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Approximation by Durrmeyer type neural network operators

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Abstract. In approximation theory, various type Durrmeyer operators have been defined and their approximation properties have been investigated. We include some of them in the article. In this article, we define a Durrmeyer type Neural Network operators within a compact interval $[a, b]$. Moreover we examine approximation properties of this operators in different function spaces.

1. Introduction

Artificial Neural-Networks are known as a mathematical model of the brain's cell, inter-neuronal communication. Generally, using some special functions σ called activation functions feed-forward artificial neural networks with a one hidden layer are defined as;

$$N_n(x) := \sum_{i=0}^{n} c_i \sigma(w_i x + b_i) \ , \ w_i \in \mathbb{R}^s, \ x \in \mathbb{R}^s, \ s \in \mathbb{N}^+,$$

where, $b_i \in \mathbb{R}$ are treshold value, $c_i \in \mathbb{R}$ are coefficents.

The importance of George Cybenko's density theorem proposed in 1989 in this field should be noted. Cybenko [17] confirmed that any continuous function defined on $[a, b]$ can be approximated by a linear and positive sequence of sigmoidal functions on $C[a, b]$. In other words, Cybenko demonstrated that artificial neural network operators activated by feed-forward one hidden layer of sigmoidal functions can approximate a continuous function.

Owing to Cybenko's idea, models of artificial neural network operators defined in 1992 with the paper of Cardaliaguet and Euvrard [8]. Subsequently, various studies have been conducted by many authors (see [3], [5], [19]).

Cardaliaguet and Euvrard's work shed light on important details between Cybenko's density theorem and approximation theory due to the availability of centered bell-shaped functions frequently used in artificial neural networks with sigmoidal functions.

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With the study by Anastassiou in 1997 [4], the approximation properties of artifical neural network operators began to be investigated. Especially in the last two decades, researchers such as G. Anastassiou ([1]), Chen Z. and Cao F. ([9]), D. Costarelli ([10] − [16]), G. Vinti ([10], [13], [16]), H. Karsli ([26]), Xiang C., Zhao Y., Wang X. and Ye P. ([35]) have defined more flexible operators and better approximation properties of neural network operators using Kantorovich, wavelet, and other forms and examining their approximation properties in various ways.

In approximation theory, many authors have defined integral-based operators like Kantorovich, Durrmeyer and quasi-interpolant to have more flexible operators and to extended them to different spaces, examining many different properties (see [18], [21] − [25], [27], [29], [30] and [31]).

In the present study, we define a Durrmeyer-Type Neural Network Operators and investigate their approximation properties in $C[a, b]$ and $L_p[a, b]$ spaces.

The second section of the this article includes preliminaries and auxiliary results and while the main results are presented in the third section.

2. Preliminaries and auxiliary results

In this section, we will introduce definitions, notations and materials related to the theory of neural network and Durrmeyer Type neural network operators. Additionally, we will denote the norms and moduli of continuity for the aforementioned function spaces.

Definition 2.1. *(Sigmoidal Function)*

A function σ defined on the $\mathbb{R}$ is termed a sigmoid functions if the following conditions,

$$\lim_{x\to-\infty}\sigma(x) = 0 \text{ and } \lim_{x\to+\infty}\sigma(x) = 1$$

are satisfied. The sigmoidal function has many known forms such as the logistic function and the hyperbolic tangent function as follows,

$$\sigma(x) = \frac{1}{1+e^{-x}} \text{ and } \sigma_{\hbar}(x) = \frac{1}{2}\left(1+\tanh x\right),\ x \in \mathbb{R}$$

It should be noted that sigmoidal functions are not a new concept. These functions were defined in population dynamics studies conducted by P.F. Verlhust in 1838 and 1845 (see [33],[34]). Verlhust provided an example of a function where the population density of species is limited by themselves.

Sigmoidal functions have a wide range of applications across various research fields. These include biology, economics, chemistry, population dynamics, statistics and probability and geoscience.

Definition 2.2. *A function $\Omega : \mathbb{R} \to \mathbb{R}$ is said to be centered bell-shaped (cbs) $\Leftrightarrow$ $\Omega \in L_1(\mathbb{R})$, its integral is nonzero, it is nondecreasing on $(-\infty, 0)$ and nonincreasing on $[0, +\infty)$.*

In 2011 Anastassiou [2, *Chapter*1] defined the following Neural-Network operators whose compact support $[a, b]$.

Let,

$$h : \mathbb{R} \to (0, 1),\ h(x) = \frac{1}{1+e^{-x}},$$

be a sigmoidal function of logistic type and centered bell-shaped (cbs) function $\hat{H}(x)$ is given by;

$$\hat{H}(x) = \frac{h(x+1) - h(x-1)}{2},\ x \in \mathbb{R}.$$

Definition 2.3. *Let $n \in \mathbb{N}$. For a bounded function f defined on $[a,b]$ and let $\lceil na \rceil \leq \lfloor nb \rfloor$. Then the corresponding neural-network operators are defined as,*

$$G_n(f;x) := \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) \hat{H}(nx-k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \hat{H}(nx-k)}.$$

Where $\hat{H}$ is a cbs function obtained from a sigmoidal function of logistic type.

Now, we summarize some well-known and remarkable properties of $\hat{H}(x)$.

Remark 2.4. *Clearly we have;*

- $\hat{H}(x) = \left[\frac{h(x+1)-h(x-1)}{2}\right] = \left(\frac{e^2-1}{2e^2}\right) \frac{1}{\left(1+e^{-x-1}\right)\left(1+e^{x-1}\right)} > 0$
- $\hat{H}(-x) = \left(\frac{e^2-1}{2e^2}\right) \frac{1}{\left(1+e^{--x-1}\right)\left(1+e^{x-1}\right)} = \hat{H}(x)$
- $\hat{H}(x) = \frac{1}{2}\left(\frac{1}{1+e^{-x-1}} - \frac{1}{1+e^{-x+1}}\right) < \frac{1}{2}\left(\frac{e^{x+1}}{1+e^{x+1}} + \frac{e^x}{e+e^x}\right) < 1.$
- $\sum_{k \in \mathbb{Z}} \hat{H}(nx-k) = 1.$
- $\int_{\mathbb{R}} \hat{H}(x)dx = 1.$
- $\hat{H}(x)$ is non-decreasing on $(-\infty, 0)$ and non-increasing on $[0, +\infty)$.

Lemma 2.5. *Let $a, b \in \mathbb{R}$ be such that $\lceil na \rceil \leq \lfloor nb \rfloor$ for $n \in \mathbb{N}^+$. Then we have,*

$$\frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \hat{H}(nx-k) \int_a^b \hat{H}(nt-k)dt} \leq \frac{1}{\left[\hat{H}(1)\right]^2 (b-a)}. \tag{1}$$

Proof. *The proof can be carried out easily in a similar manner to the proof of "Lemma ix)" on page 5 in* [2] *(see* [2]*).* □

Lemma 2.6. *Let $0 < \beta < 1$, $n \in \mathbb{N}^+$, $2 < n^{1-\beta}$*

$$\sum_{\substack{k \in \mathbb{Z} \\ |nx-k| > n^{1-\beta}}} \hat{H}(nx-k) < e^{2-n^{1-\beta}}. \tag{2}$$

(for proof see [1].*)*

Lemma 2.7. *For $[a,b]$, $n \in \mathbb{N}^+$, $k \in \mathbb{Z}$ and $0 < \beta < 1$, $k \in \mathbb{Z}$, $1 < n^{1-\beta}$, we have,*

$$\int\limits_{\substack{\mathbb{R} \\ |nt-k| > n^{1-\beta}}} \hat{H}(nt-k)dt < e^{1-n^{1-\beta}}. \tag{3}$$

Proof. Similar to proof of Lemma 2.6., using mean value theorem, we obtain the following inequality,

$$\int\limits_{\substack{\mathbb{R} \\ |nt-k| > n^{1-\beta}}} \hat{H}(|nt-k|)dt < \int\limits_{\substack{n^{1-\beta} \\ |nt-k| > n^{1-\beta}}}^{\infty} e^{1-t}dt = e^{1-n^{1-\beta}}.$$

□

Definition 2.8. *For $\delta > 0$ and $f \in C[a,b]$, the modulus of continuity of the function f is defined by;*

$$\omega(f,\delta) = \sup_{\substack{t,x \in [a,b] \\ |t-x| < \delta}} \left| f(t) - f(x) \right|. \tag{4}$$

As for $L_p[a,b]$, $(1 \le p < \infty)$ spaces, for $\delta > 0$, the modulus of continuity of the function f is defined as,

$$\omega_{L_p}(f,\delta) = \sup_{\substack{t,x \in [a,b] \\ |t-x| < \delta}} \left\{ \int_a^b \left| f(t) - f(x) \right|^p dx \right\}^{1/p} \tag{5}$$

and for $\forall\delta > 0$ and $f \in L_p[a,b]$,

$$\lim_{\delta \to 0} \omega_{L_p}(f,\delta) = 0.$$

Now we define Durrmeyer Type Neural-Network Operators.

Definition 2.9. *(Durrmeyer Type Neural-Network Operators)*

For $a,b \in \mathbb{R}$ be such that $\lceil na \rceil \le \lfloor nb \rfloor$, $n \in \mathbb{N}^+$, $f \in L_1[a,b]$, we consider the following positive and linear Durrmeyer Type Neural-Network Operators,

$$N_n^D(f;x) = \frac{\sum\limits_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \hat{H}(nx-k) \int_a^b f(t)\hat{H}(nt-k)dt}{\sum\limits_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \hat{H}(nx-k) \int_a^b \hat{H}(nt-k)dt}, x \in [a,b].$$

Where $\hat{H}$ is centered bell-shaped function. Easily can be observed that the $N_n^D(1;x) = 1$.

In the following section, we will examine the approximation properties of the $N_n^D(f;x)$ operators in the $C[a,b]$ and $L_p[a,b]$ $(1 \le p < \infty)$ spaces.

3. Fundamental properties and main results

In this section using the auxiliary results and definition from Section 1 and Seciton 2, we will provide some theorems and proofs regarding the approximation properties of $N_n^D(f;x)$ operators.

Theorem 3.1. *Let $[a,b]$ be a compact subset of $\mathbb{R}$ and $t,x \in [a,b]$, $n \in \mathbb{N}^+$, $2 < n^{1-\beta}$ and $f \in C[a,b]$,*

$$\left\|N_n^D f - f\right\|_{\sup} \leq \frac{1}{\left[\hat{H}(1)\right]^2 (b-a)} \left(\omega\left(f, \frac{1}{n^\beta}\right) + 2\left\|f\right\| \left(e^{1-n^{1-\beta}} + e^{3-2n^{1-\beta}}\right)\right)$$

Proof. Let $f \in C[a,b]$ clearly;

$$\left|N_n^D(f;x) - f(x)\right| = \left| \frac{\sum\limits_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \hat{H}(nx-k) \int\limits_a^b [f(t)-f(x)]\, \hat{H}(nt-k)dt}{\sum\limits_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \hat{H}(nx-k) \int\limits_a^b \hat{H}(nt-k)dt} \right|,$$

using (1),

$$\left|N_n^D(f;x) - f(x)\right| \leq \frac{1}{\left[\hat{H}(1)\right]^2 (b-a)} \left| \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \hat{H}(nx-k) \int_a^b [f(t)-f(x)]\, \hat{H}(nt-k)dt \right|$$

$$\leq \frac{1}{\left[\hat{H}(1)\right]^2 (b-a)} \left| \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \hat{H}(nx-k) \int_a^b [f(t)-f(x)]\, \hat{H}(nt-k)dt \right|$$

$$\leq \frac{1}{\left[\hat{H}(1)\right]^2 (b-a)} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \hat{H}(nx-k) \int_a^b \left|f(t)-f(x)\right| \hat{H}(nt-k)dt$$

For $0 < \beta < 1$ and $\left|t - \frac{k}{n}\right| + \left|x - \frac{k}{n}\right| < n^{-\beta}$ follow that $\left|t - \frac{k}{n} + \frac{k}{n} - x\right| \leq \left|t - \frac{k}{n}\right| + \left|x - \frac{k}{n}\right| < n^{-\beta}$. Therefore, this implies that $|nt-k| < n^{1-\beta}$ and $|nx-k| < n^{1-\beta}$. By using these we divide the last inequality as;

$$\left|N_n^D(f;x) - f(x)\right| \leq \frac{1}{\left[\hat{H}(1)\right]^2 (b-a)} \left(\sum_{\substack{k=\lceil na \rceil \\ |nx-k| < n^{1-\beta}}}^{\lfloor nb \rfloor} \hat{H}(nx-k) \int_{\substack{a \\ |nt-k| \leq n^{1-\beta}}}^{b} \left|f(t)-f(x)\right| \hat{H}(nt-k)dt \right.$$

$$+ \sum_{\substack{k=\lceil na \rceil \\ |nx-k| < n^{1-\beta}}}^{\lfloor nb \rfloor} \hat{H}(nx-k) \int_{\substack{a \\ |nt-k| \geq n^{1-\beta}}}^{b} \left|f(t)-f(x)\right| \hat{H}(nt-k)dt$$

$$\left. + \sum_{\substack{k=\lceil na \rceil \\ |nx-k| > n^{1-\beta}}}^{\lfloor nb \rfloor} \hat{H}(nx-k) \int_{\substack{a \\ |nt-k| \geq n^{1-\beta}}}^{b} \left|f(t)-f(x)\right| \hat{H}(nt-k)dt \right).$$

$$\left|N_n^D(f;x)-f(x)\right| \leq \frac{1}{\left[\hat{H}(1)\right]^2(b-a)}\left(\sum_{\substack{k\in\mathbb{Z}\\ |nx-k|<n^{1-\beta}}}\hat{H}(nx-k)\int_{\substack{\mathbb{R}\\ |nt-k|\leq n^{1-\beta}}}\left|f(t)-f(x)\right|\hat{H}(nt-k)dt\right.$$

$$+\sum_{\substack{k\in\mathbb{Z}\\ |nx-k|<n^{1-\beta}}}\hat{H}(nx-k)\int_{\substack{\mathbb{R}\\ |nt-k|\geq n^{1-\beta}}}\left|f(t)-f(x)\right|\hat{H}(nt-k)dt$$

$$\left.+\sum_{\substack{k\in\mathbb{Z}\\ |nx-k|>n^{1-\beta}}}\hat{H}(nx-k)\int_{\substack{\mathbb{R}\\ |nt-k|\geq n^{1-\beta}}}\left|f(t)-f(x)\right|\hat{H}(nt-k)dt\right).$$

Using the Remark 1 property (4), (5) and (2), (3), (4) we obtain;

$$\left\|N_n^D(f;x)-f(x)\right\| \leq \frac{1}{\left[\hat{H}(1)\right]^2(b-a)}\left(\omega\left(f,\frac{1}{n^\beta}\right)+2\left\|f\right\|\left(e^{1-n^{1-\beta}}+e^{3-2n^{1-\beta}}\right)\right)$$

Where $\|\circ\|$ is the classical supremum norm of $C[a,b]$. □

Theorem 3.2. *Let $[a,b]$ be a compact subset of $\mathbb{R}$, $\forall t,x\in[a,b]$, $n\in\mathbb{N}^+$, $2<n^{1-\beta}$ and $f\in C[a,b]$,*

$$\lim_{n\to\infty}\left\|N_n^D f-f\right\|=0.$$

Proof. Using the squeeze theorem we have,

$$\lim_{n\to\infty}\left\|N_n^D f-f\right\| \leq \lim_{n\to\infty}\frac{1}{\left[\hat{H}(1)\right]^2(b-a)}\left\{\omega\left(f,\frac{1}{n^\beta}\right)+2\left\|f\right\|\left(e^{1-n^{1-\beta}}+e^{3-2n^{1-\beta}}\right)\right\}=0.$$

□

Now, we will examine pointwise and uniform convergence and approximation properties of $N_n^D(f;x)$ operators in $L_p[a,b]$ $(1\leq p<\infty)$ spaces please (see [6],[10],[12],[28],[31]).

Theorem 3.3. *Let x_0 be a point of continuity of f in $[a,b]$, $f\in L_1[a,b]$, $3<n\delta$, $\delta>0$;*

$$N_n^D(f;x_0)\overset{pointwise}{\rightarrow}f(x_0).$$

Proof. Let $\forall\ t, x_0 \in [a,b]$, $f \in L_1[a,b]$ and $\forall \varepsilon > 0 \ni \delta > 0$ and let $|t - x_0| < \delta$, $\left|f(t) - f(x_0)\right| < \varepsilon \left[\hat{H}(1)\right]^2 (b-a)$.

$$\left|N_n^D(f;x) - f(x_0)\right| = \left| \frac{\sum\limits_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \hat{H}(nx_0 - k) \int\limits_a^b f(t)\hat{H}(nt-k)dt}{\sum\limits_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \hat{H}(nx_0 - k) \int\limits_a^b \hat{H}(nt-k)dt} - f(x_0) \right|$$

using the Lemma 1, we have

$$\left|N_n^D(f;x) - f(x_0)\right| \leq \frac{1}{\left[\hat{H}(1)\right]^2 (b-a)} \left| \sum_{\substack{k=\lceil na \rceil \\ |t-x_0|<\delta}}^{\lfloor nb \rfloor} \hat{H}(nx_0 - k) \int\limits_a^b \left[f(t) - f(x_0)\right] \hat{H}(nt-k)dt \right|$$

$$+ \frac{1}{\left[\hat{H}(1)\right]^2 (b-a)} \left| \sum_{\substack{k=\lceil na \rceil \\ |t-x_0|>\delta}}^{\lfloor nb \rfloor} \hat{H}(nx_0 - k) \int\limits_a^b \left[f(t) - f(x_0)\right] \hat{H}(nt-k)dt \right|$$

using the fact that f is continuous at x_0, we have

$$= \frac{\varepsilon \left[\hat{H}(1)\right]^2 (b-a)}{\left[\hat{H}(1)\right]^2 (b-a)} + \frac{1}{\left[\hat{H}(1)\right]^2 (b-a)} \left\{ \sum_{\substack{k=\lceil na \rceil \\ |t-x_0|>\delta}}^{\lfloor nb \rfloor} \hat{H}(nx_0 - k) \int\limits_a^b \left|f(t)\right| \hat{H}(nt-k)dt \right.$$

$$\left. + \sum_{\substack{k=\lceil na \rceil \\ |t-x_0|>\delta}}^{\lfloor nb \rfloor} \hat{H}(nx_0 - k) \int\limits_a^b \left|f(x_0)\right| \hat{H}(nt-k)dt \right\}$$

using the $\hat{H}(x) < 1$, and

$$\left|N_n^D(f;x) - f(x_0)\right| \leq \varepsilon + \frac{1}{\left[\hat{H}(1)\right]^2 (b-a)} \left(\left\|f\right\|_{L_1} e^{2-n\delta} + \left|f(x_0)\right| (b-a) e^{3-n\delta} \right).$$

When the limit as $n \to \infty$ is taken, the desired equality is obtained. □

Theorem 3.4. *Let $f \in L_p[a,b]$ and $1 \leq p < \infty$. Then,*

$$\left\|N_n^D(f;\circ)\right\|_p \leq \frac{1}{\left[\hat{H}(1)\right]^{2p} (b-a)^p} \left\|f\right\|_p .$$

Proof. Let,

$$\left|N_n^D(f;x)\right|^p = \left|\frac{\sum\limits_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\hat{H}(nx-k)\int\limits_a^b f(t)\hat{H}(nt-k)dt}{\sum\limits_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\hat{H}(nx-k)\int\limits_a^b \hat{H}(nt-k)dt}\right|^p$$

$$\left|N_n^D(f;x)\right|^p \le \frac{1}{\left[\left[\hat{H}(1)\right]^2(b-a)\right]^p}\left|\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\hat{H}(nx-k)\int\limits_a^b f(t)\hat{H}(nt-k)dt\right|^p$$

let $\frac{1}{p}+\frac{1}{q}=1$, using the Hölder inequality we obtain;

$$\le \frac{1}{\left[\hat{H}(1)\right]^{2p}(b-a)^p}\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\hat{H}(nx-k)\left(\int\limits_a^b f(t)\left[\hat{H}(nt-k)\right]^{\frac{1}{p}+\frac{1}{q}}dt\right)^p$$

let $\frac{1}{p}+\frac{1}{q}=1$, using the integral form of Hölder inequality;

$$\le \frac{1}{\left[\hat{H}(1)\right]^{2p}(b-a)^p}\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\hat{H}(nx-k)\left[\left(\int\limits_a^b \left|f(t)\right|^p\hat{H}(nt-k)dt\right)^{1/p}\cdot\left(\int\limits_a^b \hat{H}(nt-k)dt\right)^{1/q}\right]^p$$

using the Remark 1 property (5) we have,

$$= \frac{1}{\left[\hat{H}(1)\right]^{2p}(b-a)^p}\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\hat{H}(nx-k)\int\limits_a^b \left|f(t)\right|^p\hat{H}(nt-k)dt$$

and passing to $L_p - norm$ we get,

$$\int\limits_a^b \left|N_n^D(f;x)\right|^p dx \le \frac{1}{\left[\hat{H}(1)\right]^{2p}(b-a)^p}\int\limits_a^b\left(\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\hat{H}(nx-k)\int\limits_a^b \left|f(t)\right|^p\hat{H}(nt-k)dt\right)dx$$

$$= \frac{1}{\left[\hat{H}(1)\right]^{2p}(b-a)^p}\int\limits_a^b\int\limits_a^b\left(\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\hat{H}(nx-k)\left|f(t)\right|^p\hat{H}(nt-k)\right)dtdx.$$

Applying the Generalized Minkowski Inequality, one has

$$\int\limits_a^b \left|N_n^D(f;x)\right|^p dx \le \frac{1}{\left[\hat{H}(1)\right]^{2p}(b-a)^p}\int\limits_a^b\int\limits_a^b\left(\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\hat{H}(nx-k)\left|f(t)\right|^p\hat{H}(nt-k)\right)dxdt$$

$$= \frac{1}{\left[\hat{H}(1)\right]^{2p}(b-a)^p}\int\limits_a^b\left[\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\left(\int\limits_a^b\hat{H}(nx-k)dx\right)\left|f(t)\right|^p\hat{H}(nt-k)\right]dt$$

using the Remark 1 property (5) we have,

$$= \frac{1}{\left[\hat{H}(1)\right]^{2p}(b-a)^p}\int_a^b\left[\left|f(t)\right|^p\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\hat{H}(nt-k)\right]dt$$

using the Remark 1 property (4) we have,

$$\int_a^b\left|N_n^D(f;x)\right|^p dx \leq \frac{1}{\left[\hat{H}(1)\right]^{2p}(b-a)^p}\int_a^b\left|f(t)\right|^p dt$$

$$\left(\int_a^b\left|N_n^D(f;x)\right|^p dx\right)^{1/p} \leq \frac{1}{\left[\hat{H}(1)\right]^{2}(b-a)}\left(\int_a^b\left|f(t)\right|^p dt\right)^{1/p}$$

$$\left\|N_n^D(f;\circ)\right\|_p \leq \frac{1}{\left[\hat{H}(1)\right]^{2}(b-a)}\left\|f\right\|_p .$$

□

Theorem 3.5. *Let $f \in L_p[a,b]$ and $1 \leq p < \infty$, $2 < n^{1-\beta}$ we obtain*

$$\left\|N_n^D(f;\circ)-f\right\|_p \leq \frac{(b-a)^{\frac{1-p}{p}}}{\left[\hat{H}(1)\right]^2}\left[\left(\omega_{L_p}(f,\frac{1}{n^\beta})\right)^p + 2^p\left\|f\right\|_p^p e^{2-n^{1-\beta}}\right]^{1/p}.$$

Proof. Let $f \in L_p[a,b]$ and $1 \leq p < \infty$,

$$\left|N_n^D(f;x)-f(x)\right|^p = \left|\frac{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\hat{H}(nx-k)\int_a^b\left[f(t)-f(x)\right]\hat{H}(nt-k)dt}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\hat{H}(nx-k)\int_a^b\hat{H}(nt-k)dt}\right|^p$$

using the Lemma 1 we obtain,

$$\left|N_n^D(f;x)-f(x)\right|^p \leq \frac{1}{\left[\left[\hat{H}(1)\right]^2(b-a)\right]^p}\left|\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\hat{H}(nx-k)^{\frac{1}{p}+\frac{1}{q}}\int_a^b\left[f(t)-f(x)\right]\hat{H}(nt-k)dt\right|^p$$

let $\frac{1}{p}+\frac{1}{q}=1$, using the Hölder Inequality;

$$\leq \frac{1}{\left[\hat{H}(1)\right]^{2p}(b-a)^p}\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\hat{H}(nx-k)\left|\int_a^b\left[f(t)-f(x)\right]\hat{H}(nt-k)^{\frac{1}{p}+\frac{1}{q}}dt\right|^p$$

let $\frac{1}{p}+\frac{1}{q}=1$, using the integral form of Hölder Inequality;

$$\leq \frac{1}{\left[\hat{H}(1)\right]^{2p}(b-a)^p}\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\hat{H}(nx-k)\left[\left(\int_a^b\left|\left[f(t)-f(x)\right]\left[\hat{H}(nt-k)\right]^{1/p}\right|^p dt\right)^{1/p}\times\left(\int_a^b\left|\left[\hat{H}(nt-k)\right]^{1/q}\right|^q dt\right)^{1/q}\right]^p$$

using the Remark 1 property (5) we obtain,

$$\leq \frac{1}{\left[\hat{H}(1)\right]^{2p}(b-a)^p}\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\hat{H}(nx-k)\int_a^b\left|f(t)-f(x)\right|^p\hat{H}(nt-k)dt$$

and passing to $L_p - norm$ we get,

$$\leq \frac{1}{\left[\hat{H}(1)\right]^{2}(b-a)}\left[\int_a^b\left(\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\hat{H}(nx-k)\int_a^b\left|f(t)-f(x)\right|^p\hat{H}(nt-k)dt\right)dx\right]^{1/p}$$

where, if we applied the Generalized Minkowski Inequality (see [7]), we obtain following inequality,

$$\leq \frac{1}{\left[\hat{H}(1)\right]^{2}(b-a)}\left[\int_a^b\left(\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\hat{H}(nt-k)\int_a^b\left|f(t)-f(x)\right|^p\hat{H}(nx-k)dx\right)dt\right]^{1/p}$$

$$\left\|N_n^D(f;\circ)-f\right\|_p \leq \frac{1}{\left[\hat{H}(1)\right]^{2}(b-a)}\left[\int_a^b\left(\sum_{\substack{k=\lceil na\rceil\\|nt-k|\leq n^{1-\beta}}}^{\lfloor nb\rfloor}\hat{H}(nt-k)\int_{\substack{a\\|nx-k|\leq n^{1-\beta}}}^{b}\left|f(t)-f(x)\right|^p\hat{H}(nx-k)dx+\right.\right.$$

$$\left.\left.+\sum_{\substack{k=\lceil na\rceil\\|nt-k|>n^{1-\beta}}}^{\lfloor nb\rfloor}\hat{H}(nt-k)\int_{\substack{a\\|nx-k|>n^{1-\beta}}}^{b}\left|f(t)-f(x)\right|^p\hat{H}(nx-k)dx\right)dt\right]^{1/p}$$

the above inequality is rearranged as follows to obtain the modulus of continuity of the L_p space, and using $\hat{H}(x) < 1$ we obtain following inequality;

$$\left\|N_n^D(f;\circ)-f\right\|_p \leq \frac{1}{\left[\hat{H}(1)\right]^{2}(b-a)}\left[\int_a^b\left\{\sum_{\substack{k=\lceil na\rceil\\|nt-k|\leq n^{1-\beta}}}^{\lfloor nb\rfloor}\hat{H}(nt-k)\left(\int_{\substack{a\\|nx-k|\leq n^{1-\beta}}}^{b}\left|f(t)-f(x)\right|^p dx\right)^{p/p}+\right.\right.$$

$$\left.\left.\sum_{\substack{k=\lceil na\rceil\\|nt-k|>n^{1-\beta}}}^{\lfloor nb\rfloor}\hat{H}(nt-k)\left(\int_{\substack{a\\|nx-k|>n^{1-\beta}}}^{b}\left|f(t)-f(x)\right|^p dx\right)^{p/p}\right\}dt\right]^{1/p}$$

using the (2), (5) and Minkowski Inequality $1 \leq p < \infty$ we obtain;

$$\left\|N_n^D(f;\circ) - f\right\|_p \leq \frac{1}{\left[\hat{H}(1)\right]^2 (b-a)} \left[\int_a^b \left(\sum_{\substack{k=\lceil na \rceil \\ |nt-k| \leq n^{1-\beta}}}^{\lfloor nb \rfloor} \hat{H}(nt-k) \left(\omega_{L_p}(f, \frac{1}{n^\beta})\right)^p \right.\right.$$

$$\left. + \sum_{\substack{k=\lceil na \rceil \\ |nt-k| > n^{1-\beta}}}^{\lfloor nb \rfloor} \hat{H}(nt-k) 2^p \left\|f\right\|_p^p \right) dt^{1/p}$$

$$\left\|N_n^D(f;\circ) - f\right\|_p \leq \frac{(b-a)^{\frac{1-p}{p}}}{\left[\hat{H}(1)\right]^2} \left[\left(\omega_{L_p}(f, \frac{1}{n^\beta})\right)^p + 2^p \left\|f\right\|_p^p e^{2-n^{1-\beta}}\right]^{1/p}$$

□

Theorem 3.6. *Let $f \in L_p[a,b]$, $n \in \mathbb{N}$, $2 < n^{1-\beta}$, we have,*

$$\lim_{n\to\infty} \left\|N_n^D(f;\circ) - f\right\|_p = 0.$$

Proof. If Theorem 5 is used and the limit as $n \to \infty$ is taken, the following equality is obtained.

$$\lim_{n\to\infty} \left\|N_n^D(f;\circ) - f\right\|_p = 0.$$

□

We demonstrated pointwise and uniform convergence in L_p space. While some points were overlooked in demonstrating uniform and pointwise convergence, the properties at these points will be examined in subsequent studies (see [20]).

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