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Mutual contraction principles in strong $b_v(s)$ metric spaces: Implications for generalized metrics

Nemanja Vučićević[a], Olivera Stančić[b]

[a]*Faculty of Science, University of Kragujevac, Radoja Domanovića 12, 34000 Kragujevac, Serbia*
[b]*Faculty of Economics, University of Kragujevac, Liceja Kneževine Srbije 3, 34000 Kragujevac, Serbia*

Abstract. In this paper, we consider $b_v(s)$-metric spaces, introduced as a generalization of metric spaces, rectangular metric spaces, b-metric spaces, rectangular b-metric spaces, and v-generalized metric spaces. Next, we introduce the concept of strong $b_v(s)$-metric spaces and explore some of their properties. We provide proofs of the Banach contraction principle in strong $b_v(s)$-metric spaces. Then, we define mutual Reich contraction and present results that generalize many known results in fixed point theory. Finally, we extend these results to a set of operators and prove that equilibrium is a global attractor for any scheme presented in this paper which has numerous applications in dynamical systems.

1. Introduction

The metric space which most closely corresponds to our intuitive understanding of space is the 3-dimensional Euclidean space. In fact, the notion of "metric" is a generalization of the Euclidean metric arising from the basic long known properties of the Euclidean distance.

A metric (distance) space suggests that given two points of the space there should be a real number that measures the distance between them. Accordingly, to discuss a "metric" it is natural to begin with a pair (X, d), where X is a set and $d : X \times X \to \mathbb{R}^+$ is a mapping of the cartesian product $X \times X$ into the nonnegative reals $\mathbb{R}^+$. If $d(x, y)$ is the distance between two points $x, y \in X$, it is natural to assume that for each $x, y, z \in X$, distance d satisfied the following axioms:

$$d(x, y) = 0 \iff x = y \quad \text{(identity of indiscernibles)} \tag{1}$$

$$d(x, y) = d(y, x) \quad \text{(symmetry)} \tag{2}$$

$$d(x, y) \leq d(x, z) + d(z, y) \quad \text{(triangle inequality)} \tag{3}$$

Since the core concept is distance, it is logical to state that two metric spaces are equivalent if there exists a (necessarily one-to-one) distance-preserving mapping from one to the other. These mappings are known as isometries.

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Email addresses: nemanja.vucicevic@pmf.kg.ac.rs (Nemanja Vučićević), olivera.stancic@kg.ac.rs (Olivera Stančić)
ORCID iDs: https://orcid.org/0000-0002-4903-7280 (Nemanja Vučićević), https://orcid.org/0000-0001-7219-9029 (Olivera Stančić)

The concepts of metric and metric space have been expanded in numerous ways. For a comprehensive understanding of these extensions, one can refer to several authoritative sources, including the books referenced as [8, 12], and [20]. Additionally, detailed overviews can be found in the survey papers [3] and [13], which provide an in-depth examination of the various generalizations and their implications in the field.

In papers [2] and [5], authors introduced the concept of b-metric spaces, which generalize traditional metric spaces, and demonstrated the contraction principle within this framework. Recently, numerous researchers have derived fixed point results for both single-valued and set-valued functions within the context of b-metric spaces (see, for example, [4, 5, 16, 18, 19]).

Definition 1.1. *([2, 5]) Let X be a nonempty set and let $s \geq 1$ be a given real number. A function $d : X \times X \to [0, \infty)$ is said to be a b-metric if and only if for all $x, y, z \in X$ he following conditions are satisfied:*

$$d(x, y) = 0 \iff x = y, \tag{4}$$

$$d(x, y) = d(y, x), \tag{5}$$

$$d(x, z) \leq s[d(x, y) + d(y, z)]. \tag{6}$$

The pair (X, d) is called a b-metric space. Obviously, for $s = 1$ one obtains a metric on X.

Along with the inequality (6), called the s-relaxed triangle inequality, one considers also the s-relaxed polygonal inequality

$$d(x_0, x_n) \leq s\,[d(x_0, x_1) + d(x_1, x_2) + \cdots + d(x_{n-1}, x_n)]\,,$$

for all $x_0, x_1, \ldots, x_n \in X$ and all $n \in \mathbb{N}$ (see [2, 5, 6]).

Cobzas and Czerwik in [4] introduced the concept of strong b-metric spaces in the following way: A mapping $d : X \times X \to [0, \infty)$ s called a strong b-metric if it satisfies the conditions (4) and (5) and

$$d(x, y) \leq d(x, z) + s \cdot d(y, z), \tag{7}$$

for some $s \geq 1$ and all $x, y, z \in X$. Taking into account the symmetry of d, the inequality (7) is equivalent to

$$d(x, y) \leq \min\{s \cdot d(x, z) + d(y, z),\ d(x, z) + s \cdot d(y, z)\}, \tag{8}$$

for all $x, y, z \in X$. Also (7) implies the s-relaxed triangle inequality.

In the 1960s, Gähler in [9] introduced the concept of a 2-metric space, which stands apart topologically from other generalizations of metric spaces. Geometrically, $d(x, y, z)$ represents the area of the triangle with vertices x, y, and z in the universal set X (see [10]).

Definition 1.2. *([9]) A mapping $d : X \times X \times X \to \mathbb{R}^+$ is said to be a 2-metric on the set X if the following conditions are satisfied:*

1. *For all $x, y \in X$ $(x \neq y)$, there exists a point $z \in X$ such that $d(x, y, z) \neq 0$,*
2. *$d(x, y, z) = 0$ when at least two of x, y, z are equal,*
3. *$d(x, y, z) = d(x, z, y) = d(y, z, x)$ for all $x, y, z \in X$,*
4. *$d(x, y, z) \leq d(x, y, w) + d(x, w, z) + d(w, y, z)$ for all $x, y, z, w \in X$.*

The structure of a 2-metric space is denoted by the ordered pair (X, d). Mustafa et al. in [17] proposed a novel metric structure known as the b_2-metric, which generalizes both the 2-metric and the b-metric. Their work included the derivation of fixed point theorems under distinct contractive conditions within ordered b_2-metric spaces.

Definition 1.3. *([17]) A mapping $d : X \times X \times X \to \mathbb{R}^+$ is said to be a b_2-metric on the set X if following condition are satisfied:*

1. *For all* $x, y \in X$ $(x \neq y)$, *there exists a point* $z \in X$ *such that* $d(x, y, z) \neq 0$,
2. $d(x, y, z) = 0$ *when at least two of* x, y, z *are equal,*
3. $d(x, y, z) = d(x, z, y) = d(y, z, x)$ *for all* $x, y, z \in X$,
4. $d(x, y, z) \leq K\,(d(x, y, w) + d(x, w, z) + d(w, y, z))$ *for all* $x, y, z, w \in X$, *and for some* $K \geq 1$.

If d is a b_2-metric on X, then the ordered pair (X, d, K) is called a b_2-metric space with parameter K. It is clear that each 2-metric is a b_2-metric, and a b_2-metric coincides with a 2-metric when the parameter $K = 1$.

In [11], authors also introduced the concept of strong b_2-metric spaces in which many known properties are ensured.

Definition 1.4. *([11])A mapping* $d_s : X \times X \times X \to \mathbb{R}^+$ *is said to be a strong* b_2*-metric on the set* X *if the following conditions are satisfied:*

1. *For all* $x, y \in X$ $(x \neq y)$, *there exists a point* $z \in X$ *such that* $d_s(x, y, z) \neq 0$;
2. $d_s(x, y, z) = 0$ *when at least two of* x, y, z *are equal;*
3. $d_s(x, y, z) = d_s(y, x, z) = d_s(z, y, x)$ *for all* $x, y, z \in X$;
4. $d_s(x, y, z) \leq d_s(x, y, w) + d_s(x, w, z) + K \cdot d_s(w, y, z)$ *for all* $x, y, z, w \in X$, *where* $K \geq 1$ *is a constant.*

If d_s is a strong b_2-metric on X, then the ordered pair (X, d_s, K) is called a strong b_2-metric space with parameter K. It is obvious that strong b_2-metric is coincident with the 2-metric given in [9] for $K = 1$.

Mitrović and Radenović in [14] introduce the concept of $b_v(s)$-metric space as follows.

Definition 1.5. *Let* X *be a nonempty set,* $d : X \times X \to [0, 1)$, *and let* $v \in \mathbb{N}$. *Then,* (X, d) *is said to be a* $b_v(s)$*-metric space if for all* $x, y \in X$ *and for all distinct points* $u_1, u_2, \ldots, u_v \in X$, *each of them different from* x *and* y, *the following hold:*

$$d(x, y) = 0 \iff x = y \tag{9}$$

$$d(x, y) = d(y, x) \tag{10}$$

There exists a real number $s \geq 1$ *such that*

$$d(x, y) \leq s\left[d(x, u_1) + d(u_1, u_2) + \cdots + d(u_v, y)\right]. \tag{11}$$

Common fixed point theorems for a sequence of mappings have been studied by several authors for $b_v(s)$ [7]. Reich [23] generalized Banach fixed point theorem for single-valued as well as multivalued mappings. We are particularly interested for Banach, Kannan, and specially Reich type of mappings, which have been the focus of intensive research by many authors. Some important fixed point results for those types of mappings in the framework of complete metric spaces are proved in [14–16, 18, 21].

2. Main results

In this paper, we introduce for the first time in the literature concept of strong $b_v(s)$-metric space as follows.

Definition 2.1. *Let* X *be a nonempty set,* $d : X \times X \to [0, 1)$, *and let* $v \in \mathbb{N}$. *Then,* (X, d) *is said to be a strong* $b_v(s)$*-metric space if for all* $x, y \in X$ *and for all distinct points* $u_1, u_2, \ldots, u_v \in X$, *each of them different from* x *and* y, *the following hold:*

$$d(x, y) = 0 \iff x = y, \tag{12}$$

$$d(x, y) = d(y, x). \tag{13}$$

There exists a real number $s \geq 1$ *such that*

$$d(x, y) \leq d(x, u_1) + d(u_1, u_2) + \cdots + s \cdot d(u_v, y). \tag{14}$$

Constant s is the index of the strong $b_v(s)$ -metric space. Obviously, a $b_v(s)$-metric space is a strong $b_v(s)$-metric space taking $s = 1$. As the inequality (14) implies (11), a strong $b_v(s)$-metric space is a $b_v(s)$-metric space.

Definition 2.2. *Let (X, d_s) be a strong $b_v(s)$-metric space, $\{x_n\}$ be a sequence in X and $x \in X$.*

1. *The sequence $\{x_n\}$ is said to be convergent in (X, d_s) and converges to x, if for every $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $d_s(x_n, x) < \epsilon$ for all $n > n_0$ and this fact is represented by $\lim_{n\to\infty} x_n = x$.*
2. *The sequence $\{x_n\}$ is said to be Cauchy sequence in (X, d_s) if for every $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $d_s(x_n, x_{n+p}) < \epsilon$ for all $n > n_0, p > 0$ or equivalently, if $\lim_{n\to\infty} d_s(x_n, x_{n+p}) = 0$, for all $p > 0$.*
3. *(X, d_s) is said to be a complete strong $b_v(s)$-metric space if every Cauchy sequence in X converge to some $x \in X$.*

The following two lemmas are new, but analogous to the corresponding lemmas from the [14], for the $b_v(s)$-metric space. Consequently, the proofs of these lemmas are short and very similar to the lemmas from [14].

Lemma 2.3. *Let (X, d_s) be a strong $b_v(s)$-metric space, $T : X \to X$, and let $\{x_n\}$ be a sequence in X defined by $x_0 \in X$ and $x_{n+1} = Tx_n$, such that $x_n \neq x_{n+1}$, $(n \geq 0)$. Suppose that $\lambda \in [0, 1)$ and*

$$d_s(x_{n+1}, x_n) \leq \lambda d_s(x_n, x_{n-1}), \tag{15}$$

for all $n \in N$. Then, $x_n \neq x_m$ for all distinct $n, m \in N$.

Lemma 2.4. *Let (X, d_s) be a strong $b_v(s)$-metric space and let $\{x_n\}$ be a sequence in X such that x_n $(n \geq 0)$ are all different. Suppose that $\lambda \in [0, 1)$ and c_1, c_2 are real nonnegative numbers such that*

$$d_s(x_m, x_n) \leq \lambda d_s(x_{m-1}, x_{n-1}) + c_1\lambda^m + c_2\lambda^n, \tag{16}$$

for all $m, n \in N$. Then $\{x_n\}$ is Cauchy sequence.

In this paper, we give a proof of the Banach contraction principle in strong $b_v(s)$-metric spaces.

Theorem 2.5. *(The Banach contraction principle in strong $b_v(s)$-metric spaces) Let (X, d_s) be a complete strong $b_v(s)$-metric space and $T : X \to X$ be a mapping satisfying*

$$d_s(Tx, Ty) \leqslant \lambda d_s(x, y) \tag{17}$$

for all $x, y \in X$, where $\lambda \in [0, 1)$. Then T has a unique fixed point.

Proof. Let $x_0 \in X$ be arbitrary point. We define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for all $n \geqslant 0$.

Case I. If $x_n = x_{n+1}$ then x_n is fixed point of T and the proof holds.

Case II. Suppose that $x_n \neq x_{n+1}$ for all $n \geqslant 0$. From Lemma 2.3 we have $x_n \neq x_m$, for all distinct $n, m \in \mathbb{N}$. (15) we obtain

$$d_s(x_m, x_n) \leqslant \lambda d_s(x_{m-1}, x_{n-1}).$$

From Lemma 2.4 we obtain that $\{x_n\}$ is a Cauchy sequence in X. As (X, d_s) is complete space, there exists $x^* \in X$ scuh that

$$\lim_{n\to\infty} x_n = x^*.$$

Next we will proove that x^* is the unique fixed point of T. For any $n \in \mathbb{N}$ the following holds:

$$\begin{aligned} d_s(x^*, Tx^*) &\leqslant d_s(x^*, x_{n+1}) + d_s(x_{n+1}, x_{n+2}) + d_s(x_{n+2}, x_{n+3}) + \ldots + d_s(x_{n+\nu-2}, x_{n+\nu-1}) \\ &+ d_s(x_{n+\nu-1}, x_{n+\nu}) + s \cdot d_s(x_{n+\nu}, Tx^*) \\ &\leqslant d_s(x^*, x_{n+1}) + d_s(x_{n+1}, x_{n+2}) + d_s(x_{n+2}, x_{n+3}) + \ldots + d_s(x_{n+\nu-2}, x_{n+\nu-1}) \\ &+ d_s(x_{n+\nu-1}, x_{n+\nu}) + s \cdot d_s(Tx_{n+\nu-1}, Tx^*) \\ &\leqslant d_s(x*, x_{n+1}) + d_s(x_{n+1}, x_{n+2}) + d_s(x_{n+2}, x_{n+3}) + \ldots + d_s(x_{n+\nu-2}, x_{n+\nu-1}) \\ &+ d_s(x_{n+\nu-1}, x_{n+\nu}) + s\lambda \cdot d_s(x_{n+\nu-1}, x*) \end{aligned}$$

Since, $\lim_{n\to\infty} d_s(x^*, x_n) = 0$ and $\lim_{n\to\infty} d_s(x_n, x_{n+1}) = 0$, we have

$$d_s(x^*, Tx^*) = 0 \text{ i.e., } Tx^* = x^*.$$

Now, we will show that x^* is unique. Let us assume that there is another fixed point $\overline{x}$.

$$d_s(\overline{x}, x^*) = d_s(T\overline{x}, Tx^*) \leqslant \lambda d_s(\overline{x}, x^*) < d_s(\overline{x}, x^*).$$

Since $\lambda < 1$, it must be $d_s(\overline{x}, x^*) = 0$, i.e., $\overline{x} = x^*$.

Remark 2.6. *If $\nu = 1$ and $s = 1$ from Theorem 2.5 we obtain a Banach fixed point theorem in metric spaces.*

Remark 2.7. *If $\nu = 1$ from Theorem 2.5 we obtain a Banach fixed point theorem. (See Theorem 12.4. in [12]).*

The Banach contraction theorem is one of the most significant results in nonlinear analysis and applied mathematics. Numerous algorithms and mathematical methods have been developed using this principle, such as solutions to various types of equations: algebraic, differential, fractional differential equations, and integral ([1],[21]). In this paper, we present generalization of the classical Reich contraction by involving two operators on a metric space rather than a single map. We prove theorem regarding the existence of common fixed points when mutual contraction relations exists between operators. Furthermore, we extend this results to a family of operators. The following definition is a generalization of the Reich contraction principle, named mutually Reich contraction principle.

Definition 2.8. *Let $T_1, T_2 : X \to X$ and (X, d_s) be a strong $b_v(s)$-metric space. T_1, T_2 are mutually Reich contractive if exists $\alpha, \beta, \gamma \geqslant 0$, $\alpha + \beta + \gamma < 1$ and $\max\{\beta, \gamma\} < \frac{1}{s}$ such that for all $x, y \in X$*

$$d_s(T_1x, T_2y) \leqslant \alpha d_s(x, y) + \beta d_s(x, T_1x) + \gamma d_s(y, T_2y). \tag{18}$$

From [18] (Theorem 1.) we obtain the following variant in $b_v(s)$-metric spaces.

Theorem 2.9. *Let (X, d_s) be a complete strong $b_v(s)$-metric space and let $T_1, T_2 : X \to X$ be a mapping satisfying:*

$$d_s(T_1x, T_2y) \leqslant \alpha d_s(x, y) + \beta d_s(x, T_1x) + \gamma d_s(y, T_2y),$$

for all $x, y \in X$, where α, β, γ are nonnegative constants with $\alpha + \beta + \gamma < 1$ and $\max\{\beta, \gamma\} < \frac{1}{s}$. Then, T_1, T_2 have a unique common fixed point.

Proof. Let $x \in X$. Let us define sequence

$$x_0 = x,\ x_1 = T_1x_0, x_2 = T_2x_1, x_3 = T_1x_2, x_4 = T_2x_3, \ldots.$$

Then, according to the definition of mutual Reich contractivity (18),

$$d_s(x_1, x_2) = d_s(T_1x_0, T_2x_1) \leqslant \alpha d_s(x_0, x_1) + \beta d_s(x_0, T_1x_0) + \gamma d_s(x_1, T_2x_1) = \alpha d_s(x_0, x_1) + \beta d_s(x_0, x_1) + \gamma d_s(x_1, x_2).$$

Therefore, we obtained $d_s(x_1, x_2) \leqslant \dfrac{\alpha + \beta}{1 - \gamma} d_s(x_0, x_1)$. Consequently, we obtained $d_s(x_2, x_3) \leqslant \dfrac{\alpha + \gamma}{1 - \beta} d_s(x_1, x_2)$. Now, let

$$\lambda = \max\left\{\frac{\alpha + \beta}{1 - \gamma}, \frac{\alpha + \gamma}{1 - \beta}\right\} < 1.$$

Thus, $d_s(x_n, x_{n+1}) \leqslant \lambda^n d_s(x_0, x_1)$. We show that $\{x_n\}$ is a Cauchy sequence. Suppose that $x_n \neq x_{n+1}$ for some $n \geqslant 0$, then from Lemma 2.3 we obtain $x_n \neq x_m$ for all distinct $n, m \in \mathbb{N}$. Now,

$$\begin{aligned} d_s(x_m, x_n) &\leqslant \alpha d_s(x_{m-1}, x_{n-1}) + \beta d_s(x_{m-1}, x_m) + \gamma d_s(x_{n-1}, x_n) \\ &\leqslant \alpha d_s(x_{m-1}, x_{n-1}) + \beta \lambda^{m-1} d_s(x_0, x_1) + \gamma \lambda^{n-1} d_s(x_0, x_1) \\ &= \alpha d_s(x_{m-1}, x_{n-1}) + (\beta \lambda^{m-1} + \gamma \lambda^{n-1}) d_s(x_0, x_1) \end{aligned}$$

Since $\sum_{k=0}^{\infty} \lambda^k$ is convergent, than $\{x_n\}$ is a Cauchy sequence. By completeness of (X, d_s) there exists $x^* \in X$ such that

$$\lim_{n\to\infty} x_n = x^*.$$

We now show that x^* is a common fixed point of T_1 and T_2. For instance, for $n \in \mathbb{N}$, $(n + \nu)$ even,

$$\begin{aligned} d_s(x^*, T_1x^*) &\leqslant d_s(x^*, x_{n+1}) + d_s(x_{n+1}, x_{n+2}) + \ldots + d_s(x_{n+\nu-2}, x_{n+\nu-1}) + d_s(x_{n+\nu-1}, x_{n+\nu}) + s \cdot d_s(x_{n+\nu}, T_1x^*) \\ &\leqslant d_s(x^*, x_{n+1}) + d_s(x_{n+1}, x_{n+2}) + \ldots + d_s(x_{n+\nu-2}, x_{n+\nu-1}) + d_s(x_{n+\nu-1}, x_{n+\nu}) + s \cdot d_s(T_2x_{n+\nu-1}, T_1x^*) \\ &\leqslant d_s(x^*, x_{n+1}) + d_s(x_{n+1}, x_{n+2}) + \ldots + d_s(x_{n+\nu-1}, x_{n+\nu}) + s(\alpha d_s(x_{n+\nu-1}, x^*) + \beta d_s(x_{n+\nu-1}, T_2x_{n+\nu-1}) \\ &\quad + \gamma d_s(x^*, T_1x^*)) \\ &\leqslant d_s(x^*, x_{n+1}) + d_s(x_{n+1}, x_{n+2}) + \ldots + d_s(x_{n+\nu-1}, x_{n+\nu}) + s(\alpha d_s(x_{n+\nu-1}, x^*) + \beta d_s(x_{n+\nu-1}, x_{n+\nu}) + \gamma d_s(x^*, T_1x^*)) \end{aligned}$$

$$(1 - s\gamma)d_s(x^*, T_1x^*) \leqslant d_s(x^*, x_{n+1}) + d_s(x_{n+1}, x_{n+2}) + \ldots + d_s(x_{n+\nu-1}, x_{n+\nu}) + s(\alpha d_s(x_{n+\nu-1}, x^*) + \beta d_s(x_{n+\nu-1}, x_{n+\nu}))$$

Since, $\lim_{n\to\infty} d_s(x^*, x_n) = 0$, $\lim_{n\to\infty} d_s(x_n, x_{n+1}) = 0$, and $\max\{\beta, \gamma\} < \frac{1}{s}$, we have

$$d_s(x^*, T_1x^*) = 0 \text{ i.e., } T_1x^* = x^*.$$

The same analysis can be performed for T_2 taking an odd natural $(n + \nu)$. Now, we will show that x^* is unique. Let us assume that there is another common fixed point $\overline{x}$.

$$d_s(\overline{x}, x^*) = d_s(T_1\overline{x}, T_2x^*) \leqslant \alpha d_s(\overline{x}, x^*) + \beta d_s(\overline{x}, T_1\overline{x}) + \gamma d_s(x^*, T_2x^*) = \alpha d_s(\overline{x}, x^*).$$

Since $\alpha < 1$, $d_s(\overline{x}, x^*) = 0$, i.e., $\overline{x} = x^*$.

Remark 2.10. *Taking $T_1 = T_2 = T$ for $b_v(s)$-metric space we obtain Theorem 2.1. from [14].*

Remark 2.11. *Taking $\nu = 1$ for strong b-metric space we obtain Theorem 1. in [18].*

Definition 2.12. *Let $T_1, T_2 : X \to X$ and (X, d_s) be a strong $b_v(s)$-metric space. T_1, T_2 are mutually Banach contractive if exists $\alpha \in [0, 1)$, such that for all $x, y \in X$*

$$d_s(T_1x, T_2y) \leqslant \alpha d_s(x, y). \tag{19}$$

Definition 2.13. *Let $T_1, T_2 : X \to X$ and (X, d_s) be a strong $b_v(s)$-metric space. T_1, T_2 are mutually Kannan contractive if exists $\beta, \gamma \geqslant 0$, $\beta + \gamma < 1$, such that for all $x, y \in X$*

$$d_s(T_1x, T_2y) \leqslant \beta d_s(x, T_1x) + \gamma d_s(y, T_2y). \tag{20}$$

Remark 2.14. *The definition of a mutual Reich contraction (Definition 2.8) generalizes that of mutual Banach contraction Definition 2.12 for $\beta = 0, \gamma = 0$ and mutual Kannan contraction Definition 2.13 by taking $\alpha = 0$.*

Theorem 2.15. *Let (X, d_s) be a complete strong $b_v(s)$-metric space and $T_1, T_2 : X \to X$ are mutually Banach contractive, then T_1, T_2 have a unique common fixed point.*

Remark 2.16. *A proof is analogue to the proof of Theorem 2.9 taking $\beta = 0, \gamma = 0$. If $T_1 = T_2 = T$ in previous, we obtain Theorem 2.5.*

Theorem 2.17. *Let (X, d_s) be a complete strong $b_v(s)$-metric space and $T_1, T_2 : X \to X$ are mutually Kannan contractive, then T_1, T_2 have a unique common fixed point.*

Remark 2.18. *A proof is analogue to the proof of Theorem 2.9 taking $\alpha = 0$.*

In the following results, we provide sufficient conditions for the existence of a common fixed point of a set operators satisfying mutual relations of Reich type.

Definition 2.19. *Let* (X, d_s) *be a strong* $b_v(s)$*-metric space and* $\mathcal{F} = \{T_i : X \to X; i \in \mathcal{I}\}$*. Then,* $x^* \in X$ *is a fixed point of* $\mathcal{F}$ *if* $T_i(x^*) = x^*$*, for all* $i \in \mathcal{I}$*.*

To investigate the behavior of sequences generated by a family of operators, we consider the iterative scheme defined as for all $k \geqslant 1$, $x_0 \in X$, $T_{i_k} \in \mathcal{F}$, consider the iterative scheme

$$x_k = T_{i_k} x_{k-1}. \tag{21}$$

This framework is highly popular and used within non-autonomous discrete dynamical systems. To our knowledge, there is extensive literature on autonomous dynamical systems, specifically of the form $x_{k+1} = Tx_k$. This model is very general and suitable for a wide range of algorithms that address significant problems in applied mathematics.

Definition 2.20. *Point* $x \in X$ *is a global attractor for the scheme (21), if*

$$\lim_{n\to\infty} \omega_n(x) = x^*,$$

for all $x \in X$*, where* $\omega_n := T_{i_n} \circ T_{i_{n-1}} \circ \ldots \circ T_{i_1} \circ T_{i_1}$*.*

Theorem 2.21. *Let* (X, d_s) *be a complete strong* $b_v(s)$*-metric space and* $\mathcal{F} = \{T_i : X \to X; i \in \mathcal{I}\}$*. If there exists* $i_0 \in \mathcal{I}$ *such that for all* $i \in \mathcal{I}$*,* T_i, T_{i_0} *are mutually Reich contractive with constants* $\alpha_i, \beta_i, \gamma_i$ *such that* $\sup_i \beta_i < \frac{1}{s}$*,* $\sup_i \gamma_i < \frac{1}{s}$*, then the following hold:*

1. $\mathcal{F}$ *has a unique fixes point* $x^* \in X$*.*
2. x^* *is the only fixed point of each* T_i*, for all* $i \in \mathcal{I}$*.*

Proof. According to Theorem 2.9 (for $T_1 = T_2$), since T_{i_0} is a Reich contraction, it has a unique fixed point $x^* \in X$, $T_{i_0}(x^*) = x^*$. Let us examine if this element is a fixed point of every T_i. The definition of mutual Reich contraction 18 implies that

$$d_s(T_i(x^*), T_{i_0}(x^*)) \leqslant \beta_i d_s(x^*, T_i(x^*)) + \gamma_i d_s(x^*, T_{i_0}(x^*)),$$

then

$$d_s(T_i(x^*), x^*) \leqslant \beta_i d_s(x^*, T_i(x^*)).$$

Since $\beta_i < 1$, then $d_s(T_i(x^*), x^*) = 0$ and x^* is a fixed point of any T_i. Let us prove now that x^* is the only fixed point of T_i. If there were another fixed point, $T_i(\overline{x}) = \overline{x}$,

$$d_s(\overline{x}, T_{i_0}(\overline{x})) = d_s(T_i(\overline{x}), T_{i_0}(\overline{x})) \leqslant \gamma_i d_s(\overline{x}, T_{i_0}(\overline{x})),$$

and $\gamma_i < 1$, then $\overline{x}$ would be another fixed point of T_{i_0} and consequently $\overline{x} = x^*$.

Theorem 2.22. *Let* (X, d_s) *be a complete strong* $b_v(s)$*-metric space and* $\mathcal{F} = \{T_i : X \to X; i \in \mathcal{I}\}$*. If there exists* $i_0 \in \mathcal{I}$ *such that for all* $i \in \mathcal{I}$*,* T_i, T_j *are mutually Reich contractive with constants* $\alpha_{ij}, \beta_{ij}, \gamma_{ij}$ *such that* $\max\{\sup \beta_{ij}, \sup \gamma_{ij}\} < \frac{1}{s}$*, for any* $i, j \in \mathcal{I}$*, then* $\mathcal{F}$ *has a unique fixed point* x^* *that is a global attractor for any scheme of type (21).*

Proof. From Theorem 2.21, $\mathcal{F}$ has a unique fixed point x^*. For any $x \in X$, let us define

$$x_0 = x,\ x_1 = T_{i_1}(x_0), x_2 = T_{i_2}(x_1), \ldots, x_n = T_{i_n}(x_{n-1}).$$

Then,

$$d_s(x_1, x_2) = d_s(T_{i_1}(x_0), T_{i_2}(x_1)) \leqslant \alpha_{i_1 i_2} d_s(x_0, x_1) + \beta_{i_1 i_2} d_s(x_0, T_{i_1}(x_0)) + \gamma_{i_1 i_2} d_s(x_1, T_{i_2}(x_1))$$

and

$$d_s(x_1, x_2) \leqslant \alpha_{i_1 i_2} d_s(x_0, x_1) + \beta_{i_1 i_2} d_s(x_0, x_1) + \gamma_{i_1 i_2} d_s(x_1, x_2).$$

Let $\alpha := \sup \alpha_{ij}$, $\beta := \sup \beta_{ij}$, $\gamma := \sup \gamma_{ij}$, then

$$d_s(x_1, x_2) \leqslant \frac{\alpha + \beta}{1 - \gamma} d_s(x_0, x_1).$$

Iteretively, just like in Theorem 2.9, we obtain

$$d_s(x_n, x_{n+1}) \leqslant \lambda^n d_s(x_0, x_1), \text{ where } \lambda := \max\left\{\frac{\alpha + \beta}{1 - \gamma}, \frac{\alpha + \gamma}{1 - \beta}\right\} < 1.$$

Similar to the Theorem 2.9, we can get the required result.

3. Conclusion

In this paper, we introduced a new concept of a generalized metric space, named strong $b_v(s)$-metric space. First, we proved the existence and uniqueness of a fixed point using the Banach contraction. Then, we introduced the concept of mutual Reich contraction between operators T_1 and T_2 defined on a strong $b_v(s)$-metric space, which generalizes the structure of a metric space. Mutual Reich contractivity extends the notion of Reich contraction on a metric space from a single map to a pair of self-maps. When $T_1 = T_2$, this reduces to the classical Reich maps. We provided sufficient conditions for the existence of a common fixed point for T_1 and T_2 under mutual Reich contractivity. These results were then extended to a set $\mathcal{F}$ of operators, regardless of cardinality (finite or infinite). Additionally, we studied the convergence of iterative schemes of the form $x_k = T_{i_k} x_{k-1}$, where $x_k \in X$ and $T_{i_k} \in \mathcal{F}$. Under certain conditions, the common fixed point of $\mathcal{F}$ acts as a global attractor for these systems. In future research, the results of this study can be expanded to establish fixed points under various contraction conditions, emphasizing particularly significant insights into proximal point methods using optimization techniques, exploring an equilibrium point that generalizes the classical fixed point.

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