC-content of a given cyclic code C_i and its subcode $C_i^{(4)}$. Also, we have introduced a Gray map θ_{hom} from ($\mathcal{R}^n$, Homogeneous weight) to ($\mathbb{F}_2^{8n}$, Hamming weight). As an application of θ_{hom}, we have provided some examples of reversible cyclic codes over the ring $\mathcal{R}$ such that their corresponding binary linear codes are optimal. In the end, we have provided examples of DNA cyclic codes including their deletion distances. For future work, it would be further fascinating to construct reversible cyclic codes of even length over the ring $\mathcal{R}$.

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