Filomat 39:16 (2025), 5447–5462
https://doi.org/10.2298/FIL2516447G

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Partial sums and geometric properties of a certain unified family of harmonic mappings

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Abstract. We have introduced a new subclass of harmonic univalent functions denoted by $T_H^k(\alpha,\gamma,\beta)$, which is a harmonic analogue to the functions class $\mathcal{W}_\beta(\alpha,\gamma)$ (see [4]). It is observed that there is an analytic bridge between two classes $T_H^k(\alpha,\gamma,\beta)$ and $\mathcal{W}_\beta(\alpha,\gamma)$. Various geometric properties such as sharp coefficient bounds, growth theorem, sufficient condition, invariance property under convolution and convex combination, radius of starlikeness, convexity and close-to-convexity of the partial sums of functions are discussed.

1. Introduction and preliminaries

The study of planar harmonic mappings in the context of geometric functions theory is of interest after the pioneering work of Clunie and Sheil-Small [7], and also for their wide applications. For example, by making use of harmonic mapping on a suitable convex domain fluid flow problem has been solved in [3, 8]. Note that if f is complex-valued harmonic, then its partial sum can be treated as an approximation of f by the complex-valued harmonic polynomial. Thus, approximation of a univalent harmonic map by a univalent harmonic polynomial might lead to new applications in fluid flow problems. In [14–17], the idea of such works has been initiated by considering the sub-families of $\mathcal{S}_H$, sense-preserving univalent harmonic maps. We refer [18, 20, 22–24] and the references therein that deal with some of the recent works. Thus, it is interesting to study the various well-known subfamilies of $\mathcal{S}_H$. Let $\mathcal{H}$ be the class of complex-valued harmonic function f in the open unit disk $\mathbb{D}$ normalized by $f(0) = 0 = f_z(0) - 1$. Any function f in $\mathcal{H}$ has the canonical representation of the form $f = h + \overline{g}$. Here, both h and g are analytic functions in $\mathbb{D}$ and are called the analytic and co-analytic parts of f, respectively. In particular, for $g(z) = 0$, the class $\mathcal{H}$ reduces to the class $\mathcal{A}$, consisting of analytic functions in $\mathbb{D}$ with $f(0) = 0$ and $f'(0) = 1$. If $f \in \mathcal{H}$ then the Jacobian $J_f(z)$ of f is defined by $J_f(z) = |h'(z)|^2 - |g'(z)|^2$ and we say f is sense preserving if $J_f(z) > 0$ in $\mathbb{D}$. Let $\mathcal{S}_H$ be the subclass of $\mathcal{H}$ consisting of univalent and sense-preserving harmonic mappings in $\mathbb{D}$. If $g(z) = 0$

2020 *Mathematics Subject Classification*. Primary 30C45; Secondary 30C65, 30C20.
Keywords. Harmonic univalent mappings, coefficient estimates, convolution, partial sum, radius problems, starlikeness, convexity, close-to-convexity.
Received: 05 April 2024; Revised: 05 April 2025; Accepted: 05 May 2025
Communicated by Hari M. Srivastava
The first author was supported by OSHEC, HED, Govt. of Odisha, India, vide sanction Letter No.1040/69/OSHEC
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in $\mathbb{D}$ then the class $\mathcal{S}_H$ reduces to the sub class $\mathcal{S} \subset \mathcal{A}$, consisting of univalent analytic functions in $\mathbb{D}$. Set $\mathcal{H}_0 := \{f \in \mathcal{H} : f_{\bar{z}}(0) = 0\}$ and thus $f \in \mathcal{H}_0$ has of the form

$$f = h + \overline{g}, \text{ with } h(z) = z + \sum_{n=2}^{\infty} a_n z^n \text{ and } g(z) = \sum_{n=2}^{\infty} b_n z^n, \qquad z \in \mathbb{D}. \tag{1}$$

We also need the following definitions. Set $\mathcal{H}_0^k \subset \mathcal{H}_0$ defined by

$$\mathcal{H}_0^k := \{f = h + \overline{g} \in \mathcal{H} : h'(0) - 1 = g'(0) = h''(0) = h^{(k)}(0) = g^{(k)}(0) = 0\},$$

where $k \geq 1$. Thus, each $f \in \mathcal{H}_0^k$ has the representation

$$f = h + \overline{g}, \text{ with } h(z) = z + \sum_{n=k+1}^{\infty} a_n z^n \text{ and } g(z) = \sum_{n=k+1}^{\infty} b_n z^n, \qquad z \in \mathbb{D}. \tag{2}$$

Clearly, for $k = 1$, we have $\mathcal{H}_0^1 \equiv \mathcal{H}_0$. For $p \geq k+1$ and $q \geq k+1$, the $(p,q)-$th harmonic sections (or partial sums) of $f = h + \overline{g} \in \mathcal{H}_0^k$ is denoted by $s_{p,q}(f)$ and is given by $s_{p,q}(f) = s_p(h) + \overline{s_q(g)}$, with

$$s_p(h)(z) = z + \sum_{j=k+1}^{p} a_j z^j \text{ and } s_q(g)(z) = \sum_{j=k+1}^{q} b_j z^j, \quad z \in \mathbb{D}.$$

Following the standard notations, the subclass of $\mathcal{S}_H$ for which $f_{\bar{z}}(0) = 0$ is denoted by $\mathcal{S}_H^0$. We further denote $\mathcal{K}_H$, $\mathcal{S}_H^*$ and C_H as the sub-families of the function class $\mathcal{S}_H$ consisting of the functions f such that the image $f(\mathbb{D})$ is a convex, starlike and close-to-convex region, respectively. For a detailed treatment of the subject, we refer to the monograph by Duren[9]. Following important results due to Avci and Zlotkiewicz [6], and Clunie and Sheil-Small [7] respectively are required for our investigation.

Lemma 1.1. *Let $f = h+\overline{g} \in \mathcal{H}_0$ be of the form* (1)*. If $\sum_{n=2}^{\infty} n(|a_n|+|b_n|) \leq 1$ then $f \in \mathcal{S}_H^*$ and if $\sum_{n=2}^{\infty} n^2(|a_n|+|b_n|) \leq 1$ then $f \in \mathcal{K}_H$.*

Lemma 1.2. *If $f = h + \overline{g} \in \mathcal{H}_0$ and the function $F_\lambda = h + \lambda g$ is close-to-convex for all complex number λ with $|\lambda| = 1$, then f is close-to-convex and univalent.*

One of the classical problems in geometric function theory is to study the invariance properties of the sections of the functions of certain subfamilies of analytic functions in $\mathbb{D}$. Indeed, the partial sum of univalent and/or starlike functions need not retain the same properties in the whole domain. Szegö [33] proved that *nth* partial sums of univalent functions are univalent in the smaller disk of radius 1/4 and this constant cannot be replaced by a larger number. Though, this problem is explored for various classical subfamilies of $\mathcal{S}$, finding the largest radius of the univalence of the sections of the functions in the family $\mathcal{S}$ is still an open problem. For the more detailed survey on sections of univalent mappings found in the survey article due to Szegö [34] and also in the recent survey by Ravichandran [27]. Even later on, *nth* partial sums of functions have been addressed by many researchers in different contexts. Owa et al. [21] systematically compute the radii of stalikeness and convexity of $s_p(h)$ for $p = 3, 4$ upon taking $h(z)$ as $z/(1-z)^2$ and $z/(1-z)$. Those results were validated by the authors numerically as well as graphically using computer algebra. Notably, the classical Koebe function $z/(1-z)^2$ and $z/(1-z)$ act as extremals to various problems in univalent function theory. The radius of convexity of the sections of the functions, which are convex in a particular direction, is found in [19]. In [24] and [1], the *nth* partial sums of certain subfamilies of the close-to-convex functions have been discussed. In [32], Srivatava et al. investigated the ratio of a function related to the Hurwitz-Lerch zeta function and its sequence of partial sums in the context of meromorphic functions. Analogue to the classical problems on sections of analytic univalent functions, the problem of finding the radii of univalence, starlikeness, convexity, close-to-convexity, etc. for $s_{p,q}(f)$, where $f \in \mathcal{S}_H$ (or $\mathcal{S}_H^0$) are also open. The harmonic analogue of the problems on partial sums initially found in the work of [14, 15]. For some notable literature in this direction, we refer [16–18, 20, 22–24]. It is also important to note that q-calculus plays a significant role in geometric function theory because of its vast applications in engineering and sciences. Though it has vast applications in classical univalent theory, not much is explored in harmonic cases. Very recently, Rehman et al. [28] gave lower bounds for the ratio of some normalized q-Mittag-Leffler function and their sequences of partial sums. In [2], the authors introduced Janowski-type harmonic q-starlike functions associated with symmetrical points. In addition to the results on partial sums, other geometric properties such as distortion, convolution, and radii of univalency are also presented. Also see [13, 30, 31], wherein the authors have used q-calculus to study interesting geometric aspects of univalent analytic and harmonic mappings. In the context of convolution, we recall the proof of the famous Pólya-Schoenberg conjecture by Ruscheweyh and Sheil-Small, which guarantees that the class of univalent convex functions is closed under convolution. They also showed that starlikeness and close-to-convexity are preserved under convolution with convex univalent functions. However, a harmonic analogue of this containment does not hold and has proved to be challenging. We recall that, for $f_1 = h_1 + \overline{g}_1$ and $f_2 = h_2 + \overline{g}_2$ in $\mathcal{H}$, their convolution is defined by $f_1 * f_2 = h_1 * h_2 + \overline{g_1 * g_2}$.

In a recent paper Ali et al. [4] considered the functions class $\mathcal{W}_\beta(\alpha, \gamma)$, consisting of normalized analytic functions $h \in \mathcal{A}$ satisfying

$$\Re\Big\{(1-\alpha+2\gamma)\frac{h(z)}{z} + (\alpha-2\gamma)h'(z) + \gamma z h''(z) - \beta\Big\} > 0,$$

where $\alpha \geq 0$, $\gamma \geq 0$ and $\beta < 1$. In that paper, by making use of duality techniques, they have studied the starlikeness of a certain integral operator. They also found the necessary and sufficient conditions that assure the starlikeness of the generalized integral transform. With variants to this, few more results based on duality and order of convexity found in [5, 35, 36]. In this sequel, we introduce the following harmonic analogue of the closely related family studied in [4].

Definition 1.3. *For $\alpha, \gamma \geq 0$, $0 \leq \beta < 1$ and $k \geq 1$, a function $f = h + \overline{g} \in \mathcal{H}_0^k$ is said to be in the class $T_H^k(\alpha, \gamma, \beta)$ if for every $z \in \mathbb{D}$ it satisfies the inequality*

$$\Re\Big((1-\alpha+2\gamma)\frac{h(z)}{z} + (\alpha-2\gamma)h'(z) + \gamma z h''(z) - \beta\Big) > \Big|(1-\alpha+2\gamma)\frac{g(z)}{z} + (\alpha-2\gamma)g'(z) + \gamma z g''(z)\Big|.$$

It is observed that, the family $T_H^k(\alpha, \gamma, \beta)$ unifies several previously studied families of harmonic mappings. For examples, $T_H^k(\alpha, 0, 0) \equiv G_H^k(\alpha; r)$ (cf.[18]); $T_H^1(0, 0, \beta) \equiv g_H^0(\beta)$ (cf.[17]); $T_H^1(1, 0, \beta) \equiv \mathcal{P}_H(\beta)$ (cf.[14, 25]); $T_H^1(1+2\gamma, \gamma, \beta) \equiv \mathcal{W}_H^0(\gamma, \beta)$ (cf.[26]); $T_H^1(1+2\gamma, \gamma, 0) \equiv \mathcal{W}_H^0(\gamma)$ (cf.[11]), etc. Throughout we denote $T_H^1(\alpha, 0, \beta) \equiv G_H^1(\alpha, \beta)$ and $T_H^k(\alpha, \gamma, 0) \equiv T_H^k(\alpha, \gamma)$.

The organisation of the paper is as follows. Section 2 established one-to-one correspondence between the class $\mathcal{W}_\beta(\alpha, \gamma)$ and its harmonic analogue $T_H^k(\alpha, \gamma, \beta)$. Various basic properties, such as sharp coefficient estimates, growth theorem, and sufficient conditions for a function in the class $T_H^k(\alpha, \gamma, \beta)$ are also presented. In section 3, it is shown that the family $T_H^k(\alpha, \gamma, \beta)$ is closed under convex combinations and convolutions. The rest of the paper is devoted to studying the radii related problems in geometric function theory. In particular, for different values of the parameters p and q, we obtain radii of the sections $s_{p,q}(f)$ of functions f in $T_H^k(\alpha, \gamma, \beta)$. This includes, radii of starlikeness, convexity and close-to-convexity of the partial sums of functions. Relevant connections with known results are also pointed out.

2. Bounds on coefficients and growth estimates

The first result provides a one-to-one correspondence between the class $T_H^k(\alpha, \gamma, \beta)$ of harmonic mappings and the class $\mathcal{W}_\beta(\alpha, \gamma)$ of analytic functions.

Theorem 2.1. *The function $f = h + \overline{g}$ is in $T_H^k(\alpha, \gamma, \beta)$ if and only if for each complex number ε with $|\varepsilon| = 1$, the analytic function $h + \varepsilon g$ belongs to $\mathcal{W}_\beta(\alpha, \gamma)$.*

Proof. Let $f = h + \overline{g} \in T_H^k(\alpha,\gamma,\beta)$ and write $F_\varepsilon = h + \varepsilon g$. Then by Definition 1.3 and for each complex number ε with $|\varepsilon| = 1$, we have

$$\begin{aligned}\Re\Big[(1-\alpha+2\gamma)\frac{F_\varepsilon(z)}{z} + (\alpha-2\gamma)F'_\varepsilon(z) + \gamma z F''_\varepsilon(z)\Big] \\ \geq \Re\Big[(1-\alpha+2\gamma)\frac{h(z)}{z} + (\alpha-2\gamma)h'(z) + \gamma z h''(z)\Big] \\ - \left|\varepsilon\left((1-\alpha+2\gamma)\frac{g(z)}{z} + (\alpha-2\gamma)g'(z) + \gamma z g''(z)\right)\right| > \beta.\end{aligned}$$

Therefore, $F_\varepsilon \in \mathcal{W}_\beta(\alpha,\gamma)$. Conversely, suppose that $F_\varepsilon \in \mathcal{W}_\beta(\alpha,\gamma)$. Then for $z \in \mathbb{D}$, we have $\Re\{(1-\alpha+2\gamma)F_\varepsilon(z)/z + (\alpha-2\gamma)F'_\varepsilon(z) + \gamma z F''_\varepsilon(z)\} > \beta$ implies that

$$\begin{aligned}\Re\Big[(1-\alpha+2\gamma)\frac{h(z)}{z} + (\alpha-2\gamma)h'(z) + \gamma z h''(z) - \beta\Big] \\ > -\Re\left[\varepsilon\left\{(1-\alpha+2\gamma)\frac{g(z)}{z} + (\alpha-2\gamma)g'(z) + \gamma z g''(z)\right\}\right].\end{aligned} \tag{3}$$

Set $A := (1-\alpha+2\gamma)\frac{g(z)}{z} + (\alpha-2\gamma)g'(z) + \gamma z g''(z)$, $\theta_0 = \arg\{A\}$. Therefore, $A = |A|e^{i\theta_0}$. For each fixed $z \in \mathbb{D}$ and arbitrarily chosen complex number ε, with $|\varepsilon| = 1$, that is, $\varepsilon = e^{i(\pi-\theta_0)}$, (3) becomes,

$$\begin{aligned}\Re\Big[(1-\alpha+2\gamma)\frac{h(z)}{z} + (\alpha-2\gamma)h'(z) + \gamma z h''(z) - \beta\Big] > -\Re\Big[e^{i(\pi-\theta_0)} \cdot A\Big] \\ = -\Re\Big[e^{i(\pi-\theta_0)}|A|e^{i\theta_0}\Big] = -\Re(e^{i\pi})|A| = |A|, \quad z \in \mathbb{D}.\end{aligned}$$

This shows that $f \in T_H^k(\alpha,\gamma,\beta)$. □

In the following two theorems, we establish sharp coefficient estimates for functions in the family $T_H^k(\alpha,\gamma,\beta)$.

Theorem 2.2. *Let $f = h + \overline{g} \in T_H^k(\alpha,\gamma,\beta)$ be of the form* (2)*. Then for each $n \geq k+1$,*

$$|b_n| \leq \frac{1-\beta}{1+(n-1)\alpha+(n^2-3n+2)\gamma}. \tag{4}$$

The estimate is the best possible.

Proof. Suppose that $f = h + \overline{g} \in T_H^k(\alpha,\gamma,\beta)$. It is observed that

$$(1-\alpha+2\gamma)\frac{g(z)}{z} + (\alpha-2\gamma)g'(z) + \gamma z g''(z) = \sum_{n=k+1}^{\infty}\Big\{1+(n-1)\alpha+(n^2-3n+2)\gamma\Big\}b_n z^{n-1}.$$

Now, upon use of series expansion of $g(z)$ with $z = re^{i\theta} \in \mathbb{D}$, we get

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}\Big[(1-\alpha+2\gamma)\frac{g(re^{i\theta})}{re^{i\theta}} + (\alpha-2\gamma)g'(re^{i\theta}) + \gamma re^{i\theta}g''(re^{i\theta})\Big]e^{-i(n-1)\theta}d\theta \\ = \Big\{1+(n-1)\alpha+(n^2-3n+2)\gamma\Big\}b_n r^{n-1}.\end{aligned}$$

A simple computation implies that $\Big\{1+(n-1)\alpha+(n^2-3n+2)\gamma\Big\}|b_n|r^{n-1} \leq 1-\beta$. Taking $r \to 1^-$, we obtain the estimate (4). The bound is best possible for the function

$$f(z) = z + \frac{1-\beta}{1+(n-1)\alpha+(n^2-3n+2)\gamma}\overline{z}^n.$$

Indeed, since $f \in T_H^k(\alpha,\gamma,\beta)$, clearly $|b_n| = \dfrac{1-\beta}{1+(n-1)\alpha+(n^2-3n+2)\gamma}$. This completes the proof. □

Theorem 2.3. *Let $f = h + \overline{g} \in T_H^k(\alpha,\gamma,\beta)$ be of the form* (2) *with $k \geq 1$. Then for any $n \geq k+1$, we have*

(i) $|a_n| + |b_n| \leq \dfrac{2(1-\beta)}{1+(n-1)\alpha+(n^2-3n+2)\gamma}$

(ii) $\Big||a_n| - |b_n|\Big| \leq \dfrac{2(1-\beta)}{1+(n-1)\alpha+(n^2-3n+2)\gamma}$

(iii) $|a_n| \leq \dfrac{2(1-\beta)}{1+(n-1)\alpha+(n^2-3n+2)\gamma}.$

Equality holds true for the functions

$$f_\xi(z) = z + \sum_{j=1}^{\infty} \frac{2(1-\beta)}{1+j\xi\alpha+j\xi(j\xi-1)\gamma} z^{j\xi+1}, \text{ where } \xi = k, k+1, \cdots, 2k-1; z \in \mathbb{D}. \tag{5}$$

Proof. Let $f = h + \overline{g} \in T_H^k(\alpha,\gamma,\beta)$. Then by Theorem 2.1, we have for each ε with $|\varepsilon| = 1$, $F_\varepsilon = h(z) + \varepsilon g(z) \in \mathcal{W}_\beta(\alpha,\gamma)$. This implies that, there exists a Carathéodory function p, which is of the form $p(z) = 1 + \sum_{n=k+1}^{\infty} p_{n-1} z^{n-1}$ with $\Re(p(z)) > 0$ in $\mathbb{D}$, such that $(1-\alpha+2\gamma)F_\varepsilon/z + (\alpha-2\gamma)F'_\varepsilon + \gamma z F''_\varepsilon = \beta + (1-\beta)p(z)$. Or, equivalently we have

$$(1-\alpha+2\gamma)\frac{h(z)+\varepsilon g(z)}{z} + (\alpha-2\gamma)(h(z)+\varepsilon g(z))' + \gamma z(h(z)+\varepsilon g(z))'' = \beta + (1-\beta)p(z).$$

Upon further simplification and comparing the coefficients of z^{n-1} in both the sides of the resulting equation for all $n \geq k+1$, we get

$$(1-\beta)p_{n-1} = (a_n + \varepsilon b_n)\Big(1+(n-1)\alpha+(n^2-3n+2)\gamma\Big). \tag{6}$$

Since $|p_n| \leq 2$ for $n \geq k+1$ and ε ($|\varepsilon| = 1$) is arbitrary, it follows from (6) that,

$$(|a_n| + |b_n|)(1+(n-1)\alpha+(n^2-3n+2)\gamma) \leq 2(1-\beta).$$

This completes the proof of (i). The proof of (ii) and (iii) immediately follows from (i) as we have $\Big||a_n| - |b_n|\Big| \leq |a_n| + |b_n|$ and $|a_n| \leq |a_n| + |b_n|$. Now to show the sharpness, upon considering the functions given by (5), we have, $\Re\{(1-\alpha+2\gamma)f_\xi(z)/z + (\alpha-2\gamma)f'_\xi(z) + \gamma z f''_\xi(z) - \beta\} > 0$. Hence, $f_\xi \in T_H^\xi(\alpha,\gamma,\beta) \subset T_H^k(\alpha,\gamma,\beta)$ for $\xi = k, k+1, \cdots, 2k-1$. When $\xi = k$, for $n = jk+1$, $j = 1, 2, \cdots$, we have

$$f_n(z) = z + \sum_{n=2}^{\infty} \frac{2(1-\beta)}{1+(n-1)\alpha+(n-1)(n-2)\gamma} z^n$$

is the extremal of (i). Indeed,

$$|a_n| = \frac{2(1-\beta)}{1+jk\alpha+jk(jk-1)\gamma} = \frac{2(1-\beta)}{1+(n-1)\alpha+(n^2-3n+2)\gamma}$$

for all $n = jk+1$, where $j = 1, 2, \cdots$ and $\xi = k+1, k+2, \cdots, (2k-1)$. □

The next result gives the variation of $|f(z)|$ as z varies over $\mathbb{D}$ for the family $T_H^k(\alpha,\gamma,\beta)$. The proof directly follows from Theorem 2.3 and hence is omitted.

Theorem 2.4. *If $f \in T_H^k(\alpha,\gamma,\beta)$ is of the form* (2), *then*

$$|z| - \sum_{n=k+1}^{\infty} \frac{2(1-\beta)|z|^n}{1+(n-1)\alpha+(n-1)(n-2)\gamma} \leq |f(z)| \leq |z| + \sum_{n=k+1}^{\infty} \frac{2(1-\beta)|z|^n}{1+(n-1)\alpha+(n-1)(n-2)\gamma}. \tag{7}$$

The following result gives a sufficient condition for a function to be in the class $T_H^k(\alpha,\gamma,\beta)$.

Theorem 2.5. *If $f \in \mathcal{S}_H^0$ is of the form* (2) *for $k \geq 1$ and satisfies the condition*

$$\sum_{n=k+1}^{\infty}\Big(1+(n-1)\alpha+(n^2-3n+2)\gamma\Big)(|a_n|+|b_n|) \leq 1-\beta, \tag{8}$$

then $f \in T_H^k(\alpha,\gamma,\beta)$.

Proof. Suppose that $f = h + \overline{g} \in \mathcal{H}_0^k$ is in the class $\mathcal{S}_H^0$ and (8) holds true. Now, we find that

$$\begin{aligned}\Re\Big[(1-\alpha+2\gamma)\frac{h(z)}{z}+(\alpha-2\gamma)h'(z)+\gamma zh''(z)-\beta\Big]&\\ =\Re\Big[1+\sum_{n=k+1}^{\infty}\Big(1+(n-1)\alpha+(n^2-3n+2)\gamma\Big)a_nz^{n-1}-\beta\Big]&\\ \geq 1-\beta-\sum_{n=k+1}^{\infty}\Big(1+(n-1)\alpha+(n^2-3n+2)\gamma\Big)|a_n|.&\end{aligned}$$

Application of (8) yields

$$\begin{aligned}\Re\Big[(1-\alpha+2\gamma)\frac{h(z)}{z}+(\alpha-2\gamma)h'(z)+\gamma zh''(z)-\beta\Big] &\geq \sum_{n=k+1}^{\infty}\Big(1+(n-1)\alpha+(n^2-3n+2)\gamma\Big)|b_n|\\ &\geq \Big|(1-\alpha+2\gamma)\frac{g(z)}{z}+(\alpha-2\gamma)g'(z)+\gamma zg''(z)\Big|.\end{aligned}$$

This implies $f \in T_H^k(\alpha,\gamma,\beta)$. □

3. Convex combinations and convolutions

In this section, we establish the results which show that the family is closed under convex combinations and convolutions.

Theorem 3.1. *The functions class $T_H^1(\alpha,\gamma,\beta)$ is closed under convex combinations.*

Proof. Let $f_j = h_j + \overline{g_j} \in T_H^1(\alpha,\gamma,\beta)$, for $j = 1,2,\cdots,n$ and $\sum_{j=1}^n t_j = 1$ $(0 \leq t_j \leq 1)$. Then

$$\Re\Big[(1-\alpha+2\gamma)\frac{h_j(z)}{z}+(\alpha-2\gamma)h_j'(z)+\gamma zh_j''(z)-\beta\Big] > \Big|(1-\alpha+2\gamma)\frac{g_j(z)}{z}+(\alpha-2\gamma)g_j'(z)+\gamma zg_j''(z)\Big|.$$

Therefore, the convex combination of f_j's is of the form $\sum_{j=1}^n t_jf_j(z) =: f(z) = h(z) + \overline{g(z)}$, where $h(z) = \sum_{j=1}^n t_jh_j(z)$ and $g(z) = \sum_{j=1}^n t_jg_j(z)$. Note that, both h and g are analytic in $\mathbb{D}$ and satisfy normalization conditions $h(0) = g(0) = h'(0) - 1 = g'(0) = 0$. Subsequently, utilizing the definition of $T_H^1(\alpha,\gamma,\beta)$ upon f, the result follows. □

Theorem 3.2. *Let ϕ be in the class $\mathcal{W}_\beta(\alpha,\gamma)$, then $\Re(\phi(z)/z) > 1/(2-\beta)$.*

Proof. Let $\phi \in \mathcal{W}_\beta(\alpha,\gamma)$. Therefore, it has the Taylor-Maclaurin series expansion of the form $\phi(z) = z + \sum_{n=2}^{\infty} c_nz^n$. Thus, by Definition, we have $\Re\{(1-\alpha+2\gamma)\phi(z)/z + (\alpha-2\gamma)\phi'(z) + \gamma z\phi''(z)\} > \beta$. This implies that

$$\Re\Big[1+\sum_{n=2}^{\infty}\Big(1+(n-1)\alpha+(n^2-3n+2)\gamma\Big)c_nz^{n-1}\Big] > \beta, \qquad z \in \mathbb{D}.$$

Further simplification yields

$$\Re\Big[1+\frac{1}{2-\beta}\sum_{n=2}^{\infty}\Big(1+(n-1)\alpha+(n^2-3n+2)\gamma\Big)c_nz^{n-1}\Big] > \frac{1}{2-\beta}, \quad z\in\mathbb{D}.$$

As $0\le\beta<1$, we have $\Re\,(p(z))>1/(2-\beta)\ge 1/2$ in $\mathbb{D}$, where $p(z)=1+\{1/(2-\beta)\}\sum_{n=2}^{\infty}\{1+(n-1)\alpha+(n^2-3n+2)\gamma\}c_nz^{n-1}$. Set a sequence $\{s_n\}_{n=0}^{\infty}$ defined by $s_0=1$ and $s_{n-1}=\dfrac{2-\beta}{1+(n-1)\alpha+(n^2-3n+2)\gamma}$, for all $n\ge 2$, also when $n\to\infty$, $s_n\to 0$. It is observed that $s_{n-1}-s_n\ge s_n-s_{n+1}$ gives $\alpha^2+(3n-2)\alpha\gamma-\gamma+3n\gamma^2(n-1)\ge 0$, which is always true for each $n\in\mathbb{N}$. This shows that $\{s_n\}_{n=0}^{\infty}$ is the sequence of non-negative numbers which is a convex null sequence. Therefore, by [29, Lemma 1, pp-146], (also see [10]) the function $q(z)=\dfrac{1}{2}+\displaystyle\sum_{n=2}^{\infty}\frac{2-\beta}{1+(n-1)\alpha+(n^2-3n+2)\gamma}z^{n-1}$ is analytic and $\Re\,(q(z))>0$ in $\mathbb{D}$. Furthermore, we have

$$\frac{\phi(z)}{z}=1+\sum_{n=2}^{\infty}c_nz^{n-1}=p(z)*\Big(1+\sum_{n=2}^{\infty}\frac{2-\beta}{1+(n-1)\alpha+(n^2-3n+2)\gamma}z^{n-1}\Big).$$

Let us consider $m(z)=1+\displaystyle\sum_{n=2}^{\infty}\frac{2-\beta}{1+(n-1)\alpha+(n^2-3n+2)\gamma}z^{n-1}$. Then $m(z)$ is analytic in $\mathbb{D}$ with $m(0)=1$. Also, as $\Re(q(z))>0$, we have $\Re(m(z))>1/2$. Therefore $\Re(m(z))>1/2$ and $\Re(p(z))>1/(2-\beta)$. From application of [29, Lemma 4, pp-146], we conclude that $\Re\left[\phi(z)/z\right]>1/(2-\beta)$ for $z\in\mathbb{D}$. This completes the proof. □

Theorem 3.3. *Let ϕ_1 and ϕ_2 be in $\mathcal{W}_\beta(\alpha,\gamma)$. Then $\phi_1*\phi_2\in\mathcal{W}_\beta(\alpha,\gamma)$.*

Proof. Suppose that $\phi_1(z)=z+\sum_{n=2}^{\infty}c_nz^n$ and $\phi_2(z)=z+\sum_{n=2}^{\infty}d_nz^n$ are members of $\mathcal{W}_\beta(\alpha,\gamma)$. Since $\phi_2\in\mathcal{W}_\beta(\alpha,\gamma)$, then by Theorem 3.2 we have $\Re\big(\phi_2(z)/z\big)>1/(2-\beta)$. Set $\phi(z)=(\phi_1*\phi_2)(z)=z+\sum_{n=2}^{\infty}c_nd_nz^n$. Using the fact that $z\phi'(z)=z\phi_1'(z)*\phi_2(z)$, we obtain

$$\frac{1}{1-\beta}\Big[(1-\alpha+2\gamma)\frac{\phi_1(z)}{z}+(\alpha-2\gamma)\phi_1'(z)+\gamma z\phi_1''(z)-\beta\Big]*\frac{\phi_2(z)}{z} = \frac{1}{1-\beta}\Big[(1-\alpha+2\gamma)\frac{\phi(z)}{z}+(\alpha-2\gamma)\phi'(z)+\gamma z\phi''(z)-\beta\Big]. \quad (9)$$

Since $\phi_1\in\mathcal{W}_\beta(\alpha,\gamma)$, then the expression in the square bracket of left hand side of (9) is positive. Further, by Theorem 3.2, we have $\Re\big(\phi_2(z)/z\big)>1/(2-\beta)$. Thus, the phrase inside the square bracket of right hand side of (9) is positive through the applications of [29, Lemma 4, pp-146] (also see [10]), and hence $\phi=\phi_1*\phi_2$ is in $\mathcal{W}_\beta(\alpha,\gamma)$. □

The next result follows from Theorem 3.2. Hence we state the result without proof.

Theorem 3.4. *If f_1 and f_2 are in $T_H^1(\alpha,\gamma,\beta)$ then f_1*f_2 is in $T_H^1(\alpha,\gamma,\beta)$.*

4. Radii of starlikeness and convexity of $g_H^0(\beta)$ and $G_H^1(\alpha,\beta)$

In this section, some radii related problem such as radius of starlikeness and radius of convexity for the functions in both the classes $g_H^0(\beta)$ and $G_H^1(\alpha,\beta)$ are obtained.

Theorem 4.1. *Let $f=h+\overline{g}\in g_H^0(\beta)$, where h and g are of the form* (1)*. Then f is starlike in $|z|<r_1$, where r_1 is the smallest positive root in* $(0,1)$ *of the equation*

$$1-6r+3r^2+4r\beta-2r^2\beta=0. \quad (10)$$

Proof. Given that $f = h + \overline{g} \in g^0_H(\beta)$. For $0 < r < 1$, let $f_r(z) = r^{-1}f(rz) = r^{-1}h(rz) + r^{-1}\overline{g(rz)}$. Therefore, we have

$$f_r(z) = z + \sum_{n=2}^{\infty} a_n r^{n-1} z^n + \overline{\sum_{n=2}^{\infty} b_n r^{n-1} z^n}, \quad z \in \mathbb{D}. \tag{11}$$

Let us consider

$$P = \sum_{n=2}^{\infty} n(|a_n| + |b_n|)r^{n-1}. \tag{12}$$

To prove $g^0_H(\beta)$ is starlike, we will make use of Lemma 1.1. That means we will show that $1 - P \geq 0$ for $r < r_1$. In fact, using Theorem 2.3, we have

$$1 - P \geq 1 - \sum_{n=2}^{\infty} 2(1-\beta)nr^{n-1} = 1 - 2(1-\beta)\frac{2r - r^2}{(1-r)^2}.$$

Therefore, $1 - P \geq 0$ for $r < r_1$, whenever r_1 is the smallest positive root in $(0, 1)$ of (10). More precisely, solving (10) we get f is starlike if $|z| < r_1$, where $r_1 = 1 - (\sqrt{4\beta^2 - 10\beta + 6})/(3 - 2\beta)$. □

Taking $\beta = 0$ in Theorem 4.1, we have the following.

Corollary 4.2. *Let $f = h + \overline{g} \in g^0_H$, where h and g have the form* (1). *Then f is starlike in $|z| < r_1 \approx 0.1835034191$.*

Theorem 4.3. *Let $f = h + \overline{g} \in G^1_H(\alpha, \beta)$ with $\alpha > 0$, where h and g are of the form* (1). *Then f is starlike in $|z| < r_2$, where r_2 is the smallest positive root in* $(0, 1)$ *of the equation*

$$\frac{r}{1-r} + r\left(1 - \frac{1}{\alpha}\right)\int_0^1 \frac{u^{\frac{1}{\alpha}}}{1 - ru}du = \frac{\alpha}{2(1-\beta)}. \tag{13}$$

Proof. Let $f = h + \overline{g} \in G^1_H(\alpha, \beta)$ with $\alpha > 0$. Following the proof of Theorem 4.1 and Lemma 1.1, it suffices to show that $P \leq 1$ for $r < r_2$. In fact, we observe that for $\alpha > 0$, the coefficient inequalities from Theorem 2.3 and (12) gives

$$\begin{aligned} P &\leq 2(1-\beta)\sum_{n=2}^{\infty} \frac{nr^{n-1}}{1 + (n-1)\alpha} \leq \frac{2(1-\beta)}{\alpha}\Big(r^{1-\frac{1}{\alpha}} \sum_{n=2}^{\infty} \int_0^r t^{n-2+\frac{1}{\alpha}} dt\Big)' \\ &= \frac{2(1-\beta)}{\alpha}\Big(r^{1-\frac{1}{\alpha}} \int_0^r t^{\frac{1}{\alpha}} \sum_{n=2}^{\infty} t^{n-2} dt\Big)' \leq \frac{2(1-\beta)}{\alpha}\Big[r^{1-\frac{1}{\alpha}} \frac{r^{\frac{1}{\alpha}}}{1-r} + \left(1 - \frac{1}{\alpha}\right) r^{-\frac{1}{\alpha}} \int_0^r \frac{t^{\frac{1}{\alpha}}}{1-t} dt\Big]. \end{aligned}$$

Further, substituting $t = ru$, the above inequality gives

$$P \leq \frac{2(1-\beta)}{\alpha}\Big[\frac{r}{1-r} + r\left(1 - \frac{1}{\alpha}\right)\int_0^1 \frac{u^{\frac{1}{\alpha}}}{1 - ru}du\Big].$$

If $r < r_2$, $r_2 \in (0, 1)$, then $P \leq 1$, where r_2 is a root of (13). This completes the proof. □

Theorem 4.4. *Let $f = h + \overline{g} \in g^0_H(\beta)$, where h and g are of the form* (1). *Then f is convex in $|z| < r_3$, where r_3 is a unique root in* $(0, 1)$ *of the equation*

$$1 - (11 - 8\beta)r + 3(3 - 2\beta)r^2 - (3 - 2\beta)r^3 = 0. \tag{14}$$

Proof. For $0 < r < 1$, let $f_r(z)$ as given in (11) and set $Q = \sum_{n=2}^{\infty} n^2(|a_n| + |b_n|)r^{n-1}$. Therefore, by Lemma 1.1, it is sufficient to show that $Q \leq 1$, that is, $1 - Q \geq 0$ for $r < r_3$. In fact, from Theorem 2.3, we have

$$1 - Q \geq 1 - 2(1-\beta)\sum_{n=2}^{\infty} n^2 r^{n-1} \geq 1 - 2(1-\beta)\left(\sum_{n=2}^{\infty} nr^n\right)' = 1 - 2(1-\beta)\left(r\sum_{n=2}^{\infty} nr^{n-1}\right)'.$$

Which upon further simplifications, gives

$$\begin{aligned} 1 - Q \geq 1 - 2(1-\beta)\Big[\frac{2r^2 - r^3}{(1-r)^2}\Big]' &= 1 - 2(1-\beta)\Big(\frac{(1-r)^2(4r - 3r^2) + 2(2r^2 - r^3)(1-r)}{(1-r)^4}\Big) \\ &\geq \frac{1 - 3r + 3r^2 - r^3 - 2(1-\beta)(r^3 - 3r^2 + 4r)}{(1-r)^3}. \end{aligned}$$

Thus, $1 - Q \geq 0$ for all $r < r_3$. This shows that f is convex in $|z| < r_3$ where r_3 is a unique root in $(0, 1)$ of (14). □

Taking $\beta = 0$ in Theorem 4.4, we have the following.

Corollary 4.5. *Let $f = h + \overline{g} \in g_H^0$, where h and g are of the form* (1). *Then f is convex in $|z| < r_3 \approx 0.0986023$.*

Theorem 4.6. *Let $f = h + \overline{g} \in G_H^1(\alpha, \beta)$ with $\alpha > 0$, $0 \leq \beta < 1$ where h and g have the form* (1). *Then f is convex in $|z| < r_4$, where r_4 is the smallest positive root in $(0, 1)$ of the equation*

$$\frac{(3 - \frac{1}{\alpha})r - (2 - \frac{1}{\alpha})r^2}{(1-r)^2} + r\left(1 - \frac{1}{\alpha}\right)^2 \int_0^1 \frac{u^{\frac{1}{\alpha}}}{1 - ru} du = \frac{\alpha}{2(1-\beta)}. \tag{15}$$

Proof. Using the steps as in Theorem 4.4, the coefficient inequalities and Lemma 1.1, the result follows. □

5. Results on partial sums of functions in $T_H^k(\alpha, \gamma, \beta)$

In this section we derive the properties of the sections $s_{p,q}(f)$ of functions $f \in T_H^k(\alpha, \gamma, \beta)$.

Theorem 5.1. *Let $f \in T_H^1(\alpha, \gamma, \beta)$ be of the form* (1). *Then for each $q \geq 2$, $s_{1,q}(f) \in T_H^1(\alpha, \gamma, \beta)$ for all $|z| < 1/2$.*

Proof. Suppose that $f = h + \overline{g} \in T_H^1(\alpha, \gamma, \beta)$. Therefore, the corresponding partial sum is given by

$$s_{1,q}(f)(z) = s_1(h)(z) + \overline{s_q(g)(z)} = z + \sum_{j=2}^{q} \overline{b_j z^j} \quad (z \in \mathbb{D}).$$

Now, we have

$$\Re\Big((1 - \alpha + 2\gamma)\frac{s_1(h)(z)}{z} + (\alpha - 2\gamma)s_1'(h)(z) + \gamma z s_1''(h)(z) - \beta\Big) = 1 - \beta. \tag{16}$$

Application of Theorem 2.2 yields

$$\begin{aligned} &\left|(1 - \alpha + 2\gamma)\frac{s_q(g)(z)}{z} + (\alpha - 2\gamma)s_q'(g)(z) + \gamma z s_q''(g)(z)\right| \\ &\qquad = \left|(1 - \alpha + 2\gamma)\frac{1}{z}\sum_{j=2}^{q} b_j z^j + (\alpha - 2\gamma)\sum_{j=2}^{q} j b_j z^{j-1} + \gamma \sum_{j=2}^{q} j(j-1) b_j z^{j-1}\right| \\ &\qquad \leq \sum_{j=2}^{q} (1-\beta)|z^{j-1}| \leq (1-\beta)\frac{|z|}{1 - |z|} < 1 - \beta, \end{aligned} \tag{17}$$

for all $|z| < 1/2$. Thus, from (16) and (17), we conclude that $s_{1,q}(f) \in T_H^1(\alpha, \gamma, \beta)$ for $|z| < 1/2$. □

Theorem 5.2. *Let $f \in T_H^1(\alpha,\gamma,\beta)$ be of the form* (1)*. Then for each complex number ε with $|\varepsilon| = 1$ and $|z| < 1/2$, we have*

$$\Re\Big((1-\alpha+2\gamma)\frac{s_3(h)+\varepsilon s_3(g)}{z} + (\alpha-2\gamma)(s_3(h)+\varepsilon s_3(g))' + \gamma z(s_3(h)+\varepsilon s_3(g))'' - \beta\Big) > \frac{1}{4} - \frac{\beta}{2}.$$

Proof. Set

$$F_\varepsilon(z) := (1-\alpha+2\gamma)\frac{h(z)+\varepsilon g(z)}{z} + (\alpha-2\gamma)(h(z)+\varepsilon g(z))' + \gamma z(h(z)+\varepsilon g(z))''. \tag{18}$$

Therefore by Theorem 2.1, $F_\varepsilon \in \mathcal{W}_\beta(\alpha,\gamma)$. This implies that $\Re(F_\varepsilon(z)) > \beta \geq 0$. Further, as $f \in \mathcal{H}_0^1$, so $h(0) = 0 = h'(0) - 1$, $g(0) = 0 = g'(0)$. Therefore, we have $F_\varepsilon(0) = 1$. Now, from (18) it is a simple exercise that $F_\varepsilon(z) = 1 + \sum_{j=1}^{\infty} c_j z^j$, where $c_j = (1 + \alpha j + j(j-1)\gamma)(a_{j+1} + \varepsilon b_{j+1})$, for $j = 1, 2, \cdots$. Clearly, $s_3(h)(z) = z + a_2 z^2 + a_3 z^3$ and $s_3(g)(z) = b_2 z^2 + b_3 z^3$. Thus, we have

$$(1-\alpha+2\gamma)\frac{s_3(h)+\varepsilon s_3(g)}{z} + (\alpha-2\gamma)(s_3(h)+\varepsilon s_3(g))' + \gamma z(s_3(h)+\varepsilon s_3(g))'' = 1 + c_1 z + c_2 z^2.$$

Using [12, Problem 2.1.9, pp-35], we obtain that, $|c_2 - c_1^2/2| \leq 2(1-\beta) - |c_1^2|/2$, implies $|2c_2 - c_1^2| \leq 4(1-\beta) - |c_1^2|$. Now, let $2c_2 - c_1^2 = c$, so that $c_2 = c/2 + c_1^2/2$ and $|c| \leq 4(1-\beta) - |c_1^2|$. Also, let $c_1 z = r + is$ and $\sqrt{c}z = \eta + i\delta$, where r, s, η and δ are real numbers. Then for $|z| < 1/2$, we have $r^2 + s^2 = |c_1|^2|z|^2 \leq |c_1|^2/4$, and

$$\delta^2 = |c||z|^2 - \eta^2 \leq \frac{|c|}{4} - \eta^2 \leq \frac{4(1-\beta) - |c_1|^2}{4} - \eta^2 \leq 1 - \beta - r^2 - s^2 - \eta^2. \tag{19}$$

Therefore, we have

$$\begin{aligned}\Re(1 + c_1 z + c_2 z^2 - \beta) &= \Re\Big(1 + c_1 z + \frac{c}{2}z^2 + \frac{c_1^2}{2}z^2\Big) - \beta = 1 + r + \frac{\eta^2}{2} - \frac{\delta^2}{2} + \frac{r^2}{2} - \frac{s^2}{2} - \beta \\ &\geq \frac{1}{2} + r + \eta^2 + r^2 + \frac{\beta}{2} - \beta = \Big(r + \frac{1}{2}\Big)^2 + \frac{1}{4} + \eta^2 - \frac{\beta}{2} \geq \frac{1}{4} - \frac{\beta}{2}.\end{aligned}$$

This completes the proof. □

Theorem 5.3. *Let $f = h + \overline{g} \in T_H^1(\alpha,\gamma,\beta)$ be of the form* (1) *and suppose that p and q satisfy one of the following conditions: (i) $3 \leq p < q$; (ii) $p > q \geq 3$. Then $s_{p,q}(f) \in T_H^1(\alpha,\gamma,\beta)$ for $|z| < r_5 := \min\{R_1, R_2\}$, where R_1, R_2 respectively, are the smallest positive root in $(0,1)$ of the equations:*

$$1 - \beta - 2r + (1+\beta)r^2 - 2(1-\beta)r^p(1+r) = 0; \tag{20}$$

$$1 - \beta - 2r + (1+\beta)r^2 - (1-\beta)(1+r)(r^p + r^q) = 0. \tag{21}$$

Proof. Suppose that $f = h + \overline{g} \in T_H^1(\alpha,\gamma,\beta)$. We have $s_{p,q}(f)(z) = s_p(h)(z) + \overline{s_q(g)(z)}$ for all $z \in \mathbb{D}$. Let us consider

$$\sigma_p(h)(z) = \sum_{j=p+1}^{\infty} a_j z^j, \quad \sigma_q(g)(z) = \sum_{j=q+1}^{\infty} b_j z^j, \quad z \in \mathbb{D},$$

and $|\varepsilon| = 1$. Then $h = s_p(h) + \sigma_p(h)$ and $g = s_q(g) + \sigma_q(g)$. To claim the result, it suffices to show that $s_p(h) + \varepsilon s_q(g)$ is in the class $\mathcal{W}_\beta(\alpha,\gamma)$ for each ε with $|\varepsilon| = 1$. Indeed, if $f \in T_H^1(\alpha,\gamma,\beta)$, then

$$\begin{aligned}&\Re\Big((1-\alpha+2\gamma)\frac{s_p(h)+\varepsilon s_q(g)}{z} + (\alpha-2\gamma)(s_p(h)+\varepsilon s_q(g))' + \gamma z(s_p(h)+\varepsilon s_q(g))'' - \beta\Big) \\ &\quad\geq \Re\Big[(1-\alpha+2\gamma)\frac{h+\varepsilon g}{z} + (\alpha-2\gamma)(h+\varepsilon g)' + \gamma z(h+\varepsilon g)''\Big] \\ &\quad - \Big|(1-\alpha+2\gamma)\frac{\sigma_p(h)+\varepsilon\sigma_q(g)}{z} + (\alpha-2\gamma)(\sigma_p(h)+\varepsilon\sigma_q(g))' + \gamma z(\sigma_p(h)+\varepsilon\sigma_q(g))''\Big| - \beta.\end{aligned} \tag{22}$$

In the notion of subordination, we have

$$(1-\alpha+2\gamma)\frac{h+\varepsilon g}{z}+(\alpha-2\gamma)(h+\varepsilon g)'+\gamma z(h+\varepsilon g)'' \prec \frac{1+z}{1-z}, \quad (z\in\mathbb{D}),$$

where the symbol $\prec$ denotes the usual subordination. This implies that

$$\Re\Big((1-\alpha+2\gamma)\frac{h+\varepsilon g}{z}+(\alpha-2\gamma)(h+\varepsilon g)'+\gamma z(h+\varepsilon g)''\Big) \geq \frac{1-|z|}{1+|z|}, \quad (z\in\mathbb{D}). \tag{23}$$

We complete the proof by dividing it into the following cases.

Case (i): $3 \leq p < q$**.** Now by applications of Theorems 2.2 and 2.3, we have

$$\begin{aligned}&\left|(1-\alpha+2\gamma)\frac{\sigma_p(h)+\varepsilon\sigma_q(g)}{z}+(\alpha-2\gamma)(\sigma_p(h)+\varepsilon\sigma_q(g))'+\gamma z(\sigma_p(h)+\varepsilon\sigma_q(g))''\right|\\ &\qquad \leq \sum_{j=p+1}^{q} 2(1-\beta)|z|^{j-1}+\sum_{j=q+1}^{\infty} 2(1-\beta)|z|^{j-1}=\sum_{j=p+1}^{\infty} 2(1-\beta)|z|^{j-1}=\frac{2(1-\beta)|z|^p}{1-|z|}.\end{aligned} \tag{24}$$

Now from (22), using (23) and (24), we get

$$\begin{aligned}&\Re\Big((1-\alpha+2\gamma)\frac{s_p(h)+\varepsilon s_q(g)}{z}+(\alpha-2\gamma)(s_p(h)+\varepsilon s_q(g))'+\gamma z(s_p(h)+\varepsilon s_q(g))''-\beta\Big)\\ &\qquad \geq \frac{1-|z|}{1+|z|}-\frac{2(1-\beta)|z|^p}{1-|z|}-\beta=\frac{1-\beta-2|z|+(1+\beta)|z|^2-2(1-\beta)|z|^p(1+|z|)}{1-|z|^2}.\end{aligned}$$

The right hand side of the above inequality is greater than or equal to 0 implies $|z| = R_1$ is the smallest positive root of (20).

Case (ii): $p > q \geq 3$**.** Following the steps as in Case (i) along with applications of Theorems 2.2 and 2.3, yields

$$\begin{aligned}&\left|(1-\alpha+2\gamma)\frac{\sigma_p(h)+\varepsilon\sigma_q(g)}{z}+(\alpha-2\gamma)(\sigma_p(h)+\varepsilon\sigma_q(g))'+\gamma z(\sigma_p(h)+\varepsilon\sigma_q(g))''\right|\\ &\qquad \leq \frac{2(1-\beta)|z|^p}{1-|z|}+\frac{(1-\beta)|z|^q(1-|z|^{p-q})}{1-|z|}=\frac{(1-\beta)(|z|^p+|z|^q)}{1-|z|}.\end{aligned} \tag{25}$$

From inequality (22) using inequalities (23) and (25), we obtain

$$\begin{aligned}&\Re\Big((1-\alpha+2\gamma)\frac{s_p(h)+\varepsilon s_q(g)}{z}+(\alpha-2\gamma)(s_p(h)+\varepsilon s_q(g))'+\gamma z(s_p(h)+\varepsilon s_q(g))''-\beta\Big)\\ &\qquad \geq \frac{1-|z|}{1+|z|}-\frac{(1-\beta)(|z|^p+|z|^q)}{1-|z|}-\beta,\end{aligned} \tag{26}$$

which is greater than equal to 0 for $|z| = R_2$, where R_2 is the root of (21). Hence, $s_{p,q} \in T^1_H(\alpha,\gamma,\beta)$ for $|z| < r_5 := \min\{R_1, R_2\}$. This completes the proof. □

For the case when $\beta = 0$, from Theorem 5.3, immediately we have

Corollary 5.4. *Let $f = h + \overline{g} \in T^1_H(\alpha,\gamma)$ be of the form* (1) *and suppose that p and q satisfy one of the following conditions: (i) $3 \leq p < q$; (ii) $p > q \geq 3$. Then $s_{p,q}(f) \in T^1_H(\alpha,\gamma)$ for $|z| < 1/2$.*

It is an easy exercise to see the following results hold true.

Theorem 5.5. *Let $f = h + \overline{g} \in T^1_H(\alpha,\gamma)$ be of the form* (1)*. If $p = 2 < q$, then $s_{2,q}(f) \in T^1_H(\alpha,\gamma)$ in $|z| < r = \dfrac{3-\sqrt{5}}{2} \approx 0.381966$; if $p \geq 4$ and $q = 2$, then $s_{p,2}(f) \in T^1_H(\alpha,\gamma)$ in $|z| < r = 0.433797$.*

Theorem 5.6. *Let $f \in T_H^k(\alpha,\gamma,\beta)$ be of the form (2) with $k \geq 1$. Then for each $q \geq k+1$, $s_{1,q}(f) \in T_H^k(\alpha,\gamma,\beta)$, for $|z| < r_6$ where r_6 is the unique root in $(0,1)$ of the equation $r^k + r - 1 = 0$.*

Theorem 5.7. *Let $f = h + \overline{g} \in T_H^k(\alpha,\gamma,\beta)$ be of the form (2) with $k \geq 2$. Suppose that p and q satisfy one of the following conditions: (i) $p = q = k+1$; (ii) $p = k+1$, $q = k+2$; (iii) $q > p \geq [U]+1$; (iv) $p > q \geq [V]+1$; with $U := \left\{\ln(r_6 - r_6\beta - 2\beta - r_6^2 - r_6^2\beta) - \ln(4 - 4\beta - 2r_6 + 2r_6\beta)\right\}/\ln r_6$, and $V := \left\{\ln(r_6 - 2\beta + 3r_6\beta - r_6^2 - r_6^2\beta) - \ln(2 + r_6 - 2\beta - \beta r_6 - r_6^2 + r_6^2\beta)\right\}/\ln r_6$. Here r_6 is the unique smallest root in $(0,1)$ of the equation $r^k + r - 1 = 0$ and the symbol $[U]$ means the maximum integer no more than U. Then $s_{p,q}(f) \in T_H^k(\alpha,\gamma,\beta)$.*

Proof. Let $f = h + \overline{g} \in T_H^k(\alpha,\gamma,\beta)$ for $\alpha,\ \gamma \geq 0$. To justify our claim, we verify the statement for each of the given conditions.

Case (i): $p = q = k+1$. Clearly, we have $s_{k+1}(h) + \varepsilon s_{k+1}(g) = z + a_{k+1}z^{k+1} + \varepsilon b_{k+1}z^{k+1}$. Therefore, suitable applications of Theorem 2.3, it is observed that

$$
\begin{aligned}
&\Re\Big[(1-\alpha+2\gamma)\frac{s_{k+1}(h)+\varepsilon s_{k+1}(g)}{z} + (\alpha-2\gamma)(s_{k+1}(h)+\varepsilon s_{k+1}(g))' + \gamma z(s_{k+1}(h)+\varepsilon s_{k+1}(g))'' - \beta\Big] \\
&\qquad = \Re\Big[1 + (1 + k\alpha + (k^2-k)\gamma)(a_{k+1}+\varepsilon b_{k+1})z^k - \beta\Big] \\
&\qquad \geq 1 - \beta - \Big(1 + k\alpha + (k^2-k)\gamma\Big)\Big|(a_{k+1}+\varepsilon b_{k+1})z^k\Big| \\
&\qquad \geq 1 - \beta - 2(1-\beta)|z|^k > (1-\beta)(1-2r_6^k) = (1-\beta)(2r_6-1) > 0
\end{aligned}
$$

holds true for all $|z| < r_6$, which is the unique root of $r^k + r - 1 = 0$. Thus, $s_{k+1,k+1}(f) \in T_H^k(\alpha,\gamma,\beta)$.

Case (ii): $p = k+1,\ q = k+2$. Following the steps as in Case (i), we have

$$
\begin{aligned}
&\Re\Big[(1-\alpha+2\gamma)\frac{s_{k+1}(h)+\varepsilon s_{k+2}(g)}{z} + (\alpha-2\gamma)(s_{k+1}(h)+\varepsilon s_{k+2}(g))' + \gamma z(s_{k+1}(h)+\varepsilon s_{k+2}(g))'' - \beta\Big] \\
&\quad = \Re\Big[(1-\alpha+2\gamma)\frac{s_{k+1}(h)+\varepsilon s_{k+1}(g)}{z} + (\alpha-2\gamma)\Big(s_{k+1}(h)+\varepsilon s_{k+1}(g)\Big)' \\
&\qquad + \gamma z\Big(s_{k+1}(h)+\varepsilon s_{k+1}(g)\Big)'' + \varepsilon\Big(1 + (k+1)\alpha + (k^2+k)\gamma\Big)b_{k+2}z^{k+1} - \beta\Big] \\
&\quad \geq (1-\beta)(1 - 2|z|^k - |z|^{k+1}) > (1-\beta)(1 - 2r_6^k - r_6^{k+1}) = (1-\beta)(r_6^2 + r_6 - 1) > 0.
\end{aligned}
$$

Indeed, the right hand side of the above inequality is positive follows from the fact that $(1-\beta)(r_6^k + r_6 - 1) = 0$ implies $(1-\beta)r_6^k = (1-\beta)(1-r_6)$. That means $(1-\beta)(1-2r_6^k-r_6^{k+1}) = (1-\beta)(1-2+2r_6-r_6+r_6^2) = (1-\beta)(r_6^2+r_6-1)$. Hence, $s_{k+1,k+2}(f) \in T_H^k(\alpha,\gamma,\beta)$.

Case (iii): $q > p \geq [U]+1$. This implies that

$$
p > \left\{\ln(r_6 - r_6\beta - 2\beta - r_6^2 - r_6^2\beta) - \ln(4 - 4\beta - 2r_6 + 2r_6\beta)\right\}/\ln r_6,
$$

which upon simplification gives

$$
2(1-\beta)r_6^p < (1-r_6)\Big(\frac{r_6}{2-r_6} - \beta\Big). \tag{27}
$$

Let us consider

$$
\begin{aligned}
F_\varepsilon(z) &= (1-\alpha+2\gamma)\frac{h(z)+\varepsilon g(z)}{z} + (\alpha-2\gamma)(h(z)+\varepsilon g(z))' + \gamma z(h(z)+\varepsilon g(z))'' \\
&= 1 + \sum_{n=k+1}^{\infty}(a_n + \varepsilon b_n)\Big(1 + \alpha(n-1) + \gamma(n^2-3n+2)\Big)z^{n-1} \\
&= 1 + (1 + k\alpha + (k^2-k)\gamma)(a_{k+1}+\varepsilon b_{k+1})z^k + \cdots, \qquad z \in \mathbb{D}.
\end{aligned}
$$

Therefore, $F_\varepsilon(0) = 1$, $\Re(F_\varepsilon(z)) > 0$. Thus, application of [18, Lemma 2.6, pp-358] gives

$$\Re\Big(F_\varepsilon(z)\Big) \geq \frac{1-|z|^k}{1+|z|^k}, \qquad z \in \mathbb{D}. \tag{28}$$

From the inequality (22), using (24), (27) and (28) we have

$$\begin{aligned}&\Re\Big[(1-\alpha+2\gamma)\frac{s_p(h)+\varepsilon s_q(g)}{z} + (\alpha-2\gamma)(s_p(h)+\varepsilon s_q(g))' + \gamma z(s_p(h)+\varepsilon s_q(g))'' - \beta\Big]\\ &\quad \geq \frac{1-|z|^k}{1+|z|^k} - \frac{2(1-\beta)|z|^p}{1-|z|} - \beta > \frac{1-r_6^k}{1+r_6^k} - \frac{2r_6^p}{1-r_6} - \beta = \frac{r_6}{2-r_6} - \frac{2(1-\beta)r_6^p}{1-r_6} - \beta > 0\end{aligned}$$

holds true for all $|z| < r_6$, which is the unique root of $r^k + r - 1 = 0$. Since $\Re\Big((1-\alpha+2\gamma)\frac{s_p(h)+\varepsilon s_q(g)}{z} + (\alpha - 2\gamma)(s_p(h)+\varepsilon s_q(g))' + \gamma z(s_p(h)+\varepsilon s_q(g))''\Big)$ is harmonic, it assumes its minimum for $|z| \leq r_6$ on the circle $|z| = r_6$. Therefore, we have $s_{p,q}(f) \in T_H^k(\alpha,\gamma,\beta)$.

Case (iv): $p > q \geq [V] + 1$. This implies that

$$q > \Big\{\ln(r_6 - 2\beta + 3r_6\beta - r_6^2 - r_6^2\beta) - \ln(2 + r_6 - 2\beta - \beta r_6 - r_6^2 + r_6^2\beta)\Big\}/\ln r_6,$$

which upon simplification gives

$$(1-\beta)(1+r_6)r_6^q < (1-r_6)\Big(\frac{r_6}{2-r_6} - \beta\Big). \tag{29}$$

Therefore, for $|z| < r_6$, we have

$$\frac{|z|^p + |z|^q}{1-|z|} \leq \frac{|z|^{q+1} + |z|^q}{1-|z|} < \frac{r_6^q(r_6+1)}{1-r_6}. \tag{30}$$

Using (25), (28), (29) and (30) in (22) for $|z| < r_6$, we see that

$$\begin{aligned}&\Re\Big[(1-\alpha+2\gamma)\frac{s_p(h)+\varepsilon s_q(g)}{z} + (\alpha-2\gamma)(s_p(h)+\varepsilon s_q(g))' + \gamma z(s_p(h)+\varepsilon s_q(g))'' - \beta\Big]\\ &\quad \geq \frac{1-|z|^k}{1+|z|^k} - \frac{(1-\beta)(|z|^p+|z|^q)}{1-|z|} - \beta \geq \frac{1-r_6^k}{1+r_6^k} - \frac{(1-\beta)r_6^q(1+r_6)}{1-r_6} - \beta\\ &\quad = \frac{r_6}{2-r_6} - \frac{(1-\beta)r_6^q(1+r_6)}{1-r_6} - \beta > 0.\end{aligned}$$

This shows that $s_{p,q}(f) \in T_H^k(\alpha,\gamma,\beta)$. □

6. Radii of starlikeness and close-to-convexity of the partial sums

In this section, the value of the radius r is determined so that the partial sum $s_{1,q}(f)$ is starlike and $s_{p,q}(f)$ is close-to-convex in the disk $|z| < r$ for the functions $f \in T_H^k(\alpha,\gamma,\beta)$.

Theorem 6.1. *Let $f = h + \overline{g} \in T_H^k(\alpha,\gamma,\beta)$ be of the form (2) with $k \geq 1$, then for each $q \geq k+1$, $s_{1,q}(f)$ is starlike in $|z| < r_7$, where r_7 is the smallest positive root in $(0,1)$ of the equation*

$$1 - 3r + 3r^2 - r^3 + (1-\beta)((2k-1)r^{k+1} - kr^k - (k-1)r^{k+2}) + (1-\beta)^2(kr^{2k+1} - (k+1)r^{2k}) = 0. \tag{31}$$

Proof. By assumption, we have $s_{1,q}(f)(z) = s_1(h)(z) + \overline{s_q(g)(z)} = z + \sum_{j=k+1}^{q} \overline{b_j z^j}$. Set $H(z) = z$ and $G(z) = \sum_{j=k+1}^{q} b_j z^j$. Then $s_{1,q}(f)(z) = H(z) + \overline{G(z)}$. Now, since

$$\Re\left\{\frac{zH'(z) - \overline{zG'(z)}}{H(z) + \overline{G(z)}}\right\} = \Re\left\{\frac{z - \sum_{j=k+1}^{q} \overline{jb_j z^j}}{z + \sum_{j=k+1}^{q} \overline{b_j z^j}}\right\} \quad \text{and} \quad \lim_{z\to 0}\left\{\frac{z - \sum_{j=k+1}^{q} \overline{jb_j z^j}}{z + \sum_{j=k+1}^{q} \overline{b_j z^j}}\right\} = 1.$$

Therefore, it suffices to prove that

$$A := \Re\left\{z - \sum_{j=k+1}^{q} \overline{jb_j z^j}\right\}\left\{\bar{z} + \sum_{j=k+1}^{q} b_j z^j\right\} > 0 \text{ for } |z| = r_7.$$

In fact, application of Theorem 2.2, gives

$$A \geq |z|^2 - \sum_{n=k+1}^{q} (1-\beta)(n-1)|z|^{n+1} - \Big(\sum_{n=k+1}^{q} (1-\beta)|z|^n\Big)\Big(\sum_{n=k+1}^{q} (1-\beta)n|z|^n\Big).$$

Further simplifications yields

$$\frac{A(1-|z|)^3}{|z|^2} \geq 1 - 3|z| + 3|z|^2 - |z|^3 + (1-\beta)\Big((2k-1)|z|^{k+1} - k|z|^k - (k-1)|z|^{k+2}\Big) + (1-\beta)^2\Big(k|z|^{2k+1} - (k+1)|z|^{2k}\Big).$$

Thus, for $|z| = r_7$, where r_7 is the smallest positive root in $(0,1)$ of (31), we have $\frac{A(1-|z|)^3}{|z|^2} \geq 0$. Since $s_{1,q}(f)$ is a sense preserving harmonic mapping, then $s_{1,q}(f)$ is starlike in $|z| < r_7$. □

Theorem 6.2. *Let $f = h + \bar{g} \in T_H^k(\alpha, \gamma, \beta)$ be of the form* (2) *with $k \geq 1$. Then for each $q \geq k+2$, $s_{1,q}(f)$ is close-to-convex and univalent in $|z| < r_8$, where r_8 is the smallest positive root in* $(0,1)$ *of* $1-2r+r^2+(1-\beta)[kr^{k+1}-(k+1)r^k] = 0$.

Proof. We have $s_{1,q}(f) = s_1(h)(z) + \overline{s_q(g)(z)}$. Since $s_1'(h)(0) = 1 > s_q'(g)(0) = 0$, to apply Lemma 1.2, we will prove that for all ε with $|\varepsilon| = 1$, the function $s_{1,q}(F) = s_1(h)(z) + \varepsilon s_q(g)(z)$ is close-to-convex and univalent in $|z| < r_8$. That means, it suffices to prove

$$\Re\{s_{1,q}'(F)(z)\} = 1 + \Re\Big\{\sum_{j=k+1}^{q} \varepsilon j b_j z^{j-1}\Big\} > 0 \ \text{ for } |z| < r_8.$$

This immediately follows from Theorem 2.2. □

Theorem 6.3. *Let $f = h + \bar{g} \in T_H^k(\alpha, \beta, \gamma)$ be of the form* (2) *with $k \geq 1$. Then for each $p \geq k+1$, $q \geq k+1$, $s_{p,q}(f)$ is close-to-convex and univalent in $|z| < r_9$, where r_9 is the smallest positive root in* $(0,1)$ *of* $1-2r+r^2+2(1-\beta)[kr^{k+1} - (k+1)r^k] = 0$.

Proof. To prove that $s_{p,q}(f)$ is close-to-convex and univalent in $|z| < r_9$, by Lemma 1.2 it is enough to show that for all ε with $|\varepsilon| = 1$, the function $s_{p,q}(F) = s_p(h)(z) + \varepsilon s_q(g)(z)$ is close-to-convex and univalent in $|z| < r_9$. In fact, it suffices to prove that $\Re\left\{s_{p,q}'(F)(z)\right\} > 0$, for $|z| < r_9$. In deed, if $p < q$, then

$$\begin{aligned}\Re\{s_{p,q}'(F)(z)\} &\geq 1 - \sum_{j=k+1}^{p} |ja_j z^{j-1}| - \sum_{j=k+1}^{q} |jb_j z^{j-1}| \geq 1 - 2(1-\beta)\sum_{j=k+1}^{p} j|z|^{j-1} + (1-\beta)\sum_{j=p+1}^{q} j|z|^{j-1} \\ &\geq \frac{1 - 2|z| + |z|^2 + 2(1-\beta)(k|z|^{k+1} - (k+1)|z|^k)}{(1-|z|)^2} > 0,\end{aligned}$$

for $|z| < r_9$. The case when $p = q$, we obtain

$$\Re\{s'_{p,q}(F)(z)\} = 1 - \sum_{j=k+1}^{p} j(|a_j| + |b_j|)|z|^{j-1} \geq 1 - 2(1-\beta) \sum_{j=k+1}^{p} j|z|^{j-1}$$
$$\geq \frac{1 - 2|z| + |z|^2 + 2(1-\beta)[k|z|^{k+1} - (k+1)|z|^k]}{(1-|z|)^2} > 0,$$

for $|z| < r_9$. Finally, for $p > q$, we have

$$\Re\{s'_{p,q}(F)(z)\} \geq 1 - 2(1-\beta) \left\{ \sum_{j=k+1}^{q} j|z|^{j-1} + \sum_{j=q+1}^{p} j|z|^{j-1} \right\} = 1 - 2(1-\beta) \sum_{j=k+1}^{p} j|z|^{j-1} > 0,$$

for $|z| < r_9$. Thus, $s_{p,q}(f)(z)$ is close-to-convex and univalent in $|z| < r_9$. □

Acknowledgements

We thank the reviewer for valuable suggestions, which immensely improve the presentation of the manuscript.

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