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Nowhere-zero flows in signed grid graphs

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Abstract. In 1983, Bouchet conjectured that every flow-admissible signed graph admits a nowhere-zero 6-flow. This conjecture remains unresolved even for signed planar graphs. In this paper, we prove that any flow-admissible signed grid graph, which is a special class of planar graphs, admits a nowhere-zero 6-flow.

1. Introduction

The graphs in this paper are finite, without loops and may have multiple edges. Set $[a, b] = \{x \in \mathbb{Z} : a \leq x \leq b\}$. Terminology and notations not defined here can be found in [1, 11]. Tutte [7, 8] introduced the theory of integer flows as a dual problem of the face coloring of bridgeless plane graphs. The concept of integer flow on signed graphs arises from the study of graphs embedded on nonorientable surfaces, where a nowhere-zero flow serves as the dual of local tension. Bouchet [2] put forward the following conjecture.

Conjecture 1.1. *[2] Every flow-admissible signed graph admits a nowhere-zero 6-flow.*

This conjecture is known as Bouchet's 6-flow conjecture. It was provided by Bouchet [2] that every flow-admissible signed graph admits a nowhere-zero 216-flow, and Zýka [12] decreased the value to nowhere-zero 30-flow. Recently, DeVos et al. [3] improved the result of Zýka to nowhere-zero 11-flow. Integer flows on signed graphs have been studied for some specific graphs. Kaiser et al. [4] proved that every flow-admissible signed series-parallel graph admits a nowhere-zero 6-flow. Note that every signed outerplanar graph, which is a special class of signed planar graphs, is a signed series-parallel graph. This implies that Bouchet's 6-flow conjecture on signed outerplanar graphs has already been verified. Motivated by this result, we verify the Bouchet's 6-flow conjecture for signed grid graphs in this paper, which is a type of signed planar graphs.

Theorem 1.2. *If $(G_{n,m}, \sigma)$ is a flow-admissible signed grid graph, then $(G_{n,m}, \sigma)$ admits a nowhere-zero 6-flow.*

The present paper is organized as follows: In Section 2, we give some notations and terminology. We prove some useful lemmas in Section 3. The proof of Theorem 1.2 is presented in Section 4.

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2. Notation and terminology

A signed graph (G, σ) is an underlying graph G together with a *signature* $\sigma : E(G) \to \{\pm 1\}$. An edge e is *positive* if $\sigma(e) = 1$; otherwise it is *negative*. Let H be a signed subgraph of a signed graph G. The *sign* of H is the product of the signs of all edges in H. Additionally, a circuit is *positive (resp., negative)* if the product of the signs of all its edges equals 1 (resp., -1). For a vertex $v \in V(G)$, *switching* at v means that reversing the signs of all edges incident with v. If a signed graph (G, σ) can be transformed into (G, σ') through a series of switchings, then (G, σ) and (G, σ') are said to be *equivalent*.

Let C_n (resp., P_n) be the circuit (resp., path) with n vertices. The *length* of a path or a circuit is the number of its edges. A circuit is *balanced* if it contains an even number of negative edges; otherwise, it is *unbalanced*. If all circuits of a signed graph are balanced, then the signed graph is *balanced*; otherwise, it is *unbalanced*.

The following lemma characterizes the equivalence of two signed graphs in terms of the signs of their circuits.

Lemma 2.1. *[10] Let G be a graph. (G, σ) and (G, σ') are equivalent if and only if every circuit of G has the same sign in (G, σ) and (G, σ').*

An edge $e \in E(G)$ with two ends u and v is considered as two half edges, denoted as h_e^u and h_e^v, where h_e^u is incident to vertex u and h_e^v is incident to vertex v. Let $H_G(v)$ (or simply $H(v)$ if no confusion arises) represent the set of all half edges incident with vertex v, and let $H(G)$ denote the set of all half edges of (G, σ). An *orientation* of (G, σ) is a mapping $\tau : H(G) \to \{\pm 1\}$ such that for every $e \in E(G)$, $\tau(h_e^u)\tau(h_e^v) = -\sigma(e)$ for each $\tau(h_e^u) \in H(G)$. The τ is an assignment of orientations on $H(G)$ such that $\tau(h_e^u) = 1$ if h_e^u is oriented away from u and $\tau(h_e^u) = -1$ if h_e^u is oriented toward u.

The definition of an (integer) k-flow on a signed graph is as follows.

Definition 2.2. *Let (G, σ) be a signed graph with an orientation τ, and let f be a mapping $f : E(G) \to \mathbb{Z}$. The pair (τ, f) (or simply f) is called as an (integer) k-flow of (G, σ) if for each $v \in V(G)$, $\sum_{h \in H_G(v)} \tau(h) f(e_h) = 0$ and for each $e \in E(G)$, $|f(e)| < k$.*

Let f be a k-flow of (G, σ). The set $\{e \in E(G) : f(e) \neq 0\}$, denoted by *supp*$(f)$, is called the *support* of f. If *supp*$(f) = E(G)$, then the flow f is a *nowhere-zero k-flow*. A signed graph is *flow-admissible* if it admits a nowhere-zero k-flow for a positive integer k. For simplicity of presentation, the notation of nowhere-zero k-flow is abbreviated as k-NZF.

Note that if (G, σ) and (G, σ') are equivalent, then (G, σ) admits a k-NZF if and only if (G, σ') admits a k-NZF. Let (G, σ') be the signed graph that obtained by switching at $v \in V(G)$. If (G, σ) admits a k-NZF φ, then the orientations of the half edges $h_{e_1}^v, h_{e_2}^v, \ldots, h_{e_r}^v$ are opposite. Meanwhile, $\varphi(e_i)$ will not be changed, where $i \in [1, r]$. Therefore, φ is a k-NZF of (G, σ').

Let G be a graph and $v \in V(G)$. Define $E_G(v) = \{e \in E(G) : \text{e is incident with } v\}$ and let $F \subset E_G(v)$. *Splitting* the edges of F away from v, adding a new vertex v' and changing the end v of the edges of F to be v'. For a vertex $v \in V(G)$, the degree of v is denoted as $d_G(v)$ in G. For (G, σ), *suppressing* an induced path is replacing it with an edge e with same sign. In a signed graph (G, σ), the *subdivision* of an edge uv refers to the operation of inserting a new vertex w into the edge uv, thereby replacing uv with a path uwv, and the sign of the path is the same as the edge uv. A *subdivision* of a signed graph (G, σ) is a signed graph obtained from (G, σ) by performing a sequence of subdivisions.

Remark 2.3. *For a signed graph (G, σ), let (G^1, σ^1) be a signed graph obtained from (G, σ) by a series of splittings, and let (G', σ') be a signed graph obtained by suppressings of some induced paths in (G^1, σ^1). We claim that for a positive integer k, if (G', σ') admits a k-NZF, then (G, σ) admits a k-NZF. Let (τ, f) be a k-NZF on (G', σ'). Note that (G^1, σ^1) can be obtained from (G', σ') by a sequence of subdivisions. For any edge uv in G', there is an induced path P_{uv} in G^1, where P_{uv} is obtained from uv by subdivisions or $E(P_{uv}) = \{uv\}$. Thus, we can extend the flow (τ, f) to a k-NZF on (G^1, σ^1), as Figure 1. Since we can perform switchings on the internal vertices of P_{uv}, it follows that the assignment of signs to the edges within P_{uv} is irrelevant.*

We denote by (τ_1, f_1) the k-NZF on (G^1, σ^1). Let $S_v \subset V(G^1)$ be the set of all vertices obtained by splitting the vertex v and v itself. Then (τ_1, f_1) can be extended to a k-NZF on (G, σ), as follows. We identify all the vertices of

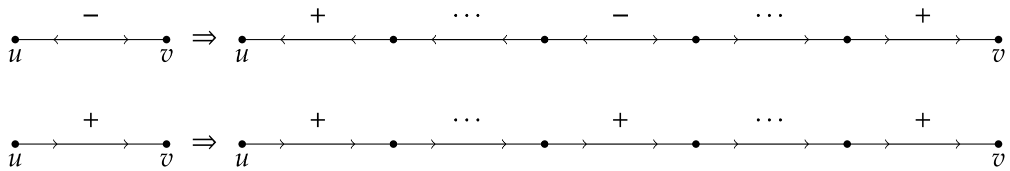

Figure 1: Extending (τ, f) from uv to P_{uv}.

S_v to be the original vertex v in (G,σ). For any vertex v' in S_v, since (τ_1, f_1) is a k-NZF on (G^1, σ^1), it follows that $\sum_{h\in H_{G^1}(v')} \tau_1(h) f_1(e_h) = 0$. Note that $E(G) = E(G^1)$ and $\sigma(e) = \sigma^1(e)$ for all $e \in E(G)$. Thus, (τ_1, f_1) is also defined on (G,σ). Because $\sum_{h\in H_G(v)} \tau(h) f(e_h) = \sum_{v'\in S_v} \sum_{h\in H_{G^1}(v')} \tau_1(h) f_1(e_h) = 0$, it follows that (τ_1, f_1) is a k-NZF on (G,σ).

Let $G_{n,m} = (V, E)$ be the *grid graph* with vertex set

$$V(G_{n,m}) = \{(i, j) : 1 \le i \le n, 1 \le j \le m\}$$

and edge set

$$E(G_{n,m}) = \{\{(i, j), (i', j')\} : |i - i'| + |j - j'| = 1\}.$$

For $G_{n,m}$, since the flow-admissible $(G_{2,2}, \sigma)$ is a balanced circuit of length 4 that admits a 2-NZF, we always assume $n > 2$ or $m > 2$.

In this paper, for a planar graph, we assume that it has already been embedded on the plane. A subset of the plane is arcwise-connected if any two of its points can be connected by a curve lying entirely within the subset. After embedding the vertices and edges of a planar graph G in the plane, G partitions the rest of the plane into a number of arcwise-connected open sets. These open sets are called the *faces* of G. A simple closed curve C partitions the plane into two open sets: the *interior region*, denoted by *int*(C), and the *exterior region* of C, denoted by *ext*(C), where exterior region contains the unbounded face of the plane graph. Since a circuit C^0 of a plane graph is a simple closed curve, saying that something is "inside" the circuit C^0 means it lies in int(C^0). The *boundary* of a face is the boundary of the open set in the usual topological sense. For simplicity, the boundary of a face is regarded as a subgraph induced by all edges embedded in the boundary of that face. The boundary of a face f^0 or a region F^0 is denoted by $\partial(f^0)$ or $\partial(F^0)$. In this paper, a graph is *nonseparable* if it is connected and has no cut vertices. Note that in a nonseparable plane graph other than K_1 or K_2, each face is bounded by a circuit [9], and grid graphs are nonseparable plane graphs.

A face is *positive* if it has an even number of negative edges in $\partial(f^0)$, otherwise, it is *negative*. For a balanced grid graph, by the 6-flow theorem of Seymour [6], it admits a 6-NZF. In fact, it admits a 3-NZF, as it is face-3-colorable. Therefore, we focus on the case that $(G_{n,m}, \sigma)$ is unbalanced.

3. Lemmas

Máčajová et al. [5] characterized the flow-admissible signed 2-edge-connected graph. Since every connected nonseparable plane graph is 2-edge-connected, their result is crucial for our conclusion.

Lemma 3.1. *[5]An unbalanced signed 2-edge-connected graph is flow-admissible except in the case when it contains an edge whose removal leaves a balanced graph.*

For a connected signed nonseparable plane graph, we have the following lemma to determine whether a circuit is balanced or unbalanced.

Lemma 3.2. *Let (G, σ) be a signed connected nonseparable plane graph, and C be a signed circuit in (G, σ). Then C is balanced if and only if there are even number of negative faces in int(C).*

Proof. Let $\mathcal{C}_k = \{C$ is a signed circuit in (G, σ) : the number of faces in int(C) is $k\}$. Since (G, σ) is a signed plane graph, it follows that C is laid out as a simple closed curve, and every face of (G, σ) is either entirely inside int(C) or entirely outside int(C). We denote the number of negative faces in int(C) by $f_N(C)$, where $C \in \mathcal{C}_k$. We proceed by induction on k. For an arbitrary circuit $C \in \mathcal{C}_k$, if $k = 1$ and C is balanced, then $f_N(C) = 0$. If $f_N(C)$ is even and $k = 1$, then $f_N(C) = 0$. Therefore, C is balanced.

Suppose that it holds for less than $k + 1$. If $\mathcal{C}_{k+1} = \emptyset$, then the result holds. If $\mathcal{C}_{k+1} \neq \emptyset$, then choose an arbitrary circuit C' in $\mathcal{C}_{k+1}$. Consider an *x,y-path* inside C', the *x,y-path* intersects only at the endpoints with C'. Since (G, σ) is a plane graph, *x,y-path* divides int(C′) into two interior regions F_1 and F_2 with boundaries $\partial(F_1)$ and $\partial(F_2)$, respectively , where $\partial(F_1)$ and $\partial(F_2)$ are circuits.

Note that C' is balanced if and only if there is an even number of negative edges in C'. Since $E(C') = (E(\partial(F_1)) \cup E(\partial(F_2))) \setminus (E(\partial(F_1)) \cap E(\partial(F_2)))$, it follows that C' is balanced if and only if both $\partial(F_1)$ and $\partial(F_2)$ are either balanced or unbalanced. By the inductive hypothesis, both $\partial(F_1)$ and $\partial(F_2)$ are balanced if and only if $f_N(\partial(F_1))$ and $f_N(\partial(F_2))$ are even. Similarly, both $\partial(F_1)$ and $\partial(F_2)$ are unbalanced if and only if $f_N(\partial(F_1))$ and $f_N(\partial(F_2))$ are odd. In either case, we have $f_N(C') = f_N(\partial(F_1)) + f_N(\partial(F_2))$ is even. Therefore, C' is balanced if and only if there are even number of negative faces in int(C). □

According to Lemma 3.2, if all faces in a connected nonseparable plane graph (G, σ) are positive, then (G, σ) is balanced. In fact, we have the following stronger lemma.

Lemma 3.3. *Let (G, σ) be a connected signed nonseparable plane graph. If the boundaries of all other faces are positive except for the unbounded face, then (G, σ) is balanced.*

Proof. For any circuit C in (G, σ), the boundaries of all faces inside C are positive. By Lemma 3.2, the circuit C is balanced. Therefore, (G, σ) is balanced. □

For an unbalanced flow-admissible $(G_{n,m}, \sigma)$, the following lemma shows that there are two negative faces whose boundaries are edge-disjoint.

Lemma 3.4. *An unbalanced flow-admissible $(G_{n,m}, \sigma)$ has two edge-disjoint negative circuits, which are boundaries of two faces.*

Proof. We denote the boundary of unbounded face of $(G_{n,m}, \sigma)$ by C. Note that the boundaries of other faces in $(G_{n,m}, \sigma)$ are circuits of length 4. Conversely, every cycle of length 4 is the boundary of some face. By Lemma 3.3, since $(G_{n,m}, \sigma)$ is unbalanced, there is at least one negative circuit of length 4 in $(G_{n,m}, \sigma)$. We will complete the proof by considering the number of negative circuit of length 4 in $(G_{n,m}, \sigma)$, denoted by $nc(G_{n,m})$.

Case 1. $nc(G_{n,m}) = 1$.

We denote the only negative circuit of length 4 by C^1. By Lemma 3.2, C is unbalanced. If $E(C^1) \cap E(C) = \emptyset$, then C and C^1 are edge-disjoint negative circuits. If $E(C^1) \cap E(C) \neq \emptyset$, then $|E(C^1) \cap E(C)| \in \{1, 2, 3\}$. If $|E(C^1) \cap E(C)| = 1$, then delete the common edges of C and C^1. Let the new signed graph be (G, σ). In this situation, except for the unbounded face of (G, σ), all other faces are positive. By Lemma 3.3, (G, σ) is balanced. According to Lemma 3.1, this is a contradiction with the fact that $(G_{n,m}, \sigma)$ is flow-admissible.

If $|E(C^1) \cap E(C)| = 2$ or 3, then $E(C^1) \cap E(C)$ induces an induced path P of $G_{n,m}$, by the structure of $G_{n,m}$. Delete one edge of $E(C^1) \cap E(C)$, and denote the resulting signed graph by (G, σ). Note that there are cut edges in (G, σ). We delete these cut edges, and the resulting signed graph is denoted by (G', σ'). Since all faces in (G', σ') except the unbounded face are positive, Lemma 3.3 implies that (G', σ') is balanced. Because cut edges are not included in any circuit and every circuit of (G', σ') is balanced, it follows that every circuit of (G, σ) is balanced. Thus, (G, σ) is balanced. According to Lemma 3.1, $(G_{n,m}, \sigma)$ is not flow-admissible, which leads to a contradiction.

Case 2. $nc(G_{n,m}) = 2$.

Let C^2 and C^3 be the two negative circuits of length 4. By Lemma 3.2, C is balanced. If $E(C^2) \cap E(C^3) = \emptyset$, then C^2 and C^3 are the edge-disjoint negative boundaries of two faces. If $E(C^2) \cap E(C^3) \neq \emptyset$, then C^2 and C^3 have only one common edge, by the structure of $G_{n,m}$. Delete the common edge of C^2 and C^3, the result signed graph denoted by (G, σ). Note that the boundaries of every face of (G, σ) is positive. By Lemma 3.3, (G, σ) is balanced. By Lemma 3.1, this contradicts the fact that $(G_{n,m}, \sigma)$ is flow-admissible.

Case 3. $nc(G_{n,m}) \geq 3$.

Without loss of generality, we assume that $nc(G_{n,m}) = 3$. We denote the three negative circuits by C^4, C^5, and C^6, respectively. We assume that any two of C^4, C^5, and C^6 have at least one common edges. For distinct $i, j \in \{4, 5, 6\}$, C^i and C^j have only one common edge. Thus, any two circuits of length 4 that share an edge with C^4 do not have common edge between themselves, see Figure 2. A contradiction. Therefore, there exist two negative circuits of length 4 that are edge-disjoint. □

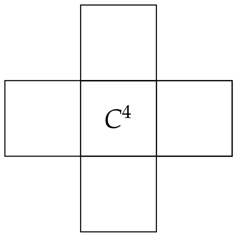

Figure 2: All circuits share an edge with C^4

4. Proof of Theorem 1.2

The outline of the proof is converting $(G_{n,m}, \sigma)$ into a flow-admissible signed series-parallel graph, more specifically, a signed outerplanar graph. Therefore, we need the following theorem.

Theorem 4.1. *[4] Every flow-admissible signed series-parallel graph has a nowhere-zero* 6*-flow.*

In order to transform $(G_{n,m}, \sigma)$ into a flow-admissible signed outerplanar graph, we perform a series of operations. First, we split vertices of degree 4 into two vertices of degree 2. Then, we suppress certain special induced paths into edges. Now, we begin to prove Theorem 1.2.

Proof. Because our proof rely on the planarity and the embedding of $(G_{n,m}, \sigma)$, unless otherwise specified, we assume that the grid graph in the proof is already embedded in the plane as Figure 3. In the following proof, edges that are not involved in the operation process are assumed unchanged for the embedding, and the other edges are redrawn so that the resulting signed graph is a plane graph.

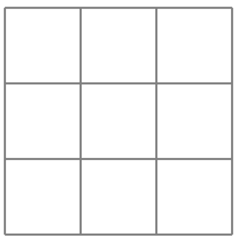

Figure 3: The embedding of $G_{4,4}$

For clarity, an example with $n = 8$ and $m = 9$ is provided at each step. Since the signs of edges do not affect the following operations, they are omitted in the following steps and figures.

Consider a circuit of length 4 of $(G_{n,m}, \sigma)$, say C_*, where $V(C_*) = \{(i, j), (i+1, j), (i, j+1), (i+1, j+1)\}$ and $i \in [1, n-1]$, $j \in [1, m-1]$. Without loss of generality, we assume that $i \leq j$. Let $s \in [1, k]$ and $t \in [1, q]$, where $k = min\{i-1, j-1\}$ and $q = min\{n-i-1, m-j-1\}$. In $(G_{n,m}, \sigma)$, let C_*^{-s} and C_*^{+t} be the two circuits of length 4 derived from C_*. Their vertex sets are defined as $V(C_*^{-s}) = \{(i-s, j-s) : (i, j) \in V(C_*)\}$ and

$V(C_*^{+t}) = \{(i+t, j+t) : (i,j) \in V(C_*)\}$. We give an example about C_0 in Figure 4, where C_0 is a circuit of length 4.

Let f^0 be the unbounded face of $(G_{n,m}, \sigma)$ and $\partial(f^0) = C$. By Lemma 3.4, $(G_{n,m}, \sigma)$ contains two edge-disjoint negative circuits, C_0 and C_1, which form the boundaries of two faces. We prove the theorem by considering whether one of C_0 and C_1 is C.

Case 1. One of C_0 and C_1 is C.

Without loss of generality, let $C_1 = C$. Then C_0 is a circuit of length 4. Let $V(C_0) = \{(i_1, j_1), (i_1+1, j_1), (i_1, j_1+1), (i_1+1, j_1+1)\}$, where $2 \le i_1 \le n-2$, $2 \le j_1 \le m-2$. Without loss of generality, we assume that $i_1 \le j_1 \le \lfloor \frac{m}{2} \rfloor$. Let $C_0P = C_0 \cup (\bigcup_{i=1}^{k} C_0^{-i})$ be a signed induced subgraph of $(G_{n,m}, \sigma)$, where $k = min\{i_1 - 1, j_1 - 1\}$. Because C_0 and C are edge-disjoint, it follows that $(i_1, j_1) \neq (1,1)$. Therefore, such a C_0P exists. Each circuit of length 4 in C_0P is a block, see Figure 4. Two blocks C_0^{-x} and C_0^{-y} have a common vertex if and only if $|x - y| = 1$.

Let $V_{C_0P}^2 = \{v \in V(C_0P) \setminus V(C) : d_{C_0P}(v) = 2\}$ be a subset of $V(C_0P)$. For any $v \in V_{C_0P}^2$, let $F_1^0(v)$ denote $E_{G_{n,m}}(v) \setminus E(C_0P)$. By splitting the edges of $F_1^0(v)$ away from v, we obtain a signed graph $(G_{n,m}^0, \sigma^0)$. We draw $(G_{n,m}^0, \sigma^0)$ on a plane such that $(G_{n,m}^0, \sigma^0)$ is a signed plane graph, as follows. Let e be an edge in $E(C) \cap E(C_0P)$. We draw $C_0P \setminus e$ on f^0, and the embedding of all other vertices and edges in $(G_{n,m}^0, \sigma^0)$ are identical to those in $(G_{n,m}, \sigma)$, see Figure 5. In $(G_{n,m}^0, \sigma^0)$, let f_1^0 denote the face that does not intersect f^0 and whose boundary contains the edges of $F_1^0(v)$, where $v \in V_{C_0P}^2$, see Figure 5. Next, a transformation process is provided. The first step is as follows.

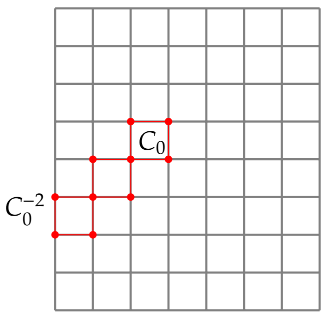

Figure 4: C_0P

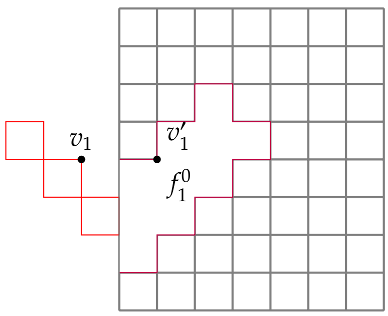

Figure 5: $(G_{8,9}^0, \sigma^0)$

Let P_1^1 be a signed path induced by $E(\partial(f_1^0)) \setminus E(C)$. For any internal vertex of P_1^1, we split the edges of $F_1^1(v)$ away from v, and then suppress P_1^1 to an edge e_1^1. We denote the resulting signed graph by $(G_{n,m}^1, \sigma^1)$. Next, we draw $(G_{n,m}^1, \sigma^1)$ on the plane as follows. We draw e_1^1 in f_1^0 such that e_1^1 has no crossings with $E(G_{n,m}^0) \setminus E(P_1^1)$, and the embedding of all other vertices and edges in $(G_{n,m}^1, \sigma^1)$ are identical to those in $(G_{n,m}^0, \sigma^0)$, see Figure 6. We denote by f_1^1 the face of $(G_{n,m}^1, \sigma^1)$ such that $E(\partial(f_1^1)) \cap E(\partial(f_1^0)) = \emptyset$. Note that some vertices in $V(P_1^1)$ have degree 2 in $G_{n,m}^0$. Thus, we perform splits on these vertices of degree 2, leading to the appearance of isolated vertices. For any flow (τ, f) and any isolated vertex $v*$, the sum $\sum_{e \in E(v*)} \tau(h_e^{v*}) f(e)$ is always 0. Hence, these isolated vertices can be omitted. This completes the first step.

Because in a certain step of the transformation process, there are more than one path as well as more than one face, it is necessary to add some subscripts. For example, in Figure 8, the subscript "1" in f_1^2 represents the first face in the *2nd* step; while "2" in P_2^3 represents the second path in the *3rd* step. In the $(s-1)$-th step, let $\mathcal{P}$ be a set of signed paths induced by $E(\bigcup_{a=1}^{k}(\partial(f_a^{s-2}))) \setminus (E_{G_{n,m}^{s-2}}(C) \cup (\bigcup_{b=1}^{h}\{e_b^{s-2}\}))$, where f_a^{s-2} are the faces appearing in the $(s-2)$-th step and e_b^{s-2} are the edges obtained from suppressing some signed paths in the $(s-2)$-th step. Let x be the number of elements in $\mathcal{P}$. We denote by $F_i^{s-1}(v)$ the edge set $E_{G_{n,m}^{s-2}}(v) \setminus (\bigcup_{i=1}^{x} E(P_i^{s-1}))$, where $P_i^{s-1} \in \mathcal{P}$ and $v \in (\bigcup_{i=1}^{x} V(P_i^{s-1}))$. For any P_i^{s-1} and any internal vertex v of P_i^{s-1}, we split the edges of $F_i^{s-1}(v)$ away from v. Then suppress the signed path P_i^{s-1} to edge e_i^{s-1}, $i \in [1, x]$. We denote the resulting signed graph by $(G_{n,m}^{s-1}, \sigma^{s-1})$. In Figure 7, we give an example of $i = 1, s = 3$ and in Figure 8, an example of $i \in [1,2], s = 4$ is given.

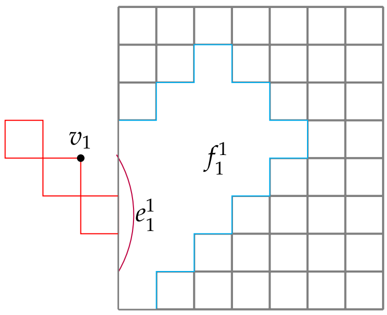

Figure 6: $(G_{8,9}^1, \sigma^1)$

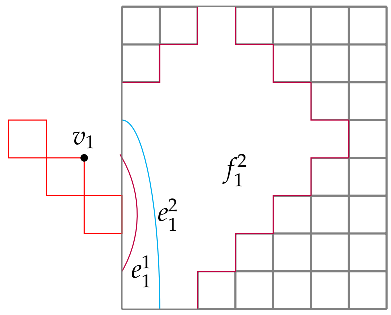

Figure 7: $(G_{8,9}^2, \sigma^2)$

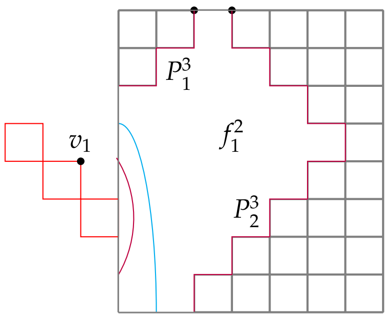

Figure 8: P_1^3 and P_2^3

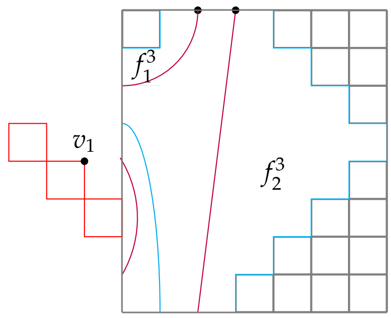

Figure 9: $(G_{8,9}^3, \sigma^3)$

Next, we draw $(G_{n,m}^{s-1}, \sigma^{s-1})$ on the plan as follows. We draw e_i^{s-1} in f_j^{s-2} such that edge e_i^{s-1} has no crossings with $E(G_{n,m}^{s-2}) \setminus (\bigcup_{i=1}^{x} E(P_i^{s-1}))$, where f_j^{s-2} is the face whose boundary containing P_i^{s-1}. And the embedding of all other vertices and edges in $(G_{n,m}^{s-1}, \sigma^{s-1})$ are identical to those in $(G_{n,m}^{s-2}, \sigma^{s-2})$. Since any two distinct paths P_y^{s-1} and P_z^{s-1} have no crossings in $(G_{n,m}^{s-2}, \sigma^{s-2})$, it follows that we can draw $(G_{n,m}^{s-1}, \sigma^{s-1})$ on the plan such that e_y^{s-1} and e_z^{s-1} have no crossings, $y, z \in [1, x]$. For any e_i^{s-1}, there is a face, denoted by f_i^{s-1}, in $(G_{n,m}^{s-1}, \sigma^{s-1})$ such that $e_i^{s-1} \in E(\partial(f_i^{s-1}))$ and $E(\partial(f_i^{s-1})) \cap E(\partial(f_j^{s-2})) = \emptyset$ for any f_j^{s-2} in $(G_{n,m}^{s-2}, \sigma^{s-2})$, see Figures 9 and 10.

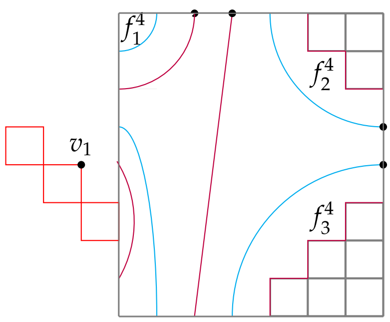

Figure 10: $(G_{8,9}^4, \sigma^4)$

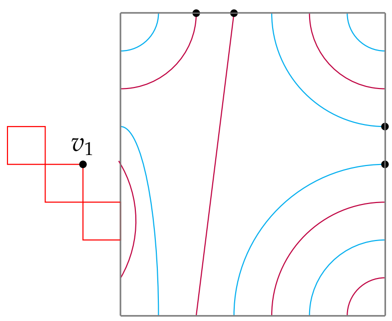

Figure 11: $(G_{8,9}^q, \sigma^q)$

Let s be the minimal positive integer such that $|E(\partial(f_i^s)) \setminus E(C)| = 1$ holds for every f_i^s. Then the transformation process terminates at the s-th step. Since $G_{n,m}$ is finite, such an s exist. Thus the final signed graph is denoted by $(G_{n,m}^s, \sigma^s)$, see Figure 11. Note that if there is a face f_y^d that satisfies $|E(\partial(f_y^d)) \setminus E(C)| = 1$ while the transformation process is still ongoing, then denote f_y^d by f_y^{d+1} in the $(d+1)$-th step.

Note that $(G^s_{n,m}, \sigma^s)$ is a signed 2-edge connected graph. Additionally, since C_0 and C are two edge-disjoint unbalanced circuits in $(G^s_{n,m}, \sigma^s)$, it follows by Lemma 3.1 that $(G^s_{n,m}, \sigma^s)$ is flow-admissible. For $(G^s_{n,m}, \sigma^s)$, because there are no vertices inside C and every vertex of C_0P is on the boundary of the unbounded face, it follows that $(G^s_{n,m}, \sigma^s)$ is a signed outerplanar graph. According to Theorem 4.1, $(G^s_{n,m}, \sigma^s)$ admits a 6-NZF. Since $(G^s_{n,m}, \sigma^s)$ is obtained from $(G_{n,m}, \sigma)$ through a series of splittings and suppressings, it follows from Remark 2.3 that $(G_{n,m}, \sigma)$ admits a 6-NZF.

Case 2. Both C_0 and C_1 are not C.

In this situation, C_0 and C_1 are the negative circuits of length 4. Let $V(C_0) = \{(i_1, j_1), (i_1 + 1, j_1), (i_1, j_1 + 1), (i_1 + 1, j_1 + 1)\}$, $V(C_1) = \{(i_2, j_2), (i_2 + 1, j_2), (i_2, j_2 + 1), (i_2 + 1, j_2 + 1)\}$. Without loss of generality, we assume that $1 \leq i_1 \leq i_2 \leq n - 1$. There are some circuits of length 4 in $G_{n,m}$ such that include a vertex of degree 2. Such circuits are referred to as the *corner* of $G_{n,m}$. For the first and second steps of the transformation process, we shall consider three subcases with respect to the number of corners in $\{C_0, C_1\}$.

Subcase 2.1. Neither C_0 nor C_1 is a corner.

Let $H_1 = C_0 \cup (\bigcup_{i=1}^{s} C_0^{-i})$ be the C_0-path, and $H_2 = C_1 \cup (\bigcup_{j=1}^{t} C_1^{+j})$ be the C_1-path, where $s = min\{j_1 - 1, i_1 - 1\}$, $t = min\{m - j_2 - 1, n - i_2 - 1\}$. Because neither C_0 nor C_1 is a corner, the two subgraphs H_1 and H_2 exist. Since C_0 and C_1 are edge-disjoint, we have $E(H_1) \cap E(H_2) = \emptyset$, see Figures 12 and 13.

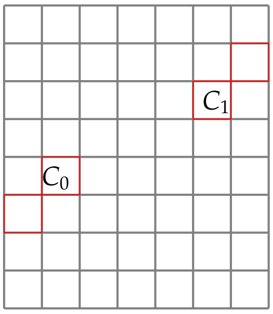

Figure 12: H_1 and H_2

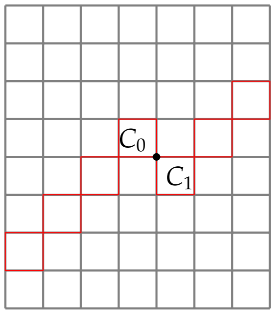

Figure 13: H_1 and H_2

We denote by $V^2_{H_i}(v)$ the set $\{v \in V(H_i) \setminus V(C) : d_{H_i}(v) = 2\}$, where $i \in \{1, 2\}$. Then we perform a series of splittings at the vertices in $V^2_{H_i}(v)$ in $(G_{n,m}, \sigma)$. For any $v \in V^2_{H_i}(v)$, let $F^0_i(v)$ denote the set $E_{G_{n,m}}(v) \setminus E(H_i)$. For every vertex $v \in V^2_{H_i}(v)$, we split the edges of $F^0_i(v)$ away from v and denote by $(G^0_{n,m}, \sigma^0)$ the resulting signed graph. Let e_i be an edge in $E(C) \cap E(H_i)$. We draw $(G^0_{n,m}, \sigma^0)$ on the plane such that $(G^0_{n,m}, \sigma^0)$ is a signed plane graph as follows. We draw $H_1 \setminus e_1$ and $H_2 \setminus e_2$ on f^0 , and the embedding of all other vertices and edges in $(G^0_{n,m}, \sigma^0)$ are identical to those in $(G_{n,m}, \sigma)$.

Note that $V(C_0) \cap V(C_1)$ may not be an empty set, as shown in Figures 12 and 13. If $V(C_0) \cap V(C_1) = \emptyset$, then we denote by f^0_i the face whose boundary includes the edges of $F^0_i(v)$ and does not intersect with f^0, where $i \in [1, 2]$, see Figure 14. If $V(C_0) \cap V(C_1) \neq \emptyset$, then we denote by f^0_1 the face whose boundary includes the edges of $F^0_1(v) \cup F^0_2(v)$ and does not intersect with f^0, see Figure 15.

In the case of $V(C_0) \cap V(C_1) \neq \emptyset$, let P^1_1 and P^1_2 be the signed paths induced by $E(\partial(f^0_1)) \setminus E(C)$. Denote by $F^1_i(v)$ the set $E_{G^0_{n,m}}(v) \setminus E(P^1_i)$, where $v \in V(P^1_i)$ and $i \in \{1, 2\}$. For any internal vertex of P^1_i, we split the edges of $F^1_i(v)$ away from v, and then suppress P^1_i to an edge e^1_i. We denote the resulting signed graph by $(G^1_{n,m}, \sigma^1)$. Next, we draw $(G^1_{n,m}, \sigma^1)$ on the plane as follows. We draw e^1_1 and e^1_2 in f^0_1 such that them have no crossings with $E(G^0_{n,m}) \setminus E(P^1_i)$ and there are no crossings between e^1_1 and e^1_2. Meanwhile, the embedding of all other vertices and edges in $(G^1_{n,m}, \sigma^1)$ are identical to those in $(G^0_{n,m}, \sigma^0)$. The transformation from $(G^0_{n,m}, \sigma^0)$ to $(G^1_{n,m}, \sigma^1)$ is analogous to that in **Case 1**. Therefore, in the remainder of the discussion, we consider the situation where $V(C_0) \cap V(C_1) = \emptyset$, see Figure 14.

Let P^1_1 be a signed path induced by $E(\partial(f^0_1)) \setminus E(C)$. For any internal vertex of P^1_1, we split the edges of $F^1_1(v)$ away from v, and then suppress P^1_1 to an edge e^1_1. We denote the resulting signed graph by $(G^1_{n,m}, \sigma^1)$. Next, we draw $(G^1_{n,m}, \sigma^1)$ on the plane as follows. We draw e^1_1 in f^0_1 such that e^1_1 has no crossings with

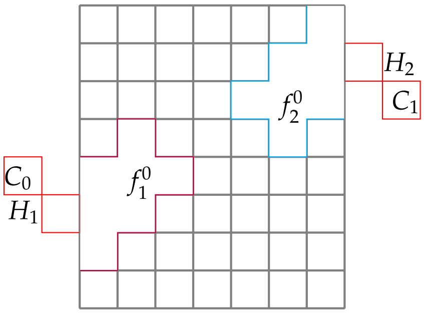

Figure 14: $(G^0_{8,9}, \sigma^0)$

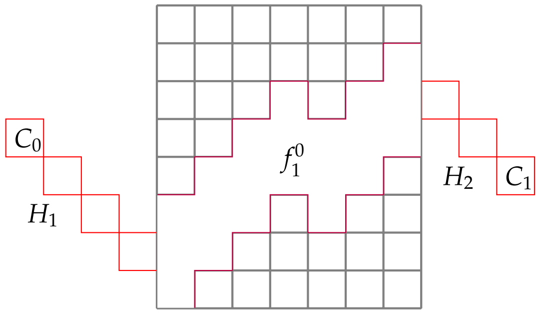

Figure 15: $(G^0_{8,9}, \sigma^0)$

$E(G^0_{n,m}) \setminus E(P^1_1)$, and the embedding of all other vertices and edges in $(G^1_{n,m}, \sigma^1)$ are identical to those in $(G^0_{n,m}, \sigma^0)$, see Figure 16. We denote by f^1_1 the face of $(G^1_{n,m}, \sigma^1)$ such that $E(\partial(f^1_1)) \cap E(\partial(f^0_1)) = \emptyset$.

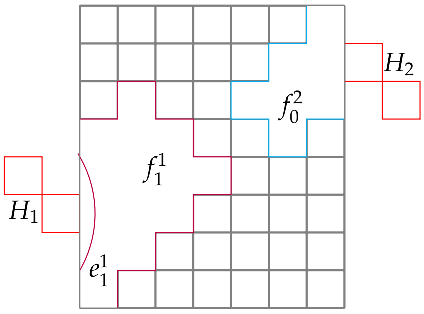

Figure 16: $(G^1_{8,9}, \sigma^1)$

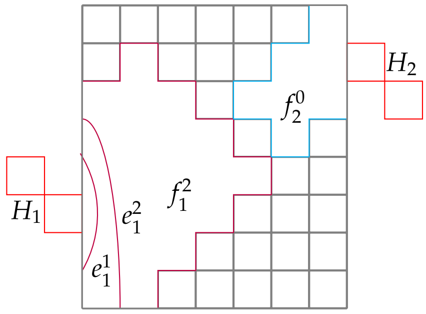

Figure 17: $(G^2_{8,9}, \sigma^2)$

Subcase 2.2. Exactly one of C_0 and C_1 is a corner.

Without loss of generality, let C_1 be the corner. We only consider $V(C_0) \cap V(C_1) = \emptyset$, as the treatment of $V(C_0) \cap V(C_1) \neq \emptyset$ is analogous and is omitted. Let $H_1 = C_0 \cup (\bigcup_{i=1}^{s} C_0^{-i})$ be the C_0-path, where $s = min\{j_1 - 1, i_1 - 1\}$. The approach for H_1 is similar to the one used in the **Subcase 2.1.** In order to avoid repetition, we omit further details. In this subcase, we only discuss the handling of C_1.

We denote by P^0 the path induced by $E(C_1) \setminus E(C)$, and let $F(v)$ denote the set $E_{G_{n,m}}(v) \setminus E_{C_1}(v)$. Then, for an internal vertex of P^0, we split the edges in $F^0(v)$ away from v and suppress P^0 to an edge e^0. We denote the resulting signed graph by $(G^{-1}_{n,m}, \sigma^{-1})$. Next, we draw $(G^{-1}_{n,m}, \sigma^{-1})$ on the plane as follows. Let f^0 denote the face with boundary C_1. We draw e^0 in f^0 such that it has no crossings with $E(G_{n,m}) \setminus E(P^0)$. Meanwhile, the embedding of all other vertices and edges in $(G^{-1}_{n,m}, \sigma^{-1})$ are identical to those in $(G_{n,m}, \sigma)$. The transformation process on H_1 is the same as in **Subcase 2.1**. Specifically, we split H_1 away from $(G^{-1}_{n,m}, \sigma^{-1})$ and redraw it on the unbounded face, see Figure 18. Let $(G^0_{n,m}, \sigma^0)$ denote the resulting signed graph. Similarly, The transformation from $(G^0_{n,m}, \sigma^0)$ to $(G^1_{n,m}, \sigma^1)$ is analogous to that in **Case 1**, see Figure 19. We denote by C'_1 the circuit of length 3, which is obtained from C_1. Note that C_0 and C'_1 are two edge-disjoint negative circuits in $(G^1_{n,m}, \sigma^1)$. Meanwhile, in $(G^1_{n,m}, \sigma^1)$, the edge e^1_1 and the faces f^1_1 and f^2_0 are defined, see Figure 19.

Subcase 2.3. Both C_0 and C_1 are corners.

For $(G_{n,m}, \sigma)$, it follows that $V(C_0) \cap V(C_1) \neq \emptyset$ if and only if $n = m = 3$. Then we split $E_{C_1}((2, 2))$ away from $(2, 2)$ in $(G_{3,3}, \sigma)$, and we suppress the induced paths inside C. It is easy to verify that the resulting signed graph is a flow-admissible signed outerplanar graph. Therefore, we consider the situation that $V(C_0) \cap V(C_1) = \emptyset$. The approaches for C_0 and C_1 are similar to the one for C_1 in the **Subcase 2.1.** In order to avoid repetition, we omit further details present $(G^1_{n,m}, \sigma^1)$ as Figure 20. In $(G^1_{n,m}, \sigma^1)$, the edge e^1_1 and the faces f^1_1 and f^2_0 are defined. This completes the first and second steps.

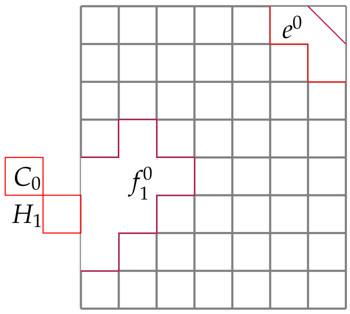

Figure 18: $(G^0_{8,9}, \sigma^0)$

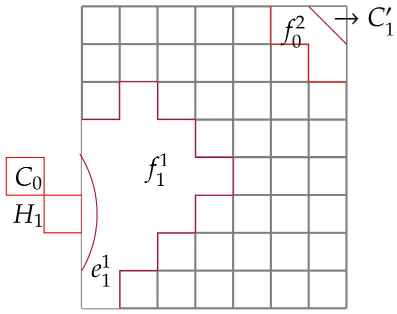

Figure 19: $(G^1_{8,9}, \sigma^1)$

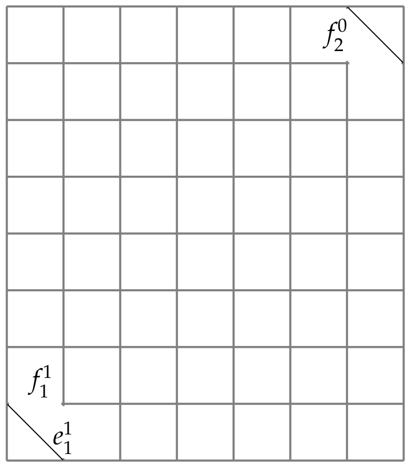

Figure 20: $(G^1_{8,9}, \sigma^1)$

In the $(s-1)$-th step, let $\mathcal{P}$ be a set of signed paths induced by $E(\bigcup_{a=1}^{k}(\partial(f_a^{s-2}))) \setminus (E_{G^{s-2}_{n,m}}(C) \cup (\bigcup_{b=1}^{h}\{e_b^{s-2}\}))$, where f_a^{s-2} are the faces appearing in the $(s-2)$-th step and e_b^{s-2} are the edges obtained from suppressing some signed paths in the $(s-2)$-th step. Let x be the number of elements in $\mathcal{P}$. We denote by $F_i^{s-1}(v)$ the edge set $E_{G^{s-2}_{n,m}}(v) \setminus (\bigcup_{i=1}^{x} E(P_i^{s-1}))$, where $P_i^{s-1} \in \mathcal{P}$ and $v \in (\bigcup_{i=1}^{x} V(P_i^{s-1}))$. For any P_i^{s-1} and any internal vertex v of P_i^{s-1}, we split the edges of $F_i^{s-1}(v)$ away from v. Then suppress the signed path P_i^{s-1} to edge e_i^{s-1}, $i \in [1, x]$. We denote the resulting signed graph by $(G^{s-1}_{n,m}, \sigma^{s-1})$. Note that there exists a positive integer l such that $V(P_i^l) \cap V(\partial(f_2^0)) \neq \emptyset$ in the l-th step of the transformation process, see Figures 17 and 21.

Next, we draw $(G^{s-1}_{n,m}, \sigma^{s-1})$ on the plan as follows. We draw e_i^{s-1} in f_j^{s-2} such that edge e_i^{s-1} has no crossings with $E(G^{s-2}_{n,m}) \setminus (\bigcup_{i=1}^{x} E(P_i^{s-1}))$, where f_j^{s-2} is the face whose boundary containing P_i^{s-1}. And the embedding of all other vertices and edges in $(G^{s-1}_{n,m}, \sigma^{s-1})$ are identical to those in $(G^{s-2}_{n,m}, \sigma^{s-2})$. Since any two distinct paths P_y^{s-1} and P_z^{s-1} have no crossings in $(G^{s-2}_{n,m}, \sigma^{s-2})$, it follows that we can draw $(G^{s-1}_{n,m}, \sigma^{s-1})$ on the plan such that e_y^{s-1} and e_z^{s-1} have no crossings, $y, z \in [1, x]$. For any e_i^{s-1}, there is a face, denoted by f_i^{s-1}, in $(G^{s-1}_{n,m}, \sigma^{s-1})$ such that $e_i^{s-1} \in E(\partial(f_i^{s-1}))$ and $E(\partial(f_i^{s-1})) \cap E(\partial(f_j^{s-2})) = \emptyset$ for any f_j^{s-2} in $(G^{s-2}_{n,m}, \sigma^{s-2})$, see Figures 22 and 23.

Let s be the minimal positive integer such that $|E(\partial(f_i^s)) \setminus E(C)| = 1$ holds for every f_i^s. Then the transformation process terminates at the s-th step. Since $G_{n,m}$ is finite, such an s exist. Thus the final signed graph is denoted by $(G^s_{n,m}, \sigma^s)$, see Figure 24. Note that if there is a face f_y^d that satisfies $|E(\partial(f_y^d)) \setminus E(C)| = 1$ while the transformation process is still ongoing, then denote f_y^d by f_y^{d+1} in the $(d+1)$-th step.

Similarly to **Case 1**, $(G^s_{n,m}, \sigma^s)$ is a flow-admissible signed outerplanar graph. Thus, $(G^s_{n,m}, \sigma^s)$ admits a 6-

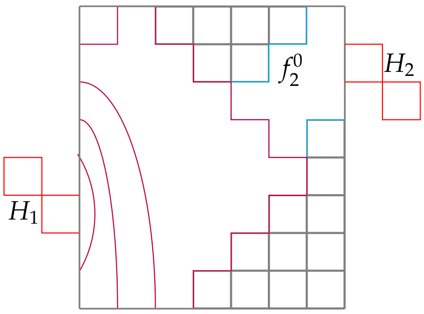

Figure 21: $E(P_1^4) \cap E(C) \neq \emptyset$

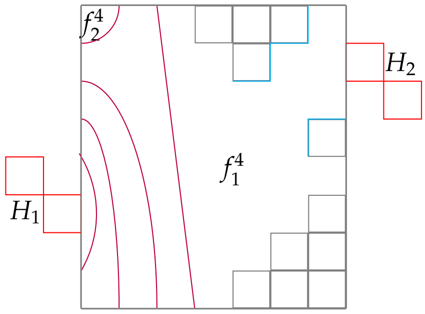

Figure 22: $(G_{8,9}^4, \sigma^4)$

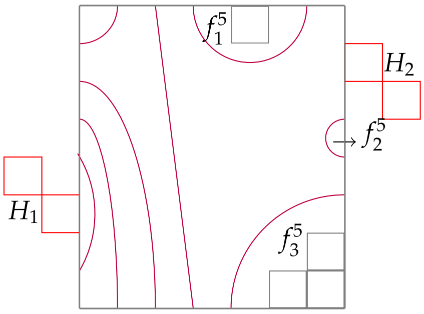

Figure 23: $(G_{8,9}^5, \sigma^5)$

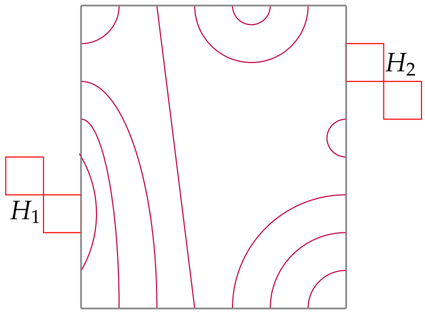

Figure 24: $(G_{8,9}^q, \sigma^q)$

NZF. Furthermore, since $(G_{n,m}^s, \sigma^s)$ is obtained from $(G_{n,m}, \sigma)$ through a series of splittings and suppressings, Remark 2.3 implies that $(G_{n,m}, \sigma)$ also admits a 6-NZF.

Consequently, we infer that every flow-admissible $(G_{n,m}, \sigma)$ admits a 6-NZF. □

Acknowledgements

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