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New bounds on spectral radius and minimum eigenvalue of structured matrices

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Abstract. The eigenvalue problem of structured matrices has always been a significant research topic in matrix analysis. In this paper, we present a new upper bound on the spectral radius of nonnegative matrices involving the Hadamard product, which generalizes and improves some existing ones. For two $n \times n$ nonsingular M-matrices A and B, some new lower bounds for the minimum eigenvalue related to the Fan product of A and B are obtained, and meanwhile, the detailed analysis and theoretical comparison between the newly proposed lower bounds and some existing results are also investigated. These estimations that only depend on the entries of the given matrices are not difficult to implement. To further illustrate the effectiveness of the main results, some numerical examples are given.

1. Introduction

For a positive integer n, the letter N denotes the set $\{1, 2, \ldots, n\}$. We say that a matrix $A = (a_{ij})$ is nonnegative if all its entries are nonnegative, i.e., $a_{ij} \geq 0$ for all $i, j = 1, 2, \ldots, n$. An $n \times n$ matrix A is reducible if there exists an $n \times n$ permutation matrix P such that

$$P^T AP = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}$$

where A_{11} is an $r \times r$ $(1 \leq r < n)$ submatrix and A_{22} is an $(n-r) \times (n-r)$ submatrix. Otherwise, the matrix A is said to be irreducible. The Hadamard product of an $n \times n$ matrix $A = (a_{ij})$ and an $n \times n$ matrix $B = (b_{ij})$, denoted by $A \circ B$, can be defined as an $n \times n$ matrix with its entries $a_{ij}b_{ij}$ for all $i, j = 1, 2, \ldots, n$ (see [5]). In particular, $A^{[2]} = A \circ A$. The spectral radius of A is defined as $\rho(A) = \max\{|\lambda| : \lambda \in \sigma(A)\}$, where $\sigma(A)$ denotes the set of all eigenvalues of A. As the special class of matrix eigenvalue, the estimate on the spectral radius for the Hadamard product of nonnegative matrices is one of the important research contents in matrix theory due to its extensive theoretical and practical prospects [5].

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Given any two $n \times n$ nonnegative matrices $A = (a_{ij})$ and $B = (b_{ij})$. Two classical inequalities for the spectral radius of the Hadamard product of nonnegative matrices A and B are independently proposed in [5] and [3], and we restate them below:

$$\rho(A \circ B) \leq \rho(A)\rho(B) \tag{1.1}$$

and

$$\rho(A \circ B) \leq \max_{i \in N}\{2a_{ii}b_{ii} + \rho(A)\rho(B) - a_{ii}\rho(B) - b_{ii}\rho(A)\}. \tag{1.2}$$

Obviously, the inequalities (1.1) and (1.2) are simple and beautiful in form and structure, and the upper bound for $\rho(A \circ B)$ in the inequality (1.2) is sharper than the one in the inequality (1.1), but they may be relatively rough in concrete applications.

Define $J_A = D_1^{-1}L_A$ with $D_1 = diag(d_{11}, d_{22}, \ldots, d_{nn})$ and $J_B = D_2^{-1}L_B$ with $D_2 = diag(s_{11}, s_{22}, \ldots, s_{nn})$, where

$$d_{ii} = \begin{cases} a_{ii}, & \text{if } a_{ii} \neq 0, \\ 1, & \text{if } a_{ii} = 0, \end{cases} \qquad s_{ii} = \begin{cases} b_{ii}, & \text{if } b_{ii} \neq 0, \\ 1, & \text{if } b_{ii} = 0, \end{cases}$$

and $L_A = A - D_A$ with $D_A = diag(a_{ii})$, $L_B = B - D_B$ with $D_B = diag(b_{ii})$. According to the different cases between the zero and the diagonal elements of two nonnegative matrices A and B, Huang [7] established some inequalities of the spectral radius $\rho(A \circ B)$ for two nonnegative matrices A and B, which can be described as follows:

(1) If $a_{ii}b_{ii} \neq 0$ for all $i \in N$, then

$$\rho(A \circ B) \leq \max_{i \in N}\{a_{ii}b_{ii} + a_{ii}b_{ii}\rho(J_A)\rho(J_B)\}. \tag{1.3}$$

(2) If $a_{i_0i_0} \neq 0$ or $b_{i_0i_0} \neq 0$ for some i_0, but $a_{ii}b_{ii} = 0$ for all $i \in N$, then

$$\rho(A \circ B) \leq \rho(J_A)\rho(J_B)\max_{i \in N}\{a_{ii}, b_{ii}\}. \tag{1.4}$$

(3) If $a_{ii} = 0$ and $b_{ii} = 0$ for all $i \in N$, then

$$\rho(A \circ B) \leq \rho(J_A)\rho(J_B). \tag{1.5}$$

(4) If $a_{i_0i_0}b_{i_0i_0} \neq 0$ and $a_{j_0j_0}b_{j_0j_0} = 0$ for some $i_0, j_0 \in N$, then the upper bound on $\rho(A \circ B)$ is the maximum value of the upper bounds of (1.3)-(1.5).

In [8], Liu and Chen applied the inequality for the Perron root of nonnegative matrices, and then presented the following inequality (1.6) depicting the upper bound of the spectral radius $\rho(A \circ B)$ for two nonnegative matrices $A = (a_{ij})$ and $B = (b_{ij})$:

$$\rho(A \circ B) \leq \max_{\substack{i,j \in N, \\ i \neq j}} \frac{1}{2}\Big\{a_{ii}b_{ii} + a_{jj}b_{jj} + \Big((a_{ii}b_{ii} - a_{jj}b_{jj})^2 + 4(\rho(A) - a_{ii})(\rho(B) - b_{ii}) \\ (\rho(A) - a_{jj})(\rho(B) - b_{jj})\Big)^{\frac{1}{2}}\Big\}. \tag{1.6}$$

In [9], Liu et al. provided some improved upper bounds for the spectral radius $\rho(A \circ B)$ of two nonnegative matrices $A = (a_{ij})$ and $B = (b_{ij})$, which are recalled as follows:

(1) If $a_{ii}b_{ii} \neq 0$ for all $i \in N$, then

$$\rho(A \circ B) \quad \leq \max_{\substack{i,j \in N, \\ i \neq j}} \tfrac{1}{2}\Big\{a_{ii}b_{ii} + a_{jj}b_{jj} + \Big((a_{ii}b_{ii} - a_{jj}b_{jj})^2 + 4a_{ii}b_{ii}a_{jj}b_{jj}\rho^2(J_A)\rho^2(J_B)\Big)^{\frac{1}{2}}\Big\}. \tag{1.7}$$

(2) If $a_{i_0i_0} \neq 0$ or $b_{i_0i_0} \neq 0$ for some i_0, but $a_{ii}b_{ii} = 0$ for all $i \in N$, then

$$\rho(A \circ B) \leq \rho(J_A)\rho(J_B)\max_{i \in N}\{\sqrt{a_{ii}a_{jj}}, \sqrt{b_{ii}b_{jj}}\}. \tag{1.8}$$

(3) If $a_{ii} = 0$ and $b_{ii} = 0$ for all $i \in N$, then the upper bound for $\rho(A \circ B)$ is the same as the upper bound in (1.5).

(4) If $a_{i_0i_0}b_{i_0i_0} \neq 0$ and $a_{j_0j_0}b_{j_0j_0} = 0$ for some $i_0, j_0 \in N$, then the upper bound on $\rho(A \circ B)$ is the maximum value of the upper bounds of (1.5) and (1.7)-(1.8).

Consider an $n \times n$ nonnegative matrix $A = (a_{ij})$, we define the following notations

$$R_i = \sum_{k \in N, k \neq i} |a_{ik}|,\ d_i = \frac{R_i}{|a_{ii}|}, m_{ji} = a_{ji}h_j, m_i = \max_{j \neq i}\{m_{ji}\}, h_j = \begin{cases} d_j, & d_j \neq 0, \\ 1, & d_j = 0, \end{cases}$$

for all $i, j \in N$, which are useful in later analysis. With the help of the notations defined above, Li et al. [12] derived an upper bound of the spectral radius $\rho(A \circ B)$ for nonnegative matrices A and B, which is the following theorem.

Theorem 1.1. [12] *Let $A = (a_{ij})$ and $B = (b_{ij})$ be two $n \times n$ nonnegative matrices. Then*

$$\rho(A \circ B) \leq \max_{i \in N} \left\{ a_{ii}b_{ii} + m_i \sum_{k \in N, k \neq i} \frac{|b_{ki}|}{h_k} \right\}. \tag{1.9}$$

Subsequently, Zhou and Li [15] obtained the following new upper bound for the spectral radius $\rho(A \circ B)$, which generalized the inequality (1.9) in Theorem 1.1.

Theorem 1.2. [15] *Let $A = (a_{ij})$ and $B = (b_{ij})$ be two $n \times n$ nonnegative matrices, $0 \leq \alpha \leq 1$. Then*

$$\rho(A \circ B) \leq \max_{i \in N} \left\{ a_{ii}b_{ii} + \left(\frac{1}{m_i} \sum_{k \in N, k \neq i} m_k |a_{ik}b_{ik}| \right)^{\alpha} \left(m_i \sum_{k \in N, k \neq i} \frac{|b_{ki}|}{h_k} \right)^{1-\alpha} \right\}. \tag{1.10}$$

Recently, Li and Hai [10] proposed two improved inequalities to characterize the upper bound of the spectral radius $\rho(A \circ B)$ for two nonnegative matrices $A = (a_{ij})$ and $B = (b_{ij})$, and one of the two inequalities is restated as below:

(1) If $a_{ii}b_{ii} \neq 0$ for all $i \in N$, then

$$\rho(A \circ B) \leq \max_{i \in N} \left\{ a_{ii}b_{ii} + a_{ii}b_{ii}\rho^{\frac{1}{2}}\left(J_A^{[2]}\right)\rho^{\frac{1}{2}}\left(J_B^{[2]}\right) \right\}. \tag{1.11}$$

(2) If $a_{i_0i_0} \neq 0$ or $b_{i_0i_0} \neq 0$ for some i_0, but $a_{ii}b_{ii} = 0$ for all $i \in N$, then

$$\rho(A \circ B) \leq \rho^{\frac{1}{2}}\left(J_A^{[2]}\right)\rho^{\frac{1}{2}}\left(J_B^{[2]}\right) \max_{i \in N}\{a_{ii}, b_{ii}\}. \tag{1.12}$$

(3) If $a_{ii} = 0$ and $b_{ii} = 0$ for all $i \in N$, then

$$\rho(A \circ B) \leq \rho^{\frac{1}{2}}\left(J_A^{[2]}\right)\rho^{\frac{1}{2}}\left(J_B^{[2]}\right). \tag{1.13}$$

(4) If $a_{i_0i_0}b_{i_0i_0} \neq 0$ and $a_{j_0j_0}b_{j_0j_0} = 0$ for some $i_0, j_0 \in N$, then the upper bound on $\rho(A \circ B)$ is the maximum value of the upper bounds of (1.11)-(1.13).

Also in [10], the following inequalities gave some upper bounds of the spectral radius $\rho(A \circ B)$ for two nonnegative matrices $A = (a_{ij})$ and $B = (b_{ij})$, which theoretically improve ones in (1.11)-(1.13).

(1) If $a_{ii}b_{ii} \neq 0$ for all $i \in N$, then

$$\rho(A \circ B) \leq \max_{\substack{i,j \in N,\\ i \neq j}} \frac{1}{2} \left\{ a_{ii}b_{ii} + a_{jj}b_{jj} + \left((a_{ii}b_{ii} - a_{jj}b_{jj})^2 + 4a_{ii}b_{ii}a_{jj}b_{jj}\rho\left(J_A^{[2]}\right)\rho\left(J_B^{[2]}\right) \right)^{\frac{1}{2}} \right\}. \tag{1.14}$$

(2) If $a_{i_0i_0} \neq 0$ or $b_{i_0i_0} \neq 0$ for some i_0, but $a_{ii}b_{ii} = 0$ for all $i \in N$, then

$$\rho(A \circ B) \leq \rho^{\frac{1}{2}}\left(J_A^{[2]}\right)\rho^{\frac{1}{2}}\left(J_B^{[2]}\right) \max_{i \in N}\{\sqrt{a_{ii}a_{jj}}, \sqrt{b_{ii}b_{jj}}\}. \tag{1.15}$$

(3) If $a_{ii} = 0$ and $b_{ii} = 0$ for all $i \in N$, then the upper bound for $\rho(A \circ B)$ is the same as the upper bound in (1.13).

(4) If $a_{i_0i_0}b_{i_0i_0} \neq 0$ and $a_{j_0j_0}b_{j_0j_0} = 0$ for some $i_0, j_0 \in N$, then the upper bound on $\rho(A \circ B)$ is the maximum value of the upper bounds of (1.13)-(1.15).

In 1964, Ky Fan [4] introduced the definition of the Fan product for matrices, which is a useful concept in investigating the bound for the minimum eigenvalue of nonsingular M-matrices. The Fan product of an $n \times n$ matrix $A = (a_{ij})$ and an $n \times n$ matrix $B = (b_{ij})$ is defined as an $n \times n$ matrix $A \star B = (c_{ij})$, where

$$c_{ij} = \begin{cases} -a_{ij}b_{ij}, & i \neq j, \\ a_{ij}b_{ij}, & i = j. \end{cases}$$

We say that a matrix A is a Z-matrix if all of its off-diagonal entries are nonpositive. An $n\times n$ matrix A is called an M-matrix if there exists an $n \times n$ nonnegative matrix B and a real number $\eta \geq \rho(B)$ such that $A = \eta I - B$, where I denotes the $n \times n$ identity matrix. If $\eta > \rho(B)$, then the matrix A is a nonsingular M-matrix [1]. Under the Fan product of matrices, two M-matrices are closed [5]. Denote $\tau(A) = \min\{Re(\lambda) : \lambda \in \sigma(A)\}$. It should be pointed out that the minimum eigenvalue for the Fan product of two nonsingular M-matrices has caused the concern and attention of many scholars. The well-known inequality of the minimum eigenvalue $\tau(A \star B)$ for two $n \times n$ nonsingular M-matrices A and B, which can be found in [5], is obtained as follows:

$$\tau(A \star B) \geq \tau(A)\tau(B). \tag{1.16}$$

Observing the inequality (1.16), we find in form that there are some similarities with the inequality (1.1), which motivates ones to investigate the minimum eigenvalue for the Fan product of two nonsingular M-matrices by analogy with the results on the spectral radius of the Hadamard product of two nonnegative matrices. For an $n \times n$ nonsingular M-matrix $A = (a_{ij})$, we define $J'_A = D^{-1}N$, where $N = D - A$ and $D = diag(a_{11}, a_{22}, \ldots, a_{nn})$. Based on the above facts, the inequalities (1.17)-(1.20) depicting the lower bound of the minimum eigenvalue $\tau(A \star B)$ for two $n \times n$ nonsingular M-matrices $A = (a_{ij})$ and $B = (b_{ij})$ were given as follows, and the readers can refer to the relevant literature [3, 7–9].

$$\tau(A \star B) \geq \min_{i\in N}\{a_{ii}\tau(B) + b_{ii}\tau(A) - \tau(A)\tau(B)\}, \tag{1.17}$$

$$\tau(A \star B) \geq \left(1 - \rho(J'_A)\rho(J'_B)\right)\min_{i\in N}\{a_{ii}b_{ii}\}, \tag{1.18}$$

$$\begin{aligned}\tau(A \star B) \geq \min_{\substack{i,j\in N,\\ i\neq j}} \frac{1}{2}\Big\{&a_{ii}b_{ii} + a_{jj}b_{jj} - \Big((a_{ii}b_{ii} - a_{jj}b_{jj})^2 + 4(a_{ii} - \tau(A))(b_{ii} - \tau(B))\\ &(a_{jj} - \tau(A))(b_{jj} - \tau(B))\Big)^{\frac{1}{2}}\Big\}\end{aligned} \tag{1.19}$$

and

$$\tau(A \star B) \quad \geq \min_{\substack{i,j\in N,\\ i\neq j}} \tfrac{1}{2}\Big\{a_{ii}b_{ii} + a_{jj}b_{jj} - \Big((a_{ii}b_{ii} - a_{jj}b_{jj})^2 + 4a_{ii}b_{ii}a_{jj}b_{jj}\rho^2(J'_A)\rho^2(J'_B)\Big)^{\frac{1}{2}}\Big\}. \tag{1.20}$$

In [12], Li et al. presented a new lower bound of the minimum eigenvalue $\tau(A \star B)$ for two $n \times n$ nonsingular M-matrices A and B, which is deduced from the weighted Geršgorin localization set of matrices.

Theorem 1.3. [12] *Let $A = (a_{ij})$ and $B = (b_{ij})$ be two $n \times n$ nonsingular M-matrices. Then*

$$\tau(A \star B) \geq \min_{i\in N}\left\{a_{ii}b_{ii} - m_i \sum_{k\in N, k\neq i} \frac{|b_{ki}|}{h_k}\right\}. \tag{1.21}$$

To theoretically improve the inequalities (1.18) and (1.20), two inequalities on the minimum eigenvalue $\tau(A \star B)$ for two $n \times n$ nonsingular M-matrices A and B were recently derived in [11], and in the following, we revisit these result.

$$\tau(A \star B) \geq \min_{i \in N} \left\{ a_{ii}b_{ii} - a_{ii}b_{ii}\rho^{\frac{1}{2}}\left(J_A^{[2]}\right)\rho^{\frac{1}{2}}\left(J_B^{[2]}\right) \right\} \tag{1.22}$$

and

$$\tau(A \star B) \geq \min_{\substack{i,j \in N, \\ i \neq j}} \frac{1}{2}\left\{ a_{ii}b_{ii} + a_{jj}b_{jj} - \left((a_{ii}b_{ii} - a_{jj}b_{jj})^2 + 4a_{ii}b_{ii}a_{jj}b_{jj}\rho\left(J_A^{[2]}\right)\rho\left(J_B^{[2]}\right)\right)^{\frac{1}{2}} \right\}. \tag{1.23}$$

The purpose of this paper is continue to focus on the spectral estimate for the Hadamard product and the Fan product of matrices. In Section 2, we present an improved upper bound on the spectral radius for the Hadamard product of nonnegative matrices, and establish the theoretical comparison between the newly proposed upper bound and the existing result in [15]. Secondly, the lower bounds of the minimum eigenvalue $\tau(A \star B)$ for two nonsingular M-matrices A and B are obtained, and then the detailed analysis and theoretical comparison between the newly proposed lower bounds for the minimum eigenvalue $\tau(A \star B)$ are also discussed. The above facts are well elaborated in Section 3. Finally, the effectiveness and validity of the obtained results can be well verified by some numerical examples in Section 4.

2. The upper bound for the spectral radius of the Hadamard product of nonnegative matrices

Before proving the main results of this section, we first collect some existing results as the following lemmas which are useful for later discussion and investigation.

Lemma 2.1. [6] *Let $A = (a_{ij})$ be an $n \times n$ irreducible nonnegative matrix. Then the spectral radius $\rho(A)$ is a positive eigenvalue with a positive eigenvector $x \in \mathbb{C}^n$, i.e., $x_i > 0$ for all $i \in [n]$, corresponding to it.*

Lemma 2.2. [14] *Let $A = (a_{ij})$ be an $n \times n$ matrix with $n > 2$, and $0 \leq \alpha \leq 1$. Then if $z \in \mathbb{C}$ is an eigenvalue of the matrix A, there exists a pair (i, j) of positive integers with $i \neq j$ $(i, j \in N)$ such that*

$$|z - a_{ii}||z - a_{jj}| \leq \left(\sum_{k \in N, k \neq i} |a_{ik}| \sum_{l \in N, l \neq j} |a_{jl}| \right)^{\alpha} \left(\sum_{k \in N, k \neq i} |a_{ki}| \sum_{l \in N, l \neq j} |a_{lj}| \right)^{1-\alpha}.$$

Based on the theory of diagonal similarity and Lemma 2.2, we easily obtain the following result.

Lemma 2.3. *Let $A = (a_{ij})$ be an $n \times n$ matrix with $n \geq 2$, $0 \leq \alpha \leq 1$, and $x_1, x_2, \ldots, x_n$ be n positive real numbers. If $z \in \mathbb{C}$ is an eigenvalue of the matrix A, then there exists a pair (i, j) of positive integers with $i \neq j (1 \leq i, j \leq n)$ such that*

$$|z - a_{ii}||z - a_{jj}| \leq \left(\frac{1}{x_i x_j} \sum_{k \in N, k \neq i} x_k |a_{ik}| \sum_{l \in N, l \neq j} x_l |a_{jl}| \right)^{\alpha} \left(x_i x_j \sum_{k \in N, k \neq i} \frac{|a_{ki}|}{x_k} \sum_{l \in N, l \neq j} \frac{|a_{lj}|}{x_l} \right)^{1-\alpha}.$$

Lemma 2.4. [5] *Let $A = (a_{ij})$ be an $n \times n$ nonnegative matrix. If the matrix A_k is a principal submatrix of the matrix A, then $\rho(A_k) \leq \rho(A)$. Especially, if the matrix A is irreducible and $A_k \neq A$, then $\rho(A_k) < \rho(A)$.*

Theorem 2.5. *Let $A = (a_{ij})$ and $B = (b_{ij})$ be two $n \times n$ nonnegative matrices, and $0 \leq \alpha \leq 1$. Then*

$$\rho(A \circ B) \leq \max_{\substack{i,j \in N, \\ i \neq j}} \frac{1}{2}\Bigg\{ a_{ii}b_{ii} + a_{jj}b_{jj} + \Bigg((a_{ii}b_{ii} - a_{jj}b_{jj})^2 + 4\Big(\frac{1}{m_i m_j} \sum_{k \in N, k \neq i} |a_{ik}b_{ik}| m_k \sum_{l \in N, l \neq j} |a_{jl}b_{jl}| m_l \Big)^{\alpha}$$
$$\Big(m_i m_j \sum_{k \in N, k \neq i} \frac{|b_{ki}|}{h_k} \sum_{l \in N, l \neq j} \frac{|b_{lj}|}{h_l} \Big)^{1-\alpha} \Bigg)^{\frac{1}{2}} \Bigg\}. \tag{2.1}$$

Proof. Case I. If $A \circ B$ is irreducible. Obviously, A and B are two nonnegative irreducible matrices, hence $h_k \neq 0$, $h_l \neq 0$, $m_i > 0$ and $m_j > 0$. According to Lemma 2.1, the spectral radius $\rho(A \circ B)$ is a positive eigenvalue of the matrix $A \circ B$. From Lemma 2.3, it follows that there is a pair (i, j) of positive integers with $i \neq j (1 \leq i, j \leq n)$ such that

$$\begin{aligned}
&|\rho(A \circ B) - a_{ii}b_{ii}||\rho(A \circ B) - a_{jj}b_{jj}| \\
\leq &\left(\frac{1}{m_i m_j} \sum_{k\in N, k\neq i} |a_{ik}b_{ik}| m_k \sum_{l\in N, l\neq j} |a_{jl}b_{jl}| m_l\right)^{\alpha} \left(m_i m_j \sum_{k\in N, k\neq i} \frac{|a_{ki}b_{ki}|}{m_k} \sum_{l\in N, l\neq j} \frac{|a_{lj}b_{lj}|}{m_l}\right)^{1-\alpha} \\
\leq &\left(\frac{1}{m_i m_j} \sum_{k\in N, k\neq i} |a_{ik}b_{ik}| m_k \sum_{l\in N, l\neq j} |a_{jl}b_{jl}| m_l\right)^{\alpha} \left(m_i m_j \sum_{k\in N, k\neq i} \frac{|a_{ki}b_{ki}|}{|a_{ki}| h_k} \sum_{l\in N, l\neq j} \frac{|a_{lj}b_{lj}|}{|a_{lj}| h_l}\right)^{1-\alpha} \\
= &\left(\frac{1}{m_i m_j} \sum_{k\in N, k\neq i} |a_{ik}b_{ik}| m_k \sum_{l\in N, l\neq j} |a_{jl}b_{jl}| m_l\right)^{\alpha} \left(m_i m_j \sum_{k\in N, k\neq i} \frac{|b_{ki}|}{h_k} \sum_{l\in N, l\neq j} \frac{|b_{lj}|}{h_l}\right)^{1-\alpha}.
\end{aligned} \tag{2.2}$$

By Lemma 2.4, the following is obvious:

$$\rho(A \circ B) - a_{ii}b_{ii} \geq 0 \text{ for all } i \in N$$

and

$$\rho(A \circ B) - a_{jj}b_{jj} \geq 0 \text{ for all } j \in N.$$

Hence, the inequality (2.2) can be equivalently written as the following:

$$\begin{aligned}
&(\rho(A \circ B) - a_{ii}b_{ii})(\rho(A \circ B) - a_{jj}b_{jj}) \\
\leq &\left(\frac{1}{m_i m_j} \sum_{k\in N, k\neq i} |a_{ik}b_{ik}| m_k \sum_{l\in N, l\neq j} |a_{jl}b_{jl}| m_l\right)^{\alpha} \left(m_i m_j \sum_{k\in N, k\neq i} \frac{|b_{ki}|}{h_k} \sum_{l\in N, l\neq j} \frac{|b_{lj}|}{h_l}\right)^{1-\alpha}.
\end{aligned} \tag{2.3}$$

The inequality (2.3) is a univariate quadratic inequality with the spectral radius $\rho(A \circ B)$ as the unknown quantity, and by solving the inequality (2.3), we obtain

$$\begin{aligned}
\rho(A \circ B) \leq &\frac{1}{2}\Bigg\{a_{ii}b_{ii} + a_{jj}b_{jj} + \Bigg((a_{ii}b_{ii} - a_{jj}b_{jj})^2 + 4\Big(\frac{1}{m_i m_j} \sum_{k\in N, k\neq i} |a_{ik}b_{ik}| m_k \sum_{l\in N, l\neq j} |a_{jl}b_{jl}| m_l\Big)^{\alpha} \\
&\Big(m_i m_j \sum_{k\in N, k\neq i} \frac{|b_{ki}|}{h_k} \sum_{l\in N, l\neq j} \frac{|b_{lj}|}{h_l}\Big)^{1-\alpha}\Bigg)^{\frac{1}{2}}\Bigg\} \\
\leq &\max_{\substack{i,j\in N,\\ i\neq j}} \frac{1}{2}\Bigg\{a_{ii}b_{ii} + a_{jj}b_{jj} + \Bigg((a_{ii}b_{ii} - a_{jj}b_{jj})^2 + 4\Big(\frac{1}{m_i m_j} \sum_{k\in N, k\neq i} |a_{ik}b_{ik}| m_k \sum_{l\in N, l\neq j} |a_{jl}b_{jl}| m_l\Big)^{\alpha} \\
&\Big(m_i m_j \sum_{k\in N, k\neq i} \frac{|b_{ki}|}{h_k} \sum_{l\in N, l\neq j} \frac{|b_{lj}|}{h_l}\Big)^{1-\alpha}\Bigg)^{\frac{1}{2}}\Bigg\}.
\end{aligned}$$

Case II. If $A \circ B$ is reducible, then there exists a positive number ϵ such that $A + \epsilon P$ and $B + \epsilon P$ are irreducible nonnegative matrices, where $P = (p_{ij})$ is a permutation matrix with $p_{12} = p_{23} = \ldots = p_{n-1,n} = p_{n,1} = 1$ and the remaining $p_{ij} = 0$, Now we substitute $A + \epsilon P$ and $B + \epsilon P$ for A and B in the irreducible case, respectively. Let $\epsilon \to 0$, the inequality (2.1) will be obtained by continuity. The proof is completed. □

Remark 2.6. (1) *Take $\alpha = 0$ in Theorem 2.5, we easily obtain*

$$\rho(A \circ B) \leq \max_{\substack{i,j\in N,\\ i\neq j}} \frac{1}{2}\left\{a_{ii}b_{ii} + a_{jj}b_{jj} + \left((a_{ii}b_{ii} - a_{jj}b_{jj})^2 + 4m_i m_j \sum_{k\in N, k\neq i} \frac{|b_{ki}|}{h_k} \sum_{l\in N, l\neq j} \frac{|b_{lj}|}{h_l}\right)^{\frac{1}{2}}\right\}, \tag{2.4}$$

which can also be given in [2, 16]. Therefore, it is not difficult to conclude that Theorem 2.5 can be regarded as the generalization of Theorem 3.1 in [2] and Theorem 2.2 in [16].

(2) *As shown in [2, 16], the inequality (2.4) theoretically improves the inequality (1.9) in Theorem 1.1. In particular, the inequality (2.1) in Theorem 2.5 is sharper than the inequality (1.9) in Theorem 1.1 when* $\alpha = 0$ *in Theorem 2.5.*

Theorem 2.7. *Let* $A = (a_{ij})$ *and* $B = (b_{ij})$ *be two* $n \times n$ *nonnegative matrices, and* $0 \leq \alpha \leq 1$. *Then*

$$\rho(A \circ B) \leq \max_{\substack{i,j \in N, \\ i \neq j}} \frac{1}{2} \Bigg\{ a_{ii}b_{ii} + a_{jj}b_{jj} + \Big((a_{ii}b_{ii} - a_{jj}b_{jj})^2 + 4 \Big(\frac{1}{m_i m_j} \sum_{k \in N, k \neq i} |a_{ik}b_{ik}| m_k \sum_{l \in N, l \neq j} |a_{jl}b_{jl}| m_l \Big)^{\alpha} \Big(m_i m_j \sum_{k \in N, k \neq i} \frac{|b_{ki}|}{h_k} \sum_{l \in N, l \neq j} \frac{|b_{lj}|}{h_l} \Big)^{1-\alpha} \Big)^{\frac{1}{2}} \Bigg\}$$
$$\leq \max_{i \in N} \Bigg\{ a_{ii}b_{ii} + \Big(\frac{1}{m_i} \sum_{k \in N, k \neq i} |a_{ik}b_{ik}| m_k \Big)^{\alpha} \Big(m_i \sum_{k \in N, k \neq i} \frac{|b_{ki}|}{h_k} \Big)^{1-\alpha} \Bigg\}.$$

Proof. Without loss of generality, for all $i, j \in N$ and $i \neq j$, we assume that

$$a_{ii}b_{ii} + \left(\frac{1}{m_i} \sum_{k \neq i} m_k |a_{ik}b_{ik}| \right)^{\alpha} \left(m_i \sum_{k \in N, k \neq i} \frac{|b_{ki}|}{h_k} \right)^{1-\alpha}$$
$$\geq a_{jj}b_{jj} + \left(\frac{1}{m_j} \sum_{l \in N, l \neq j} m_l |a_{jl}b_{jl}| \right)^{\alpha} \left(m_j \sum_{l \in N, l \neq j} \frac{|b_{lj}|}{h_l} \right)^{1-\alpha},$$

which implies that

$$\left(\frac{1}{m_j} \sum_{l \in N, l \neq j} m_l |a_{jl}b_{jl}| \right)^{\alpha} \left(m_j \sum_{l \in N, l \neq j} \frac{|b_{lj}|}{h_l} \right)^{1-\alpha} \leq \left(\frac{1}{m_i} \sum_{k \in N, k \neq i} m_k |a_{ik}b_{ik}| \right)^{\alpha} \left(m_i \sum_{k \in N, k \neq i} \frac{|b_{ki}|}{h_k} \right)^{1-\alpha} + a_{ii}b_{ii} - a_{jj}b_{jj}. \tag{2.5}$$

By the inequalities (2.1) and (2.5), we derive

$$\begin{aligned}
&\frac{1}{2}\Bigg\{a_{ii}b_{ii}+a_{jj}b_{jj}+\Big((a_{ii}b_{ii}-a_{jj}b_{jj})^2+4\Big(\frac{1}{m_im_j}\sum_{k\in N,k\neq i}|a_{ik}b_{ik}|m_k\sum_{l\in N,l\neq j}|a_{jl}b_{jl}|m_l\Big)^{\alpha}\\
&\Big(m_im_j\sum_{k\in N,k\neq i}\frac{|b_{ki}|}{h_k}\sum_{l\in N,l\neq j}\frac{|b_{lj}|}{h_l}\Big)^{1-\alpha}\Big)^{\frac{1}{2}}\Bigg\}\\
&=\frac{1}{2}\Bigg\{a_{ii}b_{ii}+a_{jj}b_{jj}+\Big((a_{ii}b_{ii}-a_{jj}b_{jj})^2+4\Big(\frac{1}{m_i}\sum_{k\in N,k\neq i}|a_{ik}b_{ik}|m_k\Big)^{\alpha}\Big(m_i\sum_{k\in N,k\neq i}\frac{|b_{ki}|}{h_k}\Big)^{1-\alpha}\\
&\Big(\frac{1}{m_j}\sum_{l\in N,l\neq j}|a_{jl}b_{jl}|m_l\Big)^{\alpha}\Big(m_j\sum_{l\in N,l\neq j}\frac{|b_{lj}|}{h_l}\Big)^{1-\alpha}\Big)^{\frac{1}{2}}\Bigg\}\\
&\leq\frac{1}{2}\Bigg\{a_{ii}b_{ii}+a_{jj}b_{jj}+\Big((a_{ii}b_{ii}-a_{jj}b_{jj})^2+4\Big(\frac{1}{m_i}\sum_{k\in N,k\neq i}|a_{ik}b_{ik}|m_k\Big)^{\alpha}\Big(m_i\sum_{k\in N,k\neq i}\frac{|b_{ki}|}{h_k}\Big)^{1-\alpha}\\
&\Big(\Big(\frac{1}{m_i}\sum_{k\in N,k\neq i}m_k|a_{ik}b_{ik}|\Big)^{\alpha}\Big(m_i\sum_{k\in N,k\neq i}\frac{|b_{ki}|}{h_k}\Big)^{1-\alpha}+a_{ii}b_{ii}-a_{jj}b_{jj}\Big)\Big)^{\frac{1}{2}}\Bigg\}\\
&=\frac{1}{2}\Bigg\{a_{ii}b_{ii}+a_{jj}b_{jj}+\Big(\Big(a_{ii}b_{ii}-a_{jj}b_{jj}+2\Big(\frac{1}{m_i}\sum_{k\in N,k\neq i}|a_{ik}b_{ik}|m_k\Big)^{\alpha}\Big(m_i\sum_{k\in N,k\neq i}\frac{|b_{ki}|}{h_k}\Big)^{1-\alpha}\Big)^2\Big)^{\frac{1}{2}}\Bigg\}\\
&=\frac{1}{2}\Bigg\{a_{ii}b_{ii}+a_{jj}b_{jj}+\Big(a_{ii}b_{ii}-a_{jj}b_{jj}+2\Big(\frac{1}{m_i}\sum_{k\in N,k\neq i}|a_{ik}b_{ik}|m_k\Big)^{\alpha}\Big(m_i\sum_{k\in N,k\neq i}\frac{|b_{ki}|}{h_k}\Big)^{1-\alpha}\Big)\Bigg\}\\
&=a_{ii}b_{ii}+\Big(\frac{1}{m_i}\sum_{k\in N,k\neq i}|a_{ik}b_{ik}|m_k\Big)^{\alpha}\Big(m_i\sum_{k\in N,k\neq i}\frac{|b_{ki}|}{h_k}\Big)^{1-\alpha}.
\end{aligned}\tag{2.6}$$

From the inequalities (2.1) and (2.6), it follows that

$$\begin{aligned}
\rho(A\circ B)\leq&\max_{\substack{i,j\in N,\\ i\neq j}}\frac{1}{2}\Bigg\{a_{ii}b_{ii}+a_{jj}b_{jj}+\Big((a_{ii}b_{ii}-a_{jj}b_{jj})^2+4\Big(\frac{1}{m_im_j}\sum_{k\in N,k\neq i}|a_{ik}b_{ik}|m_k\sum_{l\in N,l\neq j}|a_{jl}b_{jl}|m_l\Big)^{\alpha}\\
&\Big(m_im_j\sum_{k\in N,k\neq i}\frac{|b_{ki}|}{h_k}\sum_{l\in N,l\neq j}\frac{|b_{lj}|}{h_l}\Big)^{1-\alpha}\Big)^{\frac{1}{2}}\Bigg\}\\
\leq&\max_{i\in N}\left\{a_{ii}b_{ii}+\Big(\frac{1}{m_i}\sum_{k\in N,k\neq i}|a_{ik}b_{ik}|m_k\Big)^{\alpha}\Big(m_i\sum_{k\in N,k\neq i}\frac{|b_{ki}|}{h_k}\Big)^{1-\alpha}\right\}.
\end{aligned}$$

The proof is completed. □

Remark 2.8. *From Theorem 2.7, it clearly demonstrates that the upper bound characterized by the inequality (2.1) in Theorem 2.5 is sharper than the one depicted by the inequality (1.10) in Theorem 1.2, and therefore Theorem 2.5 theoretically improves Theorem 2.1 in [15].*

3. The lower bound for the minimum eigenvalue of the Fan product of nonsingular *M*-matrices

In this section, some lower bounds on the minimum eigenvalue for the Fan product of two nonsingular *M*-matrices are obtained.

Lemma 3.1. [5] *Let $A=(a_{ij})$ be an $n\times n$ nonsingular M-matrix. Then*
(1) the minimum eigenvalue $\tau(A)$ is a positive real eigenvalue of the matrix A.
(2) $\tau(A)\leq\min_{i\in N}a_{ii}$.

Lemma 3.2. [13] *Let $A = (a_{ij})$ be an $n \times n$ matrix with $0 \le \alpha \le 1$, and $x_1, x_2, \ldots, x_n$ be n positive real numbers. If $z \in \mathbb{C}$ is an eigenvalue of the matrix A, then there exists a positive integer i with $1 \le i \le n$ such that*

$$|z - a_{ii}| \le \left(\frac{1}{x_i} \sum_{k \in N, k \ne i} x_k |a_{ik}| \right)^{\alpha} \left(x_i \sum_{k \in N, k \ne i} \frac{|a_{ki}|}{x_k} \right)^{1-\alpha}.$$

Theorem 3.3. *Let $A = (a_{ij})$ and $B = (b_{ij})$ be two $n \times n$ nonsingular M-matrices, and $0 \le \alpha \le 1$. Then*

$$\tau(A \star B) \ge \min_{i \in N} \left\{ a_{ii}b_{ii} - \left(\frac{1}{m_i} \sum_{k \in N, k \ne i} |a_{ik}b_{ik}| m_k \right)^{\alpha} \left(m_i \sum_{k \in N, k \ne i} \frac{|b_{ki}|}{h_k} \right)^{1-\alpha} \right\}. \tag{3.1}$$

Proof. Case I. If $A \star B$ is irreducible. Then A and B are two irreducible M-matrices, which implies that $h_k \ne 0$ and $m_i > 0$. From Lemma 3.1, we know that the minimum eigenvalue $\tau(A)$ is a positive real eigenvalue of the matrix A, and hence it follows from Lemma 3.2 that there is a positive integer i with $1 \le i \le n$ such that

$$\begin{aligned} |\tau(A \star B) - a_{ii}b_{ii}| &\le \left(\frac{1}{m_i} \sum_{k \in N, k \ne i} |a_{ik}b_{ik}| m_k \right)^{\alpha} \left(m_i \sum_{k \in N, k \ne i} \frac{|a_{ki}b_{ki}|}{m_k} \right)^{1-\alpha} \\ &\le \left(\frac{1}{m_i} \sum_{k \in N, k \ne i} |a_{ik}b_{ik}| m_k \right)^{\alpha} \left(m_i \sum_{k \in N, k \ne i} \frac{|a_{ki}b_{ki}|}{|a_{ki}| h_k} \right)^{1-\alpha} \\ &= \left(\frac{1}{m_i} \sum_{k \in N, k \ne i} |a_{ik}b_{ik}| m_k \right)^{\alpha} \left(m_i \sum_{k \in N, k \ne i} \frac{|b_{ki}|}{h_k} \right)^{1-\alpha}. \end{aligned} \tag{3.2}$$

By Lemma 3.1, we have

$$\tau(A \star B) - a_{ii}b_{ii} \le 0 \text{ for all } i \in N,$$

and then the inequality (3.2) is equivalent to the following:

$$a_{ii}b_{ii} - \tau(A \star B) \le \left(\frac{1}{m_i} \sum_{k \in N, k \ne i} |a_{ik}b_{ik}| m_k \right)^{\alpha} \left(m_i \sum_{k \in N, k \ne i} \frac{|b_{ki}|}{h_k} \right)^{1-\alpha}. \tag{3.3}$$

Observing the inequality (3.3), we obtain

$$\begin{aligned} \tau(A \star B) &\ge a_{ii}b_{ii} - \left(\frac{1}{m_i} \sum_{k \in N, k \ne i} |a_{ik}b_{ik}| m_k \right)^{\alpha} \left(m_i \sum_{k \in N, k \ne i} \frac{|b_{ki}|}{h_k} \right)^{1-\alpha} \\ &\ge \min_{i \in N} \left\{ a_{ii}b_{ii} - \left(\frac{1}{m_i} \sum_{k \in N, k \ne i} |a_{ik}b_{ik}| m_k \right)^{\alpha} \left(m_i \sum_{k \in N, k \ne i} \frac{|b_{ki}|}{h_k} \right)^{1-\alpha} \right\}. \end{aligned}$$

Case II. If $A \star B$ is reducible. As pointed in [1], a Z-matrix is a nonsingular M-matrix if and only if all its leading principal minors are positive. Define an $n \times n$ permutation matrix $P = (p_{ij})$ with its entry as follows:

$$p_{12} = p_{23} = \ldots = p_{n-1,n} = p_{n,1} = 1$$

and other $p_{ij} = 0$. Then $A - \epsilon P$ and $B - \epsilon P$ are two $n \times n$ irreducible nonsingular M-matrices, where ϵ is any sufficiently small positive real number such that all the leading principal minors of both $A - \epsilon P$ and $B - \epsilon P$ are positive. Substitute A and B in the irreducible case with $A - \epsilon P$ and $B - \epsilon P$, respectively. Let $\epsilon \to 0$, the inequality (3.1) can be derived by continuity. The proof is completed. □

Remark 3.4. *If $\alpha = 0$ in Theorem 3.3, then*

$$\tau(A \star B) \geq \min_{i\in N}\left\{a_{ii}b_{ii} - m_i \sum_{k\in N, k\neq i} \frac{|b_{ki}|}{h_k}\right\},$$

which is the inequality (1.21) in Theorem 1.3. Therefore, Theorem 3.3 can be regarded as a generalization of Theorem 3.1 in [12].

Setting $\alpha = \frac{1}{2}$ in Theorem 3.3, we easily obtain the following corollary, which can also be found in [13].

Corollary 3.5. *Let $A = (a_{ij})$ and $B = (b_{ij})$ be two $n \times n$ nonsingular M-matrices. Then*

$$\tau(A \star B) \geq \min_{i\in N}\left\{a_{ii}b_{ii} - \left(\frac{1}{m_i}\sum_{k\in N, k\neq i}|a_{ik}b_{ik}|m_k\right)^{\frac{1}{2}}\left(m_i \sum_{k\in N, k\neq i}\frac{|b_{ki}|}{h_k}\right)^{\frac{1}{2}}\right\}.$$

Note that the result in Theorem 3.3 generalizes the one of Theorem 4 in [13].

Theorem 3.6. *Let $A = (a_{ij})$ and $B = (b_{ij})$ be two $n \times n$ nonsingular M-matrices, and $0 \leq \alpha \leq 1$. Then*

$$\tau(A \star B) \geq \min_{\substack{i,j\in N,\\ i\neq j}} \frac{1}{2}\Bigg\{a_{ii}b_{ii} + a_{jj}b_{jj} - \Bigg((a_{ii}b_{ii} - a_{jj}b_{jj})^2 + 4\Big(\frac{1}{m_i m_j}\sum_{k\in N, k\neq i}|a_{ik}b_{ik}|m_k \sum_{l\in N, l\neq j}|a_{jl}b_{jl}|m_l\Big)^{\alpha}$$
$$\Big(m_i m_j \sum_{k\in N, k\neq i}\frac{|b_{ki}|}{h_k}\sum_{l\in N, l\neq j}\frac{|b_{lj}|}{h_l}\Big)^{1-\alpha}\Bigg)^{\frac{1}{2}}\Bigg\}. \tag{3.4}$$

Proof. Case I. If $A \star B$ is irreducible, then A and B are two irreducible M-matrices. Therefore, $h_k \neq 0$, $h_l \neq 0$, $m_i > 0$ and $m_j > 0$. By Lemma 3.1, the minimum eigenvalue $\tau(A)$ is a positive real eigenvalue of the matrix A. It follows from Lemma 2.1 that there is a pair (i, j) of positive integers with $i \neq j (1 \leq i, j \leq n)$ such that

$$\begin{aligned}
&|\tau(A \star B) - a_{ii}b_{ii}||\tau(A \star B) - a_{jj}b_{jj}| \\
&\leq \left(\frac{1}{m_i m_j}\sum_{k\in N, k\neq i}|a_{ik}b_{ik}|m_k \sum_{l\in N, l\neq j}|a_{jl}b_{jl}|m_l\right)^{\alpha}\left(m_i m_j \sum_{k\in N, k\neq i}\frac{|a_{ki}b_{ki}|}{m_k}\sum_{l\in N, l\neq j}\frac{|a_{lj}b_{lj}|}{m_l}\right)^{1-\alpha} \\
&\leq \left(\frac{1}{m_i m_j}\sum_{k\in N, k\neq i}|a_{ik}b_{ik}|m_k \sum_{l\in N, l\neq j}|a_{jl}b_{jl}|m_l\right)^{\alpha}\left(m_i m_j \sum_{k\in N, k\neq i}\frac{|a_{ki}b_{ki}|}{|a_{ki}|h_k}\sum_{l\in N, l\neq j}\frac{|a_{lj}b_{lj}|}{|a_{lj}|h_l}\right)^{1-\alpha} \\
&= \left(\frac{1}{m_i m_j}\sum_{k\in N, k\neq i}|a_{ik}b_{ik}|m_k \sum_{l\in N, l\neq j}|a_{jl}b_{jl}|m_l\right)^{\alpha}\left(m_i m_j \sum_{k\in N, k\neq i}\frac{|b_{ki}|}{h_k}\sum_{l\in N, l\neq j}\frac{|b_{lj}|}{h_l}\right)^{1-\alpha}.
\end{aligned} \tag{3.5}$$

From Lemma 3.1, it yields that

$$\tau(A \star B) - a_{ii}b_{ii} \leq 0 \text{ for all } i \in N$$

and

$$\tau(A \star B) - a_{jj}b_{jj} \leq 0 \text{ for all } j \in N.$$

Then it is evident that inequality (3.5) becomes:

$$\begin{aligned}
&(\tau(A \star B) - a_{ii}b_{ii})(\tau(A \star B) - a_{jj}b_{jj}) \\
&\leq \left(\frac{1}{m_i m_j}\sum_{k\in N, k\neq i}|a_{ik}b_{ik}|m_k \sum_{l\in N, l\neq j}|a_{jl}b_{jl}|m_l\right)^{\alpha}\left(m_i m_j \sum_{k\in N, k\neq i}\frac{|b_{ki}|}{h_k}\sum_{l\in N, l\neq j}\frac{|b_{lj}|}{h_l}\right)^{1-\alpha}.
\end{aligned} \tag{3.6}$$

Solving the inequality (3.6), we have

$$\tau(A\star B)\geq \frac{1}{2}\Big\{a_{ii}b_{ii}+a_{jj}b_{jj}-\Big((a_{ii}b_{ii}-a_{jj}b_{jj})^2+4\Big(\frac{1}{m_im_j}\sum_{k\in N,k\neq i}|a_{ik}b_{ik}|m_k\sum_{l\in N,l\neq j}|a_{jl}b_{jl}|m_l\Big)^{\alpha}$$

$$\Big(m_im_j\sum_{k\in N,k\neq i}\frac{|b_{ki}|}{h_k}\sum_{l\in N,l\neq j}\frac{|b_{lj}|}{h_l}\Big)^{1-\alpha}\Big)^{\frac{1}{2}}\Big\}$$

$$\geq \min_{\substack{i,j\in N,\\ i\neq j}}\frac{1}{2}\Big\{a_{ii}b_{ii}+a_{jj}b_{jj}-\Big((a_{ii}b_{ii}-a_{jj}b_{jj})^2+4\Big(\frac{1}{m_im_j}\sum_{k\in N,k\neq i}|a_{ik}b_{ik}|m_k\sum_{l\in N,l\neq j}|a_{jl}b_{jl}|m_l\Big)^{\alpha}$$

$$\Big(m_im_j\sum_{k\in N,k\neq i}\frac{|b_{ki}|}{h_k}\sum_{l\in N,l\neq j}\frac{|b_{lj}|}{h_l}\Big)^{1-\alpha}\Big)^{\frac{1}{2}}\Big\}.$$

Case II. If $A\star B$ is reducible. Similar to the proof of Case II in Theorem 3.3, we easily obtain the inequality (3.4). The proof is completed. □

Remark 3.7. *Theorem 3.6 will reduce to the following:*

$$\tau(A\star B)\geq \min_{\substack{i,j\in N,\\ i\neq j}}\frac{1}{2}\left\{a_{ii}b_{ii}+a_{jj}b_{jj}-\left((a_{ii}b_{ii}-a_{jj}b_{jj})^2+4m_im_j\sum_{k\in N,k\neq i}\frac{|b_{ki}|}{h_k}\sum_{l\in N,l\neq j}\frac{|b_{lj}|}{h_l}\right)^{\frac{1}{2}}\right\} \tag{3.7}$$

when $\alpha=0$ in Theorem 3.6, which happens to be Theorem 2.1 in [2] and Theorem 3.1 in [16]. Therefore, Theorem 3.6 can be regarded as the generalization of Theorem 2.1 in [2] and Theorem 3.1 in [16].

Similar to the conclusion of Corollary 3.5, we easily obtain the following corollary, which is deduced from taking $\alpha=\frac{1}{2}$ in Theorem 3.6.

Corollary 3.8. *Let $A=(a_{ij})$ and $B=(b_{ij})$ be two $n\times n$ nonsingular M-matrices. Then*

$$\tau(A\star B)\geq \min_{\substack{i,j\in N,\\ i\neq j}}\frac{1}{2}\Big\{a_{ii}b_{ii}+a_{jj}b_{jj}-\Big((a_{ii}b_{ii}-a_{jj}b_{jj})^2+4\Big(\sum_{k\in N,k\neq i}|a_{ik}b_{ik}|m_k\sum_{l\in N,l\neq j}|a_{jl}b_{jl}|m_l\Big)^{\frac{1}{2}}$$

$$\Big(\sum_{k\in N,k\neq i}\frac{|b_{ki}|}{h_k}\sum_{l\in N,l\neq j}\frac{|b_{lj}|}{h_l}\Big)^{\frac{1}{2}}\Big)^{\frac{1}{2}}\Big\}.$$

Theorem 3.9. *Let $A=(a_{ij})$ and $B=(b_{ij})$ be two $n\times n$ nonsingular M-matrices, and $0\leq\alpha\leq 1$. Then*

$$\tau(A\star B)\geq \min_{\substack{i,j\in N,\\ i\neq j}}\frac{1}{2}\Big\{a_{ii}b_{ii}+a_{jj}b_{jj}-\Big((a_{ii}b_{ii}-a_{jj}b_{jj})^2+4\Big(\frac{1}{m_im_j}\sum_{k\in N,k\neq i}|a_{ik}b_{ik}|m_k\sum_{l\in N,l\neq j}|a_{jl}b_{jl}|m_l\Big)^{\alpha}$$

$$\Big(m_im_j\sum_{k\in N,k\neq i}\frac{|b_{ki}|}{h_k}\sum_{l\in N,l\neq j}\frac{|b_{lj}|}{h_l}\Big)^{1-\alpha}\Big)^{\frac{1}{2}}\Big\}$$

$$\geq \min_{i\in N}\left\{a_{ii}b_{ii}-\Big(\frac{1}{m_i}\sum_{k\in N,k\neq i}|a_{ik}b_{ik}|m_k\Big)^{\alpha}\Big(m_i\sum_{k\in N,k\neq i}\frac{|b_{ki}|}{h_k}\Big)^{1-\alpha}\right\}.$$

Proof. Without loss of generality, suppose that

$$a_{ii}b_{ii}-\left(\frac{1}{m_i}\sum_{k\in N,k\neq i}m_k|a_{ik}b_{ik}|\right)^{\alpha}\left(m_i\sum_{k\in N,k\neq i}\frac{|b_{ki}|}{h_k}\right)^{1-\alpha}$$

$$\leq a_{jj}b_{jj}-\left(\frac{1}{m_j}\sum_{l\in N,l\neq j}m_l|a_{jl}b_{jl}|\right)^{\alpha}\left(m_j\sum_{l\in N,l\neq j}\frac{|b_{lj}|}{h_l}\right)^{1-\alpha}$$

for all $i, j \in N$ and $i \neq j$. This implies that

$$\left(\frac{1}{m_j}\sum_{l\in N, l\neq j} m_l|a_{jl}b_{jl}|\right)^{\alpha}\left(m_j\sum_{l\in N, l\neq j}\frac{|b_{lj}|}{h_l}\right)^{1-\alpha} \leq \left(\frac{1}{m_i}\sum_{k\in N, k\neq i} m_k|a_{ik}b_{ik}|\right)^{\alpha}\left(m_i\sum_{k\in N, k\neq i}\frac{|b_{ki}|}{h_k}\right)^{1-\alpha} + a_{jj}b_{jj} - a_{ii}b_{ii} \tag{3.8}$$

for all $i, j \in N$ and $i \neq j$. Observing the inequalities (3.1), (3.4) and (3.8), we have

$$\begin{aligned}
&\frac{1}{2}\Big\{a_{ii}b_{ii} + a_{jj}b_{jj} - \Big((a_{ii}b_{ii} - a_{jj}b_{jj})^2 + 4\Big(\frac{1}{m_im_j}\sum_{k\in N, k\neq i}|a_{ik}b_{ik}|m_k\sum_{l\in N, l\neq j}|a_{jl}b_{jl}|m_l\Big)^{\alpha}\\
&\quad\Big(m_im_j\sum_{k\in N, k\neq i}\frac{|b_{ki}|}{h_k}\sum_{l\in N, l\neq j}\frac{|b_{lj}|}{h_l}\Big)^{1-\alpha}\Big)^{\frac{1}{2}}\Big\}\\
&= \frac{1}{2}\Big\{a_{ii}b_{ii} + a_{jj}b_{jj} - \Big((a_{ii}b_{ii} - a_{jj}b_{jj})^2 + 4\Big(\frac{1}{m_i}\sum_{k\in N, k\neq i}|a_{ik}b_{ik}|m_k\Big)^{\alpha}\Big(m_i\sum_{k\in N, k\neq i}\frac{|b_{ki}|}{h_k}\Big)^{1-\alpha}\\
&\quad\Big(\frac{1}{m_j}\sum_{l\in N, l\neq j}|a_{jl}b_{jl}|m_l\Big)^{\alpha}\Big(m_j\sum_{l\in N, l\neq j}\frac{|b_{lj}|}{h_l}\Big)^{1-\alpha}\Big)^{\frac{1}{2}}\Big\}\\
&\geq \frac{1}{2}\Big\{a_{ii}b_{ii} + a_{jj}b_{jj} - \Big((a_{ii}b_{ii} - a_{jj}b_{jj})^2 + 4\Big(\frac{1}{m_i}\sum_{k\in N, k\neq i}|a_{ik}b_{ik}|m_k\Big)^{\alpha}\Big(m_i\sum_{k\in N, k\neq i}\frac{|b_{ki}|}{h_k}\Big)^{1-\alpha}\\
&\quad\Big(\Big(\frac{1}{m_i}\sum_{k\in N, k\neq i}m_k|a_{ik}b_{ik}|\Big)^{\alpha}\Big(m_i\sum_{k\in N, k\neq i}\frac{|b_{ki}|}{h_k}\Big)^{1-\alpha} + a_{jj}b_{jj} - a_{ii}b_{ii}\Big)\Big)^{\frac{1}{2}}\Big\}\\
&= \frac{1}{2}\Big\{a_{ii}b_{ii} + a_{jj}b_{jj} - \Big(\Big(a_{jj}b_{jj} - a_{ii}b_{ii} + 2\Big(\frac{1}{m_i}\sum_{k\in N, k\neq i}|a_{ik}b_{ik}|m_k\Big)^{\alpha}\Big(m_i\sum_{k\in N, k\neq i}\frac{|b_{ki}|}{h_k}\Big)^{1-\alpha}\Big)^2\Big)^{\frac{1}{2}}\Big\}\\
&= \frac{1}{2}\Big\{a_{ii}b_{ii} + a_{jj}b_{jj} - \Big(a_{jj}b_{jj} - a_{ii}b_{ii} + 2\Big(\frac{1}{m_i}\sum_{k\in N, k\neq i}|a_{ik}b_{ik}|m_k\Big)^{\alpha}\Big(m_i\sum_{k\in N, k\neq i}\frac{|b_{ki}|}{h_k}\Big)^{1-\alpha}\Big)\Big\}\\
&= a_{ii}b_{ii} - \Big(\frac{1}{m_i}\sum_{k\in N, k\neq i}|a_{ik}b_{ik}|m_k\Big)^{\alpha}\Big(m_i\sum_{k\in N, k\neq i}\frac{|b_{ki}|}{h_k}\Big)^{1-\alpha}.
\end{aligned} \tag{3.9}$$

According to the inequality (3.9), we have

$$\begin{aligned}
\tau(A \star B) &\leq \min_{\substack{i,j\in N,\\ i\neq j}}\frac{1}{2}\Big\{a_{ii}b_{ii} + a_{jj}b_{jj} - \Big((a_{ii}b_{ii} - a_{jj}b_{jj})^2 + 4\Big(\frac{1}{m_im_j}\sum_{k\in N, k\neq i}|a_{ik}b_{ik}|m_k\sum_{l\in N, l\neq j}|a_{jl}b_{jl}|m_l\Big)^{\alpha}\\
&\qquad\Big(m_im_j\sum_{k\in N, k\neq i}\frac{|b_{ki}|}{h_k}\sum_{l\in N, l\neq j}\frac{|b_{lj}|}{h_l}\Big)^{1-\alpha}\Big)^{\frac{1}{2}}\Big\}\\
&\leq \min_{i\in N}\Big\{a_{ii}b_{ii} - \Big(\frac{1}{m_i}\sum_{k\in N, k\neq i}|a_{ik}b_{ik}|m_k\Big)^{\alpha}\Big(m_i\sum_{k\in N, k\neq i}\frac{|b_{ki}|}{h_k}\Big)^{1-\alpha}\Big\}.
\end{aligned}$$

The proof is completed. □

Remark 3.10. *Theorem 3.9 evidently indicates that the lower bound for $\tau(A \star B)$ in Theorem 3.6 is theoretically sharper than the one in Theorem 3.3. Similarly, the low bound of Corollary 3.8 also improves the one of Theorem 4 in [13] when $\alpha = \frac{1}{2}$ in Theorem 3.6.*

4. Numerical examples

In this section, some numerical examples are given to verify the effectiveness and validity of our theoretical results by comparing them to the existing ones.

Example 4.1. *Let*

$$A = \begin{pmatrix} 4 & 1 & 1 & 1 \\ 1 & 5 & 2 & 1 \\ 1 & 1 & 4 & 1 \\ 1 & 1 & 1 & 4 \end{pmatrix}, \ B = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 3 & 2 & 0 \\ 0 & 2 & 4 & 2 \\ 0 & 0 & 1 & 5 \end{pmatrix}.$$

Obviously, A and B are two nonnegative matrices. Using the MATLAB software, it yields that

$$\rho(A) = 7.5616, \rho(B) = 6.5172, \rho(J_A) = 0.7623,$$

$$\rho(J_B) = 0.8488, \rho\left(J_A^{[2]}\right) = 0.2000, \rho\left(J_B^{[2]}\right) = 0.4768.$$

According to the inequality (1.1) (i.e, Observation 5.7.3 in [5]), we obtain

$$\rho(A \circ B) \leq \rho(A)\rho(B) = 49.2805.$$

By the inequality (1.2) (i.e, Theorem 4 in [3]), we have

$$\rho(A \circ B) \leq \max_{i \in N}\{2a_{ii}b_{ii} + \rho(A)\rho(B) - a_{ii}\rho(B) - b_{ii}\rho(A)\} = 25.4037.$$

Using the inequality (1.3) (i.e, Theorem 6 in [7]), we derive

$$\rho(A \circ B) \leq \max_{i \in N}\{a_{ii}b_{ii} + a_{ii}b_{ii}\rho(J_A)\rho(J_B)\} = 32.9408.$$

By the inequality (1.6) (i.e, Theorem 4 in [8]), we get

$$\rho(A \circ B) \leq \max_{\substack{i,j \in N, \\ i \neq j}} \frac{1}{2}\Big\{a_{ii}b_{ii} + a_{jj}b_{jj} + \Big((a_{ii}b_{ii} - a_{jj}b_{jj})^2 + 4(\rho(A) - a_{ii})(\rho(B) - b_{ii})$$
$$(\rho(A) - a_{jj})(\rho(B) - b_{jj})\Big)^{\frac{1}{2}}\Big\} = 25.2419.$$

From the inequality (1.7) (i.e., Theorem 3 in [9]), it follows that

$$\rho(A \circ B) \quad \leq \max_{\substack{i,j \in N, \\ i \neq j}} \tfrac{1}{2}\Big\{a_{ii}b_{ii} + a_{jj}b_{jj} + \Big((a_{ii}b_{ii} - a_{jj}b_{jj})^2 + 4a_{ii}b_{ii}a_{jj}b_{jj}\rho^2(J_A)\rho^2(J_B)\Big)^{\frac{1}{2}}\Big\} = 29.7461.$$

Applying the inequality (1.9) (i.e., Theorem 4.1 in [12]), we gain

$$\rho(A \circ B) \leq \max_{i \in N}\left\{a_{ii}b_{ii} + m_i \sum_{k \in N, k \neq i} \frac{|b_{ki}|}{h_k}\right\} = 22.1333.$$

Utilizing the inequalities (1.11) and (1.14) (i.e., Theorems 1 and 2 in [10]), we obtain

$$\rho(A \circ B) \leq \max_{i \in N}\left\{a_{ii}b_{ii} + a_{ii}b_{ii}\rho^{\frac{1}{2}}\left(J_A^{[2]}\right)\rho^{\frac{1}{2}}\left(J_B^{[2]}\right)\right\} = 26.1761$$

and

$$\rho(A \circ B) \quad \leq \max_{\substack{i,j \in N, \\ i \neq j}} \tfrac{1}{2}\left\{a_{ii}b_{ii} + a_{jj}b_{jj} + \left((a_{ii}b_{ii} - a_{jj}b_{jj})^2 + 4a_{ii}b_{ii}a_{jj}b_{jj}\rho\left(J_A^{[2]}\right)\rho\left(J_B^{[2]}\right)\right)^{\frac{1}{2}}\right\} = 23.8750.$$

Employing the inequality (2.4) (i.e., Theorem 3.1 in [2] and Theorem 2.2 in [16]), we have

$$\rho(A\circ B)\leq \max_{\substack{i,j\in N,\\ i\neq j}}\frac{1}{2}\left\{a_{ii}b_{ii}+a_{jj}b_{jj}+\left((a_{ii}b_{ii}-a_{jj}b_{jj})^2+4m_im_j\sum_{k\in N,k\neq i}\frac{|b_{ki}|}{h_k}\sum_{l\in N,l\neq j}\frac{|b_{lj}|}{h_l}\right)^{\frac{1}{2}}\right\}=22.1333.$$

Set $\alpha=0.5$. *By the inequality (1.10) (i.e, Theorem 2.1 in [15]), it generates the following:*

$$\rho(A\circ B)\leq \max_{i\in N}\left\{a_{ii}b_{ii}+\left(\frac{1}{m_i}\sum_{k\in N,k\neq i}m_k|a_{ik}b_{ik}|\right)^{\alpha}\left(m_i\sum_{k\in N,k\neq i}\frac{|b_{ki}|}{h_k}\right)^{1-\alpha}\right\}=22.0656.$$

According to the inequality (2.1) of Theorem 2.5, there is the following:

$$\rho(A\circ B)\leq \max_{\substack{i,j\in N,\\ i\neq j}}\frac{1}{2}\Bigg\{a_{ii}b_{ii}+a_{jj}b_{jj}+\Bigg((a_{ii}b_{ii}-a_{jj}b_{jj})^2+4\Big(\frac{1}{m_im_j}\sum_{k\in N,k\neq i}|a_{ik}b_{ik}|m_k\sum_{l\in N,l\neq j}|a_{jl}b_{jl}|m_l\Big)^{\alpha}$$

$$\Big(m_im_j\sum_{k\in N,k\neq i}\frac{|b_{ki}|}{h_k}\sum_{l\in N,l\neq j}\frac{|b_{lj}|}{h_l}\Big)^{1-\alpha}\Bigg)^{\frac{1}{2}}\Bigg\}=21.6635.$$

In fact, $\rho(A\circ B)=20.6270$. Observing and analyzing the above numerical results, it is not difficult to verify the correctness of the new upper bound for $\rho(A\circ B)$ obtained by the inequality (2.1) in Theorem 2.5, and to indicate the fact that our result improves the existing ones in [2, 3, 5, 7–10, 12, 15, 16].

Example 4.2. *Let*

$$A=\begin{pmatrix}4&-1&-1&-1\\-1&5&-2&-1\\-1&-1&4&-1\\-1&-1&-1&4\end{pmatrix},\ B=\begin{pmatrix}1&-0.5&0&0\\-0.5&1&-0.5&0\\0&-0.5&1&-0.5\\0&0&-0.5&1\end{pmatrix}.$$

Obviously, A and B are two nonsingular M-matrices. Applying the MATLAB software, there are some key quantities as follows:

$$\tau(A)=1.0000,\tau(B)=0.1910,\rho(J_A)=0.7623,$$
$$\rho(J_B)=0.8090,\rho\left(J_A^{[2]}\right)=0.2000,\rho\left(J_B^{[2]}\right)=0.4045.$$

From the inequality (1.16) (i.e, Corollary 5.7.4.1 in [5]), it yields that

$$\tau(A\star B)\geq\tau(A)\tau(B)=0.1910.$$

By the inequality (1.17) (i.e, Theorem 9 in [3]), we have

$$\tau(A\star B)\geq\min_{i\in N}\{a_{ii}\tau(B)+b_{ii}\tau(A)-\tau(A)\tau(B)\}=1.5730.$$

According the inequality (1.18) (i.e, Theorem 4 in [7]), we obtain

$$\tau(A\star B)\geq(1-\rho(J'_A)\rho(J'_B))\min_{i\in N}(a_{ii}b_{ii}))=1.5332.$$

Using the inequality (1.19) (i.e, Theorem 7 in [8]), we derive

$$\tau(A\star B)\geq\min_{\substack{i,j\in N,\\ i\neq j}}\frac{1}{2}\Big\{a_{ii}b_{ii}+a_{jj}b_{jj}-\Big((a_{ii}b_{ii}-a_{jj}b_{jj})^2+4(a_{ii}-\tau(A))(b_{ii}-\tau(B))$$

$$(a_{jj}-\tau(A))(b_{jj}-\tau(B))\Big)^{\frac{1}{2}}\Big\}=1.5730.$$

Utilizing the inequality (1.20) (i.e, Theorem 2 in [9]), we get

$$\tau(A \star B) \quad \geq \min_{\substack{i,j\in N,\\ i\neq j}} \tfrac{1}{2}\Big\{a_{ii}b_{ii} + a_{jj}b_{jj} - \Big((a_{ii}b_{ii} - a_{jj}b_{jj})^2 + 4a_{ii}b_{ii}a_{jj}b_{jj}\rho^2(J'_A)\rho^2(J'_B)\Big)^{\frac{1}{2}}\Big\} = 1.5332.$$

Employing the inequality (1.21) (i.e, Theorem 3.1 in [12]), there is the following:

$$\tau(A \star B) \geq \min_{i\in N}\left\{a_{ii}b_{ii} - m_i \sum_{k\in N, k\neq i} \frac{|b_{ki}|}{h_k}\right\} = 1.9333.$$

Using the inequalities (1.22) and (1.23) (i.e., Theorems 2.7 and 2.8 in [11]), we gain

$$\tau(A \star B) \geq \min_{i\in N}\left\{a_{ii}b_{ii} - a_{ii}b_{ii}\rho^{\frac{1}{2}}\left(J_A^{[2]}\right)\rho^{\frac{1}{2}}\left(J_B^{[2]}\right)\right\} = 2.8623$$

and

$$\tau(A \star B) \quad \geq \min_{\substack{i,j\in N,\\ i\neq j}} \tfrac{1}{2}\left\{a_{ii}b_{ii} + a_{jj}b_{jj} - \left((a_{ii}b_{ii} - a_{jj}b_{jj})^2 + 4a_{ii}b_{ii}a_{jj}b_{jj}\rho\left(J_A^{[2]}\right)\rho\left(J_B^{[2]}\right)\right)^{\frac{1}{2}}\right\} = 2.8623.$$

By the inequality (3.7) (i.e, Theorem 2.1 in [2] and Theorem 3.1 in [16]), we have

$$\tau(A \star B) \geq \min_{\substack{i,j\in N,\\ i\neq j}} \frac{1}{2}\left\{a_{ii}b_{ii} + a_{jj}b_{jj} - \left((a_{ii}b_{ii} - a_{jj}b_{jj})^2 + 4m_i m_j \sum_{k\in N, k\neq i}\frac{|b_{ki}|}{h_k}\sum_{l\in N, l\neq j}\frac{|b_{lj}|}{h_l}\right)^{\frac{1}{2}}\right\} = 2.9779.$$

Let $\alpha = 0.5$. According to the inequality (3.1) in Theorem 3.3, it generates the following:

$$\tau(A \star B) \geq \min_{i\in N}\left\{a_{ii}b_{ii} - \left(\frac{1}{m_i}\sum_{k\in N, k\neq i}|a_{ik}b_{ik}|m_k\right)^{\alpha}\left(m_i\sum_{k\in N, k\neq i}\frac{|b_{ki}|}{h_k}\right)^{1-\alpha}\right\} = 2.9835.$$

By the inequality (3.4) in Theorem 3.6, we obtain

$$\tau(A \star B) \geq \min_{\substack{i,j\in N,\\ i\neq j}} \frac{1}{2}\Big\{a_{ii}b_{ii} + a_{jj}b_{jj} - \Big((a_{ii}b_{ii} - a_{jj}b_{jj})^2 + 4\Big(\frac{1}{m_i m_j}\sum_{k\in N, k\neq i}|a_{ik}b_{ik}|m_k\sum_{l\in N, l\neq j}|a_{jl}b_{jl}|m_l\Big)^{\alpha}$$
$$\Big(m_i m_j\sum_{k\in N, k\neq i}\frac{|b_{ki}|}{h_k}\sum_{l\in N, l\neq j}\frac{|b_{lj}|}{h_l}\Big)^{1-\alpha}\Big)^{\frac{1}{2}}\Big\} = 3.1275.$$

In fact, $\tau(A \star B) = 3.2845$. According to the analysis and comparisons of the above numerical results, we easily obtain that the new lower bounds for $\tau(A \star B)$ proposed by the inequality (3.1) in Theorem 3.3 and the inequality (3.4) in Theorem 3.6 are effectiveness, and are sharper than some existing ones in [2, 3, 5, 7–9, 11, 12, 16].

As shown in Theorems 1.2, 2.5, 3.3 and 3.6, the upper bounds for $\rho(A \circ B)$ and the lower bounds for $\tau(A \star B)$ all involve the parameter $\alpha \in [0, 1]$, which implies that the various choices of the parameter α influence the upper bound for $\rho(A \circ B)$ and the lower bound for $\tau(A \star B)$. To well illustrate the effectiveness of the main results, the following numerical example is performed.

Example 4.3. *Consider two 4×4 nonnegative matrices defined in Example 4.1 and two 4×4 nonsingular M-matrices defined in Example 4.2, the numerical results of the upper bounds for $\rho(A \circ B)$ in Theorems 1.2, 2.5 and the lower bounds of $\tau(A \star B)$ in Theorems 3.3, 3.6 are summarized in Table 4.1 by choosing the distinct parameters $\alpha \in [0, 1]$.*

Table 4.1. Numerical results for Theorems 1.2, 2.5, 3.3 and 3.6

Values of α	Theorem 1.2	Theorem 2.5	Theorem 3.3	Theorem 3.6
0.0	22.4000	22.1333	1.8667	2.9779
0.1	22.1196	21.9478	2.1606	2.9924
0.2	22.1060	21.7760	2.1441	3.0030
0.3	22.0924	21.6170	2.6326	3.0700
0.4	22.0790	21.5227	2.8210	3.0991
0.5	22.0656	21.6635	2.9835	3.1275
0.6	22.0523	21.8154	3.1235	3.1551
0.7	22.0391	21.9787	3.0131	3.0966
0.8	22.6075	22.1543	2.8083	3.0226
0.9	23.5459	22.3428	2.5825	2.9125
1.0	24.6000	22.5448	2.3333	2.7922

To indicate the influences of the distinct choices for parameter α on the upper bound for $\rho(A \circ B)$ in Theorems 1.2, 2.5 and the lower bound for $\tau(A \star B)$ in Theorem 3.3, 3.6 more visually, the numerical approximations with the distinct parameter α between the estimated values of $\rho(A \circ B)$ (resp. $\tau(A \star B)$) in Theorems 1.2, 2.5 (resp. Theorems 3.3, 3.6) and the true value of $\rho(A \circ B)$ (resp. $\tau(A \star B)$) are drawn in Figure 1.

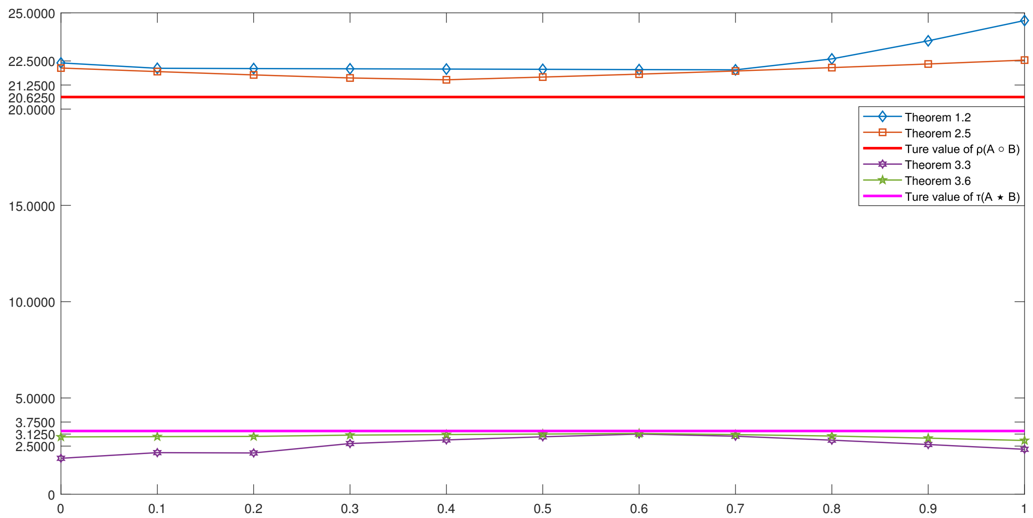

Figure 1. Numerical approximations for the estimated values and the true values

Observing and analyzing Figure 1, some conclusions are obtained as follows: 1) As the parameter α changes in the interval [0,1], the upper bounds of $\rho(A \circ B)$ in Theorems 1.2 and 2.5 can well approximate the true value of $\rho(A \circ B)$; 2) For the various choices of the parameter $\alpha \in [0, 1]$, the upper bound of $\rho(A \circ B)$ in Theorem 2.5 is relatively stable, and the new upper bound of $\rho(A \circ B)$ in Theorem 2.5 is sharper than the one in Theorem 1.2 (i.e., Theorem 2.1 in [15]), which verifies the theoretical comparison in Theorem 2.7; 3) Taking the distinct parameters $\alpha \in [0, 1]$, the numerical approximation between the lower bounds of $\tau(A \star B)$ in Theorems 3.3, 3.6 and the true value of $\tau(A \star B)$ is rather wonderful and stable; 4) The lower bound of $\tau(A \star B)$ in Theorem 3.6 is sharper than the one in Theorem 3.3, and then it shows the theoretical comparison in Theorem 3.9.

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