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Claw-decomposition of generalized Kneser graph $GKG_{n,3,1}$

K. Arthi[a], R. Sangeetha[a,*], C. Sankari[a]

[a]*Department of Mathematics, A. V. V. M. Sri Pushpam College (Affiliated to Bharathidasan University), Poondi-613 503, Thanjavur(Dt.), Tamil Nadu, India*

Abstract. A star with three edges is called a claw. The Generalized Kneser Graph $GKG_{n,k,r}$ is the graph whose vertices are the k-element subsets of n-elements, in which two vertices are adjacent if and only if they intersect in precisely r elements. In this paper, we prove that the graph $GKG_{n,3,1}$ has a claw-decomposition for all $n \geq 6$.

1. Introduction

A *star* with k edges is denoted by S_k and $S_k \cong K_{1,k}$. If k=3, then the graph $K_{1,3}$ is called a *claw*. If a graph G has no edges, then it is called a *null graph*. The degree of a vertex v of G, denoted by $D_G v$ is the number of edges incident with v in G. Let k be a positive integer. A graph G is said to be *k-regular*, if $D_G v$=k, for all $v \in V(G)$. Let $S \subset V(G)$, then the subgraph induced by S is denoted by $\langle S \rangle$. Let T be a subgraph of G. The graph obtained by deleting the edges of T is denoted by $G \setminus T$. A *cycle* and a *path* with k edges is denoted by C_k and P_k, respectively. If $H_1, H_2, ..., H_l$ are edge disjoint subgraphs of a graph G such that $E(G)$=$\bigcup_{i=1}^{l} E(H_i)$, then we say that $H_1, H_2, ..., H_l$ *decompose* G and we denote it by G=$\oplus_{i=1}^{l} H_i$. If $H_i \cong S_k$ for i=1, 2, ..., l, then we say that G has an S_k-*decomposition* or a k-star decomposition and we denote it by $S_k|G$. Let $A = \{1, 2, 3, ..., n\}$ and let $\mathcal{P}_k(A)$ denotes the set of all k-element subsets of A. The *Kneser Graph* $KG_{n,k}$ is defined as follows: $V(KG_{n,k})$=$\mathcal{P}_k(A)$ and $E(KG_{n,k})$=$\{XY|X, Y \in \mathcal{P}_k(A) \text{ and } |X \cap Y| = \emptyset\}$. The *Generalized Kneser Graph* $GKG_{n,k,r}$ is defined as follows: $V(GKG_{n,k,r})$=$\mathcal{P}_k(A)$ and $E(GKG_{n,k,r})$=$\{XY|X, Y \in \mathcal{P}_k(A) \text{ and } |X \cap Y| = r\}$.

In 1955, M. Kneser [4] introduced the Kneser graph. In 2015, Rodger and Whitt [5] established the necessary and sufficient conditions for a P_3-decomposition of the Kneser graph $KG_{n,2}$ and the Generalized Kneser Graph $GKG_{n,3,1}$. In 2015, Whitt and Rodger [9] proved that the Kneser graph $KG_{n,2}$ is P_4-decomposable if and only if $n \equiv 0, 1, 2, 3 (mod\, 16)$. In 2018, Ganesamurthy and Paulraja [3] proved that if $n \equiv 0, 1, 2, 3 (mod\, 8k)$, $k \geq 2$, then the Kneser graph $KG_{n,2}$ can be decomposed into paths of length $2k$. In the same paper they also proved that, for $k = 2^l$, $l \geq 1$, $KG_{n,2}$ has a P_{2k}-decomposition if and only if $n \equiv 0, 1, 2, 3 (mod\, 2^{l+3})$. In 2024, Cecily Sahai

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* Corresponding author: R. Sangeetha
Email addresses: arthi1505@gmail.com (K. Arthi), jaisangmaths@yahoo.com (R. Sangeetha), sankari9791@gmail.com (C. Sankari)
ORCID iDs: https://orcid.org/0000-0002-0399-1343 (K. Arthi), https://orcid.org/0000-0003-2726-0956 (R. Sangeetha), https://orcid.org/0000-0003-3325-7163 (C. Sankari)

et al. [1] proved that if $n \equiv 0, 1, 2, 3 (mod\, 4k)$, where $k \geq 1$, then the Kneser graph $KG_{n,2}$ is P_k-decomposable. In 2025, Cecily Sahai *et al.* [2] proved that the necessary and sufficient conditions for the existence of a 4-cycle decomposition of λ-fold Kneser graphs $\lambda KG_{n,2}$ and λ-fold Bipartite Kneser graphs $\lambda BKG_{n,2}$. Recently, the authors proved that $KG_{n,2}$ is claw-decomposable, for all $n \geq 6$, see [6] and $KG_{n,2}$ is S_5-decomposable if and only if $n \geq 7$, $n \equiv 0, 1, 2, 3 (mod\, 5)$, see [7]. Recently the authors [8] proved that $KG_{n,3}$ is claw-decomposable if and only if $n \geq 9$ and $n \equiv 0, 1, 2, 3, 4, 5 (mod\, 9)$. In this paper, we prove that the Generalized Kneser Graph $GKG_{n,3,1}$ is claw-decomposable for all $n \geq 6$.

Let G be a graph on n vertices and $\{1, 2, 3, ..., k\} \subset V(G)$. The notation $(1; 2, 3, ..., k)$ denotes a star with a center vertex 1 and $k - 1$ pendent edges $12, 13, ..., 1k$. Let X and Y be two disjoint subsets of $V(G)$. Then $E(X, Y)$ denotes the set of edges in G, whose one end vertex is in X and the other end vertex is in Y. The notation $\langle E(X, Y)\rangle$ denotes the graph induced by the edges of $E(X, Y)$. If the degree of each vertex of X (or Y) is $3r$, where r is any positive integer, then by fixing each vertex of X (or Y) as a center vertex r times, we get a claw-decomposition in $\langle E(X, Y)\rangle$.

To prove our results we use the following:

Theorem 1.1. *(Sankari et al.[6]) For all $n \geq 6$, the graph $KG_{n,2}$ is claw-decomposable.*

2. Claw-decomposition of $GKG_{n,3,1}$

As we are looking for a claw-decomposition, we have $|V(GKG_{n,3,1})| \geq 4$. Therefore $n \geq 4$. The graph $GKG_{4,3,1}$ is a null graph. We know that $|E(GKG_{n,3,1})| = \frac{1}{2}\binom{n}{3}\binom{3}{1}\binom{n-3}{2}$, and is divisible by 3 for all $n \geq 5$. The graph $GKG_{5,3,1}$ is the Petersen graph (see Figure 1), which doesn't admit a claw-decomposition [6].

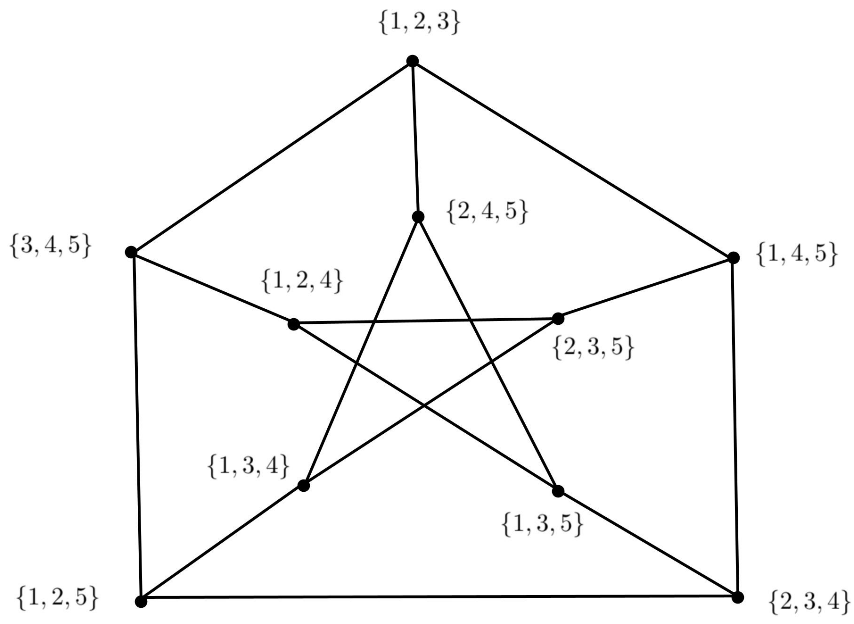

Figure 1: $GKG_{5,3,1}$

Therefore, we look for a claw-decomposition in $GKG_{n,3,1}$, when $n \geq 6$.

Lemma 2.1. *The graph $GKG_{6,3,1}$ is claw-decomposable.*

Proof. We define $V(GKG_{6,3,1})$=$X \cup Y \cup Z$, where

$$\begin{aligned} X &= X_1 \cup X_2, \text{ where } X_1 = \{1,2,3\} \text{ and } X_2 = \{4,5,6\} \\ Y &= \{\{a,b,c\} | a,b \in X_1,\ a < b \text{ and } c \in X_2\} \\ Z &= \{\{a,b,c\} | a,b \in X_2,\ a < b \text{ and } c \in X_1\} \end{aligned}$$

So, $|X_1|$=$|X_2|$=1, $|Y|$=$|Z|$=9. Note that, the graph $GKG_{6,3,1}$=$\langle Y\rangle \cup \langle Z\rangle \cup \langle E(Y,Z)\rangle \cup \langle E(X_1, Y \cup Z)\rangle \cup \langle E(X_2, Y \cup Z)\rangle$. We prove that each one of these subgraphs has a claw-decomposition. The graph $\langle Y\rangle$ can be decomposed into six copies of claws as follows: $(\{1,2,4\};\{1,3,5\},\{2,3,5\},\{1,3,6\})$, $(\{1,2,5\};\{1,3,6\},\{2,3,6\},\{1,3,4\})$, $(\{1,2,6\};\{1,3,4\},\{2,3,4\},\{1,3,5\})$, $(\{2,3,4\};\{1,2,5\},\{1,3,5\},\{1,3,6\})$, $(\{2,3,5\};\{1,2,6\},\{1,3,6\},\{1,3,4\})$, $(\{2,3,6\};\{1,2,4\},\{1,3,4\},\{1,3,5\})$. We see that $\langle Z\rangle \cong \langle Y\rangle$ and hence has a claw-decomposition. In $\langle E(Y,Z)\rangle$, consider the three stars S^1 : $(\{4,5,3\};\{1,2,5\},\{1,3,6\},\{2,3,6\})$, S^2 : $(\{4,6,3\};\{1,2,4\},\{1,3,5\},\{2,3,5\})$ and S^3 : $(\{5,6,3\};\{1,2,6\},\{1,3,4\},\{2,3,4\})$. Now, $D_{\langle E(Y,Z)\rangle \setminus \bigcup_{i=1}^{3} S^i}\{a,b,c\}$=3, for all $\{a,b,c\} \in Y$. Hence $\langle E(Y,Z)\rangle \setminus \bigcup_{i=1}^{3} S^i$ has a claw-decomposition. The subgraphs $\langle E(X_1, Y \cup Z)\rangle \cong \langle E(X_2, Y \cup Z)\rangle \cong S_9$ and hence have a claw-decomposition. □

Let $n = 6m$, where $m \geq 2$ be a positive integer. Let $n_1 = 6$ and $n_2 = n - n_1$. Let N_1=$\{1,2,...,n_1\}$ and N_2=$\{n_1+1,...,n\}$. Define $V(GKG_{n,3,1})$=$A_1 \cup A_2 \cup A_3 \cup A_4$, where A_1=$\{\{a,b,c\}|(a,b,c) \in \mathcal{P}_3(N_1)\}$, A_2=$\{\{a,b,c\}|(a,b,c) \in \mathcal{P}_3(N_2)\}$, A_3=$\{\{a,b,c\}|(a,b) \in \mathcal{P}_2(N_1), c \in N_2\}$ and A_4=$\{\{a,b,c\}|(a,b) \in \mathcal{P}_2(N_2), c \in N_1\}$. So, $|A_1|$=$\binom{n_1}{3}$, $|A_2|$=$\binom{n_2}{3}$, $|A_3|$=$n_2\binom{n_1}{2}$ and $|A_4|$=$n_1\binom{n_2}{2}$. We define the graphs $H_i \subset GKG_{n,3,1}, 1 \leq i \leq 7$ as follows: $H_1 = \langle A_1\rangle(\cong GKG_{6,3,1})$, $H_2 = \langle A_2\rangle(\cong GKG_{n_2,3,1})$, $H_3 = \langle A_3\rangle$, $H_4 = \langle A_4\rangle$, $H_5 = \langle E(A_3, A_4)\rangle$, $H_6 = \langle E(A_1, A_3 \cup A_4)\rangle$, $H_7 = \langle E(A_2, A_3 \cup A_4)\rangle$.

Remark 2.2. $GKG_{n,3,1}$=$\oplus_{i=1}^{7} H_i$.

Therefore, to prove $GKG_{n,3,1}$ is claw-decomposable, it is enough to prove that each H_i is claw-decomposable.

Remark 2.3. *The graph H_1 is claw-decomposable, by Lemma 2.1.*

Lemma 2.4. *The graph H_3 is claw-decomposable.*

Proof. For $n_1 + 1 \leq t \leq n$, let A_{3t}=$\{\{a,b,t\}|(a,b) \in \mathcal{P}_2(N_1)\}$. Then $\langle A_{3t}\rangle \cong KG_{6,2}$, for each t. We write A_3= $\bigcup_{n_1+1 \leq t \leq n} A_{3t}$ and the graph H_3=$n_2KG_{6,2} \oplus \bigcup_{(a,b) \in \mathcal{P}_2(N_1)} \langle E(\{a,b,t_1\},\{a,b,t_2\})\rangle$, $n_1 + 1 \leq t_1 < t_2 \leq n$. By Theorem 1.1, the graph $KG_{6,2}$ is claw-decomposable. Now we prove that, the graph $H_3 \setminus n_2KG_{6,2}$ has a claw-decomposition. Consider the two set of stars from $H_3 \setminus n_2KG_{6,2}$ as follows:
$S^{'}$ = $\{(\{1,6,t_2\};\{1,2,t_1\},\{1,3,t_1\},\{1,4,t_1\}), (\{2,6,t_2\};\{2,3,t_1\},\{2,4,t_1\},\{2,5,t_1\}), (\{3,6,t_2\};\{2,6,t_1\},\{3,5,t_1\},\{5,6,t_1\}), (\{4,6,t_2\};\{1,6,t_1\},\{3,4,t_1\},\{3,6,t_1\}), (\{5,6,t_2\};\{1,5,t_1\},\{4,5,t_1\},\{4,6,t_1\}), n_1 + 1 \leq t_1 < t_2 \leq n\}$
$S^{''}$ = $\{(\{1,2,t_2\};\{1,4,t_1\},\{1,5,t_1\},\{1,6,t_1\}), (\{2,3,t_2\};\{2,4,t_1\},\{2,5,t_1\},\{2,6,t_1\}), (\{2,6,t_2\};\{1,2,t_1\},\{4,6,t_1\},\{5,6,t_1\}), (\{3,4,t_2\};\{3,5,t_1\},\{3,6,t_1\},\{4,5,t_1\}), (\{3,6,t_2\};\{1,3,t_1\},\{2,3,t_1\},\{3,4,t_1\}), n_1 + 1 \leq t_1 < t_2 \leq n\}$

Now, remove the stars $S^{'}$ and $S^{''}$ from $H_3 \setminus n_2KG_{6,2}$. It is denoted by $[H_3 \setminus n_2KG_{6,2}] \setminus (S^{'} \cup S^{''})$. Consider the subgraph $\langle E(\{a,b,t_1\},\{a,b,t_2\})\rangle$, $n_1 + 1 \leq t_1 < t_2 \leq n$ in $[H_3 \setminus n_2KG_{6,2}] \setminus (S^{'} \cup S^{''})$. The degree of the vertices $\{a,b,t_1\}$ in each subgraph $\langle E(\{a,b,t_1\},\{a,b,t_2\})\rangle$, $n_1 + 1 \leq t_1 < t_2 \leq n$ is exactly 6, see Figure 2.

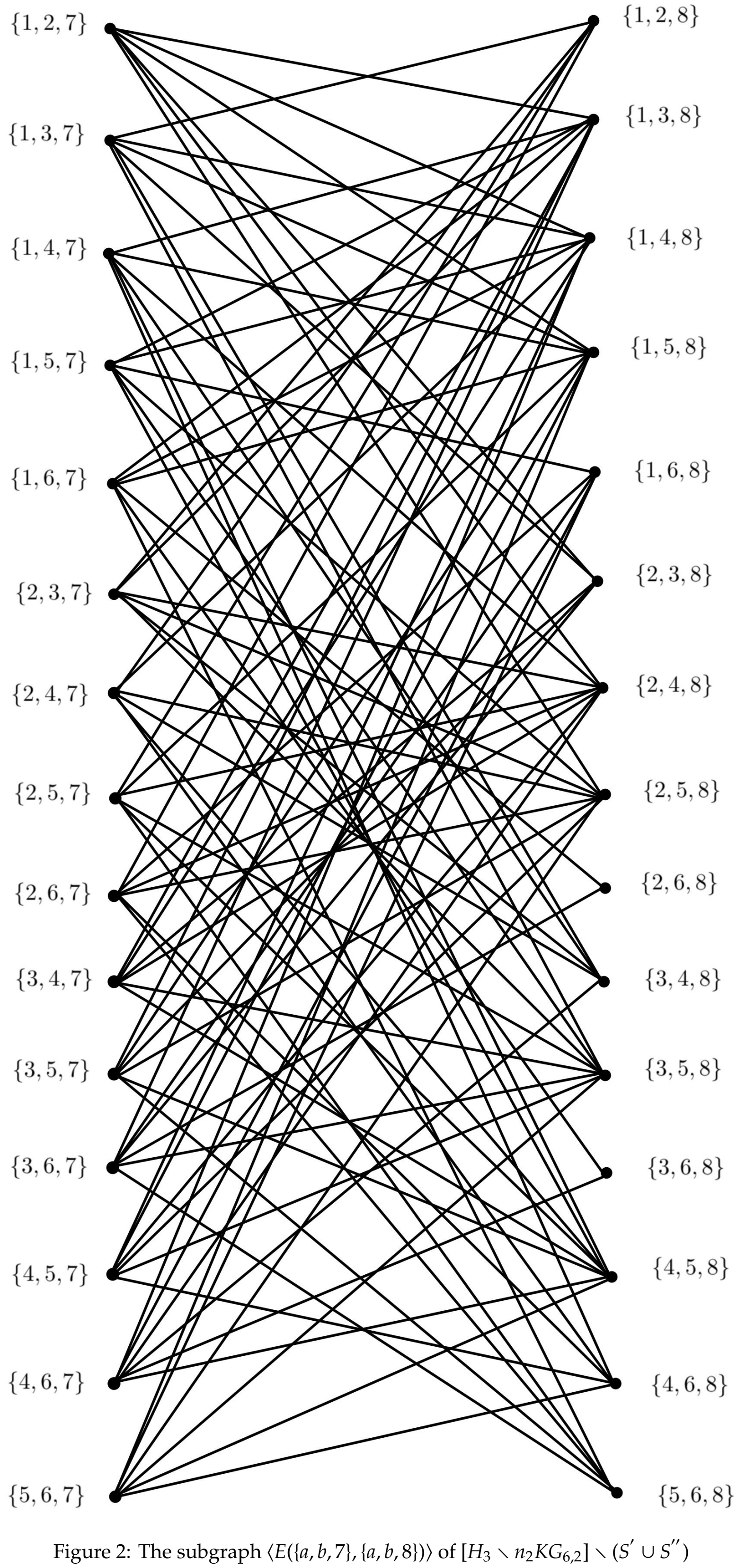

Figure 2: The subgraph $\langle E(\{a, b, 7\}, \{a, b, 8\})\rangle$ of $[H_3 \setminus n_2KG_{6,2}] \setminus (S' \cup S'')$

Hence each subgraph $\langle E(\{a,b,t_1\},\{a,b,t_2\})\rangle$, $n_1+1 \le t_1 < t_2 \le n$ has a claw-decomposition. Hence H_3 is claw-decomposable. □

Lemma 2.5. *The graph H_4 is claw-decomposable.*

Proof. Let $n = 6m$, where $m \ge 2$ be a positive integer. For $1 \le t \le 6$, let A_{4t}=$\{\{a,b,t\}|(a,b) \in \mathcal{P}_2(N_2)\}$. Then $\langle A_{4t}\rangle \cong KG_{n_2,2}$, for each t. We write A_4= $\bigcup_{1\le t\le 6} A_{4t}$ and the graph H_4=$6KG_{n_2,2} \oplus \bigcup_{(a,b)\in\mathcal{P}_2(N_2)} \langle E(\{a,b,t_1\},\{a,b,t_2\})\rangle$, $1 \le t_1 < t_2 \le 6$. By Theorem 1.1, the graph $KG_{n_2,2}$ is claw-decomposable. Now, we prove that the graph $H_4 \setminus 6KG_{n_2,2}$ has a claw-decomposition. If $m = 2$, then the graph $H_4 \setminus 6KG_{n_2,2} \cong H_3 \setminus n_2KG_{6,2}$ has a claw-decomposition, by Lemma 2.4. Now take $m > 2$. Consider the subgraph $\bigcup_{(a,b)\in\mathcal{P}_2(N_2)} \langle E(\{a,b,1\},\{a,b,2\})\rangle$. We partition the vertex set of this subgraph as follows: For i=1,2, P_i=$\{\{a,b,i\}|(a,b) \in \mathcal{P}_2(\{7,8,\ldots,n-6\})\}$, Q_i=$\{\{a,b,i\}|a \in \{7,8,\ldots,n-6\}, b \in \{n-5,\ldots,n\}$ *and* $a < b\}$ and R_i=$\{\{a,b,i\}|(a,b) \in \mathcal{P}_2(\{n-5,\ldots,n\})\}$. We write $\langle E(\{a,b,1\},\{a,b,2\})\rangle$=$\langle E(P_1,P_2)\rangle \cup \langle E(P_1,Q_2)\rangle \cup \langle E(P_1,R_2)\rangle \cup \langle E(Q_1,P_2)\rangle \cup \langle E(Q_1,Q_2)\rangle \cup \langle E(Q_1,R_2)\rangle \cup \langle E(R_1,P_2)\rangle \cup \langle E(R_1,Q_2)\rangle \cup \langle E(R_1,R_2)\rangle$.

The graph $\langle E(R_1,P_2)\rangle$ is a null graph. In $\langle E(R_1,Q_2)\rangle$, $D_{\langle E(R_1,Q_2)\rangle}\{a,b,1\}$=$2(n_2-6)$, for all $\{a,b,1\} \in R_1$. Which is a multiple of 3, and hence has a claw-decomposition in $\langle E(R_1,Q_2)\rangle$. The graph $\langle E(R_1,R_2)\rangle \cong H_3 \setminus n_2KG_{6,2}$ and hence has a claw-decomposition, by Lemma 2.4.

In $\langle E(Q_1,P_2)\rangle$, $D_{\langle E(Q_1,P_2)\rangle}\{a,b,2\}$=12, for all $\{a,b,2\} \in P_2$. Hence $\langle E(Q_1,P_2)\rangle$ has a claw-decomposition. In $\langle E(Q_1,Q_2 \cup R_2)\rangle$, $D_{\langle E(Q_1,Q_2\cup R_2)\rangle}\{a,b,1\}$= $(n_2-7)+10$=n_2+3 and $n_2 \equiv 0(mod\ 6)$, for all $\{a,b,1\} \in Q_1$. Which is a multiple of 3, and hence has a claw-decomposition in $\langle E(Q_1,Q_2 \cup R_2)\rangle$.

The graph $\langle E(P_1,R_2)\rangle$ is a null graph. In $\langle E(P_1,Q_2)\rangle$, $D_{\langle E(P_1,Q_2)\rangle}\{a,b,1\}$=12, for all $\{a,b,1\} \in P_1$. Hence $\langle E(P_1,Q_2)\rangle$ has a claw-decomposition. Now, we prove that the graph $\langle E(P_1,P_2)\rangle$ has a claw-decomposition. If m=3, then the graph $\langle E(P_1,P_2)\rangle \cong \bigcup_{(a',b')\in\{7,8,\ldots,n-6\}} \langle E(\{a',b',1\},\{a',b',2\})\rangle$ and hence has a claw-decomposition, by Lemma 2.4. Hence if m=3, the graph $\langle E(\{a,b,1\},\{a,b,2\})\rangle$ has a claw-decomposition. If $m = 4$, then the graph $\langle E(P_1,P_2)\rangle \cong \langle E(\{a',b',1\},\{a',b',2\})\rangle$, where $(a',b') \in \mathcal{P}_2(N_2)$ has a claw-decomposition, by previous case as $\langle E(\{a',b',1\},\{a',b',2\})\rangle \cong H_4 \setminus 6KG_{12,2}$ and n_2=12. Assume that the graph has a claw-decomposition, when $4 < m < k$. Now we prove that the result is true for $m = k$. Note that, the graph $\langle E(P_1,P_2)\rangle \cong \langle E(\{a',b',1\},\{a',b',2\})\rangle$, where $(a',b') \in \mathcal{P}_2(N_2)$ (here n_2=$6(k-2)$) has a claw-decomposition by our assumption as $\langle E(\{a',b',1\},\{a',b',2\})\rangle \cong H_4 \setminus 6KG_{n_2-6,2}$ and $n_2 = 6(k-2)$. Hence, the graph $\langle E(P_1,P_2)\rangle$ has a claw-decomposition. Therefore, the graph $\langle E(\{a,b,1\},\{a,b,2\})\rangle$ has a claw-decomposition.

Similarly, each subgraph $\langle E(\{a,b,t_1\},\{a,b,t_2\})\rangle$, $1 \le t_1 < t_2 \le 6$, $(a,b) \in \mathcal{P}_2(N_2)$ has a claw-decomposition. Hence H_4 is claw-decomposable. □

Lemma 2.6. *The graph H_5 is claw-decomposable.*

Proof. Let $n = 6m$, where $m \ge 2$ be a positive integer. For $n_1+1 \le c \le n$, we write H_5=$\langle E(\bigcup_{(a,b)\in\mathcal{P}_2(N_1)} \{a,b,c\}, A_4)\rangle$. First we prove that the subgraph $\langle E(\bigcup_{(a,b)\in\mathcal{P}_2(N_1)} \{a,b,c\}, A_4)\rangle$ has a claw-decomposition, when c=n_1+1. Consider the five stars as follows: For x=$c+1$ and y=$c+2$, S^1 : $(\{x,y,1\};\{1,2,c\},\{1,3,c\},\{1,4,c\})$, S^2 : $(\{x,y,2\};\{2,3,c\},\{2,4,c\},\{2,5,c\})$, S^3 : $(\{x,y,3\};\{3,4,c\},\{3,5,c\},\{3,6,c\})$, S^4 : $(\{x,y,5\};\{1,5,c\},\{4,5,c\},\{5,6,c\})$ and S^5 : $(\{x,y,6\};\{1,6,c\},\{2,6,c\},\{4,6,c\})$. The stars S^i, $1 \le i \le 5$, where x=8, y=9 are shown in Figure 3. In $\langle E(\bigcup_{(a,b)\in\mathcal{P}_2(N_1)} \{a,b,c\}, A_4)\rangle$, the degree of each vertex of $\bigcup_{(a,b)\in\mathcal{P}_2(N_1)} \{a,b,c\}$ is $[4(n_2-1) + 2[\binom{n_2}{2} - (n_2-1)]] + 1$=$(n_2-1)(n_2+2)+1$. Now remove the stars $\bigcup_{i=1}^5 S^i$ from $\langle E(\bigcup_{(a,b)\in\mathcal{P}_2(N_1)} \{a,b,c\}, A_4)\rangle$ and it is denoted by $\langle E(\bigcup_{(a,b)\in\mathcal{P}_2(N_1)} \{a,b,c\}, A_4)\rangle \setminus \bigcup_{i=1}^5 S^i$.

In $\langle E(\bigcup_{(a,b)\in\mathcal{P}_2(N_1)} \{a,b,c\}, A_4)\rangle \setminus \bigcup_{i=1}^5 S^i$, the degree of each vertex of $\bigcup_{(a,b)\in\mathcal{P}_2(N_1)} \{a,b,c\}$ is reduced by one, which is a multiple of 3. Hence there exists a claw-decomposition in $\langle E(\bigcup_{(a,b)\in\mathcal{P}_2(N_1)} \{a,b,c\}, A_4)\rangle \setminus \bigcup_{i=1}^5 S^i$.

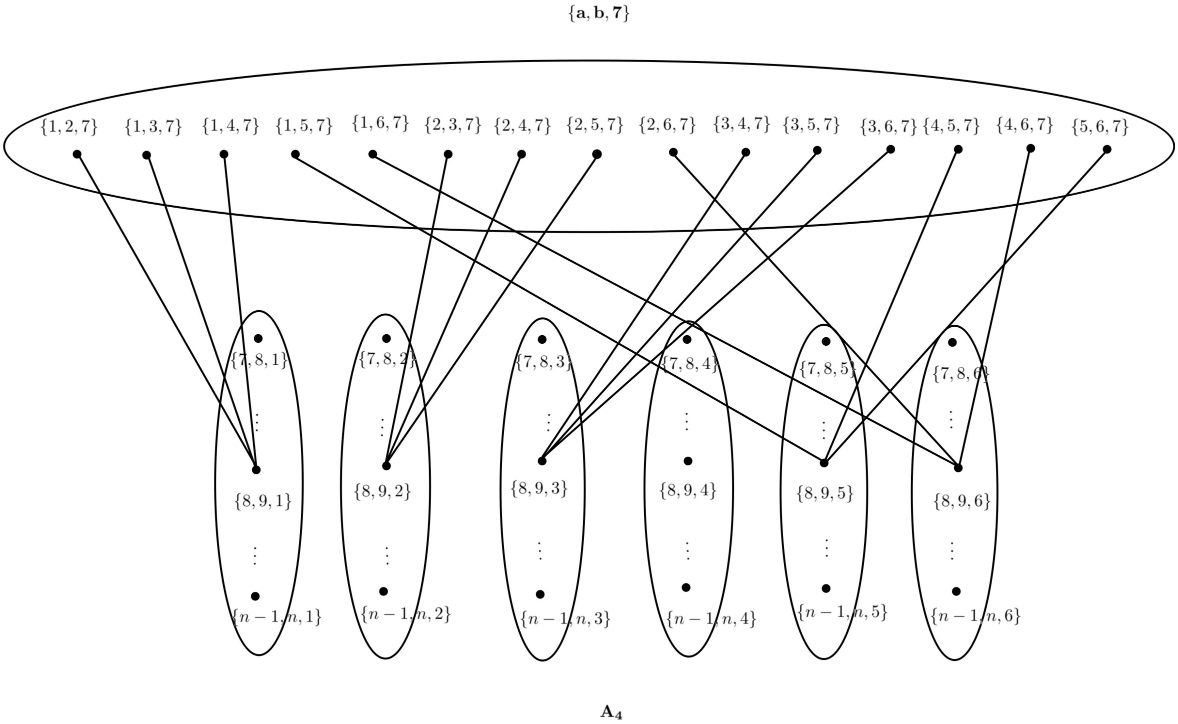

Figure 3: The stars S^i, $1 \leq i \leq 5$, where x=8, y=9

Similarly, we proceed the same process if $n_1 + 2 \leq c \leq n - 2$, to get a claw-decomposition in $\langle E(\bigcup_{(a,b)\in\mathcal{P}_2(N_1)} \{a,b,c\}, A_4)\rangle$. If $n-1 \leq c \leq n$, then take

$$(x, y) = \begin{cases} (7, n) & \text{if } c = n-1 \\ (7, 8) & \text{if } c = n \end{cases}$$

in the above construction, to get a claw-decomposition in $\langle E(\bigcup_{(a,b)\in\mathcal{P}_2(N_1)} \{a,b,c\}, A_4)\rangle$. Hence the graph $\langle E(\bigcup_{(a,b)\in\mathcal{P}_2(N_1)} \{a,b,c\}, A_4)\rangle$, $n_1+1 \leq c \leq n$ has a claw-decomposition. Therefore, there exists a claw-decomposition in H_5. □

Lemma 2.7. *The graph H_6 is claw-decomposable.*

Proof. Let $n = 6m$, where $m \geq 2$ be a positive integer. In $\langle E(A_1, A_3)\rangle$, $D_{\langle E(A_1,A_3)\rangle}\{a,b,c\}$=$9n_2$, for all $\{a,b,c\} \in A_1$. In $\langle E(A_1, A_4)\rangle$, $D_{\langle E(A_1,A_4)\rangle}\{a,b,c\}$= $3\binom{n_2}{2}$, for all $\{a,b,c\} \in A_1$. In H_6, the degree of each vertex of A_1 is $9n_2 + 3\binom{n_2}{2}$=$3[3n_2 + \binom{n_2}{2}]$. Which is a multiple of 3, and hence has a claw-decomposition in H_6. □

Lemma 2.8. *The graph H_7 is claw-decomposable.*

Proof. Let $n = 6m$, where $m \geq 2$ be a positive integer. In $\langle E(A_2, A_3)\rangle$, $D_{\langle E(A_2,A_3)\rangle}\{a,b,c\}$=$3\binom{n_1}{2}$, for all $\{a,b,c\} \in A_2$. In $\langle E(A_2, A_4)\rangle$, $D_{\langle E(A_2,A_4)\rangle}\{a,b,c\}$ =$18(n_2 - 3)$, for all $\{a,b,c\} \in A_2$. In H_7, the degree of each vertex of A_2 is $3\binom{n_1}{2} + 18(n_2 - 3)$=$3[\binom{n_1}{2} + 6(n_2 - 3)]$. Which is a multiple of 3, and hence has a claw-decomposition in H_7. □

Theorem 2.9. *If $n \equiv 0 (mod\ 6)$, then $GKG_{n,3,1}$ is claw-decomposable.*

Proof. Let $n = 6m$, where $m \geq 1$ be a positive integer. Let $n_1 = 6$ and $n_2=6(m-1)$. We apply mathematical induction on m, to prove the theorem. If $m=1$, then the graph $GKG_{6,3,1}$ has a claw-decomposition, by Lemma 2.1. Therefore, the result is true for $m=1$. Assume that the result is true for all $1 < m < k$. Now, we prove that the result is true for $m=k$. The graph H_1 has a claw-decomposition, by Remark 2.3. The graph $H_2 = GKG_{n_2,3,1} \cong GKG_{6(k-1),3,1}$, has a claw-decomposition, by our assumption. The graphs H_3, H_4, H_5, H_6 and H_7 have a claw-decomposition, by Lemma 2.4, 2.5, 2.6, 2.7 and 2.8 respectively. By remark 2.2, the graph $GKG_{6k,3,1}$ is claw-decomposable. □

Theorem 2.10. *For $n > 6$ and $n \equiv 1, 2, 3, 4, 5 (mod\ 6)$, the graph $GKG_{n,3,1}$ is claw-decomposable.*

Proof. Let $A=\{1, 2, ..., n\}$, $A_1=\{\{1, b, c\}|(b, c) \in \mathcal{P}_2(A \setminus \{1\})\}$ and $A_2=\{\{a, b, c\}|(a, b, c) \in \mathcal{P}_3(A \setminus \{1\})\}$. Then A_1 and A_2 are disjoint subsets of $V(GKG_{n,3,1})$ and $V(GKG_{n,3,1})=A_1 \cup A_2$. The graph $GKG_{n,3,1}=\langle A_1\rangle \cup \langle A_2\rangle \cup \langle E(A_1, A_2)\rangle$. Observe that, the graph $\langle A_1\rangle \cong KG_{n-1,2}$, and hence has a claw-decomposition, by Theorem 1.1.

Now, we prove that the graph $\langle E(A_1, A_2)\rangle$ has a claw-decomposition. If $n \equiv 1, 3, 4 (mod\ 6)$, the degree of each vertex of A_1 in $\langle E(A_1, A_2)\rangle$ is exactly $2\binom{n-3}{2}$, which is a multiple of 3. If $n \equiv 2, 5 (mod\ 6)$, the degree of each vertex of A_2 in $\langle E(A_1, A_2)\rangle$ is exactly $3(n-4)$. Hence $\langle E(A_1, A_2)\rangle$ has a claw-decomposition.

Observe that, the graph $\langle A_2\rangle \cong GKG_{n-1,3,1}$. If $n \equiv 1 (mod\ 6)$, the graph $GKG_{n-1,3,1}$ has a claw-decomposition by Theorem 2.9. Hence by the above arguments, the graph $GKG_{n,3,1}$ has a claw-decomposition. If $n \equiv 2 (mod\ 6)$, then $\langle A_2\rangle$ has a claw-decomposition by the previous case. Hence the graph $GKG_{n,3,1}$ has a claw-decomposition. Similarly, if $n \equiv 3, 4, 5 (mod\ 6)$, then apply the above procedure recursively to get a claw-decomposition in $\langle A_2\rangle$. Thus $GKG_{n,3,1}$ is claw-decomposable. □

By combining Remark 2.3, Lemma 2.1 to 2.8, Theorem 2.9 and 2.10, we get the following:

Theorem 2.11. *For all $n \geq 6$, the Generalized Kneser Graph $GKG_{n,3,1}$ is claw-decomposable.*

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