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Several generalizations and variations of Chu-Vandermonde identity

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Abstract. In this paper we prove some combinatorial identities which can be considered as generalizations and variations of remarkable Chu-Vandermonde identity. These identities are proved by using an elementary combinatorial-probabilistic approach to the expressions for the k-th moments ($k = 1, 2$) of some particular cases of investigated discrete random variables by the author of this paper [16]. As applications of one of these Chu-Vandermonde-type identities, we prove two congruences modulo p^4 and modulo p^5, where $p \geq 5$ is a prime number.

1. Introduction and Preliminaries

As noticed in [1, Section 1.1], the probabilistic method is a powerful tool in tackling many problems in Discrete Mathematics (Combinatorics, Graph Theory, Number Theory and Combinatorial Geometry). More recently, it has been applied in the development of efficient algorithmic techniques and in the study of various computational problems.

In this paper we present three combinatorial identities whose proofs are based on a simple probability technique consisting on calculations of k-th moments ($k = 1, 2$) of some discrete random variables. Our proofs consist of showing that these identities essentially compute the moments of order k ($k = 1, 2$) of the discrete random variable defined in [16]. Notice that this random variable is a generalization of the complex-valued discrete random variable defined in [20] by providing a statistical analysis for efficient detection of signal components when missing data samples are present (cf. [21]). On the other hand, the author of this paper continued the research on the mentioned complex-valued discrete random variables [16].

Notice that combinatorial identities and combinatorial problems appear in many areas of mathematics, notably in Number Theory, Probability Theory, Topology, Geometry, Mathematical Optimization, Computer Science, Ergodic Theory and Statistical Physics.

As usually, throughout our considerations we use the term "multiset" (often written as "set") to mean "a totality having possible multiplicities"; so that two (multi)sets will be counted as equal if and only if they have the same elements with identical multiplicities. Let $\mathbb{C}$ and $\mathbb{R}$ denote the fields of complex and real numbers, respectively. For a given positive integer N, let $\mathcal{M}_N$ denote the collections of all multisets of

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the form

$$\Phi_N = \{z_1, z_2, \ldots, z_N : z_1, z_2, \ldots, z_N \in \mathbb{C}\}. \tag{1}$$

Furthermore, denote by $\mathcal{M}$ the set consisting of all multisets of the form (1), i.e.,

$$\mathcal{M} = \bigcup_{N=1}^{\infty} \mathcal{M}_N.$$

Following Definition 1.2 from [16], the random variable $X(m, \Phi_N)$ was generalized in [16] as follows.

Definition 1.1. *([16, Definition 1.1]) Let N and m be arbitrary nonnegative integers such that $1 \le m \le N$. For given not necessarily distinct complex numbers $z_1, z_2, \ldots, z_N$, let $\Phi_N \in \mathcal{M}_N$ be a multiset defined by (1). Define the discrete complex-valued random variable $X(m, \Phi_N)$ as*

$$\begin{aligned} &\operatorname{Prob}\left(X(m,\Phi_N) = \sum_{i=1}^{m} z_{n_i}\right) \\ &= \frac{1}{\binom{N}{m}} \cdot \left|\{\{t_1, t_2, \ldots, t_m\} \subset \{1, 2, \ldots, N\} : \sum_{i=1}^{m} z_{t_i} = \sum_{i=1}^{m} z_{n_i}\}\right| \\ &=: \frac{q(n_1, n_2, \ldots, n_m)}{\binom{N}{m}}, \end{aligned} \tag{2}$$

where $\{n_1, n_2, \ldots, n_m\}$ is an arbitrary fixed subset of $\{1, 2, \ldots, N\}$ such that $1 \le n_1 < n_2 < \cdots < n_m \le N$. Moreover, $q(n_1, n_2, \ldots, n_m)$ is the cardinality of a collection of all subsets $\{t_1, t_2, \ldots, t_m\}$ of the set $\{1, 2, \ldots, N\}$ such that $\sum_{i=1}^{m} z_{t_i} = \sum_{i=1}^{m} z_{n_i}$.

Notice that the above definition is correct taking into account that there are $\binom{N}{m}$ index sets $T \subset \{1, 2, \ldots, N\}$ with m elements. Moreover, a very short, but not strongly exact version of Definition 1.1 is given as follows (cf. [16, Definition 1.2′]).

Definition 1.1′. Let N and m be arbitrary nonnegative integers such that $1 \le m \le N$. For given not necessarily distinct complex numbers $z_1, z_2, \ldots, z_N$, let $\Phi_N \in \mathcal{M}_N$ be a multiset defined by (1). Choose a random subset S of size m (the so-called m-element subset) without replacement from the set $\{1, 2, \ldots, N\}$. Then the complex-valued discrete random variable $X(m, \Phi_N)$ is defined as a sum

$$X(m, \Phi_N) = \sum_{n \in S} z_n.$$

It was proved in [16] the following result (cf. [16, proof of Theorem 2.1] as a particular case).

Theorem 1.2. *([16, the expressions (3) and (5) of Theorem 1.2]). Let N and m be positive integers such that $N \ge 2$ and $1 \le m \le N$. Let $\Phi_N = \{z_1, z_2, \ldots, z_N\}$ be any multiset with $z_1, z_2, \ldots, z_N \in \mathbb{C}$. Then the expected value of the random variable $X(m, \Phi_N)$ from Definition* 1.1 *and the second moment of the random variable $|X(m, \Phi_N)|$ are respectively given by*

$$\mathbb{E}[X(m, \Phi_N)] = \frac{m}{N} \sum_{i=1}^{N} z_i, \tag{3}$$

and

$$\mathbb{E}[|X(m, \Phi_N)|^2] = \frac{m}{N(N-1)} \left((N-m) \sum_{i=1}^{N} |z_i|^2 + (m-1)\left|\sum_{i=1}^{N} z_i\right|^2\right). \tag{4}$$

Notice that in the case when $X(m,\Phi_N)$ is a real-valued random variable (i.e., if in Definition 1.1 $z_1,z_2,\ldots,z_n$ are real numbers), then a related expression for the third moment $\mathbb{E}[(X(m,\Phi_N))^3]$ of $X(m,\Phi_N)$ can be proved similarly as the above expression (4) given in [16, Theorem 2.1].

It was indicated in [16, Section 3] that for some particular cases of sets $\Phi(N)$ (given by (1)) and some values m, the expressions (3) and (4) concerning the associated random variables $X(m,\Phi_N)$ yield some combinatorial identities. For a comprehensive list of combinatorial identities, see [6] (also see [17] and [8, Chapter 5]). Motivated by this fact, by using some other particular cases of the random variables $X(m,\Phi_N)$ from Definition 1.1, in the next section we deduce some new and some known combinatorial identities which can be considered as generalizations of Chu-Vandermonde identity. Notice that Chu-Vandermonde identity is often called Vandermonde's identity or sometimes Vandermonde's formula.

2. Chu-Vandermonde-type identities and their proofs

We start with the following identity.

Identity 2.1. *Let $n_1,n_2,\ldots,n_s$ be arbitrary positive integers and le $z_1,z_2,\ldots,z_s$ be arbitrary complex numbers ($s\geq 2$). If m is a a positive integer such that $m\leq\sum_{i=1}^s n_i$, then*

$$\begin{aligned}&\sum_{\substack{\sum_{i=1}^s k_i=m\\ k_1\le n_1,k_2\le n_2,\ldots,k_s\le n_s}}\binom{n_1}{k_1}\binom{n_2}{k_2}\cdots\binom{n_s}{k_s}(k_1z_1+k_2z_2+\cdots+k_sz_s)\\ &=\binom{\sum_{i=1}^s n_i}{m}\frac{m(\sum_{i=1}^s n_iz_i)}{\sum_{i=1}^s n_i},\end{aligned}\tag{5}$$

where the summation ranges over all nonnegative integers k_i ($i=1,2,\ldots,s$) such that $k_1\leq n_1,k_2\leq n_2,\ldots,k_s\leq n_s$ and $\sum_{i=1}^s k_i=m$.

Proof. Put $\sum_{i=1}^s n_i=N$ and consider the multiset Φ_N defined by

$$\Phi_N=\{\underbrace{z_1,\ldots,z_1}_{n_1},\underbrace{z_2,\ldots,z_2}_{n_2},\ldots,\underbrace{z_s,\ldots,z_s}_{n_s}\}.$$

Now consider the random variable $X(m,\Phi_N)$ given by Definition 1.1. Then by the expression (3) of Theorem 1.2, we have

$$\mathbb{E}[X(m,N)]=\frac{m(\sum_{i=1}^s n_iz_i)}{\sum_{i=1}^s n_i}.\tag{6}$$

On the other hand, for each s-tuple $(k_1,k_2,\ldots,k_s)$ of nonnegative integers $k_1,k_2,\ldots,k_s$ such that $k_1\leq n_1,k_2\leq n_2,\ldots,k_s\leq n_s$ and $\sum_{i=1}^s k_i=m$, by (3), we get

$$\mathrm{Prob}\left(X(m,\Phi_N)=\sum_{i=1}^s k_iz_i\right)=\frac{1}{\binom{\sum_{i=1}^s n_i}{m}}\binom{n_1}{k_1}\binom{n_2}{k_2}\cdots\binom{n_s}{k_s}.\tag{7}$$

Then by the definition of the expectation of a complex-valued discrete random variable, from (7) we find that

$$\begin{aligned}&\mathbb{E}[X(m,N)]\\ &=\frac{1}{\binom{\sum_{i=1}^s n_i}{m}}\sum_{\substack{\sum_{i=1}^s k_i=m\\ k_1\le n_1,k_2\le n_2,\ldots,k_s\le n_s}}\binom{n_1}{k_1}\binom{n_2}{k_2}\cdots\binom{n_s}{k_s}(k_1z_1+k_2z_2+\cdots+k_sz_s),\end{aligned}\tag{8}$$

where the summation ranges over all nonnegative integers k_i ($i=1,2,\ldots,s$) such that $k_1\leq n_1,k_2\leq n_2,\ldots,k_s\leq n_s$ and $\sum_{i=1}^s k_i=m$.

Finally, comparing the equalities (6) and (8), we immediately obtain (5). □

Remark 2.2. *Recently, by using functional equations of the generating function of certain class of polynomials, a new Chu-Vandermonde-type identity (Vandermonde type convolution formula) is derived in [10, Theorem 5.4 of Section 5]. As a special case of this result, is the following identity [10, Corollary 5.5 of Section 5]:*

$$\sum_{v_1=0}^{n}\binom{k_1+v_1-1}{v_1}\binom{k_2+n-v_1-1}{n-v_1}=\binom{k_1+k_2+n-1}{n},$$

where $k_1 \geq 1$, $k_2 \geq 1$ and n are nonnegative integers. Another generalization of Chu-Vandermonde identity was given in [9]. Moreover, two diferent interpretations of this identity are recently considered in [19], as an identity for polynomials, and as an identity for infinite matrices.

Remark 2.3. *If $P_{s-1}(z) = \sum_{i=1}^{s} n_i z^{i-1}$ is a complex polynomial of the variable z of degree $s-1$ with integer coefficients $n_1, n_2, \dots, n_s \geq 0$ ($n_s \neq 0$), then taking $z_i = z^{i-1}$ ($i = 1, 2, \dots, s$) into the identity (5), it becomes*

$$\sum_{\substack{\sum_{i=1}^{s} k_i=m\\ k_1\le n_1,k_2\le n_2,\dots,k_s\le n_s}}\binom{n_1}{k_1}\binom{n_2}{k_2}\cdots\binom{n_s}{k_s}(k_1+k_2z+\cdots+k_sz^{s-1})$$
$$=\frac{m}{\sum_{i=1}^{s} n_i}\binom{\sum_{i=1}^{s} n_i}{m}P_{s-1}(z),$$

where the summation ranges over all nonnegative integers k_i ($i = 1, 2, \dots, s$) such that $k_1 \leq n_1, k_2 \leq n_2, \dots, k_s \leq n_s$ and $\sum_{i=1}^{s} k_i = m$.

Remark 2.4. *As usually, if we use the convention that $\binom{a}{b} = 0$ for all nonnegative integers a and b such that $b > a$, then the conditions $k_1 \leq n_1, k_2 \leq n_2, \dots, k_s \leq n_s$ which appear under the first sum $\sum \cdot$ of (5) can be omitted.*

A particular case of the Identity 2.1 is the Identity 2.5 given below, which is a well known "multinomial" generalization of the Vandermonde identity (often called *Vandermonde convolution formula* or *Chu-Vandermonde convolution*) (see, e.g., [17]).

Identity 2.5. *Let $n_1, n_2, \dots, n_s$ ($s \geq 2$) be arbitrary positive integers. If m is a positive integer such that $m \leq \sum_{i=1}^{s} n_i$, then*

$$\sum_{\substack{\sum_{i=1}^{s} k_i=m\\ k_1\le n_1,k_2\le n_2,\dots,k_s\le n_s}}\binom{n_1}{k_1}\binom{n_2}{k_2}\cdots\binom{n_s}{k_s}=\binom{\sum_{i=1}^{s} n_i}{m}, \tag{9}$$

where the summation ranges over all nonnegative integers k_i ($i = 1, 2, \dots, s$) such that $k_1 \leq n_1, k_2 \leq n_2, \dots, k_s \leq n_s$ and $\sum_{i=1}^{s} k_i = m$.

Proof. Taking $z_1 = z_2 = \cdots = z_s = 1$ into the equality (5), we immediately obtain the equality (9). □

Remark 2.6. *There are well known algebraic and combinatorial proofs of the identity (9) (see, e.g., [25]). Notice also that for $s = 2$, the identity (9) with $k_1 = k$, $n_1 = a$ and $n_2 = b$ simplifies to the Chu-Vandermonde identity given by (see, e.g., [2, p. 67])*

$$\sum_{k=0}^{m}\binom{a}{k}\binom{b}{m-k}=\binom{a+b}{m}, \tag{10}$$

which also holds for any complex numbers a and b. Notice that the identity (10) is named after A.T. Vandermonde (1772), although it was already known in 1303 by the Chinese mathematician Zhu Shijie (Chu Shih-Chieh) (see [3, pp. 59–60] for the history). This identity plays an important role in Combinatorics, Combinatorial Number Theory and Probability Theory (see [8], [7] and [17]). As indicated in [17, p. 8], Vandermonde convolution formula is perhaps the most widely used combinatorial identity. In the literature there are many proofs of this identity and its several generalizations. A proof given in [23] was established by giving probabibilstic interpretations to the summands.

Taking $s = 2$ and $z_1/z_2 = z$ ($z_2 \neq 0$) into (5), it simplifies to the following Vandermonde-type convolution formula.

Identity 2.7. *Let n_1 and n_2 be arbitrary positive integers and let z be any complex numbers. If m is a positive integer such that $m \leq n_1 + n_2$, then*

$$\sum_{k=0}^{m} \binom{n_1}{k}\binom{n_2}{m-k}(kz + (m-k)) = \binom{n_1+n_2}{m}\frac{2m(n_1 z + n_2)}{n_1+n_2}. \tag{11}$$

Remark 2.8. *Observe that taking $z = 1$, $n_1 = a$ and $n_2 = b$ into (11), it immediately reduces to the Chu-Vandermonde identity given by (10).*

Substituting $s = 3$ $z_1/z_3 = z$ and $z_2/z_3 = w$ ($z_3 \neq 0$) into (5), it reduces to the following Vandermonde-type convolution formula.

Identity 2.9. *Let n_1, n_2 and n_3 be arbitrary positive integers and let z and w be arbitrary complex numbers. If m is a positive integer such that $m \leq n_1 + n_2 + n_3$, then*

$$\begin{aligned}&\sum_{\substack{0\leq k_1+k_2\leq m\\ k_1\geq 0, k_2\geq 0}} \binom{n_1}{k_1}\binom{n_2}{k_2}\binom{n_3}{m-k_1-k_2}(k_1 z + k_2 w + (m-k_1-k_2))\\ &= \binom{n_1+n_2+n_3}{m}\frac{2m(n_1 z + n_2 w + n_3)}{n_1+n_2+n_3}.\end{aligned} \tag{12}$$

Taking $z = w = 1/2$ into (12), we obtain the following identity.

Identity 2.10. *Let n_1, n_2 and n_3 be arbitrary positive integers and let z_1 and z_2 be arbitrary complex numbers. If m is a positive integer such that $m \leq n_1 + n_2 + n_3$, then*

$$\begin{aligned}&\sum_{\substack{0\leq k_1+k_2\leq m\\ k_1\geq 0, k_2\geq 0}} (2m-k_1-k_2)\binom{n_1}{k_1}\binom{n_2}{k_2}\binom{n_3}{m-k_1-k_2}\\ &= \frac{2m(n_1+n_2+2n_3)}{n_1+n_2+n_3}\binom{n_1+n_2+n_3}{m}.\end{aligned} \tag{13}$$

Another special case of Identity 2.1 is given as follows.

Identity 2.11. *Let s and l be arbitrary positive integers, and let m be a positive integer such that $m \leq sl$. Then*

$$\sum_{\substack{\sum_{i=1}^{s} k_i = m\\ k_1\leq l, k_2\leq l,\dots,k_s\leq l}} \binom{l}{k_1}\binom{l}{k_2}\cdots\binom{l}{k_s}(k_1 + 2k_2 + \cdots + sk_s) = \frac{m(s+1)}{2}\binom{sl}{m}, \tag{14}$$

where the summation ranges over all nonnegative integers k_i ($i = 1, 2, \dots, s$) such that $\sum_{i=1}^{s} k_i = m$ and $k_1 \leq l, k_2 \leq l, \dots, k_s \leq l$.

Proof. Substituting $z_i = i$ for all $i = 1, 2, \dots, s$ and $n_1 = n_2 = \cdots = n_s = l$ into (5), it immediately reduces to the equality (14). □

As a consequence of Identity 2.11, we obtain the following "supercongruence" closely related to the remarkable *Wolstenhlme's theorem* which asserts that

$$\binom{2p-1}{p-1} \equiv 1 \pmod{p^3}$$

for all primes $p \geq 3$ ([24]; also see [12, p. 3] and [14]).

Congruence 2.12. *Let $p \geq 5$ be a prime. Then for each positive integer s there holds*

$$\sum_{\substack{\sum_{i=1}^{s} k_i = p \\ k_1, k_2, \dots, k_s \geq 0}} \binom{p}{k_1}\binom{p}{k_2}\cdots\binom{p}{k_s}(k_1 + 2k_2 + \cdots + sk_s) \equiv \frac{s(s+1)p}{2} \pmod{p^4}. \tag{15}$$

In particular, we have

$$\sum_{\substack{\sum_{i=1}^{s} k_i = p \\ k_1, k_2, \dots, k_s \geq 0}} \binom{p}{k_1}\binom{p}{k_2}\cdots\binom{p}{k_s}(k_1 + 2k_2 + \cdots + sk_s) \equiv 0 \pmod{p}. \tag{16}$$

Proof. If we substitute $l = m = p$ into the equality (14), then its right hand side is equal to $\frac{(s+1)p}{2}\binom{sp}{p}$. Since by the classical Glaisher's congruence [5, p. 21] (or more general, Ljunggren's congruence [4]; also see [12, the congruences (15), p. 7, (35) and (36), p. 11] (cf. [15, Section 3.3] and [13]), for any prime $p \geq 5$ and a positive integer s, we have

$$\binom{sp}{p} \equiv s \pmod{p^3},$$

and hence,

$$\frac{(s+1)p}{2}\binom{sp}{p} \equiv \frac{s(s+1)p}{2} \pmod{p^4}.$$

Substituting the above congruence into (14) with $l = m = p$, we immediately obtain the congruence (15). Finally, reducing the modulus in (18) to (mod p), implies the congruence (16). □

Let us recall that a prime p is said to be a *Wolstenholme prime* (see, e.g., [11] and [12, Section 7]; this is Sloane's sequence A088164 from [18]) if it satisfies the congruence

$$\binom{2p-1}{p-1} \equiv 1 \pmod{p^4}.$$

It is well known (see [5, p. 21] and [12, p. 14]) that p is a Wolstenholme prime if and only if p divides he numerator of the *Bernoulli number* B_{p-3}. It can be shown that for any Wolstenholme prime, the congruence (15) holds modulo p^5, i.e., we have the following assertion.

Congruence 2.13. *Let p be a Wolstenholme prime. Then for each positive integer s there holds*

$$\sum_{\substack{\sum_{i=1}^{s} k_i = p \\ k_1, k_2, \dots, k_s \geq 0}} \binom{p}{k_1}\binom{p}{k_2}\cdots\binom{p}{k_s}(k_1 + 2k_2 + \cdots + sk_s) \equiv \frac{s(s+1)p}{2} \pmod{p^5}. \tag{17}$$

Proof. Notice that by a classical result of Glaisher ([5, p. 21]; also see [12, the conguence (15), p. 7] and the *Jacobsthal's congruence* [4]), for any Wolstenholme prime p,

$$\binom{sp}{p} \equiv s \pmod{p^4}.$$

Then the rest of the proof is quite similar to that of the previous Congruence 2.12, and hence may be omitted. □

Another consequence of Identity 2.1 is given as follows.

Identity 2.14. *Let $n \geq 2$ and s be fixed positive integers and let $k = k_1 + k_2 n + \cdots + k_s n^{s-1}$ be the base n representation of a positive integer $k < n^s$ (with $0 \leq k_1, k_2, \ldots, k_s \leq n-1$). If m is a positive integer such that $m \leq s(n-1)$, then*

$$\begin{aligned}&\sum_{\substack{\sum_{i=1}^{s} k_i = m\\ k_1 \leq n-1, k_2 \leq n-1, \ldots, k_s \leq n-1}} \binom{n-1}{k_1}\binom{n-1}{k_2}\cdots\binom{n-1}{k_s}(k_1 + k_2 n + \cdots + k_s n^{s-1})\\ &= (n^s - 1)\binom{s(n-1)-1}{m-1}.\end{aligned} \tag{18}$$

Proof. Setting $n_1 = n_2 = \cdots = n_s = n-1$ and $z_i = n^{i-1}$ $(i = 1, 2, \ldots, s)$ into the identity (5) and using the identity $\binom{s(n-1)}{m} = \frac{s(n-1)}{m}\binom{s(n-1)-1}{m-1}$, immediately gives the identity (18). □

The binary case of Identity 2.14 can be reformulated as follows.

Corollary 2.15. *Let s and m be positive integers such that $m \leq s$. Then the sum of all positive integers less than 2^s whose binary representation contains exactly m 1's is equal to $(2^s - 1)\binom{s-1}{m-1}$ (as usually, it is assumed that $\binom{0}{0} = 1$).*

Proof. Taking $n = 2$ into (21), we have

$$\sum_{\substack{\sum_{i=1}^{s} k_i = m\\ k_1, k_2, \ldots, k_s \in \{0,1\}}} (k_1 + 2k_2 + \cdots + 2^{s-1}k_s) = (2^s - 1)\binom{s-1}{m-1}.$$

□

Remark 2.16. *Note that Corollary 2.15 can be easily proved by induction on $s \geq 1$ and also by using a simple counting argument.*

A quadratic analogue of Identity 2.1 is given as follows.

Identity 2.17. *Let $n_1, n_2, \ldots, n_s$ be arbitrary positive integers and let $z_1, z_2, \ldots, z_s$ be arbitrary complex numbers ($s \geq 2$). If m is a positive integer such that $2 \leq m \leq \sum_{i=1}^{s} n_i$, then*

$$\begin{aligned}&\sum_{\substack{\sum_{i=1}^{s} k_i = m\\ k_1 \leq n_1, k_2 \leq n_2, \ldots, k_s \leq n_s}} \binom{n_1}{k_1}\binom{n_2}{k_2}\cdots\binom{n_s}{k_s}|k_1 z_1 + k_2 z_2 + \cdots + k_s z_s|^2\\ &= \binom{\sum_{i=1}^{s} n_i - 2}{m-1}\sum_{i=1}^{s} n_i |z_i|^2 + \binom{\sum_{i=1}^{s} n_i - 2}{m-2}\Big|\sum_{i=1}^{s} n_i z_i\Big|^2,\end{aligned} \tag{19}$$

where the summation ranges over all nonnegative integers k_i $(i = 1, 2, \ldots, s)$ such that $k_1 \leq n_1, k_2 \leq n_2, \ldots, k_s \leq n_s$ and $\sum_{i=1}^{s} k_i = m$.

Proof. Put $\sum_{i=1}^{s} n_i = N$ and as in the proof of Identity 2.1, consider the multiset Φ_N defined by

$$\Phi_N = \{\underbrace{z_1, \ldots, z_1}_{n_1}, \underbrace{z_2, \ldots, z_2}_{n_2}, \ldots, \underbrace{z_s, \ldots, z_s}_{n_s}\}.$$

Now consider the random variable $X(m, \Phi_N)$ given by Definition 1.1. Then by the expression (4) of Theorem 1.2, we have

$$\mathbb{E}[|X(m, \Phi_N)|^2] = \frac{m(N-m)}{N(N-1)}\Big(\sum_{i=1}^{s} n_i |z_i|^2 + \frac{m(m-1)}{N(N-1)}\Big|\sum_{i=1}^{s} n_i z_i\Big|^2\Big). \tag{20}$$

On the other hand, for each s-tuple $(k_1, k_2, \dots, k_s)$ of nonnegative integers $k_1, k_2, \dots, k_s$ such that $k_1 \le n_1, k_2 \le n_2, \dots, k_s \le n_s$ and $\sum_{i=1}^s k_i = m$, by (2) we get

$$\mathrm{Prob}\left(X(m, \Phi_N) = \sum_{i=1}^s k_i z_i\right) = \frac{1}{\binom{N}{m}}\binom{n_1}{k_1}\binom{n_2}{k_2}\cdots\binom{n_s}{k_s}. \tag{21}$$

Then by definition of the expectation of a discrete random variable, from (21) we find that

$$\begin{aligned}&\mathbb{E}[|X(m, \Phi_N)|^2]\\ &= \frac{1}{\binom{N}{m}} \sum_{\substack{\sum_{i=1}^s k_i = m\\ k_1 \le n_1, k_2 \le n_2, \dots, k_s \le n_s}} \binom{n_1}{k_1}\binom{n_2}{k_2}\cdots\binom{n_s}{k_s} |k_1 z_1 + k_2 z_2 + \cdots + k_s z_s|^2,\end{aligned} \tag{22}$$

where the summation ranges over all nonnegative integers k_i $(i = 1, 2, \dots, s)$ such that $k_1 \le n_1, k_2 \le n_2, \dots, k_s \le n_s$ and $\sum_{i=1}^s k_i = m$.

Note that if $m = \sum_{i=1}^s n_i := N$, then the sum on the left hand side of (19) consists of one term (with $k_1 = n_1, k_2 = n_2, \dots, k_s = n_s$) equals to $|\sum_{i=1}^s n_i z_i|^2$, which is (because of $\binom{m-2}{m-1} = 0$) identically equal to the the right hand side of (19). Finally, if $m \le N - 1$, then comparing the equalities (20) and (22), using the identities $\binom{N}{m} = \frac{N(N-1)}{m(m-1)}\binom{N-2}{m-2}$ and $\binom{N}{m} = \frac{N(N-1)}{m(N-m)}\binom{N-2}{m-1}$, we immediately obtain (19). □

In particular, Identity 2.17 implies the following one.

Identity 2.18. *Let s and l be arbitrary positive integers, and let m be a positive integer such that $m \le sl$. Then*

$$\begin{aligned}&\sum_{\substack{\sum_{i=1}^s k_i = m\\ k_1 \le l, k_2 \le l, \dots, k_s \le l}} \binom{l}{k_1}\binom{l}{k_2}\cdots\binom{l}{k_s}(k_1 + 2k_2 + \cdots + sk_s)^2\\ &= \frac{m(s+1)(3s^2lm + 3slm + s^2l - 4sm - sl - 2m)}{12(sl-1)}\binom{sl}{m},\end{aligned} \tag{23}$$

where the summation ranges over all nonnegative integers k_i $(i = 1, 2, \dots, s)$ such that $\sum_{i=1}^s k_i = m$ and $k_1 \le l, k_2 \le l, \dots, k_s \le l$.

Proof. Substituting $z_i = i$ for all $i = 1, 2, \dots, s$ and $n_1 = n_2 = \cdots = n_s = l$ into (19), and taking $\sum_{i=1}^s i = s(s+1)/2$ and $\sum_{i=1}^s i^2 = s(s+1)(2s+1)/6$, it immediately reduces to the identity (23). □

Furthermore, notice that by linearity, Identity 2.1 can be immediadely extended in matrix forms as follows.

Identity 2.19. *Denote by $\mathbb{K}^{M\times N}$ the vector space of all matrices over the field $\mathbb{K}$ with M rows and N columns ($\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$ and $M, N \ge 1$). Let $n_1, n_2, \dots, n_s$ be arbitrary positive integers and let $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_s \in \mathbb{K}^{M\times N}$ be arbitrary $M \times N$ matrices. If m is a positive integer such that $m \le \sum_{i=1}^s n_i$, then*

$$\begin{aligned}&\sum_{\substack{\sum_{i=1}^s k_i = m\\ k_1 \le n_1, k_2 \le n_2, \dots, k_s \le n_s}} \binom{n_1}{k_1}\binom{n_2}{k_2}\cdots\binom{n_s}{k_s}(k_1\mathbf{A}_1 + k_2\mathbf{A}_2 + \cdots + k_s\mathbf{A}_s)\\ &= \binom{\sum_{i=1}^s n_i}{m}\frac{m(\sum_{i=1}^s n_i\mathbf{A}_i)}{\sum_{i=1}^s n_i},\end{aligned} \tag{24}$$

where the summation ranges over all nonnegative integers k_i $(i = 1, 2, \dots, s)$ such that $k_1 \le n_1, k_2 \le n_2, \dots, k_s \le n_s$ and $\sum_{i=1}^s k_i = m$.

Identity 2.20. *Denote by $\mathbb{K}^{N\times N}$ the algebra of all square matrices of order N ($N \geq 1$) over the field $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$. Let $n_1, n_2, \dots, n_s$ be arbitrary positive integers and let $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_s \in \mathbb{K}^{N\times N}$ be arbitrary square matrices of order N. If m is a positive integer such that $m \leq \sum_{i=1}^{s} n_i$, then*

$$\begin{aligned}&\sum_{\substack{\sum_{i=1}^{s} k_i = m \\ k_1\leq n_1, k_2\leq n_2,\dots,k_s\leq n_s}} \binom{n_1}{k_1}\binom{n_2}{k_2}\cdots\binom{n_s}{k_s}(k_1\mathbf{A}_1 + k_2\mathbf{A}_2 + \cdots + k_s\mathbf{A}_s)^2 \\ &= \binom{\sum_{i=1}^{s} n_i - 2}{m-1}\sum_{i=1}^{s} n_i\mathbf{A}_i^2 + \binom{\sum_{i=1}^{s} n_i - 2}{m-2}(\sum_{i=1}^{s} n_i\mathbf{A}_i)^2,\end{aligned} \tag{25}$$

where the summation ranges over all nonnegative integers k_i ($i = 1, 2, \dots, s$) such that $k_1 \leq n_1, k_2 \leq n_2, \dots, k_s \leq n_s$ and $\sum_{i=1}^{s} k_i = m$.

Identity 2.21. *Denote by $\mathbb{K}^{M\times N}$ the vector space of all matrices over the field $\mathbb{K}$ with M rows and N columns ($\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$ and $M, N \geq 1$). Let $\mathbf{A}^* \in \mathbb{K}^{N\times M}$ be the conjugate transpose (Hermitian transpose) of a matrix $\mathbf{A} \in \mathbb{K}^{M\times N}$. Let $n_1, n_2, \dots, n_s$ be arbitrary positive integers and let $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_s \in \mathbb{K}^{M\times N}$ be arbitrary $M \times N$ matrices. If m is a positive integer such that $m \leq \sum_{i=1}^{s} n_i$, then*

$$\begin{aligned}&\sum_{\substack{\sum_{i=1}^{s} k_i = m \\ k_1\leq n_1, k_2\leq n_2,\dots,k_s\leq n_s}} \binom{n_1}{k_1}\binom{n_2}{k_2}\cdots\binom{n_s}{k_s}(\sum_{i=1}^{s} k_i\mathbf{A}_i)(\sum_{i=1}^{s} k_i\mathbf{A}_i^*) \\ &= \binom{\sum_{i=1}^{s} n_i - 2}{m-1}\sum_{i=1}^{s} n_i\mathbf{A}_i\mathbf{A}_i^* + \binom{\sum_{i=1}^{s} n_i - 2}{m-2}(\sum_{i=1}^{s} n_i\mathbf{A}_i)(\sum_{i=1}^{s} n_i\mathbf{A}_i^*),\end{aligned} \tag{26}$$

where the summation ranges over all nonnegative integers k_i ($i = 1, 2, \dots, s$) such that $k_1 \leq n_1, k_2 \leq n_2, \dots, k_s \leq n_s$ and $\sum_{i=1}^{s} k_i = m$.

3. Conclusion

We believe that it is possible to define and to investigate the multivariate random variable analogue of the random variable $X(m, \Phi_N)$. Then it would be most likely be possible to generalize several obtained identities in this paper.

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