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Convergence analysis of the improved θ-scheme for two-dimensional double singular stochastic Volterra integral equations

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Abstract. This study contributes to the understanding of the behavior of two-dimensional double singular stochastic Volterra integral equations (DSSVIEs) and provides a comprehensive analysis of their properties. The evaluation of well-posedness, convergence, and the introduction of a numerical technique such as the improved θ-scheme (ITS) for solving the equation offer valuable insights for future research in this field. The proposed improvement utilizing sum-of-exponentials (SOE) approximation promises enhanced computational efficiency, which is crucial for practical applications. Overall, this study advances the knowledge of two-dimensional DSSVIEs and provides a solid foundation for further investigation and development in this area. An example is presented to demonstrate the effectiveness of the results achieved.

1. Introduction

Two-dimensional DSSVIEs are based on the concepts of stochastic Volterra integral equations (SVIEs) with weakly singular kernels. Their ability to capture memory effects and pervasive noise disturbances has led to extensive theoretical exploration and real-world use. In two-dimensional cases, the structure commonly displayed by DSSVIEs is as follows:

$$x(s,t) = x_0 + \int_0^t \int_0^s K_1(s,t,\tau_1,\tau_2) f(x(\tau_1,\tau_2)) d\tau_1 d\tau_2 + \int_0^t \int_0^s K_2(s,t,\tau_1,\tau_2) g(x(\tau_1,\tau_2)) dW(\tau_1) dW(\tau_2). \quad (1)$$

In the Eq. (1), the unknown function denoted as $x(s,t)$ is referred to as the solution, $(s,t) \in [0,T] \times [0,T]$, $\tau_1 < s$ and $\tau_2 < t$. This characteristic makes SVIEs well-suited for representing phenomena involving memory and noise across diverse scientific and technological domains, including biological population models [2, 3], mathematical finance models [4, 7, 14], and beyond [8]. In mathematics, numerous pertinent investigations have been conducted, as seen in [9, 10, 22–25]. Simultaneously, numerous researchers have concentrated on numerical analyses for SVIEs [16]-[19]. It's essential to understand the significant implications of the kernel functions $K_i(s,t,\tau_1,\tau_2), i = 1,2$ in Eq. (1) to grasp the characteristics of these equations. For instance, the Euler-Maruyama (EM) scheme's strong convergence order for regular SVIEs is $\frac{1}{2}$ [19]. However, for weakly singular SVIEs, this order diminishes [11, 37].

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As closed-form solutions for SVIEs are rarely available, numerical approximations have emerged as crucial tools for understanding solution behavior. The majority of numerical techniques are designed for SVIEs featuring smooth kernels. Various methods have been cited in literature, such as the EM method [12, 15, 19–21, 26, 27], enhanced rectangular method [28], the operational matrix technique [29–31] and Picard iteration approach [32, 33]. Regarding weakly singular SVIEs, Wang [34] formulated an existence and uniqueness theorem amidst non-Lipschitz prerequisites, alongside a linear growth criterion and specific integrable criteria. Under certain conditions and specific integrable prerequisites, Zhang [36] delved into the convergence analysis of the EM scheme. In a recent study [1], we established the convergence rate of the proposed scheme for one-dimensional DSSVIEs, specifically applied to the rough Heston model. In this paper, our emphasis shifts to two-dimensional DSSVIEs as detailed below:

$$\begin{aligned} x(s,t) = x_0(s,t) &+ \int_0^t \int_0^s (s-\tau_1)^{-\alpha_1}(t-\tau_2)^{-\alpha_1}\tau_1^{-\beta_1}\tau_2^{-\beta_1} f(x(\tau_1,\tau_2))d\tau_1 d\tau_2 \\ &+ \int_0^t \int_0^s (s-\tau_1)^{-\alpha_2}(t-\tau_2)^{-\alpha_2}\tau_1^{-\beta_2}\tau_2^{-\beta_2} g(x(\tau_1,\tau_2))dW(\tau_1)dW(\tau_2), \quad (s,t)\in[0,T]\times[0,T], \end{aligned} \tag{2}$$

where

- The function $x_0(s,t)$ is a known deterministic function that satisfies $\mathbb{E}\Big(|x_0(s,t)|^2\Big) < +\infty$. The functions f and g are both nonlinear and measurable functions.
- The expression $\int_0^t \int_0^s (s-\tau_1)^{-\alpha_2}(t-\tau_2)^{-\alpha_2}\tau_1^{-\beta_2}\tau_2^{-\beta_2} g(x(\tau_1,\tau_2))dW(\tau_1)dW(\tau_2)$ represents the double Itô integral, where $W(\tau_1)$ and $W(\tau_2)$ are two independent Brownian motions defined on the complete filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$.
- α_i and β_i $(i=1,2)$ are non-negative, and they satisfy the conditions $0<\alpha_1+\beta_1<1$, and $0<\alpha_2+\beta_2<\frac{1}{2}$.

To tackle the numerical solution of two-dimensional DSSVIEs (2), one might contemplate devising high-order numerical methods. However, the absence of the key Itô formula, a crucial tool in studying SDEs, poses a challenge. Therefore, advanced techniques such as the Milstein method [38], utilizing Itô-Taylor expansion, are not suitable for constructing effective solutions for Eq. (2). As a result, our focus turns to the improved θ-scheme based on the SOE approach. As evidenced in [37, 39], the presence of weakly singular kernels diminishes the convergence rate of EM method, necessitating the adoption of sufficiently small step sizes to ensure reasonable accuracy in concrete computations. Consequently, the paper will primarily concentrate on the subsequent three items:

- Our initial goal is to examine the well-posedness of two-dimensional DSSVIEs (2) under the conditions of global Lipschitz and linear growth. Furthermore, we will explore the boundedness and Holder continuity of the analytical solution, considering its reliance on the initial value.
- Secondly, we will introduce the θ-scheme for two-dimensional DSSVIEs (2) and determine its strong convergence rate.
- The incorporation of the SOE approximation in the suggested method provides substantial benefits concerning computational efficiency. By decreasing the computing expenses and storage demands to $O(N\log N)$ and $O(\log N)$ respectively for large T values, and even further to $O(N\log^2 N)$ and $O(\log^2 N)$ for T approaching 1, the method proves to be highly effective in handling the computational complexity associated with solving SVIEs.

The outlined structure of the remaining sections provides a clear roadmap for the study, starting with the introduction of essential assumptions and lemmas in Section 2. The subsequent sections explore the examination of well-posedness in Section 3, the determination of strong convergence properties of the proposed scheme in Section 4, and the implementation of the θ-scheme using the SOE approximation in Section 5. Section 6 focuses on presenting numerical experiments to validate the theoretical findings from Sections 4 and 5, while Section 7 offers a summary of the work conducted in the study.

2. Mathematical preliminaries

In this paper, $|\cdot|$ represents the Euclidean norm, where for a $x \in \mathbb{R}^d$, $|x| = \left(\sum_{i=0}^{d} x_i^2\right)^{\frac{1}{2}}$; and $\|\cdot\|$ denotes the trace norm of a matrix, that is, for $A \in \mathbb{R}^{d\times r}$, $\|A\| = \sqrt{trace(A^T A)}$. Consider a full probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ that meets standard criteria $\Big($i.e. it is right continuous, increasing and contains all the $\mathbb{P}$-null sets$\Big)$. The indicator function for a set S is symbolized as 1_S, defined as $1_S(x) = 1$ if $x \in S$ and 0 otherwise. For any pair of real numbers a and b, we denote the maximum as $a \vee b := \max\{a, b\}$ and the minimum as $a \wedge b := \min\{a, b\}$. Furthermore, the uppercase letter C stands for a general positive constant, the specific value of which may differ between instances but remains constant regardless of the step size h.

Assumption 2.1. *f and g comply with the following criteria:*

- *In accordance with the global Lipschitz condition, there is $L > 0$ that guarantees the validity of the following inequality:*

$$|f(u) - f(v)|^2 \vee \|g(u) - g(v)\|^2 \leq L|u - v|^2, \qquad \forall\, u, v \in \mathbb{R}^d. \tag{3}$$

- *In accordance with the linear growth condition, there is $K > 0$ that guarantees the validity of the following inequality:*

$$|f(u)|^2 \vee \|g(u)\|^2 \leq K(1 + |u|^2), \qquad \forall\, u \in \mathbb{R}^d. \tag{4}$$

We will frequently utilize the Beta function $\mathbf{B}(p, q) := \int_0^1 (1 - s)^{p-1} s^{q-1} ds$, which is well defined for $p, q > 0$. It is known that $\mathbf{B}(p, q) = \dfrac{\Gamma(p)\Gamma(q)}{\Gamma(p + q)}$, where Γ is the Gamma function defined as $\Gamma(p) := \int_0^\infty s^{p-1} \exp(-s) ds$. The following lemma is classical, and we present a concise proof for thoroughness since it is relevant to our discussion.

Lemma 2.2. *Let $0 \leq t_1 < t$, $\alpha > 0$ and $\beta < 1$. Then for $t > t_1$ we have:*

$$\int_{t_1}^{t} (t - s)^{-\alpha}(s - t_1)^{-\beta} ds = (t - t_1)^{1-(\alpha+\beta)} \mathbf{B}\Big(1 - \alpha, 1 - \beta\Big). \tag{5}$$

Proof. Change the variable of integration from s to u where $s = t_1 + u(t - t_1)$ and the integral becomes:

$$\begin{aligned} &\int_0^1 \Big((1 - u)(t - t_1)\Big)^{-\alpha} \Big(u(t - t_1)\Big)^{-\beta} (t - t_1) du \\ &\quad = (t - t_1)^{1-(\alpha+\beta)} \int_0^1 \Big(1 - u\Big)^{-\alpha} u^{-\beta} du = (t - t_1)^{1-(\alpha+\beta)} \mathbf{B}\Big(1 - \alpha, 1 - \beta\Big). \end{aligned} \tag{6}$$

□

Corollary 2.3. *If α and β are positive constants such that $\alpha + \beta \in (0, 1)$, it implies that for all $(s, t) \in [s_n, s_{n+1}] \times [t_m, t_{m+1}]$, where $n, m = 1, 2, \cdots, N - 1$.*

$$\begin{aligned} &\int_0^{t_m} \int_0^{s_n} (s_n - \tau_1)^{-\alpha}(t_m - \tau_2)^{-\alpha} \tau_1^{-\beta} \tau_2^{-\beta} d\tau_1 d\tau_2 \\ &\qquad = \int_0^1 \int_0^1 (s_n - s_n v)^{-\alpha}(t_m - t_m u)^{-\alpha}(s_n v)^{-\beta}(t_m u)^{-\beta} s_n t_m dv du \\ &\qquad = \int_0^1 (s_n - s_n v)^{-\alpha}(s_n v)^{-\beta} s_n dv \cdot \int_0^1 (t_m - t_m u)^{-\alpha}(t_m u)^{-\beta} t_m du \\ &\qquad = s_n^{1-(\alpha+\beta)} t_m^{1-(\alpha+\beta)} \mathbf{B}^2\Big(1 - \alpha, 1 - \beta\Big). \end{aligned} \tag{7}$$

Lemma 2.4. *([35]) Let $a \geq 0$ and $b > 0$ be constants and suppose that $\alpha > 0$, $\beta \geq 0$ and $\alpha + \beta < 1$. If u is an element of the space $L_{+}^{\infty}[0,T]$ and fulfills the inequality:*

$$u(t) \leq a + b\int_0^t (t-s)^{-\alpha} s^{-\beta} u(s)ds. \tag{8}$$

We define $\mathbf{B_0} = \mathbf{B}\Big(1-\alpha, 1-\beta\Big)$. For $r > 0$ let $t_r := \Big(\frac{r}{b\mathbf{B_0}}\Big)^{1/(1-\alpha-\beta)}$, and let $r_0 := b\mathbf{B_0}T^{1-\alpha-\beta}$ so that $t_r \leq T$ for $r \leq r_0$. Then, if $r \leq r_0$ and also $r < 1$ we have:

$$u(t) \leq \frac{a}{1-r}\exp\Big(\frac{bt_r^{-\alpha}}{(1-r)(1-\beta)}t^{1-\beta}\Big). \tag{9}$$

Moreover, for $r_1 = \frac{\alpha}{1-\beta}$, we have:

$$u(t) \leq \frac{a(1-\beta)}{1-\alpha-\beta}\exp\Big(\frac{bt_{r_1}^{-\alpha}}{(1-\alpha-\beta)}t^{1-\beta}\Big), \quad \text{for a.e. } t \in [0,T]. \tag{10}$$

Specifically, there is a defined constant $C(b,\alpha,\beta,T)$ such that $u(t) \leq aC(b,\alpha,\beta,T)$ for nearly all t within the interval $[0,T]$.

Lemma 2.5. *Let G be a stochastic process with $\mathbb{E}(|G|^2) < \infty$. Define:*

$$x(t) = \int_0^t G(s)dW(s), \qquad A(t) = \int_0^t |G(s)|^2 ds,$$

then for every $p > 0$, there exists a universal positive constant C_p (depending only on p), such that:

$$\mathbb{E}\Big(\sup_{0\leq s\leq t}|x(s)|^p\Big) \leq C_p\mathbb{E}\Big(|A(t)|^{\frac{p}{2}}\Big),$$

for all $t \geq 0$ (see [13], Theorem 7.3).

3. Well-posedness of two-dimensional DSSVIEs

In this section, the investigation includes examining the boundedness of the pth moment and studying how the solution of two-dimensional DSSVIEs (2) depends on the initial data.

Theorem 3.1. *Considering $x(s,t)$ as the solution of two-dimensional DSSVIEs (2), it can be stated that, subject to condition (4), $x(s,t)$ fulfills the following:*

$$\sup_{0\leq s,t\leq T}\mathbb{E}\Big(|x(s,t)|^p\Big) \leq C_p, \tag{11}$$

for some constant C_p.

Proof. For every integer $l \geq 1$, define the stopping time:

$$\rho_l = \inf\{(s,t) \in [0,T]\times[0,T] : |x(s,t)| \geq l\} \wedge T. \tag{12}$$

It is obvious that $\rho \to T$ almost surely when $l \to +\infty$. Concurrently, we define $x_l(s,t) = x(s\wedge\rho_l, t\wedge\rho_l)$ for $(s,t) \in [0,T]\times[0,T]$ and derive:

$$\mathbb{E}|x_l(s,t)|^p$$
$$\leq 3^{p-1}\mathbb{E}|x_0|^p + 3^{p-1}\mathbb{E}\Big|\int_0^{t\wedge\rho_l}\int_0^{s\wedge\rho_l}(s\wedge\rho_l-\tau_1)^{-\alpha_1}(t\wedge\rho_l-\tau_2)^{-\alpha_1}\tau_1^{-\beta_1}\tau_2^{-\beta_1}f\Big(x_l(\tau_1,\tau_2)\Big)d\tau_1 d\tau_2\Big|^p$$

$$+ 3^{p-1}\mathbb{E}\Big|\int_0^{t\wedge\rho_l}\int_0^{s\wedge\rho_l}(s\wedge\rho_l-\tau_1)^{-\alpha_2}(t\wedge\rho_l-\tau_2)^{-\alpha_2}\tau_1^{-\beta_2}\tau_2^{-\beta_2}g\Big(x_l(\tau_1,\tau_2)\Big)dW\tau_1 dW\tau_2\Big|^p$$

$$\leq 3^{p-1}\Big(\mathbb{E}|x_0|^p + C_1\Big(\int_0^{t\wedge\rho_l}\int_0^{s\wedge\rho_l}(s\wedge\rho_l-\tau_1)^{\frac{-(\alpha_1+\beta_1)p+q}{p-1}}(t\wedge\rho_l-\tau_2)^{\frac{-(\alpha_1+\beta_1)p+q}{p-1}}\Big(\tau_1\tau_2\Big)^{\frac{-(\alpha_1+\beta_1)p+q}{p-1}}d\tau_1 d\tau_2\Big)^{p-1}$$

$$\cdot\int_0^{t\wedge\rho_l}\int_0^{s\wedge\rho_l}(s\wedge\rho_l-\tau_1)^{-q}(t\wedge\rho_l-\tau_2)^{-q}\tau_1^{-q}\tau_2^{-q}\mathbb{E}|f\Big(x_l(\tau_1,\tau_2)\Big)|^p d\tau_1 d\tau_2$$

$$+ C_2\Big(\int_0^{t\wedge\rho_l}\int_0^{s\wedge\rho_l}(s\wedge\rho_l-\tau_1)^{\frac{-2(\alpha_2+\beta_2)p+2q}{p-2}}(t\wedge\rho_l-\tau_2)^{\frac{-2(\alpha_2+\beta_2)p+2q}{p-2}}\Big(\tau_1\tau_2\Big)^{\frac{-2(\alpha_2+\beta_2)p+2q}{p-2}}d\tau_1 d\tau_2\Big)^{p-2}$$

$$\cdot\int_0^{t\wedge\rho_l}\int_0^{s\wedge\rho_l}(s\wedge\rho_l-\tau_1)^{-q}(t\wedge\rho_l-\tau_2)^{-q}\tau_1^{-q}\tau_2^{-q}\mathbb{E}\|g\Big(x_l(\tau_1,\tau_2)\Big)\|^p d\tau_1 d\tau_2\Big)$$

$$\leq C_3\Big(1+\int_0^{t\wedge\rho_l}\int_0^{s\wedge\rho_l}(s\wedge\rho_l-\tau_1)^{-q}(t\wedge\rho_l-\tau_2)^{-q}\tau_1^{-q}\tau_2^{-q}\mathbb{E}|x_l(\tau_1,\tau_2)|^p d\tau_1 d\tau_2\Big), \tag{13}$$

where we used elementary inequality, Hölder's inequality, Lemma 2.5 and condition (4) for any $q \in \Big(1-p\min\{1-(\alpha_1+\beta_1), \frac{1}{2}-(\alpha_2+\beta_2)\}, 1\Big)$. It is concluded that:

$$\sup_{0\leq r_1\leq s,\ 0\leq r_2\leq t}\mathbb{E}\Big(|x_l(r_1,r_2)|^p\Big)$$

$$\leq C_p\Big(1+\sup_{0\leq r_1\leq s,\ 0\leq r_2\leq t}\int_0^{r_2\wedge\rho_l}\int_0^{r_1\wedge\rho_l}(r_1\wedge\rho_l-\tau_1)^{-q}(r_2\wedge\rho_l-\tau_2)^{-q}\tau_1^{q}\tau_2^{-q}$$

$$\cdot\sup_{0\leq\eta_1\leq\tau_1,\ 0\leq\eta_2\leq\tau_2}\mathbb{E}\Big(|x_l(\eta_1,\eta_2)|^p\Big)d\tau_1 d\tau_2\Big)$$

$$= C_p\Big(1+\sup_{0\leq r_1\leq s,\ 0\leq r_2\leq t}(r_1\wedge\rho_l)^{1-2q}(r_2\wedge\rho_l)^{1-2q}\int_0^1\int_0^1(1-v)^{-q}(1-u)^{-q}v^{-q}u^{-q}$$

$$\cdot\sup_{0\leq\eta_1\leq(r_1\wedge\rho_l)v,\ 0\leq\eta_2\leq(r_2\wedge\rho_l)u}\mathbb{E}\Big(|x_l(\eta_1,\eta_2)|^p\Big)dvdu\Big)$$

$$\leq C_p\Big(1+s^{1-2q}t^{1-2q}\int_0^1\int_0^1(1-v)^{-q}(1-u)^{-q}v^{-q}u^{-q}\sup_{0\leq\eta_1\leq sv,\ 0\leq\eta_2\leq tu}\mathbb{E}\Big(|x_l(\eta_1,\eta_2)|^p\Big)dvdu\Big)$$

$$= C_p\Big(1+\int_0^t\int_0^s(s-\tau_1)^{-q}(t-\tau_2)^{-q}\tau_1^{-q}\tau_2^{-q}\sup_{0\leq\eta_1\leq\tau_1,\ 0\leq\eta_2\leq\tau_2}\mathbb{E}\Big(|x_l(\eta_1,\eta_2)|^p\Big)d\tau_1 d\tau_2\Big). \tag{14}$$

By letting $v=\dfrac{\tau_1}{r_1\wedge\rho_l}$ and $u=\dfrac{\tau_2}{r_2\wedge\rho_l}$, we can utilize Lemma 2.4 and Fatou's lemma as l approaches infinity to finalize the proof. □

Theorem 3.2. *Given that the solution of relation* (2) *exhibits continuous dependency on the initial value for every $\xi > 0$ and any $\epsilon > 0$, in accordance with Assumption 2.1, the following relationship is established:*

$$\mathbb{E}|x(s,t)-z(s,t)|^2 < \epsilon, \tag{15}$$

where $\mathbb{E}|x_0-z_0|^2 < \xi$.

Proof. We introduce two stopping times as:

$$\mu_l = \inf\{(s,t)\in[0,T]\times[0,T] : |x(s,t)| > l\}, \quad \nu_l = \inf\{(s,t)\in[0,T]\times[0,T] : |z(s,t)| > l\}. \tag{16}$$

Then, we define $\vartheta_l = \mu_l\wedge\nu_l$ and $\psi(s,t) = x(s,t)-z(s,t)$. By choosing any positive number δ and $r > 2$, we have:

$$\mathbb{E}|\psi(s,t)|^2 = \mathbb{E}\Big(|\psi(s,t)|^2 1_{\{\vartheta_l > T\}}\Big) + \mathbb{E}\Big(|\psi(s,t)|^2 1_{\{\vartheta_l\leq T\}}\Big)$$

$$\leq \mathbb{E}\Big(|\psi(s\wedge\vartheta_l, t\wedge\vartheta_l)|^2 1_{\{\vartheta_l>T\}}\Big) + \frac{2\delta}{r}\mathbb{E}|\psi(s,t)|^r + \frac{r-2}{r\delta^{\frac{2}{r-2}}}\mathbb{P}\Big(\vartheta_l \leq T\Big), \tag{17}$$

where we have used the Young's inequality. Using Theorem 3.1, we obtain:

$$\mathbb{E}|\psi(s,t)|^r \leq 2^{r-1}\Big(\mathbb{E}|x(s,t)|^r + \mathbb{E}|z(s,t)|^r\Big) \leq 2^r C_r. \tag{18}$$

Furthermore, we have:

$$\begin{aligned}\mathbb{P}\Big(\vartheta_l \leq T\Big) &\leq \mathbb{P}\Big(\mu_l \leq T\Big) + \mathbb{P}\Big(\nu_l \leq T\Big)\\ &\leq \mathbb{E}\Big(1_{\{\mu_l\leq T\}}\frac{|x(\mu_l)|^r}{l^r}\Big) + \mathbb{E}\Big(1_{\{\nu_l\leq T\}}\frac{|z(\nu_l)|^r}{l^r}\Big)\\ &\leq \frac{1}{l^r}\Big(\mathbb{E}|x\Big(\mu_l\wedge T\Big)| + \mathbb{E}|z\Big(\nu_l\wedge T\Big)|\Big) \leq \frac{2C_r}{l^r}.\end{aligned} \tag{19}$$

According to (18) and (19), we have:

$$\mathbb{E}|\psi(s,t)|^2 = \mathbb{E}|\psi(s\wedge\vartheta_l, t\wedge\vartheta_l)|^2 + \frac{2^{r+1}C_r\delta}{r} + \frac{2C_r(r-2)}{rl^r\delta^{\frac{2}{r-2}}}. \tag{20}$$

We can explore:

$$\begin{aligned}&\mathbb{E}|\psi(s\wedge\vartheta_l, t\wedge\vartheta_l)|^2\\ &\leq CL\Big(\mathbb{E}|x_0 - z_0|^2 + \int_0^{t\wedge\vartheta_l}\int_0^{s\wedge\vartheta_l}(s\wedge\vartheta_l - \tau_1)^{-\alpha_1}(t\wedge\vartheta_l - \tau_2)^{-\alpha_1}\tau_1^{-\beta_1}\tau_2^{-\beta_1}\mathbb{E}|\psi(\tau_1,\tau_2)|^2 d\tau_1 d\tau_2\\ &+ \int_0^{t\wedge\vartheta_l}\int_0^{s\wedge\vartheta_l}(s\wedge\vartheta_l - \tau_1)^{-2\alpha_2}(t\wedge\vartheta_l - \tau_2)^{-2\alpha_2}\tau_1^{-2\beta_2}\tau_2^{-2\beta_2}\mathbb{E}|\psi(\tau_1,\tau_2)|^2 d\tau_1 d\tau_2\Big),\end{aligned} \tag{21}$$

where we have used the condition (3) and the Cauchy-Schwarz's inequality. Therefore, we achieve the following:

$$\begin{aligned}\mathbb{E}|\psi(s\wedge\vartheta_l, t\wedge\vartheta_l)|^2 &\leq 2C_l\mathbb{E}|x_0 - z_0|^2\\ &+ 2C_l\int_0^{t\wedge\vartheta_l}\int_0^{s\wedge\vartheta_l}(s\wedge\vartheta_l - \tau_1)^{-\alpha}(t\wedge\vartheta_l - \tau_2)^{-\alpha}\tau_1^{-\beta}\tau_2^{-\beta}\mathbb{E}|\psi(\tau_1,\tau_2)|^2 d\tau_1 d\tau_2.\end{aligned} \tag{22}$$

For parameters $\alpha, \beta > 0$ and $\alpha + \beta < 1$, employing Lemma 2.4 and substituting (22) into (20) results in:

$$\mathbb{E}|\psi(s,t)|^2 = C_l\mathbb{E}|x_0 - z_0|^2 + \frac{2^{r+1}C_r\delta}{r} + \frac{2C_r(r-2)}{rl^r\delta^{\frac{2}{r-2}}}. \tag{23}$$

By selecting suitable constants $\frac{2^{r+1}C_r\delta}{r} \leq \frac{1}{3}\epsilon$ and $\frac{2C_r(r-2)}{rl^r\delta^{\frac{2}{r-2}}} \leq \frac{1}{3}\epsilon$, the proof is finalized when $\mathbb{E}|x_0 - z_0|^2 < \xi$ and $C_l\xi < \frac{1}{3}\epsilon$. □

4. Convergence analysis

Within this section, we investigate the convergence analysis of a proposed method for two-dimensional DSSVIEs (2). The approach entails the approximation of (2) through a θ-scheme on a grid $I_h =: \{s_n := nh, t_m := mh : n, m = 0, 1, \ldots, N\}$ in the following manner:

$$x(s_n, t_m) = x_0 + \int_0^{t_m}\int_0^{s_n}(s_n - \tau_1)^{-\alpha_1}(t_m - \tau_2)^{-\alpha_1}\tau_1^{-\beta_1}\tau_2^{-\beta_1}f(x(\tau_1,\tau_2))d\tau_1 d\tau_2$$

$$+ \int_0^{t_m}\int_0^{s_n}(s_n-\tau_1)^{-\alpha_2}(t_m-\tau_2)^{-\alpha_2}\tau_1^{-\beta_2}\tau_2^{-\beta_2}g(x(\tau_1,\tau_2))dW(\tau_1)dW(\tau_2)$$

$$= x_0 + \theta\int_0^{t_m}\int_0^{s_n}(s_n-\tau_1)^{-\alpha_1}(t_m-\tau_2)^{-\alpha_1}\tau_1^{-\beta_1}\tau_2^{-\beta_1}f(x(\tau_1,\tau_2))d\tau_1 d\tau_2$$

$$+ (1-\theta)\int_0^{t_m}\int_0^{s_n}(s_n-\tau_1)^{-\alpha_1}(t_m-\tau_2)^{-\alpha_1}\tau_1^{-\beta_1}\tau_2^{-\beta_1}f(x(\tau_1,\tau_2))d\tau_1 d\tau_2$$

$$+ \int_0^{t_m}\int_0^{s_n}(s_n-\tau_1)^{-\alpha_2}(t_m-\tau_2)^{-\alpha_2}\tau_1^{-\beta_2}\tau_2^{-\beta_2}g(x(\tau_1,\tau_2))dW(\tau_1)dW(\tau_2)$$

$$\approx x_0 + \theta\sum_{j=0}^{m-1}\sum_{i=0}^{n-1}(s_n-s_i)^{-\alpha_1}(t_m-t_j)^{-\alpha_1}s_{i+1}^{-\beta_1}t_{j+1}^{-\beta_1}f\Big(x(s_{i+1},t_{j+1})\Big)h$$

$$+ (1-\theta)\sum_{j=0}^{m-1}\sum_{i=0}^{n-1}(s_n-s_i)^{-\alpha_1}(t_m-t_j)^{-\alpha_1}s_{i+1}^{-\beta_1}t_{j+1}^{-\beta_1}f\Big(x(s_i,t_j)\Big)h$$

$$+ \sum_{j=0}^{m-1}\sum_{i=0}^{n-1}(s_n-s_i)^{-\alpha_2}(t_m-t_j)^{-\alpha_2}s_{i+1}^{-\beta_2}t_{j+1}^{-\beta_2}g\Big(x(s_i,t_j)\Big)\Delta W_{i,j}. \tag{24}$$

Here $\theta \in [0,1]$ and $\Delta W_{i,j}$ represents two-dimensional Brownian motion. The θ-scheme for two-dimensional DSSVIEs (2) is introduced as follows:

$$X_{n,m} = x_0 + \theta h\sum_{j=0}^{m-1}\sum_{i=0}^{n-1}(s_n-s_i)^{-\alpha_1}(t_m-t_j)^{-\alpha_1}s_{i+1}^{-\beta_1}t_{j+1}^{-\beta_1}f(X_{i+1,j+1})$$

$$+ (1-\theta)h\sum_{j=0}^{m-1}\sum_{i=0}^{n-1}(s_n-s_i)^{-\alpha_1}(t_m-t_j)^{-\alpha_1}s_{i+1}^{-\beta_1}t_{j+1}^{-\beta_1}f(X_{i,j})$$

$$+ \sum_{j=0}^{m-1}\sum_{i=0}^{n-1}(s_n-s_i)^{-\alpha_2}(t_m-t_j)^{-\alpha_2}s_{i+1}^{-\beta_2}t_{j+1}^{-\beta_2}g(X_{i,j})\Delta W_{i,j}. \tag{25}$$

Considering $X_{n,m}$ as $x(s_n,t_m)$, the continuous version of relation (25) is formulated as follows:

$$X(s,t) = x_0 + \theta\int_0^t\int_0^s(s-\underline{\tau_1})^{-\alpha_1}(t-\underline{\tau_2})^{-\alpha_1}\overline{\tau_1}^{-\beta_1}\overline{\tau_2}^{-\beta_1}f(\overline{X}(\tau_1,\tau_2))d\tau_1 d\tau_2$$

$$+ (1-\theta)\int_0^t\int_0^s(s-\underline{\tau_1})^{-\alpha_1}(t-\underline{\tau_2})^{-\alpha_1}\overline{\tau_1}^{-\beta_1}\overline{\tau_2}^{-\beta_1}f(\underline{X}(\tau_1,\tau_2))d\tau_1 d\tau_2$$

$$+ \int_0^t\int_0^s(s-\underline{\tau_1})^{-\alpha_2}(t-\underline{\tau_2})^{-\alpha_2}\overline{\tau_1}^{-\beta_2}\overline{\tau_2}^{-\beta_2}g(\underline{X}(\tau_1,\tau_2))dW\tau_1 dW\tau_2, \tag{26}$$

where $\underline{\tau_1} := s_i$ for $\tau_1 \in [s_i, s_{i+1})$, $\underline{\tau_2} := t_j$ for $\tau_2 \in [t_j, t_{j+1})$ denote the left endpoints, $\overline{\tau_1} := s_{i+1}$ for $\tau_1 \in (s_i, s_{i+1}]$, $\overline{\tau_2} := t_{j+1}$ for $\tau_2 \in (t_j, t_{j+1}]$ denote the right endpoints, $\overline{X}(\tau_1,\tau_2) := X_{i+1,j+1}$ for $\tau_1 \in (s_i, s_{i+1}]$, $\tau_2 \in (t_j, t_{j+1}]$ and $\underline{X}(\tau_1,\tau_2) := X_{i,j}$ for $\tau_1 \in [s_i, s_{i+1})$, $\tau_2 \in [t_j, t_{j+1})$ stand for the piecewise constants interpolation of the proposed method. We introduce the following lemmas that indicate the rate of strong convergence of (26).

Lemma 4.1. *For $\alpha_1, \beta_1 > 0$ and $\alpha_1 + \beta_1 < 1$, there is a constant $C > 0$ not dependent on h, such that:*

$$\int_0^t\int_0^s |(s-\tau_1)^{-\alpha_1}(t-\tau_2)^{-\alpha_1}\tau_1^{-\beta_1}\tau_2^{-\beta_1} - (s-\underline{\tau_1})^{-\alpha_1}(t-\underline{\tau_2})^{-\alpha_1}\overline{\tau_1}^{-\beta_1}\overline{\tau_2}^{-\beta_1}|d\tau_1 d\tau_2 \le Ch^{2-2(\alpha_1+\beta_1)}. \tag{27}$$

Proof. Let $t \in [0, t_3]$ and $s \in [0, s_3]$. Using the relation (7) yields:

$$\begin{aligned}
&\int_0^t \int_0^s |(s-\tau_1)^{-\alpha_1}(t-\tau_2)^{-\alpha_1}\tau_1^{-\beta_1}\tau_2^{-\beta_1} - (s-\underline{\tau_1})^{-\alpha_1}(t-\underline{\tau_2})^{-\alpha_1}\overline{\tau_1}^{-\beta_1}\overline{\tau_2}^{-\beta_1}| d\tau_1 d\tau_2 \\
&\leq \int_0^t \int_0^s (s-\tau_1)^{-\alpha_1}(t-\tau_2)^{-\alpha_1}\tau_1^{-\beta_1}\tau_2^{-\beta_1} d\tau_1 d\tau_2 \\
&= \int_0^s (s-\tau_1)^{-\alpha_1}\tau_1^{-\beta_1} d\tau_1 \cdot \int_0^t (t-\tau_2)^{-\alpha_1}\tau_2^{-\beta_1} d\tau_2 \\
&= s^{1-(\alpha_1+\beta_1)} t^{1-(\alpha_1+\beta_1)} B^2(1-\alpha_1, 1-\beta_1) \leq Ch^{2-2(\alpha_1+\beta_1)}.
\end{aligned} \tag{28}$$

For $s \in (s_3, T]$ and $t \in (t_3, T]$, there are distinct positive integers $n, m \geq 3$ such that $s \in [s_n, s_{n+1})$ and $t \in [t_m, t_{m+1})$. The integral can be divided into three parts:

$$\begin{aligned}
&\int_0^t \int_0^s |(s-\tau_1)^{-\alpha_1}(t-\tau_2)^{-\alpha_1}\tau_1^{-\beta_1}\tau_2^{-\beta_1} - (s-\underline{\tau_1})^{-\alpha_1}(t-\underline{\tau_2})^{-\alpha_1}\overline{\tau_1}^{-\beta_1}\overline{\tau_2}^{-\beta_1}| d\tau_1 d\tau_2 \\
&= \int_0^{t_1} \int_0^{s_1} |(s-\tau_1)^{-\alpha_1}(t-\tau_2)^{-\alpha_1}\tau_1^{-\beta_1}\tau_2^{-\beta_1} - (s-\underline{\tau_1})^{-\alpha_1}(t-\underline{\tau_2})^{-\alpha_1}\overline{\tau_1}^{-\beta_1}\overline{\tau_2}^{-\beta_1}| d\tau_1 d\tau_2 \\
&+ \int_{t_1}^{t_{m-1}} \int_{s_1}^{s_{n-1}} |(s-\tau_1)^{-\alpha_1}(t-\tau_2)^{-\alpha_1}\tau_1^{-\beta_1}\tau_2^{-\beta_1} - (s-\underline{\tau_1})^{-\alpha_1}(t-\underline{\tau_2})^{-\alpha_1}\overline{\tau_1}^{-\beta_1}\overline{\tau_2}^{-\beta_1}| d\tau_1 d\tau_2 \\
&+ \int_{t_{m-1}}^{t} \int_{s_{n-1}}^{s} |(s-\tau_1)^{-\alpha_1}(t-\tau_2)^{-\alpha_1}\tau_1^{-\beta_1}\tau_2^{-\beta_1} - (s-\underline{\tau_1})^{-\alpha_1}(t-\underline{\tau_2})^{-\alpha_1}\overline{\tau_1}^{-\beta_1}\overline{\tau_2}^{-\beta_1}| d\tau_1 d\tau_2 \\
&:= I_1(s,t) + I_2(s,t) + I_3(s,t),
\end{aligned} \tag{29}$$

where we have:

$$\begin{aligned}
I_1(s,t) &= \int_0^{t_1} \int_0^{s_1} |(s-\tau_1)^{-\alpha_1}(t-\tau_2)^{-\alpha_1}\tau_1^{-\beta_1}\tau_2^{-\beta_1} - (s-\underline{\tau_1})^{-\alpha_1}(t-\underline{\tau_2})^{-\alpha_1}\overline{\tau_1}^{-\beta_1}\overline{\tau_2}^{-\beta_1}| d\tau_1 d\tau_2 \\
&\leq \int_0^{t_1} \int_0^{s_1} (s-\tau_1)^{-\alpha_1}(t-\tau_2)^{-\alpha_1}\tau_1^{-\beta_1}\tau_2^{-\beta_1} d\tau_1 d\tau_2 \\
&\leq \int_0^{s_1} (s_1-\tau_1)^{-\alpha_1}\tau_1^{-\beta_1} d\tau_1 \cdot \int_0^{t_1} (t_1-\tau_2)^{-\alpha_1}\tau_2^{-\beta_1} d\tau_2 \leq Ch^{2-2(\alpha_1+\beta_1)},
\end{aligned} \tag{30}$$

and the second term exibits:

$$\begin{aligned}
I_2(s,t) &= \int_{t_1}^{t_{m-1}} \int_{s_1}^{s_{n-1}} |(s-\tau_1)^{-\alpha_1}(t-\tau_2)^{-\alpha_1}\tau_1^{-\beta_1}\tau_2^{-\beta_1} - (s-\underline{\tau_1})^{-\alpha_1}(t-\underline{\tau_2})^{-\alpha_1}\overline{\tau_1}^{-\beta_1}\overline{\tau_2}^{-\beta_1}| d\tau_1 d\tau_2 \\
&= \int_{t_1}^{t_{m-1}} \int_{s_1}^{s_{n-1}} (s-\tau_1)^{-\alpha_1}(t-\tau_2)^{-\alpha_1}\Big(\tau_1^{-\beta_1}\tau_2^{-\beta_1} - \overline{\tau_1}^{-\beta_1}\overline{\tau_2}^{-\beta_1}\Big) d\tau_1 d\tau_2 \\
&+ \int_{t_1}^{t_{m-1}} \int_{s_1}^{s_{n-1}} \Big((s-\tau_1)^{-\alpha_1}(t-\tau_2)^{-\alpha_1} - (s-\underline{\tau_1})^{-\alpha_1}(t-\underline{\tau_2})^{-\alpha_1}\Big)\overline{\tau_1}^{-\beta_1}\overline{\tau_2}^{-\beta_1} d\tau_1 d\tau_2 \\
&:= I_{21}(s,t) + I_{22}(s,t),
\end{aligned} \tag{31}$$

such that:

$$\begin{aligned}
I_{21}(s,t) &\leq \sum_{j=1}^{m-2}\sum_{i=1}^{n-2} \int_{t_j}^{t_{j+1}} \int_{s_i}^{s_{i+1}} (s_n - s_{i+1})^{-\alpha_1}(t_m - t_{j+1})^{-\alpha_1}\Big(s_i^{-\beta_1} t_j^{-\beta_1} - s_{i+1}^{-\beta_1} t_{j+1}^{-\beta_1}\Big) d\tau_1 d\tau_2 \\
&= s_1^{-\beta_1} t_1^{-\beta_1}(s_n - s_2)^{-\alpha_1}(t_m - t_2)^{-\alpha_1} h^2 - s_{n-1}^{-\beta_1} t_{m-1}^{-\beta_1}(s_n - s_{n-1})^{-\alpha_1}(t_m - t_{m-1})^{-\alpha_1} h^2
\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{j=2}^{m-2}\sum_{i=2}^{n-2} s_i^{-\beta_1} t_j^{-\beta_1}\Big((s_n-s_{i+1})^{-\alpha_1}(t_m-t_{j+1})^{-\alpha_1}-(s_n-s_i)^{-\alpha_1}(t_m-t_j)^{-\alpha_1}\Big)h^2\\
&\le s_1^{-\beta_1}t_1^{-\beta_1}(s_n-s_2)^{-\alpha_1}(t_m-t_2)^{-\alpha_1}h^2\\
&+ s_2^{-\beta_1}t_2^{-\beta_1}\sum_{j=2}^{m-2}\sum_{i=2}^{n-2}\Big((s_n-s_{i+1})^{-\alpha_1}(t_m-t_{j+1})^{-\alpha_1}-(s_n-s_i)^{-\alpha_1}(t_m-t_j)^{-\alpha_1}\Big)h^2\\
&= s_1^{-\beta_1}t_1^{-\beta_1}(s_n-s_2)^{-\alpha_1}(t_m-t_2)^{-\alpha_1}h^2\\
&+ s_2^{-\beta_1}t_2^{-\beta_1}\sum_{j=2}^{m-2}\sum_{i=2}^{n-2}\Big((s_n-s_{i+1})^{-\alpha_1}-(s_n-s_i)^{-\alpha_1}\Big)(t_m-t_j)^{-\alpha_1}h^2\\
&+ s_2^{-\beta_1}t_2^{-\beta_1}\sum_{j=2}^{m-2}\sum_{i=2}^{n-2}\Big((t_m-t_{j+1})^{-\alpha_1}-(t_m-t_j)^{-\alpha_1}\Big)(s_n-s_{i+1})^{-\alpha_1}h^2\\
&\le s_1^{-\beta_1}t_1^{-\beta_1}(s_3-s_2)^{-\alpha_1}(t_3-t_2)^{-\alpha_1}h^2 + s_2^{-\beta_1}t_2^{-\beta_1}(s_n-s_{n-1})^{-\alpha_1}\sum_{j=2}^{m-2}(t_m-t_j)^{-\alpha_1}h^2\\
&+ s_2^{-\beta_1}t_2^{-\beta_1}(t_m-t_{m-1})^{-\alpha_1}\sum_{i=2}^{n-2}(s_n-s_{i+1})^{-\alpha_1}h^2 \le Ch^{2-2(\alpha_1+\beta_1)}, \qquad (32)
\end{aligned}
$$

and

$$
\begin{aligned}
I_{22}(s,t) &\le s_1^{-\beta_1}t_1^{-\beta_1}\sum_{j=1}^{m-2}\sum_{i=1}^{n-2}\int_{t_j}^{t_{j+1}}\int_{s_i}^{s_{i+1}}\Big((s_n-s_{i+1})^{-\alpha_1}(t_m-t_{j+1})^{-\alpha_1}-(s_{n+1}-s_i)^{-\alpha_1}(t_{m+1}-t_j)^{-\alpha_1}\Big)d\tau_1 d\tau_2\\
&= s_1^{-\beta_1}t_1^{-\beta_1}h^2\sum_{j=1}^{m-2}\sum_{i=1}^{n-2}\Big((s_n-s_{i+1})^{-\alpha_1}-(s_{n+1}-s_i)^{-\alpha_1}\Big)(t_m-t_{j+1})^{-\alpha_1}\\
&+ s_1^{-\beta_1}t_1^{-\beta_1}h^2\sum_{j=1}^{m-2}\sum_{i=1}^{n-2}\Big((t_m-t_{j+1})^{-\alpha_1}-(t_{m+1}-t_j)^{-\alpha_1}\Big)(s_{n+1}-s_i)^{-\alpha_1}\\
&= s_1^{-\beta_1}t_1^{-\beta_1}h^2\sum_{i=1}^{n-2}\Big((n-i-1)^{-\alpha_1}-(n-i+1)^{-\alpha_1}\Big)h^{-\alpha_1}\cdot\sum_{j=1}^{m-2}(m-j-1)^{-\alpha_1}h^{-\alpha_1}\\
&+ s_1^{-\beta_1}t_1^{-\beta_1}h^2\sum_{j=1}^{m-2}\Big((m-j-1)^{-\alpha_1}-(m-j+1)^{-\alpha_1}\Big)h^{-\alpha_1}\cdot\sum_{i=1}^{n-2}(n-i+1)^{-\alpha_1}h^{-\alpha_1}\\
&\le s_1^{-\beta_1}t_1^{-\beta_1}h^{2-2\alpha_1}\Big(1+2^{-\alpha_1}-(n-1)^{-\alpha_1}-n^{-\alpha_1}\Big)+s_1^{-\beta_1}t_1^{-\beta_1}h^{2-2\alpha_1}\Big(1-2^{-\alpha_1}-(m-1)^{-\alpha_1}-m^{-\alpha_1}\Big)\\
&\le Ch^{2-2(\alpha_1+\beta_1)}. \qquad (33)
\end{aligned}
$$

Additionally, the final term allows:

$$
\begin{aligned}
I_3(s,t) &= \int_{t_{m-1}}^{t}\int_{s_{n-1}}^{s}|(s-\tau_1)^{-\alpha_1}(t-\tau_2)^{-\alpha_1}\tau_1^{-\beta_1}\tau_2^{-\beta_1}-(s-\underline{\tau_1})^{-\alpha_1}(t-\underline{\tau_2})^{-\alpha_1}\overline{\tau_1}^{-\beta_1}\overline{\tau_2}^{-\beta_1}|d\tau_1 d\tau_2\\
&\le \int_{t_{m-1}}^{t}\int_{s_{n-1}}^{s}(s-\tau_1)^{-\alpha_1}(t-\tau_2)^{-\alpha_1}\tau_1^{-\beta_1}\tau_2^{-\beta_1}d\tau_1 d\tau_2\\
&= \int_{s_{n-1}}^{s}(s-\tau_1)^{-\alpha_1}\tau_1^{-\beta_1}d\tau_1\cdot\int_{t_{m-1}}^{t}(t-\tau_2)^{-\alpha_1}\tau_2^{-\beta_1}d\tau_2
\end{aligned}
$$

$$\leq s_{n-1}^{-\beta_1}\int_{s_{n-1}}^{s}(s-\tau_1)^{-\alpha_1}d\tau_1\cdot t_{m-1}^{-\beta_1}\int_{t_{m-1}}^{t}(t-\tau_2)^{-\alpha_1}d\tau_2\leq Ch^{2-2(\alpha_1+\beta_1)}. \tag{34}$$

The first step is based on the fact:

$$0\leq\int_{t_{m-1}}^{t}\int_{s_{n-1}}^{s}(s-\underline{\tau_1})^{-\alpha_1}(t-\underline{\tau_2})^{-\alpha_1}\overline{\tau_1}^{-\beta_1}\overline{\tau_2}^{-\beta_1}d\tau_1d\tau_2\leq\int_{t_{m-1}}^{t}\int_{s_{n-1}}^{s}(s-\tau_1)^{-\alpha_1}(t-\tau_2)^{-\alpha_1}\tau_1^{-\beta_1}\tau_2^{-\beta_1}d\tau_1d\tau_2.$$

Hence, combining (30)–(34) with (29) confirms the desired claim (27). □

Lemma 4.2. *For $\alpha_2,\beta_2>0$ and $\alpha_2+\beta_2<\frac{1}{2}$, there is a constant $C>0$ not dependent on h, such that:*

$$\int_0^{t_m}\int_0^{s_n}|(s-\underline{\tau_1})^{-\alpha_2}(t-\underline{\tau_2})^{-\alpha_2}\overline{\tau_1}^{-\beta_2}\overline{\tau_2}^{-\beta_2}-(s_n-\underline{\tau_1})^{-\alpha_2}(t_m-\underline{\tau_2})^{-\alpha_2}\overline{\tau_1}^{-\beta_2}\overline{\tau_2}^{-\beta_2}|^2d\tau_1d\tau_2\leq Ch^{2-4(\alpha_2+\beta_2)}. \tag{35}$$

Furthermore, for any $(s,t)\in[0,T]\times[0,T]$, it holds true that:

$$\int_0^{t}\int_0^{s}|(s-\tau_1)^{-\alpha_2}(t-\tau_2)^{-\alpha_2}\tau_1^{-\beta_2}\tau_2^{-\beta_2}-(s-\underline{\tau_1})^{-\alpha_2}(t-\underline{\tau_2})^{-\alpha_2}\overline{\tau_1}^{-\beta_2}\overline{\tau_2}^{-\beta_2}|^2d\tau_1d\tau_2\leq Ch^{2-4(\alpha_2+\beta_2)}. \tag{36}$$

Proof. Based on Lemma 4.1, we can write:

$$\begin{aligned}
&\int_0^{t_m}\int_0^{s_n}|(s-\underline{\tau_1})^{-\alpha_2}(t-\underline{\tau_2})^{-\alpha_2}\overline{\tau_1}^{-\beta_2}\overline{\tau_2}^{-\beta_2}-(s_n-\underline{\tau_1})^{-\alpha_2}(t_m-\underline{\tau_2})^{-\alpha_2}\overline{\tau_1}^{-\beta_2}\overline{\tau_2}^{-\beta_2}|^2d\tau_1d\tau_2\\
&=\int_0^{t_m}\int_0^{s_n}\Big((s-\underline{\tau_1})^{-2\alpha_2}(t-\underline{\tau_2})^{-2\alpha_2}+(s_n-\underline{\tau_1})^{-2\alpha_2}(t_n-\underline{\tau_2})^{-2\alpha_2}\\
&-2(s-\underline{\tau_1})^{-\alpha_2}(t-\underline{\tau_2})^{-\alpha_2}(s_n-\underline{\tau_1})^{-\alpha_2}(t_n-\underline{\tau_2})^{-\alpha_2}\Big)\overline{\tau_1}^{-2\beta_2}\overline{\tau_2}^{-2\beta_2}d\tau_1d\tau_2\\
&\leq\int_0^{t_m}\int_0^{s_n}\Big((s-\underline{\tau_1})^{-2\alpha_2}(t-\underline{\tau_2})^{-2\alpha_2}+(s_n-\underline{\tau_1})^{-2\alpha_2}(t_n-\underline{\tau_2})^{-2\alpha_2}\\
&-2(s-\underline{\tau_1})^{-2\alpha_2}(t-\underline{\tau_2})^{-2\alpha_2}\Big)\overline{\tau_1}^{-2\beta_2}\overline{\tau_2}^{-2\beta_2}d\tau_1d\tau_2\\
&=\int_0^{t_m}\int_0^{s_n}\Big((s_n-\underline{\tau_1})^{-2\alpha_2}(t_n-\underline{\tau_2})^{-2\alpha_2}-(s-\underline{\tau_1})^{-2\alpha_2}(t-\underline{\tau_2})^{-2\alpha_2}\Big)\overline{\tau_1}^{-2\beta_2}\overline{\tau_2}^{-2\beta_2}d\tau_1d\tau_2\\
&=\int_0^{t_m}\int_0^{s_n}(s_n-\underline{\tau_1})^{-2\alpha_2}(t_n-\underline{\tau_2})^{-2\alpha_2}\overline{\tau_1}^{-2\beta_2}\overline{\tau_2}^{-2\beta_2}d\tau_1d\tau_2\\
&-\int_0^{t}\int_0^{s}(s-\underline{\tau_1})^{-2\alpha_2}(t-\underline{\tau_2})^{-2\alpha_2}\overline{\tau_1}^{-2\beta_2}\overline{\tau_2}^{-2\beta_2}d\tau_1d\tau_2\\
&+\int_{t_m}^{t}\int_{s_n}^{s}(s-\underline{\tau_1})^{-2\alpha_2}(t-\underline{\tau_2})^{-2\alpha_2}\overline{\tau_1}^{-2\beta_2}\overline{\tau_2}^{-2\beta_2}d\tau_1d\tau_2\\
&\leq\int_0^{t_m}\int_0^{s_n}(s_n-\tau_1)^{-2\alpha_2}(t_n-\tau_2)^{-2\alpha_2}\tau_1^{-2\beta_2}\tau_2^{-2\beta_2}d\tau_1d\tau_2\\
&-\int_0^{t}\int_0^{s}(s-\tau_1)^{-2\alpha_2}(t-\tau_2)^{-2\alpha_2}\tau_1^{-2\beta_2}\tau_2^{-2\beta_2}d\tau_1d\tau_2\\
&+\int_{t_m}^{t}\int_{s_n}^{s}(s-\tau_1)^{-2\alpha_2}(t-\tau_2)^{-2\alpha_2}\tau_1^{-2\beta_2}\tau_2^{-2\beta_2}d\tau_1d\tau_2\\
&\leq s_n^{-2\beta_2}t_m^{-2\beta_2}\int_{s_n}^{s}(s-\tau_1)^{-2\alpha_2}d\tau_1\cdot\int_{t_m}^{t}(t-\tau_2)^{-2\alpha_2}d\tau_2\leq Ch^{2-4(\alpha_2+\beta_2)}.
\end{aligned} \tag{37}$$

Similarly, for $s \in [0, s_3]$ and $t \in [0, t_3]$ we consider (36) and obtain:

$$\begin{aligned}
&\int_0^t \int_0^s |(s-\tau_1)^{-\alpha_2}(t-\tau_2)^{-\alpha_2}\tau_1^{-\beta_2}\tau_2^{-\beta_2} - (s-\underline{\tau_1})^{-\alpha_2}(t-\underline{\tau_2})^{-\alpha_2}\overline{\tau_1}^{-\beta_2}\overline{\tau_2}^{-\beta_2}|^2 d\tau_1 d\tau_2 \\
&\quad \le \int_0^t \int_0^s \Big((s-\tau_1)^{-2\alpha_2}(t-\tau_2)^{-2\alpha_2}\tau_1^{-2\beta_2}\tau_2^{-2\beta_2} - (s-\underline{\tau_1})^{-2\alpha_2}(t-\underline{\tau_2})^{-2\alpha_2}\overline{\tau_1}^{-2\beta_2}\overline{\tau_2}^{-2\beta_2}\Big) d\tau_1 d\tau_2 \\
&\quad \le \int_0^t \int_0^s (s-\tau_1)^{-2\alpha_2}(t-\tau_2)^{-2\alpha_2}\tau_1^{-2\beta_2}\tau_2^{-2\beta_2} d\tau_1 d\tau_2 \\
&\quad = \int_0^s (s-\tau_1)^{-2\alpha_2}\tau_1^{-2\beta_2} d\tau_1 \cdot \int_0^t (t-\tau_2)^{-2\alpha_2}\tau_2^{-2\beta_2} d\tau_2 \\
&\quad = s^{1-2(\alpha_2+\beta_2)} t^{1-2(\alpha_2+\beta_2)} B^2(1-2\alpha_2, 1-2\beta_2) \le C h^{2-4(\alpha_2+\beta_2)},
\end{aligned} \tag{38}$$

and for $s \in (s_3, T]$ and $t \in (t_3, T]$, it is easy to show that:

$$\begin{aligned}
&\int_0^t \int_0^s |(s-\tau_1)^{-\alpha_2}(t-\tau_2)^{-\alpha_2}\tau_1^{-\beta_2}\tau_2^{-\beta_2} - (s-\underline{\tau_1})^{-\alpha_2}(t-\underline{\tau_2})^{-\alpha_2}\overline{\tau_1}^{-\beta_2}\overline{\tau_2}^{-\beta_2}|^2 d\tau_1 d\tau_2 \\
&= \int_0^{t_1} \int_0^{s_1} |(s-\tau_1)^{-\alpha_2}(t-\tau_2)^{-\alpha_2}\tau_1^{-\beta_2}\tau_2^{-\beta_2} - (s-\underline{\tau_1})^{-\alpha_2}(t-\underline{\tau_2})^{-\alpha_2}\overline{\tau_1}^{-\beta_2}\overline{\tau_2}^{-\beta_2}|^2 d\tau_1 d\tau_2 \\
&+ \int_{t_1}^{t_{m-1}} \int_{s_1}^{s_{n-1}} |(s-\tau_1)^{-\alpha_2}(t-\tau_2)^{-\alpha_2}\tau_1^{-\beta_2}\tau_2^{-\beta_2} - (s-\underline{\tau_1})^{-\alpha_2}(t-\underline{\tau_2})^{-\alpha_2}\overline{\tau_1}^{-\beta_2}\overline{\tau_2}^{-\beta_2}|^2 d\tau_1 d\tau_2 \\
&+ \int_{t_{m-1}}^{t} \int_{s_{n-1}}^{s} |(s-\tau_1)^{-\alpha_2}(t-\tau_2)^{-\alpha_2}\tau_1^{-\beta_2}\tau_2^{-\beta_2} - (s-\underline{\tau_1})^{-\alpha_2}(t-\underline{\tau_2})^{-\alpha_2}\overline{\tau_1}^{-\beta_2}\overline{\tau_2}^{-\beta_2}|^2 d\tau_1 d\tau_2 \\
&:= M_1(s,t) + M_2(s,t) + M_3(s,t),
\end{aligned} \tag{39}$$

where the initial term on the right-hand side is:

$$\begin{aligned}
M_1(s,t) &= \int_0^{t_1} \int_0^{s_1} |(s-\tau_1)^{-\alpha_2}(t-\tau_2)^{-\alpha_2}\tau_1^{-\beta_2}\tau_2^{-\beta_2} - (s-\underline{\tau_1})^{-\alpha_2}(t-\underline{\tau_2})^{-\alpha_2}\overline{\tau_1}^{-\beta_2}\overline{\tau_2}^{-\beta_2}|^2 d\tau_1 d\tau_2 \\
&\le \int_0^{t_1} \int_0^{s_1} (s-\tau_1)^{-2\alpha_2}(t-\tau_2)^{-2\alpha_2}\tau_1^{-2\beta_2}\tau_2^{-2\beta_2} d\tau_1 d\tau_2 \\
&\le \int_0^{s_1} (s_1-\tau_1)^{-2\alpha_2}\tau_1^{-2\beta_2} d\tau_1 \cdot \int_0^{t_1} (t_1-\tau_2)^{-2\alpha_2}\tau_2^{-2\beta_2} d\tau_2 \\
&= s_1^{1-2(\alpha_2+\beta_2)} t_1^{1-2(\alpha_2+\beta_2)} \mathbf{B}^2\Big(1-2\alpha_2, 1-2\beta_2\Big) \le C h^{2-4(\alpha_2+\beta_2)},
\end{aligned} \tag{40}$$

and the second term exhibits:

$$\begin{aligned}
M_2(s,t) &= \int_{t_1}^{t_{m-1}} \int_{s_1}^{s_{n-1}} |(s-\tau_1)^{-\alpha_2}(t-\tau_2)^{-\alpha_2}\tau_1^{-\beta_2}\tau_2^{-\beta_2} - (s-\underline{\tau_1})^{-\alpha_2}(t-\underline{\tau_2})^{-\alpha_2}\overline{\tau_1}^{-\beta_2}\overline{\tau_2}^{-\beta_2}|^2 d\tau_1 d\tau_2 \\
&\le \int_{t_1}^{t_{m-1}} \int_{s_1}^{s_{n-1}} \Big((s-\tau_1)^{-2\alpha_2}(t-\tau_2)^{-2\alpha_2}\tau_1^{-2\beta_2}\tau_2^{-2\beta_2} - (s-\underline{\tau_1})^{-2\alpha_2}(t-\underline{\tau_2})^{-2\alpha_2}\overline{\tau_1}^{-2\beta_2}\overline{\tau_2}^{-2\beta_2}\Big) d\tau_1 d\tau_2 \\
&= \int_{t_1}^{t_{m-1}} \int_{s_1}^{s_{n-1}} (s-\tau_1)^{-2\alpha_2}(t-\tau_2)^{-2\alpha_2}\Big((\tau_1\tau_2)^{-2\beta_2} - (\overline{\tau_1\tau_2})^{-2\beta_2}\Big) d\tau_1 d\tau_2 \\
&+ \int_{t_1}^{t_{m-1}} \int_{s_1}^{s_{n-1}} \Big((s-\tau_1)^{-2\alpha_2}(t-\tau_2)^{-2\alpha_2} - (s-\underline{\tau_1})^{-2\alpha_2}(t-\underline{\tau_2})^{-2\alpha_2}\Big)(\overline{\tau_1\tau_2})^{-2\beta_2} d\tau_1 d\tau_2 \\
&:= M_{21}(s,t) + M_{22}(s,t),
\end{aligned} \tag{41}$$

such that:

$$
\begin{aligned}
M_{21}(s,t) &\le \sum_{j=1}^{m-2}\sum_{i=1}^{n-2}\int_{t_j}^{t_{j+1}}\int_{s_i}^{s_{i+1}}(s_n-s_{i+1})^{-2\alpha_2}(t_m-t_{j+1})^{-2\alpha_2}\Big(s_i^{-2\beta_2}t_j^{-2\beta_2}-s_{i+1}^{-2\beta_2}t_{j+1}^{-2\beta_2}\Big)d\tau_1 d\tau_2\\
&= s_1^{-2\beta_2}t_1^{-2\beta_2}(s_n-s_2)^{-2\alpha_2}(t_m-t_2)^{-2\alpha_2}h^2 - s_{n-1}^{-2\beta_2}t_{m-1}^{-2\beta_2}(s_n-s_{n-1})^{-2\alpha_2}(t_m-t_{m-1})^{-2\alpha_2}h^2\\
&+\sum_{j=2}^{m-2}\sum_{i=2}^{n-2}s_i^{-2\beta_2}t_j^{-2\beta_2}\Big((s_n-s_{i+1})^{-2\alpha_2}(t_m-t_{j+1})^{-2\alpha_2}-(s_n-s_i)^{-2\alpha_2}(t_m-t_j)^{-2\alpha_2}\Big)h^2\\
&\le s_1^{-2\beta_2}t_1^{-2\beta_2}(s_n-s_2)^{-2\alpha_2}(t_m-t_2)^{-2\alpha_2}h^2\\
&+ s_2^{-2\beta_2}t_2^{-2\beta_2}\sum_{j=2}^{m-2}\sum_{i=2}^{n-2}\Big((s_n-s_{i+1})^{-2\alpha_2}(t_m-t_{j+1})^{-2\alpha_2}-(s_n-s_i)^{-2\alpha_2}(t_m-t_j)^{-2\alpha_2}\Big)h^2\\
&= s_1^{-2\beta_2}t_1^{-2\beta_2}(s_n-s_2)^{-2\alpha_2}(t_m-t_2)^{-2\alpha_2}h^2\\
&+ s_2^{-2\beta_2}t_2^{-2\beta_2}\sum_{j=2}^{m-2}\sum_{i=2}^{n-2}\Big((s_n-s_{i+1})^{-2\alpha_2}-(s_n-s_i)^{-2\alpha_2}\Big)(t_m-t_j)^{-2\alpha_2}h^2\\
&+ s_2^{-2\beta_2}t_2^{-2\beta_2}\sum_{j=2}^{m-2}\sum_{i=2}^{n-2}\Big((t_m-t_{j+1})^{-2\alpha_2}-(t_m-t_j)^{-2\alpha_2}\Big)(s_n-s_{i+1})^{-2\alpha_2}h^2\\
&\le s_1^{-2\beta_2}t_1^{-2\beta_2}(s_3-s_2)^{-2\alpha_2}(t_3-t_2)^{-2\alpha_2}h^2 + s_2^{-2\beta_2}t_2^{-2\beta_2}(s_n-s_{n-1})^{-2\alpha_2}\sum_{j=2}^{m-2}(t_m-t_j)^{-2\alpha_2}h^2\\
&+ s_2^{-2\beta_2}t_2^{-2\beta_2}(t_m-t_{m-1})^{-2\alpha_2}\sum_{i=2}^{n-2}(s_n-s_{i+1})^{-2\alpha_2}h^2 \le Ch^{2-4(\alpha_2+\beta_2)},
\end{aligned}
\tag{42}
$$

and

$$
\begin{aligned}
&M_{22}(s,t)\\
&\le (s_1t_1)^{-2\beta_2}\sum_{j=1}^{m-2}\sum_{i=1}^{n-2}\int_{t_j}^{t_{j+1}}\int_{s_i}^{s_{i+1}}\Big((s_n-s_{i+1})^{-2\alpha_2}(t_m-t_{j+1})^{-2\alpha_2}-(s_{n+1}-s_i)^{-2\alpha_2}(t_{m+1}-t_j)^{-2\alpha_2}\Big)d\tau_1 d\tau_2\\
&= (s_1t_1)^{-2\beta_2}h^2\sum_{j=1}^{m-2}\sum_{i=1}^{n-2}\Big((s_n-s_{i+1})^{-2\alpha_2}-(s_{n+1}-s_i)^{-2\alpha_2}\Big)(t_m-t_{j+1})^{-2\alpha_2}\\
&+ (s_1t_1)^{-2\beta_2}h^2\sum_{j=1}^{m-2}\sum_{i=1}^{n-2}\Big((t_m-t_{j+1})^{-2\alpha_2}-(t_{m+1}-t_j)^{-2\alpha_2}\Big)(s_{n+1}-s_i)^{-2\alpha_2}\\
&= (s_1t_1)^{-2\beta_2}h^2\sum_{i=1}^{n-2}\Big((n-i-1)^{-2\alpha_2}-(n-i+1)^{-2\alpha_2}\Big)h^{-2\alpha_2}\cdot\sum_{j=1}^{m-2}(m-j-1)^{-2\alpha_2}h^{-2\alpha_2}\\
&+ (s_1t_1)^{-2\beta_2}h^2\sum_{j=1}^{m-2}\Big((m-j-1)^{-2\alpha_2}-(m-j+1)^{-2\alpha_2}\Big)h^{-2\alpha_2}\cdot\sum_{i=1}^{n-2}(n-i+1)^{-2\alpha_2}h^{-2\alpha_2}\\
&\le (s_1t_1)^{-2\beta_2}h^{2-4\alpha_2}\Big(1+2^{-2\alpha_2}-(n-1)^{-2\alpha_2}-n^{-2\alpha_2}\Big)+(s_1t_1)^{-2\beta_2}h^{2-4\alpha_2}\Big(1-2^{-2\alpha_2}-(m-1)^{-2\alpha_2}-m^{-2\alpha_2}\Big)\\
&\le Ch^{2-4(\alpha_2+\beta_2)}.
\end{aligned}
\tag{43}
$$

Furthermore, the final term allows:

$$
\begin{aligned}
M_3(s,t) &= \int_{t_{m-1}}^{t}\int_{s_{n-1}}^{s} |(s-\tau_1)^{-\alpha_2}(t-\tau_2)^{-\alpha_2}\tau_1^{-\beta_2}\tau_2^{-\beta_2} - (s-\underline{\tau_1})^{-\alpha_2}(t-\underline{\tau_2})^{-\alpha_2}\overline{\tau_1}^{-\beta_2}\overline{\tau_2}^{-\beta_2}|^2 d\tau_1 d\tau_2 \\
&\le \int_{t_{m-1}}^{t}\int_{s_{n-1}}^{s} \Big((s-\tau_1)^{-2\alpha_2}(t-\tau_2)^{-2\alpha_2}\tau_1^{-2\beta_2}\tau_2^{-2\beta_2} - (s-\underline{\tau_1})^{-2\alpha_2}(t-\underline{\tau_2})^{-2\alpha_2}\overline{\tau_1}^{-2\beta_2}\overline{\tau_2}^{-2\beta_2}\Big) d\tau_1 d\tau_2 \\
&\le \int_{t_{m-1}}^{t}\int_{s_{n-1}}^{s} (s-\tau_1)^{-2\alpha_2}(t-\tau_2)^{-2\alpha_2}\tau_1^{-2\beta_2}\tau_2^{-2\beta_2} d\tau_1 d\tau_2 \\
&\le s_{n-1}^{-2\beta_2} t_{m-1}^{-2\beta_2}\int_{t_{m-1}}^{t}\int_{s_{n-1}}^{s} (s-\tau_1)^{-2\alpha_2}(t-\tau_2)^{-2\alpha_2} d\tau_1 d\tau_2 \le Ch^{2-4(\alpha_2+\beta_2)}. \qquad (44)
\end{aligned}
$$

By replacing equations (40)–(44) into equation (39), we achieve the intended result (36). This completes the proof of Lemma 4.2. □

Lemma 4.3. *Given that Assumption 2.1 is met, we obtain:*

$$
\sup_{0\le s,t\le T}\mathbb{E}\Big(|\underline{X}(s,t)|^2\Big) \le \sup_{0\le s,t\le T}\mathbb{E}\Big(|\overline{X}(s,t)|^2\Big) \le \sup_{0\le s,t\le T}\mathbb{E}\Big(|X(s,t)|^2\Big) \le C_1, \qquad (45)
$$

and

$$
\mathbb{E}\Big(|X(s,t)-\underline{X}(s,t)|^2\Big) \vee \mathbb{E}\Big(|X(s,t)-\overline{X}(s,t)|^2\Big) \le Ch^{\min\left\{4-4(\alpha_1+\beta_1),2-4(\alpha_2+\beta_2)\right\}}. \qquad (46)
$$

where C_1 and C_2 are positive constants independent of h.

Drawing from Lemmas 4.1 and 4.2, the process of establishing Lemma 4.3 bears resemblance to the method employed in proving Theorem 3.1.

Theorem 4.4. *Let $x(s,t)$ and $X(s,t)$ denote the exact and approximate solution respectively for two-dimensional DSSVIEs* (2)*, then based on Assumptions 2.1 and Theorem 3.2 we have:*

$$
\Big(\mathbb{E}\Big(|x(s,t)-X(s,t)|^2\Big)\Big)^{\frac{1}{2}} \le Ch^{\min\{2-2(\alpha_1+\beta_1),1-2(\alpha_2+\beta_2)\}}. \qquad (47)
$$

Proof. We can split $x(s,t)-X(s,t)$ in the form:

$$
x(s,t)-X(s,t) = Y_1(s,t)+Y_2(s,t)+Y_3(s,t), \qquad (48)
$$

where

$$
\begin{aligned}
Y_1(s,t) &= \big(1-\theta\big)\Big(\int_0^t\int_0^s (s-\tau_1)^{-\alpha_1}(t-\tau_2)^{-\alpha_1}\tau_1^{-\beta_1}\tau_2^{-\beta_1}\Big(f(x(\tau_1,\tau_2)) - f(\underline{X}(\tau_1,\tau_2))\Big)d\tau_1 d\tau_2 \\
&\quad + \int_0^t\int_0^s \Big((s-\tau_1)^{-\alpha_1}(t-\tau_2)^{-\alpha_1}\tau_1^{-\beta_1}\tau_2^{-\beta_1} - (s-\underline{\tau_1})^{-\alpha_1}(t-\underline{\tau_2})^{-\alpha_1}\overline{\tau_1}^{-\beta_1}\overline{\tau_2}^{-\beta_1}\Big)f(\underline{X}(\tau_1,\tau_2))d\tau_1 d\tau_2\Big) \\
&:= \big(1-\theta\big)\Big(Y_{11}(s,t)+Y_{12}(s,t)\Big), \qquad (49)
\end{aligned}
$$

$$
\begin{aligned}
Y_2(s,t) &= \theta\Big(\int_0^t\int_0^s (s-\tau_1)^{-\alpha_1}(t-\tau_2)^{-\alpha_1}\tau_1^{-\beta_1}\tau_2^{-\beta_1}\Big(f(x(\tau_1,\tau_2)) - f(\overline{X}(\tau_1,\tau_2))\Big)d\tau_1 d\tau_2 \\
&\quad + \int_0^t\int_0^s \Big((s-\tau_1)^{-\alpha_1}(t-\tau_2)^{-\alpha_1}\tau_1^{-\beta_1}\tau_2^{-\beta_1} - (s-\underline{\tau_1})^{-\alpha_1}(t-\underline{\tau_2})^{-\alpha_1}\overline{\tau_1}^{-\beta_1}\overline{\tau_2}^{-\beta_1}\Big)f(\overline{X}(\tau_1,\tau_2))d\tau_1 d\tau_2\Big) \\
&:= \theta\Big(Y_{21}(s,t)+Y_{22}(s,t)\Big), \qquad (50)
\end{aligned}
$$

$$Y_3(s,t) = \int_0^t \int_0^s (s-\tau_1)^{-\alpha_2}(t-\tau_2)^{-\alpha_2}\tau_1^{-\beta_2}\tau_2^{-\beta_2}\Big(g(x(\tau_1,\tau_2)) - g(\underline{X}(\tau_1,\tau_2))\Big)dW_{\tau_1}dW_{\tau_2}$$
$$+ \int_0^t \int_0^s \Big((s-\tau_1)^{-\alpha_2}(t-\tau_2)^{-\alpha_2}\tau_1^{-\beta_2}\tau_2^{-\beta_2} - (s-\underline{\tau_1})^{-\alpha_2}(t-\underline{\tau_2})^{-\alpha_2}\overline{\tau_1}^{-\beta_2}\overline{\tau_2}^{-\beta_2}\Big)g(\underline{X}(\tau_1,\tau_2))dW_{\tau_1}dW_{\tau_2}$$
$$:= Y_{31}(s,t) + Y_{32}(s,t). \tag{51}$$

For $Y_{11}(s,t)$ and $Y_{21}(s,t)$, we have:

$$\mathbb{E}\Big(|Y_{11}(s,t)|^2\Big)$$
$$\leq 2(1-\theta)^2\mathbb{E}\Big(\int_0^t \int_0^s (s-\tau_1)^{-\alpha_1}(t-\tau_2)^{-\alpha_1}\tau_1^{-\beta_1}\tau_2^{-\beta_1}d\tau_1 d\tau_2$$
$$\cdot \int_0^t \int_0^s (s-\tau_1)^{-\alpha_1}(t-\tau_2)^{-\alpha_1}\tau_1^{-\beta_1}\tau_2^{-\beta_1}\Big(f(x(\tau_1,\tau_2)) - f(\underline{X}(\tau_1,\tau_2))\Big)^2 d\tau_1 d\tau_2\Big)$$
$$\leq 2L(1-\theta)^2 s^{1-(\alpha_1+\beta_1)}t^{1-(\alpha_1+\beta_1)}\mathbf{B}^2(1-\alpha_1, 1-\beta_1)$$
$$\cdot\,\mathbb{E}\Big(\int_0^t \int_0^s (s-\tau_1)^{-\alpha_1}(t-\tau_2)^{-\alpha_1}\tau_1^{-\beta_1}\tau_2^{-\beta_1}\Big(f(x(\tau_1,\tau_2)) - f(\underline{X}(\tau_1,\tau_2))\Big)^2 d\tau_1 d\tau_2\Big)$$
$$\leq C\int_0^t \int_0^s (s-\tau_1)^{-\alpha_1}(t-\tau_2)^{-\alpha_1}\tau_1^{-\beta_1}\tau_2^{-\beta_1}\Big(\mathbb{E}|x(\tau_1,\tau_2) - X(\tau_1,\tau_2)|^2 + \mathbb{E}|X(\tau_1,\tau_2) - \underline{X}(\tau_1,\tau_2)|^2\Big)d\tau_1 d\tau_2, \tag{52}$$

and

$$\mathbb{E}\Big(|Y_{21}(s,t)|^2\Big)$$
$$\leq 2\theta^2\mathbb{E}\Big(\int_0^t \int_0^s (s-\tau_1)^{-\alpha_1}(t-\tau_2)^{-\alpha_1}\tau_1^{-\beta_1}\tau_2^{-\beta_1}d\tau_1 d\tau_2$$
$$\cdot \int_0^t \int_0^s (s-\tau_1)^{-\alpha_1}(t-\tau_2)^{-\alpha_1}\tau_1^{-\beta_1}\tau_2^{-\beta_1}\Big(f(x(\tau_1,\tau_2)) - f(\overline{X}(\tau_1,\tau_2))\Big)^2 d\tau_1 d\tau_2\Big)$$
$$\leq 2L\theta^2 s^{1-(\alpha_1+\beta_1)}t^{1-(\alpha_1+\beta_1)}\mathbf{B}^2(1-\alpha_1, 1-\beta_1)$$
$$\cdot\,\mathbb{E}\Big(\int_0^t \int_0^s (s-\tau_1)^{-\alpha_1}(t-\tau_2)^{-\alpha_1}\tau_1^{-\beta_1}\tau_2^{-\beta_1}\Big(f(x(\tau_1,\tau_2)) - f(\underline{X}(\tau_1,\tau_2))\Big)^2 d\tau_1 d\tau_2\Big)$$
$$\leq C\int_0^t \int_0^s (s-\tau_1)^{-\alpha_1}(t-\tau_2)^{-\alpha_1}\tau_1^{-\beta_1}\tau_2^{-\beta_1}\Big(\mathbb{E}|x(\tau_1,\tau_2) - X(\tau_1,\tau_2)|^2 + \mathbb{E}|X(\tau_1,\tau_2) - \underline{X}(\tau_1,\tau_2)|^2\Big)d\tau_1 d\tau_2, \tag{53}$$

where we have used the Cauchy-Schwarz's inequality along with Lemma 2.2 and condition (3). analogously based on Assumption 2.1, we can calculate:

$$\mathbb{E}\Big(|Y_{12}(s,t)|^2\Big)$$
$$\leq 2(1-\theta)^2\mathbb{E}\Big(\int_0^t \int_0^s |(s-\tau_1)^{-\alpha_1}(t-\tau_2)^{-\alpha_1}\tau_1^{-\beta_1}\tau_2^{-\beta_1} - (s-\underline{\tau_1})^{-\alpha_1}(t-\underline{\tau_2})^{-\alpha_1}\overline{\tau_1}^{-\beta_1}\overline{\tau_2}^{-\beta_1}|d\tau_1 d\tau_2$$
$$\cdot \int_0^t \int_0^s |(s-\tau_1)^{-\alpha_1}(t-\tau_2)^{-\alpha_1}\tau_1^{-\beta_1}\tau_2^{-\beta_1} - (s-\underline{\tau_1})^{-\alpha_1}(t-\underline{\tau_2})^{-\alpha_1}\overline{\tau_1}^{-\beta_1}\overline{\tau_2}^{-\beta_1}||f(\underline{X}(\tau_1,\tau_2)|^2 d\tau_1 d\tau_2\Big)$$
$$\leq 2K(1-\theta)^2\Big(1 + \mathbb{E}|\underline{X}(\tau_1,\tau_2)|^2\Big)h^{4-4(\alpha_1+\beta_1)} \leq Ch^{4-4(\alpha_1+\beta_1)}, \tag{54}$$

$$\mathbb{E}\Big(|Y_{22}(s,t)|^2\Big)$$

$$\leq 2\theta^2 \mathbb{E}\Big(\int_0^t \int_0^s |(s-\tau_1)^{-\alpha_1}(t-\tau_2)^{-\alpha_1}\tau_1^{-\beta_1}\tau_2^{-\beta_1} - (s-\underline{\tau_1})^{-\alpha_1}(t-\underline{\tau_2})^{-\alpha_1}\overline{\tau_1}^{-\beta_1}\overline{\tau_2}^{-\beta_1}| d\tau_1 d\tau_2$$
$$\cdot \int_0^t \int_0^s |(s-\tau_1)^{-\alpha_1}(t-\tau_2)^{-\alpha_1}\tau_1^{-\beta_1}\tau_2^{-\beta_1} - (s-\underline{\tau_1})^{-\alpha_1}(t-\underline{\tau_2})^{-\alpha_1}\overline{\tau_1}^{-\beta_1}\overline{\tau_2}^{-\beta_1}| \|f(\overline{X}(\tau_1,\tau_2)|^2 d\tau_1 d\tau_2\Big)$$
$$\leq 2K\theta^2\Big(1+\mathbb{E}|\overline{X}(\tau_1,\tau_2)|^2\Big)h^{4-4(\alpha_1+\beta_1)} \leq Ch^{4-4(\alpha_1+\beta_1)}. \tag{55}$$

Moreover, using Lemma 2.5 for $Y_{31}(s,t)$ and $Y_{32}(s,t)$ allows us to infer that:

$$\mathbb{E}\Big(|Y_{31}(s,t)|^2\Big)$$
$$\leq 2\mathbb{E}\Big(\int_0^t \int_0^s (s-\tau_1)^{-2\alpha_2}(t-\tau_2)^{-2\alpha_2}\tau_1^{-2\beta_2}\tau_2^{-2\beta_2}\|g(x(\tau_1,\tau_2)) - g(\underline{X}(\tau_1,\tau_2))\|^2 d\tau_1 d\tau_2\Big)$$
$$\leq C\int_0^t \int_0^s (s-\tau_1)^{-2\alpha_2}(t-\tau_2)^{-2\alpha_2}\tau_1^{-2\beta_2}\tau_2^{-2\beta_2}\Big(\mathbb{E}|x(\tau_1,\tau_2) - X(\tau_1,\tau_2)|^2 + \mathbb{E}|X(\tau_1,\tau_2) - \underline{X}(\tau_1,\tau_2)|^2\Big)d\tau_1 d\tau_2, \tag{56}$$

and

$$\mathbb{E}\Big(|Y_{32}(s,t)|^2\Big)$$
$$\leq 2\mathbb{E}\Big(\int_0^t \int_0^s \Big((s-\tau_1)^{-\alpha_2}(t-\tau_2)^{-\alpha_2}\tau_1^{-\beta_2}\tau_2^{-\beta_2} - (s-\underline{\tau_1})^{-\alpha_2}(t-\underline{\tau_2})^{-\alpha_2}\overline{\tau_1}^{-\beta_2}\overline{\tau_2}^{-\beta_2}\Big)^2 \|g(\underline{X}(\tau_1,\tau_2))\|^2 d\tau_1 d\tau_2\Big)$$
$$\leq 2K\Big(1+\mathbb{E}|\underline{X}(\tau_1,\tau_2)|^2\Big)h^{2-4(\alpha_2+\beta_2)} \leq Ch^{2-4(\alpha_2+\beta_2)}. \tag{57}$$

Utilizing relationships (52)-(57) along with Lemma 4.3 allows us to confirm the existence of a pair (α,β) where $\alpha+\beta<1$. This implies that:

$$\mathbb{E}\Big(|x(s,t) - X(s,t)|^2\Big) \leq C\Big(\int_0^t \int_0^s (s-\tau_1)^{-\alpha}(t-\tau_2)^{-\alpha}\tau_1^{-\beta}\tau_2^{-\beta}\mathbb{E}\Big(|x(\tau_1,\tau_2) - X(\tau_1,\tau_2)|^2\Big)d\tau_1 d\tau_2$$
$$+ h^{\min\big\{4-4(\alpha_1+\beta_1), 2-4(\alpha_2+\beta_2)\big\}}\Big). \tag{58}$$

Hence, the application of Lemma 2.4 allows us to effectively finalize the proof. □

5. Improved θ-scheme using the SOE approach

This section highlights the obstacles posed by the reduced order of the θ-scheme (25). This diminished order results in considerable computational expenses and storage needs necessary to attain acceptable precision. To elevate computational efficiency, we introduce a novel technique named the SOE approximation. Experimental findings indicate a marked decrease in total computation time when utilizing this new approach.

Lemma 5.1. *([1]) Given α in the range of $(0,1)$ and a designated time threshold δ, we have:*

$$|t^{-\alpha} - \sum_{l=1}^{K_{\exp}} \omega_l e^{-\tau_l t}| \leq \epsilon, \qquad \forall\, t \in [\delta, T]. \tag{59}$$

Here, τ_l and ω_l, with $l = 1,2,\ldots,K_{\exp}$, represent the quadrature and weight nodes, respectively. Additionally, $K_{\exp}$ satisfies the condition:

$$K_{\exp} = O\Big(\log\frac{1}{\epsilon}\Big(\log\log\frac{1}{\epsilon} + \log\frac{T}{\delta}\Big) + \log\frac{1}{\delta}\Big(\log\log\frac{1}{\epsilon} + \log\frac{1}{\delta}\Big)\Big). \tag{60}$$

Put differently, when we fix the precision at ϵ and use a cut-off time of $\delta = h$, the value of K_{exp} is given by $K_{\text{exp}} = \begin{cases} O(\log N), & T \gg 1, \\ O(\log^2 N), & T \approx 1. \end{cases}$

By using the SOE approach (59) and assuming a very small tolerance ϵ, we can rearrange the θ-scheme (25) as follows:

$$\begin{aligned} X_{n,m} = x_0 &+ \theta h \sum_{j=0}^{m-1} \sum_{i=0}^{n-1} \sum_{l=1}^{K_{\text{exp},1}} w_{l,1} e^{-\tau_{l,1}\left((s_n - s_i) + (t_m - t_j)\right)} s_{i+1}^{-\beta_1} t_{j+1}^{-\beta_1} f(X_{i+1,j+1}) \\ &+ (1-\theta) h \sum_{j=0}^{m-SOE1} \sum_{i=0}^{n-1} \sum_{l=1}^{K_{\text{exp},1}} w_{l,1} e^{-\tau_{l,1}\left((s_n - s_i) + (t_m - t_j)\right)} s_{i+1}^{-\beta_1} t_{j+1}^{-\beta_1} f(X_{i,j}) \\ &+ \sum_{j=0}^{m-1} \sum_{i=0}^{n-1} \sum_{l=1}^{K_{\text{exp},2}} w_{l,2} e^{-\tau_{l,2}\left((s_n - s_i) + (t_m - t_j)\right)} s_{i+1}^{-\beta_2} t_{j+1}^{-\beta_2} g(X_{i,j}) \Delta W_{i,j}, \end{aligned} \tag{61}$$

where $(s_n - s_i)^{-\alpha_1}(t_m - t_j)^{-\alpha_1}$ and $(s_n - s_i)^{-\alpha_2}(t_m - t_j)^{-\alpha_2}$ are replaced by $\sum_{l=1}^{K_{\text{exp},k}} w_{l,k} e^{-\tau_{l,k}\left((s_n - s_i) + (t_m - t_j)\right)}$, $k = 1, 2$, respectively. In addition, based on Eq. (61) we have:

$$X_{n,m} = x_0 + \theta \sum_{l=1}^{K_{\text{exp},1}} w_{l,1} M_{1,l}(s_n, t_m) + (1-\theta) \sum_{l=1}^{K_{\text{exp},1}} w_{l,1} M_{2,l}(s_n, t_m) + \sum_{l=1}^{K_{\text{exp},2}} w_{l,2} M_{3,l}(s_n, t_m), \tag{62}$$

where

$$\begin{aligned} M_{1,l}(s_n, t_m) &= h \sum_{j=0}^{m-1} \sum_{i=0}^{n-1} e^{-\tau_{l,1}\left((s_n - s_i) + (t_m - t_j)\right)} s_{i+1}^{-\beta_1} t_{j+1}^{-\beta_1} f(X_{i+1,j+1}), \\ M_{2,l}(s_n, t_m) &= h \sum_{j=0}^{m-1} \sum_{i=0}^{n-1} e^{-\tau_{l,1}\left((s_n - s_i) + (t_m - t_j)\right)} s_{i+1}^{-\beta_1} t_{j+1}^{-\beta_1} f(X_{i,j}), \\ M_{3,l}(s_n, t_m) &= \sum_{j=0}^{m-1} \sum_{i=0}^{n-1} e^{-\tau_{l,2}\left((s_n - s_i) + (t_m - t_j)\right)} s_{i+1}^{-\beta_2} t_{j+1}^{-\beta_2} g(X_{i,j}) \Delta W_{i,j}. \end{aligned} \tag{63}$$

This implies that the computational expense reduces from $O(N^2)$ to $O\left(NK_{\text{exp}}\right)$.

6. Numerical experiments

In this section, we investigate the error illustrations at final point $T = 1$ for a two-dimensional system of DSSVIEs. We let:

$$e_h = \Big(\frac{1}{M} \sum_{i=1}^{M} |x^{(i)}(s_N, t_N) - X^{(i)}(s_N, t_N)|^2\Big)^{\frac{1}{2}}. \tag{64}$$

In each sample path i, $X^{(i)}(s_N, t_N)$ represents the numerical solution, where $x^{(i)}(s_N, t_N)$ denotes the exact solution. We set $h^* = 2^{-14}$ and as the reference solution is saved based on the proposed step size and examine the errors based on $h = 16h^*, 32h^*, 64h^*, 128h^*$ and $M = 5000$ is the number of simulations.

Example 6.1. *Let us consider the following two-dimensional system of DSSVIEs:*

$$\begin{cases} dx_1(s,t) = \Big[\int_0^t \int_0^s (s-\tau_1)^{-\alpha_{11}}(t-\tau_2)^{-\alpha_{11}}\tau_1^{-\beta_{11}}\tau_2^{-\beta_{11}} \sin\Big(x_1(\tau_1,\tau_2)+x_2(\tau_1,\tau_2)\Big)d\tau_1 d\tau_2\Big] \\ \qquad\qquad +\Big[\int_0^t \int_0^s (s-\tau_1)^{-\alpha_{12}}(t-\tau_2)^{-\alpha_{12}}\tau_1^{-\beta_{12}}\tau_2^{-\beta_{12}} x_1(\tau_1,\tau_2)dW(\tau_1,\tau_2)\Big], \\ dx_2(s,t) = \Big[\int_0^t \int_0^s (s-\tau_1)^{-\alpha_{21}}(t-\tau_2)^{-\alpha_{21}}\tau_1^{-\beta_{21}}\tau_2^{-\beta_{21}} \cos\Big(x_1(\tau_1,\tau_2)+x_2(\tau_1,\tau_2)\Big)d\tau_1 d\tau_2\Big] \\ \qquad\qquad +\Big[\int_0^t \int_0^s (s-\tau_1)^{-\alpha_{22}}(t-\tau_2)^{-\alpha_{22}}\tau_1^{-\beta_{22}}\tau_2^{-\beta_{22}} \cos(2x_2(\tau_1,\tau_2))dW(\tau_1,\tau_2)\Big], \\ \Big(x_1(0,0),x_2(0,0)\Big)^T = \Big(1,1\Big)^T, \quad (s,t)\in[0,1]\times[0,1]. \end{cases} \tag{65}$$

We examine two cases involving the positive parameters α_{ij} and β_{ij} $(i,j=1,2)$.

- *Case I: $\alpha_{11}=\alpha_{21}=0.4$, $\beta_{11}=\beta_{21}=0.4$, $\alpha_{12}=\alpha_{22}=0.2$, and $\beta_{12}=\beta_{22}=0.1$,*

- *Case II: $\alpha_{11}=\alpha_{21}=0.7$, $\beta_{11}=\beta_{21}=0.1$, $\alpha_{12}=\alpha_{22}=0.1$, and $\beta_{12}=\beta_{22}=0.1$.*

In both cases, the functions f and g fulfill the conditions of Theorem 4.4. The numerical outcomes for these two cases are presented in Tables 1 and 2, along with Figures 1 and 2, demonstrating varying step sizes h. We generated two plots depicting the mean-square error relative to the time step size h. Remarkably, for both $\theta=0.1$ and $\theta=0.9$ across cases I and II, the results closely align with each other, showcasing minimal deviation from the reference solution.

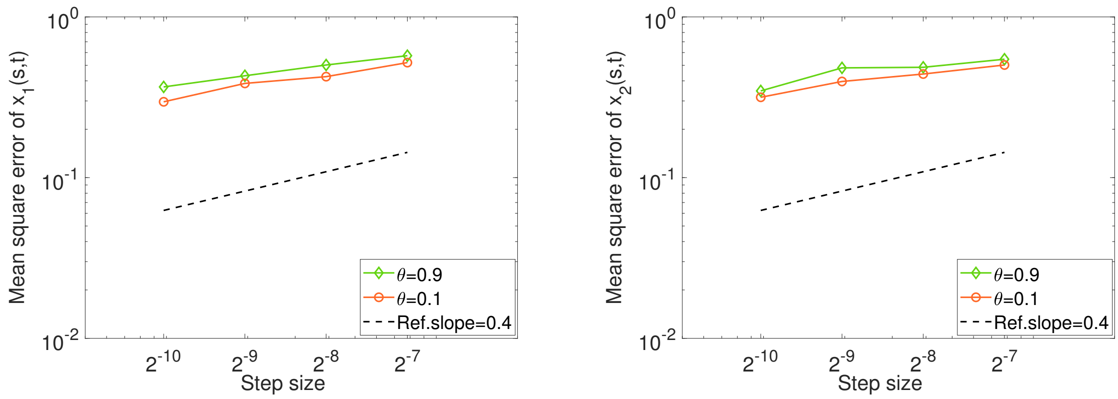

Figure 1: Illustrations of errors based on various step sizes for system (65) in case I.

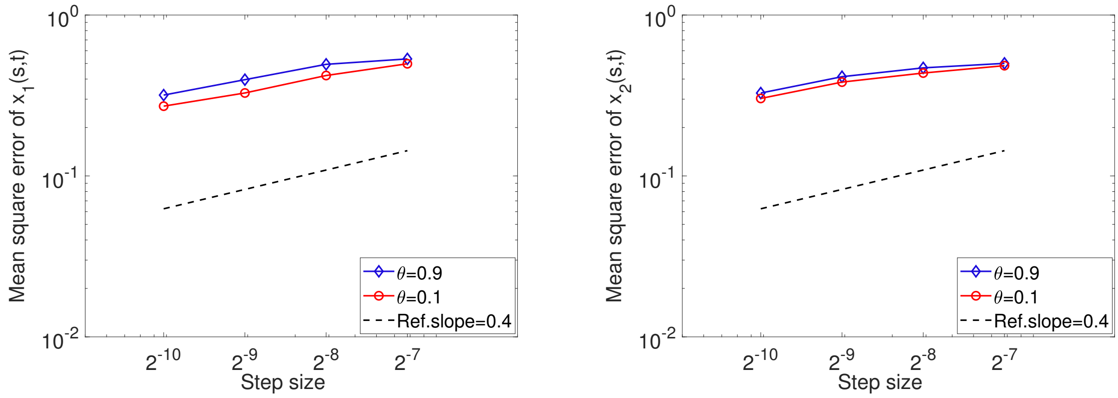

Figure 2: Illustrations of errors based on various step sizes for system (65) in case II.

In tables 1 and 2, a single-digit level of accuracy is achieved for the singular SVIEs using various step sizes h and θ values. Additionally, our findings are juxtaposed with classical techniques like Galerkin and EM methods for comparison. The findings demonstrate that our method is more time-efficient compared to the traditional techniques under consideration. For instance, in case I with $h = 2^{-9}$ and $\theta = 0.9$, our method requires 8.06 *units of time, whereas the Galerkin and EM methods demand* 264.51 *and* 243.08 *units of time, respectively.*

Table 1: The error and its corresponding CPU times(s) based on different values of θ and h for $x_1(s,t)$.

		ITS with $\theta = 0.1$		ITS with $\theta = 0.5$		ITS with $\theta = 0.9$		Galerkin method		Euler method	
	h	e_h	run time	e_h	run time	e_h	run time	e_h	run time	e_h	run time
Case I	2^{-7}	3.84E-02	0.34	3.91E-02	0.45	4.24E-02	0.67	3.62E-02	40.02	3.28E-02	36.53
	2^{-8}	3.14E-02	1.94	3.46E-02	2.38	3.71E-02	2.79	3.29E-02	73.22	3.04E-02	59.07
	2^{-9}	2.85E-02	7.61	2.92E-02	7.82	3.18E-02	8.06	3.14E-02	264.51	2.91E-02	243.08
	2^{-10}	2.19E-02	13.64	2.30E-02	14.06	2.71E-02	14.82	2.41E-02	366.19	2.25E-02	351.40
	order	0.2915	19.41	0.3045	19.76	0.3127	20.13	0.3231	539.11	0.3044	514.68
Case II	2^{-7}	3.65E-02	0.26	3.71E-02	0.31	3.87E-02	0.37	4.02E-02	51.46	3.86E-02	48.11
	2^{-8}	3.11E-02	1.75	3.32E-02	2.17	3.65E-02	2.66	3.84E-02	96.68	3.53E-02	81.62
	2^{-9}	2.57E-02	6.47	2.76E-02	7.32	2.93E-02	7.85	3.63E-02	298.11	3.22E-02	262.33
	2^{-10}	2.08E-02	12.43	2.18E-02	11.56	2.35E-02	12.19	3.14E-02	437.24	2.93E-02	385.03
	order	0.2869	18.33	0.2931	19.36	0.3074	21.07	0.3124	519.82	0.2965	496.67

Table 2: The error and its corresponding CPU times(s) based on different values of θ and h for $x_2(s,t)$.

		ITS with $\theta = 0.1$		*ITS with* $\theta = 0.5$		*ITS with* $\theta = 0.9$		*Galerkin method*		*Euler method*	
	h	e_h	*run time*	e_h	*run time*	e_h	*run time*	e_h	*run time*	e_h	*run time*
	2^{-7}	3.71E-02	0.31	3.82E-02	0.37	4.03E-02	0.45	3.83E-02	45.74	3.62E-02	43.66
	2^{-8}	3.26E-02	0.74	3.38E-02	0.81	3.59E-02	1.12	3.91E-02	103.25	3.40E-02	94.53
Case I	2^{-9}	2.93E-02	9.02	3.21E-02	9.67	3.56E-02	11.26	3.86E-02	251.13	3.14E-02	236.40
	2^{-10}	2.41E-02	16.19	2.45E-02	17.02	2.56E-02	20.54	2.62E-02	473.17	2.31E-02	438.42
	order	0.3149	21.77	0.3164	22.37	0.3196	22.88	0.3218	561.09	0.3105	508.32
	2^{-7}	3.58E-02	0.21	3.63E-02	0.25	3.70E-02	0.31	3.66E-02	44.89	3.63E-02	42.01
	2^{-8}	3.37E-02	1.16	3.42E-02	1.27	3.47E-02	1.32	3.54E-02	86.46	3.41E-02	78.92
Case II	2^{-9}	2.83E-02	5.57	2.87E-02	6.27	2.91E-02	7.18	3.48E-02	234.26	3.45E-02	237.62
	2^{-10}	2.24E-02	9.89	2.39E-02	10.46	2.42E-02	11.25	2.57E-02	502.21	2.66E-02	481.05
	order	0.3064	16.58	0.3271	20.62	0.3264	19.74	0.3321	574.92	0.3405	583.16

7. Conclusion

This paper extends the one-dimensional DSSVIEs to the two-dimensional case. Our focus lies in exploring foundational concepts related to the continuous dependency of the solution on the initial data. Following this, we establish well-posedness and conduct a convergence analysis. Additionally, we assess computational complexity through SOE approximation. Our method is put to the test and compared to relevant approaches to demonstrate its validity and superiority. The technique of establishing strong convergence may be extended to DSSVIEs with time change or jump. Additionally, one can use the split-step θ-Milstein scheme [5, 6] and the split-step θ-Milstein scheme with Poisson jump [17] to modify the θ-scheme defined in (25).

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Conflict of interest. The authors declare no conflict of interest.

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