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Inequalities for a rational function with restricted zeros

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Abstract. Recently, several authors have investigated the extremal problems for rational functions in the supremum norm on the unit disk in the plane. These investigations primarily focus on Bernstein and Turán-type norm estimates, which provide non-trivial generalizations and refinements of various polynomial inequalities. In this paper, we focus on obtaining bounds for the derivative of rational functions in both directions, which leads to more precise versions of certain polynomial inequalities for derivatives and polar derivatives. Furthermore, our findings generalize several well-known results from the classical literature on rational inequalities.

1. Introduction

Geometric function theory studies complex interactions between rational functions, going beyond simple polynomial comparisons and play a crucial role in approximation theory. These inequalities have been extensively studied in a good number of recent papers, emphasizing their crucial role in approximation theory (see, e.g., [5]-[8], [10], [13], [19]). In their investigations, the researchers have often used the unit disk in the complex plane as a prototype for bounded domains while studying the extremal features of rational functions and their derivatives. In order to determine the maximum modulus of an analytic function or its derivative in the unit disk, one only needs to examine the values on the boundary due to the maximum modulus principle. We denote by $\mathcal{P}_n$ the class of all complex polynomials $P(z) := \sum_{\nu=0}^{n} c_\nu z^\nu$ of degree n, and $P'(z)$ is the derivative of $P(z)$. If $P \in \mathcal{P}_n$, the estimate of $|P'(z)|$ on $|z| = 1$ is as follows:

$$\max_{|z|=1} |P'(z)| \leq n \max_{|z|=1} |P(z)|. \tag{1}$$

Inequality (1) is a classical result of Bernstein [3], who established it in 1912. One may easily notice that an improvement in (1) is implied by the restriction on the zeros of $P(z)$. It was conjectured by Erdös and later proved by Lax [9] that if $P \in \mathcal{P}_n$ and $P(z) \neq 0$ in $|z| < 1$, then (1) can be replaced by

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{2} \max_{|z|=1} |P(z)|. \tag{2}$$

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The reverse inequality of (2) was established by Turán in 1939. Specifically, Turán [12] demonstrated that if $P \in \mathcal{P}_n$ and $P(z)$ has all its zeros within $|z| \leq 1$, then

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{2} \max_{|z|=1} |P(z)|. \tag{3}$$

Recently, Ahangar and Shah [1] generalized (2) by proving that if $P \in \mathcal{P}_n$ and $P(z) \neq 0$ in $|z| < k$, $k \geq 1$, then for $|z| = 1$,

$$|P'(z)| \leq \frac{n\|P\|}{2}\left\{1 - \frac{k-1}{k+1}\,\frac{|P(z)|^2}{\|P\|^2}\right\}, \tag{4}$$

where $\|P(z)\| = \max_{|z|=1} |P(z)|$.
Equality in (4) holds for $P(z) = (z+k)^n$, evaluated at $z = 1$.
Geometric function theory investigates extremal problems involving complex-variable functions while expanding on classical polynomial inequalities in various ways. Another approach is to create rational counterparts to classical polynomial inequalities, the most well-known examples being Bernstein and Turán-type norm estimates. This type of research has a significant impact on the development of classical inequalities and their practical applications, such as those in physical systems.
Let $W(z) := \prod_{\nu=1}^{n}(z - b_\nu)$ and we consider the set of rational functions

$$\mathcal{R}_n := \mathcal{R}_n(b_1, b_2, ..., b_n) = \left\{\frac{P(z)}{W(z)} : P \in \mathcal{P}_n\right\}.$$

The Blaschke product

$$B(z) := \prod_{\nu=1}^{n}\left(\frac{1 - \overline{b}_\nu z}{z - b_\nu}\right),$$

is defined for every collection of poles $b_\nu \in \mathbb{C}$, with $|b_\nu| \neq 1;\ \nu = 1, 2, ..., n$. Note that $B(z) \in \mathcal{R}_n$ and $|B(z)| = 1$ for $|z| = 1$.
Borwein and Erdélyi [4] (see also Li, Mohapatra, and Rodriguez [11] or Rusak [16]) provided a new perspective on the inequalities (1)-(4) and extended them to rational functions with specified poles. For $r \in \mathcal{R}_n$, we have the following Bernstein-type inequality for rational functions, similar to (1):

$$|r'(z)| \leq |B'(z)| \max_{|z|=1} |r(z)| \text{ for } |z| = 1. \tag{5}$$

The above inequality (5) was first mentioned in Rusak's paper [16]. Subsequently, Borwein and Erdélyi [4] rigorously proved it with refinements, and independently, Li, Mohapatra, and Rodriguez [11] arrived at the same result. In the same paper (see also Borwein and Erdélyi [4]), Li, Mohapatra, and Rodriguez proved that if $r \in \mathcal{R}_n$ and $r(z) \neq 0$ in $|z| < 1$, then for $|z| = 1$, we have

$$|r'(z)| \leq \frac{|B'(z)|}{2} \max_{|z|=1} |r(z)|, \tag{6}$$

whereas, if $r \in \mathcal{R}_n$ has all its zeros in $|z| \leq 1$, then for $|z| = 1$, we have

$$|r'(z)| \geq \frac{|B'(z)|}{2} \max_{|z|=1} |r(z)|. \tag{7}$$

Since the establishment of inequalities (6) and (7), a large number of papers on rational approximation theory have been published. As a generalization of (6), Aziz and Zargar [2] proved that if $r \in \mathcal{R}_n$ and $r(z) \neq 0$ in $|z| < k$, $k \geq 1$, then for $|z| = 1$, we have

$$|r'(z)| \leq \frac{1}{2}\left\{|B'(z)| - n\left(\frac{k-1}{k+1}\right)\frac{|r(z)|^2}{\|r\|^2}\right\}\|r(z)\|, \tag{8}$$

where here and throughout $\|r(z)\| = \max_{|z|=1} |r(z)|$.
Wali and Shah [19] improved (7) and proved that if $r \in \mathcal{R}_n$ and $r(z)$ has all its zeros in $|z| \le 1$, then for $|z| = 1$, we have

$$\max_{|z|=1} |r'(z)| \ge \frac{1}{2}\left\{|B'(z)| + \left(\frac{\sqrt{|c_n|} - \sqrt{|c_0|}}{\sqrt{|c_n|}}\right)\right\}|r(z)|. \tag{9}$$

The growing interest of these inequalities in geometric function theory in the recent years is due to their wide range of applications in various fields of science and engineering and discrete dynamical systems. We refer the interested readers to the works listed in ([14], [15], [18], [20],) for the latest insights on rational inequalities and their applications.
The purpose of this work is to prove various refinements of the above inequalities and related conclusions, resulting in more precise versions of certain polynomial inequalities for derivatives and polar derivatives. This framework unifies and simplifies the derivation of these generalizations, resulting in new and old inequalities for a variety of polynomial operators.

2. Main results

It is common to look into the geometric properties of rational functions in general, specifically the norm estimates of their derivatives on the unit disk. The primary goal here is to look into the extensions of various results found in the literature on this subject. We will assume that all poles $b_1, b_2, ..., b_n$, of the function $r(z)$ are located on $|z| > 1$. We start by refining (8). Furthermore, the findings strengthen some existing inequalities.

Theorem 2.1. *Let $r(z) = \frac{P(z)}{W(z)} \in \mathcal{R}_n$, where $P(z) = z^m(c_0 + c_1 z + \cdots + c_{n-m}z^{n-m})$, $0 \le m < n$ and $r(z)$ have no zeros in $|z| < k$, $k \ge 1$ except a zero of order m at the origin. Then for all z on $|z| = 1$, we have*

$$|r'(z)| \le \frac{\|r(z)\|}{2}\left\{|B'(z)| - \left[\frac{n(k-1) - 2mk}{1+k} + \frac{2}{1+k}\left(\frac{\sqrt{|c_0|} - \sqrt{k^{n-m}|c_{n-m}|}}{\sqrt{|c_0|}}\right)\right]\frac{|r(z)|^2}{\|r(z)\|^2}\right\}. \tag{10}$$

The result is best possible and equality in (10) *holds for*

$$r(z) = \frac{z^m(z+k)^{n-m}}{(z-b)^n} \text{ and } B(z) = \left(\frac{1-bz}{z-b}\right)^n, \text{ at } z = 1,\ b > 1.$$

Taking $m = 0$, we get the following refinement of (8):

Corollary 2.2. *Let $r(z) = \frac{P(z)}{W(z)} \in \mathcal{R}_n$, where $P(z) = c_0 + c_1 z + \cdots + c_n z^n$, and $r(z)$ has no zeros in $|z| < k$, $k \ge 1$. Then for all z on $|z| = 1$, we have*

$$|r'(z)| \le \frac{\|r(z)\|}{2}\left\{|B'(z)| - \left[\frac{n(k-1)}{1+k} + \frac{2}{1+k}\left(\frac{\sqrt{|c_0|} - \sqrt{k^n|c_n|}}{\sqrt{|c_0|}}\right)\right]\frac{|r(z)|^2}{\|r(z)\|^2}\right\}. \tag{11}$$

The result is best possible and equality in (11) *holds for*

$$r(z) = \frac{(z+k)^n}{(z-b)^n} \text{ and } B(z) = \left(\frac{1-bz}{z-b}\right)^n, \text{ at } z = 1,\ b > 1.$$

Remark 2.3. The bound achieved in (11) yields a consistently sharper bound compared to that derived from (8), except in the case when all zeros of $r(z)$ lie on the unit circle $|z| = 1$.

For $k = 1$ in (11), we get the following refinement of (6):

Corollary 2.4. *Let $r(z) = \frac{P(z)}{W(z)} \in \mathcal{R}_n$, where $P(z) = c_0 + c_1 z + \cdots + c_n z^n$, and $r(z)$ has no zeros in $|z| < 1$. Then for all z on $|z| = 1$, we have*

$$|r'(z)| \leq \frac{\|r(z)\|}{2} \left\{ |B'(z)| - \left(\frac{\sqrt{|c_0|} - \sqrt{|c_n|}}{\sqrt{|c_0|}} \right) \frac{|r(z)|^2}{\|r(z)\|^2} \right\}.$$

Remark 2.5. If we assume that $r(z)$ has a pole of order n at $z = \alpha$ with $|\alpha| > 1$, then $W(z) = (z - \alpha)^n$ and $B(z) = \left(\frac{1-\overline{\alpha}z}{z-\alpha}\right)^n$.

With this substitution, we have

$$r(z) = \frac{P(z)}{(z-\alpha)^n},$$
$$r'(z) = \left(\frac{P'(z)}{(z-\alpha)^n} - \frac{nP(z)}{(z-\alpha)^{n+1}} \right)$$
$$\text{and } \; B'(z) = \frac{n(1-\overline{\alpha}z)^{n-1}(|\alpha|^2 - 1)}{(z-\alpha)^{n+1}}.$$

Using these observations in Theorem 2.1 and letting $|\alpha| \to \infty$, we get the following result:

Corollary 2.6. *Let $P(z) = z^m(c_0 + c_1 z + \cdots + c_{n-m} z^{n-m})$, $0 \leq m < n$ and $P(z)$ has no zeros in $|z| < k$, $k \geq 1$ except a zero of order m at the origin. Then for all z on $|z| = 1$, we have*

$$|P'(z)| \leq \frac{\|P(z)\|}{2} \left\{ n - \left[\frac{n(k-1) - 2mk}{1+k} + \frac{2}{1+k} \left(\frac{\sqrt{|c_0|} - \sqrt{k^{n-m}|c_{n-m}|}}{\sqrt{|c_0|}} \right) \right] \frac{|P(z)|^2}{\|P(z)\|^2} \right\}. \tag{12}$$

The result is best possible and equality in (12) holds for $P(z) = z^m(z+k)^{n-m}$, evaluated at $z = 1$.

Taking $m = 0$ in (12), we get the following refinement of (4):

Corollary 2.7. *Let $P \in \mathcal{P}_n$ and $P(z)$ has no zeros in $|z| < k$, $k \geq 1$. Then for all z on $|z| = 1$, we have*

$$|P'(z)| \leq \frac{\|P(z)\|}{2} \left\{ n - \left[\frac{n(k-1)}{1+k} + \frac{2}{1+k} \left(\frac{\sqrt{|c_0|} - \sqrt{k^n|c_n|}}{\sqrt{|c_0|}} \right) \right] \frac{|P(z)|^2}{\|P(z)\|^2} \right\}. \tag{13}$$

The result is best possible and equality in (13) holds for $P(z) = (z+k)^n$, evaluated at $z = 1$.

For $k = 1$ in (13), we get the following refinement of (2):

Corollary 2.8. *Let $P \in \mathcal{P}_n$ and $P(z)$ has no zeros in $|z| < 1$. Then for all z on $|z| = 1$, we have*

$$|P'(z)| \leq \frac{\|P(z)\|}{2} \left\{ n - \left(\frac{\sqrt{|c_0|} - \sqrt{|c_n|}}{\sqrt{|c_0|}} \right) \frac{|P(z)|^2}{\|P(z)\|^2} \right\}.$$

Next, we prove the following extension of (9).

Theorem 2.9. *Let $r(z) = \frac{P(z)}{W(z)} \in \mathcal{R}_n$, where $P(z) = z^m(c_0 + c_1 z + \cdots + c_{n-m} z^{n-m})$, $0 \leq m \leq n$ and $r(z)$ has all its zeros in $|z| \leq k$, $k \leq 1$. Then for all z on $|z| = 1$, we have*

$$\max_{|z|=1} |r'(z)| \geq \frac{1}{2} \left\{ |B'(z)| + \frac{n(1-k) + 2mk}{1+k} + \frac{2k}{1+k} \left(\frac{\sqrt{k^{n-m}|c_{n-m}|} - \sqrt{|c_0|}}{\sqrt{k^{n-m}|c_{n-m}|}} \right) \right\} |r(z)|. \tag{14}$$

The result is best possible and equality in (14) holds for

$$r(z) = \frac{z^m(z+k)^{n-m}}{(z-b)^n} \;\text{ and }\; B(z) = \left(\frac{1-bz}{z-b} \right)^n, \text{ at } z = 1, \; b > 1.$$

Taking $m = 0$, we get the following generalization of (9):

Corollary 2.10. *Let* $r(z) = \frac{P(z)}{W(z)} \in \mathcal{R}_n$*, and* $r(z)$ *has all its zeros in* $|z| \le k$*,* $k \le 1$*. Then for all* z *on* $|z| = 1$*, we have*

$$\max_{|z|=1} |r'(z)| \ge \frac{1}{2}\left\{|B'(z)| + \frac{n(1-k)}{1+k} + \frac{2k}{1+k}\left(\frac{\sqrt{k^n|c_n|} - \sqrt{|c_0|}}{\sqrt{k^n|c_n|}}\right)\right\}|r(z)|. \tag{15}$$

The result is best possible and equality in (15) *holds for*

$$r(z) = \frac{(z+k)^n}{(z-b)^n} \text{ and } B(z) = \left(\frac{1-bz}{z-b}\right)^n, \text{ at } z = 1,\ b > 1.$$

Remark 2.11. If we assume that $r(z)$ has a pole of order n at $z = \alpha$ with $|\alpha| > 1$, then $W(z) = (z-\alpha)^n$ and $B(z) = \left(\frac{1-\overline{\alpha}z}{z-\alpha}\right)^n$. With this substitution, we have

$$\begin{aligned} r'(z) &= \frac{(z-\alpha)^n P'(z) - n(z-\alpha)^{n-1}P(z)}{(z-\alpha)^{2n}} \\ &= -\left\{\frac{nP(z) - (z-\alpha)P'(z)}{(z-\alpha)^{n+1}}\right\} \\ &= -\frac{D_\alpha P(z)}{(z-\alpha)^{n+1}}, \end{aligned}$$

where

$$D_\alpha P(z) := nP(z) - (z-\alpha)P'(z),$$

is the polar derivative of $P(z)$ with respect to the complex number α. It generalizes the ordinary derivative in the following sense:

$$P'(z) = \lim_{\alpha\to\infty} \frac{D_\alpha P(z)}{\alpha - z},$$

uniformly with respect to z for $|z| \le R,\ R > 0$. Further, as in Remark 2.5, we have

$$B(z) = \left(\frac{1-\overline{\alpha}z}{z-\alpha}\right)^n,$$

implying that

$$B'(z) = \frac{n(1-\overline{\alpha}z)^{n-1}(|\alpha|^2-1)}{(z-\alpha)^{n+1}}.$$

With this choice, from (14) for $|z| = 1$, we get

$$\begin{aligned} |D_\alpha P(z)| &\ge \frac{1}{2}\left\{\frac{n(|\alpha|^2-1)}{|z-\alpha|} + \frac{n(1-k)+2mk}{1+k}|z-\alpha| \right. \\ &\quad \left. + \frac{2k}{1+k}|z-\alpha|\left(\frac{\sqrt{k^{n-m}|c_{n-m}|} - \sqrt{|c_0|}}{\sqrt{k^{n-m}|c_{n-m}|}}\right)\right\}|P(z)| \\ &\ge \frac{1}{2}\left\{\frac{n(|\alpha|-1)(|\alpha|+1)}{|\alpha|+1} + \frac{n(1-k)+2mk}{1+k}(|\alpha|-1) \right. \\ &\quad \left. + \frac{2k(|\alpha|-1)}{1+k}\left(\frac{\sqrt{k^{n-m}|c_{n-m}|} - \sqrt{|c_0|}}{\sqrt{k^{n-m}|c_{n-m}|}}\right)\right\}|P(z)| \\ &= \frac{|\alpha|-1}{2}\left\{n + \frac{n(1-k)+2mk}{1+k} \right. \\ &\quad \left. + \frac{2k}{1+k}\left(\frac{\sqrt{k^{n-m}|c_{n-m}|} - \sqrt{|c_0|}}{\sqrt{k^{n-m}|c_{n-m}|}}\right)\right\}|P(z)|. \end{aligned}$$

Thus, from Theorem 2.9, we immediately get the following result for the polar derivative of polynomial $P(z)$:

Corollary 2.12. *Let $P(z) = z^m(c_0 + c_1 z + \cdots + c_{n-m}z^{n-m})$ and $P(z)$ has all its zeros in $|z| \le k$, $k \le 1$ with $m-$fold zeros at the origin. Then for all z on $|z| = 1$, and for every complex number α with $|\alpha| \ge 1$, we have*

$$\max_{|z|=1} |D_\alpha P(z)| \ge n\left(\frac{|\alpha|-1}{1+k}\right)\left\{1 + \frac{k}{n}\left(m + \frac{\sqrt{k^{n-m}|c_{n-m}|} - \sqrt{|c_0|}}{\sqrt{k^{n-m}|c_{n-m}|}}\right)\right\}|P(z)|. \tag{16}$$

For $k = 1$ in (16), we get the following result which is a generalization of a result due to Wali and Shah [19].

Corollary 2.13. *Let $P(z) = z^m(c_0 + c_1 z + \cdots + c_{n-m}z^{n-m})$ and $P(z)$ has all its zeros in $|z| \le 1$ with $m-$fold zeros at the origin. Then for all z on $|z| = 1$, and for every complex number α with $|\alpha| \ge 1$, we have*

$$\max_{|z|=1} |D_\alpha P(z)| \ge \left(\frac{|\alpha|-1}{2}\right)\left\{n + m + \frac{\sqrt{|c_{n-m}|} - \sqrt{|c_0|}}{\sqrt{|c_{n-m}|}}\right\}|P(z)|.$$

3. Auxiliary results

For the proofs of our results, we need the following lemmas.

Lemma 3.1. *If $P \in \mathcal{P}_n$ and $P(z)$ has no zeros in $|z| < k$, $k \ge 1$, then for all z on $|z| = 1$ other than the zeros of $P(z)$, we have*

$$Re\left(\frac{zP'(z)}{P(z)}\right) \le \frac{1}{1+k}\left\{n - \left(\frac{\sqrt{|c_0|} - \sqrt{k^n|c_n|}}{\sqrt{|c_0|}}\right)\right\}. \tag{17}$$

Proof of Lemma 3.1. Without loss of generality, we will prove the above inequality with the assumption that $c_n = 1$, by the use of principle of mathematical induction on the degree n, and for this, we first verify that the result holds for $n = 1$.
If $n = 1$, then $P(z) = z - w$ with $|w| \ge k$, $k \ge 1$, and therefore for all z on $|z| = 1$ other than the zeros of $P(z)$, we have

$$Re\left(\frac{zP'(z)}{P(z)}\right) = Re\left(\frac{z}{z-w}\right) \le \frac{1}{1+|w|}.$$

Since $k \ge 1$ and $|w| \ge k$, then it follows easily

$$\frac{1}{1+|w|} \le \frac{1}{1+k}\left\{1 - \left(\frac{\sqrt{|w|} - \sqrt{k}}{\sqrt{|w|}}\right)\right\}.$$

So,

$$Re\left(\frac{zP'(z)}{P(z)}\right) \le \frac{1}{1+k}\left\{1 - \left(\frac{\sqrt{|w|} - \sqrt{k}}{\sqrt{|w|}}\right)\right\}.$$

which is nothing but (17) when $n = 1$.
Assume that the result holds for the polynomials of degree n. Let $Q(z) = (z - w)P(z)$ with $|w| \ge k$, $k \ge 1$ where $P(z)$ is a polynomial of degree n having all its zeros in $|z| \ge k$, $k \ge 1$. Then for z on $|z| = 1$ other than the zeros of $Q(z)$, we get by using induction hypothesis

$$\begin{aligned} Re\left(\frac{zQ'(z)}{Q(z)}\right) &= Re\left(\frac{z}{z-w}\right) + Re\left(\frac{zP'(z)}{P(z)}\right) \\ &\le \frac{1}{1+k}\left\{1 - \left(\frac{\sqrt{|w|} - \sqrt{k}}{\sqrt{|w|}}\right)\right\} + \frac{1}{1+k}\left\{n - \left(\frac{\sqrt{|c_0|} - \sqrt{k^n}}{\sqrt{|c_0|}}\right)\right\}. \end{aligned}$$

To complete the induction principle, we need to show that for all $|z| = 1$ other than the zeros of $Q(z)$,

$$Re\Big(\frac{zQ'(z)}{Q(z)}\Big) \leq \frac{1}{1+k}\Big\{n+1-\Big(\frac{\sqrt{|w||c_0|}-\sqrt{k^{n+1}}}{\sqrt{|w||c_0|}}\Big)\Big\}. \tag{18}$$

Clearly, the inequality (18) holds if

$$\frac{\sqrt{|w|}-\sqrt{k}}{\sqrt{|w|}}+\frac{\sqrt{|c_0|}-\sqrt{k^n}}{\sqrt{|c_0|}} \geq \frac{\sqrt{|w||c_0|}-\sqrt{k^{n+1}}}{\sqrt{|w||c_0|}},$$

which is equivalent to

$$1-\sqrt{\frac{k}{w}}+\sqrt{\frac{k^{n+1}}{|w||c_0|}} \geq \sqrt{\frac{k^n}{|c_0|}},$$

which on simplification and using the fact that all the zeros of $P(z)$ lie in $|z| \geq k,\ k \geq 1$ gives the desired assertion. This proves Lemma 3.1 completely.

Lemma 3.2. *If $P(z) = z^m(c_0 + c_1z + \cdots + c_{n-m}z^{n-m})$ is a polynomial of degree n having all its zeros in $|z| \geq k, k \geq 1$, except a zero of order $m \geq 0$ at the origin, then for all z on $|z| = 1$ other than the zeros of $P(z)$, we have*

$$Re\Big(\frac{zP'(z)}{P(z)}\Big) \leq \frac{n}{1+k}+\frac{k}{1+k}\Big\{m-\frac{1}{k}\Big(\frac{\sqrt{|c_0|}-\sqrt{k^{n-m}|c_{n-m}|}}{\sqrt{|c_0|}}\Big)\Big\}.$$

Proof of Lemma 3.2. Let $P(z) = z^mH(z)$, where $H(z) = b_0 + b_1z \cdots + b_{n-m}z^{n-m}$, $m \geq 0$, then for all z on $|z| = 1$ other than the zeros of $P(z)$, we have ,

$$Re\Big(\frac{zP'(z)}{P(z)}\Big) = m + Re\Big(\frac{zH'(z)}{H(z)}\Big). \tag{19}$$

Since $H(z)$ is a polynomial of degree $n - m$ having no zero in $|z| < k$, $k \geq 1$, therefore, on applying Lemma 3.1 to $H(z)$ in (19), we get for all z on $|z| = 1$ other than the zeros of $P(z)$, that

$$Re\Big(\frac{zP'(z)}{P(z)}\Big) \leq m+\frac{1}{1+k}\Big\{n-m-\Big(\frac{\sqrt{|c_0|}-\sqrt{k^{n-m}|c_{n-m}|}}{\sqrt{|c_0|}}\Big)\Big\},$$

which is equivalent to

$$Re\Big(\frac{zP'(z)}{P(z)}\Big) \leq \frac{n}{1+k}+\frac{k}{1+k}\Big\{m-\frac{1}{k}\Big(\frac{\sqrt{|c_0|}-\sqrt{k^{n-m}|c_{n-m}|}}{\sqrt{|c_0|}}\Big)\Big\}.$$

This completes the proof of Lemma 3.2.

Lemma 3.3. If $|z| = 1$, then

$$Re\left(\frac{zW'(z)}{W(z)}\right) = \frac{n-|B'(z)|}{2},$$

where $W(z) = \prod_{v=1}^{n}(z - b_v)$.

The above lemma is due to Aziz and Zargar [2].

Lemma 3.4. *If $r \in \mathcal{R}_n$, and $r^*(z) = B(z)\overline{r\left(\frac{1}{\bar{z}}\right)}$, then for $|z| = 1$, we have*

$$|r'(z)| + |(r^*(z))'| \leq |B'(z)| \max_{|z|=1} |r(z)|.$$

The above lemma is due to Li, Mohapatra and Rodriguez ([11], Theorem 2).

Lemma 3.5. *If $P \in \mathcal{P}_n$ and $P(z)$ has all its zeros in $|z| \leq k, k \leq 1$, then for all z on $|z| = 1$ other than zeros of $P(z)$, we have*

$$Re\left(\frac{zP'(z)}{P(z)}\right) \geq \frac{1}{1+k}\left\{n + k\left(\frac{\sqrt{k^n|c_n|} - \sqrt{|c_0|}}{\sqrt{k^n|c_n|}}\right)\right\}. \tag{20}$$

The above lemma is due to Singha and Chanam [17].

4. Proofs of the main results

Proof of Theorem 2.1. Recall that $r \in \mathcal{R}_n$ and $r(z)$ has no zeros in $|z| < k$, $k \geq 1$, except a zero of order m, $0 \leq m < n$, at the origin. That is, $r(z) = \frac{P(z)}{W(z)}$ where $P(z) = z^m H(z)$ and $h(z) = c_{n-m} \prod_{j=1}^{n-m}(z - z_j)$, $c_{n-m} \neq 0$, $|z_j| \geq k \geq 1$, $j = 1, 2, ..., n - m$. Suppose first $k > 1$, then for all z on $|z| = 1$, we have

$$Re\left(\frac{zr'(z)}{r(z)}\right) = Re\left(\frac{zP'(z)}{P(z)}\right) - Re\left(\frac{zW'(z)}{W(z)}\right).$$

On using Lemma 3.2 and Lemma 3.3, we have for all z on $|z| = 1$, that

$$\begin{aligned} Re\left(\frac{zr'(z)}{r(z)}\right) &\leq \frac{n}{1+k} + \frac{k}{1+k}\left(m - \frac{1}{k}\left(\frac{\sqrt{|c_0|} - \sqrt{k^{n-m}|c_{n-m}|}}{\sqrt{|c_0|}}\right)\right) - \frac{n - |B'(z)|}{2} \\ &= \frac{n + mk}{1+k} - \frac{1}{1+k}\left(\frac{\sqrt{|c_0|} - \sqrt{k^{n-m}|c_{n-m}|}}{\sqrt{|c_0|}}\right) - \frac{n - |B'(z)|}{2} \\ &= \frac{1}{2}\left\{|B'(z)| - \left[\frac{n(k-1) - 2mk}{1+k} + \frac{2}{1+k}\left(\frac{\sqrt{|c_0|} - \sqrt{k^{n-m}|c_{n-m}|}}{\sqrt{|c_0|}}\right)\right]\right\}. \end{aligned} \tag{21}$$

If $r^*(z) = B(z)\overline{r(\frac{1}{\bar{z}})}$, then it easily follows for $|z| = 1$, that

$$\begin{aligned} \left|\frac{z(r^*(z))'}{r(z)}\right|^2 &= \left||B'(z)| - \frac{zr'(z)}{r(z)}\right|^2 \\ &= |B'(z)|^2 + \left|\frac{z(r'(z))}{r(z)}\right|^2 - 2|B'(z)|Re\left(\frac{zr'(z)}{r(z)}\right), \end{aligned}$$

which on using (21) for $|z| = 1$, gives

$$\begin{aligned} \left|\frac{z(r^*(z))'}{r(z)}\right|^2 &\geq |B'(z)|^2 + \left|\frac{z(r'(z))}{r(z)}\right|^2 \\ &\quad - |B'(z)|\left\{|B'(z)| - \left[\frac{n(k-1) - 2mk}{1+k} + \frac{2}{1+k}\left(\frac{\sqrt{|c_0|} - \sqrt{k^{n-m}|c_{n-m}|}}{\sqrt{|c_0|}}\right)\right]\right\} \\ &= \left|\frac{z(r'(z))}{r(z)}\right|^2 + |B'(z)|\left[\frac{n(k-1) - 2mk}{1+k} + \frac{2}{1+k}\left(\frac{\sqrt{|c_0|} - \sqrt{k^{n-m}|c_{n-m}|}}{\sqrt{|c_0|}}\right)\right]. \end{aligned}$$

Equivalently for $|z| = 1$, we have

$$\left\{|(r'(z))|^2 + |B'(z)||r(z)|^2\left[\frac{n(k-1)-2mk}{1+k} + \frac{2}{1+k}\left(\frac{\sqrt{|c_0|} - \sqrt{k^{n-m}|c_{n-m}|}}{\sqrt{|c_0|}}\right)\right]\right\}^{\frac{1}{2}} \leq |r^*(z))'|,$$

which in combination with Lemma 3.4, gives

$$|(r'(z))|^2 + |B'(z)||r(z)|^2\left[\frac{n(k-1)-2mk}{1+k} + \frac{2}{1+k}\left(\frac{\sqrt{|c_0|} - \sqrt{k^{n-m}|c_{n-m}|}}{\sqrt{|c_0|}}\right)\right] \leq |B'(z)|^2\|r\|^2 + |r'(z)|^2 - 2|B'(z)||r'(z)|\|r\|,$$

which after simplification gives

$$|r'(z)| \leq \frac{\|r(z)\|}{2}\left\{|B'(z) - \left[\frac{n(k-1)-2mk}{1+k} + \frac{2}{1+k}\left(\frac{\sqrt{|c_0|} - \sqrt{k^{n-m}|c_{n-m}|}}{\sqrt{|c_0|}}\right)\right]\frac{|r(z)|^2}{\|r(z)\|^2}\right\}, \text{ for } |z| = 1. \tag{22}$$

For $k = 1$, the above inequality (22) is trivially true for z with $|z| = 1$ such that $r(z) = 0$ by (6). This completes the proof of Theorem 2.1.

Proof of Theorem 2.9. Since $r \in \mathcal{R}_n$ and $r(z)$ has all its zeros in $|z| \leq k$, $k \leq 1$. That is, $r(z) = \frac{P(z)}{W(z)}$ where $P(z) = z^m H(z)$ and $H(z) = c_{n-m}\prod_{j=1}^{n-m}(z - z_j)$, $c_{n-m} \neq 0$, $|z_j| \leq k \leq 1$, $j = 1, 2, ..., n-m$. By direct calculation, we obtain for $|z| = 1$, other than the zeros of $r(z)$, that

$$Re\left(\frac{zr'(z)}{r(z)}\right) = Re\left(\frac{zP'(z)}{P(z)}\right) - Re\left(\frac{zW'(z)}{W(z)}\right) = m + Re\left(\frac{zh'(z)}{h(z)}\right) - Re\left(\frac{zW'(z)}{W(z)}\right).$$

On using Lemmas 3.3 and 3.5, we get

$$Re\left(\frac{zr'(z)}{r(z)}\right) \geq m + \frac{1}{1+k}\left\{n - m + k\left(\frac{\sqrt{k^{n-m}|c_{n-m}|} - \sqrt{|c_0|}}{\sqrt{k^{n-m}|c_{n-m}|}}\right)\right\} - \left(\frac{n - |B'(z)|}{2}\right) = \frac{1}{2}\left\{|B'(z)| + \frac{n(1-k)+2mk}{1+k} + \frac{2k}{1+k}\left(\frac{\sqrt{k^{n-m}|c_{n-m}|} - \sqrt{|c_0|}}{\sqrt{k^{n-m}|c_{n-m}|}}\right)\right\},$$

for all z on $|z| = 1$, other than the zeros of $r(z)$. Hence, we have

$$|r'(z)| \geq \frac{1}{2}\left\{|B'(z)| + \frac{n(1-k)+2mk}{1+k} + \frac{2k}{1+k}\left(\frac{\sqrt{k^{n-m}|c_{n-m}|} - \sqrt{|c_0|}}{\sqrt{k^{n-m}|c_{n-m}|}}\right)\right\}|r(z)|, \tag{23}$$

for all z on $|z| = 1$, other than the zeros of $r(z)$. Since (23) is trivially true for z on $|z| = 1$ such that $r(z) = 0$. It follows that for all z on $|z| = 1$,

$$\max_{|z|=1}|r'(z)| \geq \frac{1}{2}\left\{|B'(z)| + \frac{n(1-k)+2mk}{1+k} + \frac{2k}{1+k}\left(\frac{\sqrt{k^{n-m}|c_{n-m}|} - \sqrt{|c_0|}}{\sqrt{k^{n-m}|c_{n-m}|}}\right)\right\}|r(z)|,$$

which is (14), and this completes the proof of Theorem 2.9.

Remark 4.1. In fact, excepting the case when all the zeros of $P(z)$ lie on $|z| = k$, the bound obtained in Corollary 2.7 is always sharper than the bound obtained in (4). As an illustration we consider the following polynomial of degree 4 to compare the bounds.
Let $P(z) = z^4 + z^3 + 2z^2 + z + 81$. Clearly the zeros of $P(z)$ are $-2.2724 \pm 2.2107i$ and $1.7724 \pm 2.2176i$, which all lie in $|z| > 2$. By taking $k = 2$, we obtain from (4) for $|z| = 1$, that

$$|P'(z)| \leq \frac{n\|P\|}{2}\left\{1 - \frac{1}{3}\frac{|P(z)|^2}{\|P\|^2}\right\},$$

while as Corollary 2.7 gives

$$|P'(z)| \leq \frac{n\|P\|}{2}\left\{1 - \frac{23}{54}\frac{|P(z)|^2}{\|P\|^2}\right\},$$

showing that (13) gives a considerable improvement over the bound obtained from (4).

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References

[1] U. M. Ahangar, W. M. Shah, *Inequalities for the derivative of a polynomial with restricted zeros,* J. Anal., **29** (2021), 1367–1374.
[2] A. Aziz, B. A. Zargar, *Some properties of rational functions with prescribed poles,* Canad. Math. Bull., **42** (1999), 417-426.
[3] S. Bernstein, *Sur l'ordre de la meilleure approximation des fonctions continues par des polynômes de degré donné*, Mem. Acad. R. Belg., **4** (1912), 1-103.
[4] P. Borwein, T. Erdélyi, *Polynomials and Polynomial Inequalities,* Springer (1995).
[5] V. N. Dubinin, *Sharp inequalities for rational functions on a circle*, Math. Notes, **110** (2021), 41-47.
[6] V. N. Dubinin, *Methods of geometric function theory in classical and modern problems for polynomials*, Russian Math. Surveys, **67** (2012), 599-684.
[7] R. B. Gardner, N. K. Govil, G. V. Milovanović, *Extremal Problems and Inequalities of Markov-Bernstein type for Algebraic Polynomials*, Elsevier, Academic Press, London (2022).
[8] S. I. Kalmykov, *About multipoint distortion theorems for rational functions*, Siberian Math. J., **61** (2020), 85-94.
[9] P. D. Lax, *Proof of a conjecture of P. Erdös on the derivative of a polynomial*, Bull. Amer. Math. Soc., **50** (1944), 509-513.
[10] X. Li, *A comparison inequality for rational functions*, Proc. Amer. Math. Soc., **139** (2011), 1659-1665.
[11] X. Li, R. N. Mohapatra, R. S. Rodriguez, *Bernstein-type inequalities for rational functions with prescribed poles,* J. London Math. Soc., **51** (1995), 523-531.
[12] G. V. Milovanović, D. S. Mitrinović, Th. M. Rassias, *Topics in Polynomials: Extremal Problems, Inequalities, Zeros*, World Sci., Publ., Singapore (1994).
[13] A. Mir, S. Hans, *Inequalities concerning rational functions in the complex domain*, Siberian Math. J., **63** (2022), 1012-1022.
[14] M. Y. Mir, S. L. Wali, W. M. Shah, *Extremal problems of Turán-type involving the location of all zeros of a class of rational functions*, Vestn. St.-Peterbg. Univ. Mat. Mekh. Astron., **57** (2024), 219-225.
[15] N. A. Rather, A. Iqbal, I. Dar, *Inequalities for rational functions with prescribed poles*, Math. Notes, **114** (2023), 593–607.
[16] V. N. Rusak, *Rational Functions as Technique of Approximation,* Bellorussia State Univ. Press, Minsk (1979). (Russian)
[17] N. K. Singha, B. Chanam, *Some refinements on lower bound estimates for polynomial with restricted zeros*, Rend. Circ. Mat. Palermo, Ser. II., **73** (2024), 1747–1762.
[18] R. Soraisam, K. B. Devi, B. Chanam, *Some inequalities for rational function with prescribed poles and restricted zeros*, Demonstr. Math., **58** (2025) (Art. 20250102), pp-1-15.
[19] S. L. Wali, W. M. Shah, *Some applications of Dubinin's lemma to rational functions with prescribed poles,* J. Math. Anal. Appl., **450** (2017), 769-779.
[20] S. L. Wali, *Inequalities for maximum modulus of rational functions with prescribed poles*, Kragujevac J. Math., **47** (2023), 865–875.