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Duality for nonsmooth multiobjective semi-infinite programming problems with equilibrium constraints on Hadamard manifolds

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Abstract. This article is concerned with a class of nonsmooth multiobjective semi-infinite programming problems with equilibrium constraints in the setting of Hadamard manifolds (abbreviated as, (NSIMPEC)). We formulate Wolfe as well as Mond-Weir type dual problems related to (NSIMPEC) and derive several duality results that relate (NSIMPEC) and the corresponding dual models. Non-trivial numerical examples are incorporated to demonstrate the validity of the results established in this paper. To the best of our knowledge, this is for the first time that Wolfe and Mond-Weir dual models have been considered for (NSIMPEC) in the setting of Hadamard manifolds.

1. Introduction

In the last few decades, it has been observed that numerous real-life problems emerging in various areas related to engineering, technology and science can be formulated in a more effective way on manifold setting instead of Euclidean space, see [4, 15]. Further, extending and generalizing the methods of optimization from the setting of Euclidean spaces to the setting of manifolds have several crucial advantages. For instance, by appropriately using the notions of the Riemannian geometry, several constrained mathematical optimization problems can be conveniently converted into unconstrained problems. Apart from this, numerous non-convex optimization problems can be converted into convex problems by employing the Riemannian geometry perspective (see, for instance, [25, 27]). Furthermore, it is a common observation that numerous important constraints which naturally arise in certain mathematical programming problems have a relative interior that can be viewed as Hadamard manifolds, for instance, the hypercube $(0,1)^n$ (see, for instance, [24]) endowed with the metric $Z^{-2}(I-Z)^{-2} = \operatorname{diag}\left(z_1^{-2}(1-z_1)^{-2}, \ldots, z_n^{-2}(1-z_n)^{-2}\right)$ and the set containing every symmetric positive definite matrix $\mathcal{S}^n_{++}$ (see, for instance, [20]) with the metric $-\log\det X$ are Hadamard manifolds. As a result, a wider range of mathematical programming problems can be solved by formulating the problems in the framework of Riemannian and Hadamard manifolds. In recent times, various other notions and concepts involved in mathematical programming have been extended from Euclidean spaces to Riemannian and Hadamard manifolds by several authors; see, for instance, [4, 23, 32, 33, 36, 37, 42–44].

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In theory of mathematical programming, an optimization problem accompanied by certain complementarity constraints or variational inequality constraints is referred to as a mathematical programming problem with equilibrium constraints (in brief, (MPEC)). One of the first attempts in investigating such optimization problems is due to Harker and Pang [13], who explored existence of efficient solutions for (MPECs). Due to its immense scope of applicability in numerous fields of science, technology and engineering (see, for instance, [28, 29]), (MPECs) have been studied by numerous authors in recent years. For further details and updated survey of (MPEC) and its applications, we refer the readers to [9, 21, 31, 41] and the references cited therein.

Multiobjective semi-infinite programming comprises of those optimization problems in which more than one objective functions are optimized simultaneously over some feasible set that is determined by infinitely many constraints. In case there is only one objective function, such problems reduce to a standard semi-infinite programming problem (abbreviated as, (SIP)). These problems have wide applications in various branches of mathematics, physics and engineering, see [7, 8, 11, 16] and the references cited therein.

Several regularity and optimality criteria for (MPEC) was investigated by Chen and Florian [6]. Ye [45] studied necessary as well as sufficient criteria of optimality for (MPEC). Flegel and Kanzow [9] studied Abadie-type and Slater-type constraint qualifications for (MPEC). First-order optimality criteria for (MPEC) was derived by Flegel and Kanzow [10] by using Guignard constraint qualification. Optimality criteria and several duality relations for (MPEC) were deduced by Singh and Mishra [31]. Optimality conditions for multiobjective (MPEC) was explored by Ardali et al. [1] by using the notion of convexificators. Duality models for (MPEC) were studied by Pandey and Mishra [22]. Duality for nonsmooth (MPEC) was explored bu Guu et al. [12]. Recently, Treanţă et al. [33] studied optimality conditions for multiobjective (MPEC) on Hadamard manifolds.

Motivated by the results derived in [6, 19, 31, 33, 34, 45], nonsmooth multiobjective semi-infinite programming problems in the setting of Hadamard manifolds (abbreviated as, (NSIMPEC)) is investigated in this paper. We formulate the Wolfe type and Mond-Weir type dual models related to (NSIMPEC). We derive several duality results, namely, the weak duality, strong duality as well as strict converse duality results relating (NSIMPEC) and the corresponding dual models. Non-trivial numerical examples are incorporated to demonstrate the validity of results established in this paper.

The novelty and the contributions of the present paper are twofold. Firstly, the results that are derived in this article generalize the corresponding results deduced by [34] on Hadamard manifolds, which is a wider space, and for more general category of mathematical programming problems, that is, (NSIMPEC). Secondly, the results derived by [12, 22, 31] are extended to the setting of semi-infinite optimization problems on Hadamard manifolds by the results that are deduced in this article. To the best of our knowledge, this is for the first time that duality models for (NSIMPEC) have been investigated in the setting of Hadamard manifolds.

The remaining portion of the article unfolds in the following manner. We recall some basic definitions and mathematical preliminaries that will be helpful in Section 2. We formulate (NSIMPEC) in the Hadamard manifold setting and introduce (ACQ) for (NSIMPEC) in Section 3. Further, we present KKT type necessary criteria of optimality employing (ACQ). In Section 4, we formulate Wolfe and Mond-Weir dual problems related to (NSIMPEC) and derive several duality results relating (NSIMPEC) and the corresponding dual models. In Section 5, we draw conclusions to our work in this article and further discuss some future research directions.

2. Notations and mathematical preliminaries

We shall use the standard symbols $\mathbb{N}$ and $\mathbb{R}^n$ to signify the set consisting of every natural number and the Euclidean space having dimension n, respectively. The non-negative orthant of $\mathbb{R}^n$, denoted by $\mathbb{R}^n_+$, is defined as:

$$\mathbb{R}^n_+ := \{(z_1, z_2, ..., z_n) : z_j \geq 0, \quad \forall j = 1, 2, ..., n\}.$$

Let $\mathcal{A}$ be any arbitrary infinite set. The vector space denoted by the symbol $\mathbb{R}^{|\mathcal{A}|}$ is the set given by:

$$\mathbb{R}^{|\mathcal{A}|} := \{(\eta_l)_{l\in\mathcal{A}} : \eta_l = 0 \text{ for every } l \in \mathcal{A}, \text{ except } \eta_l \neq 0 \text{ for finitely many } l \in \mathcal{A}\}.$$

We use the symbol $\mathbb{R}^{|\mathcal{A}|}_{+}$ to denote the positive cone of the linear space $\mathbb{R}^{|\mathcal{A}|}$. That is, set theoretically, we have

$$\mathbb{R}^{|\mathcal{A}|}_{+} := \{\eta = (\eta_l)_{l\in\mathcal{A}} \in \mathbb{R}^{|\mathcal{A}|} : \eta_l \geq 0, \ \forall l \in \mathcal{A}\}.$$

Let $c, d \in \mathbb{R}^n$ be an arbitrary pair of vectors in $\mathbb{R}^n$. We shall use the following notation for inequalities in the sequel:

$$c < d \Longleftrightarrow c_j < d_j, \quad \forall j = 1, 2, \ldots, n.$$
$$c \leq d \Longleftrightarrow \begin{cases} c_j \leq d_j, & \forall j = 1, 2, \ldots, n, \\ c_l < d_l, & \text{for at least one } l \in \{1, 2, \ldots, n\}. \end{cases}$$

Let us consider any subset $\mathcal{B} \subset \mathbb{R}^n$. The linear hull, the closure and the convex hull of the set $\mathcal{B}$ in $\mathbb{R}^n$ is signified by the symbols span($\mathcal{B}$), cl($\mathcal{B}$) and co($\mathcal{B}$), respectively. Further, the notation pos($\mathcal{B}$) is employed to signify the positive conic hull of $\mathcal{B}$. The following sets will be employed in the sequel (see, [14]):

$$\mathcal{B}^{-} := \{u \in \mathbb{R}^n : u^T v \leq 0, \ \forall v \in \mathcal{B}\},$$
$$\mathcal{B}^{s} := \{u \in \mathbb{R}^n : u^T v < 0, \ \forall v \in \mathcal{B}\},$$
$$\mathcal{B}^{\perp} := \{u \in \mathbb{R}^n : u^T v = 0, \ \forall v \in \mathcal{B}\}.$$

Let $\mathcal{B}_1, \mathcal{B}_2 \subset \mathbb{R}^n$. Then the following relations are well-known:

$$\text{pos}(\mathcal{B}_1 \cup \mathcal{B}_2) = \text{pos}(\mathcal{B}_1) + \text{pos}(\mathcal{B}_2), \quad \text{span}(\mathcal{B}_1 \cup \mathcal{B}_2) = \text{span}(\mathcal{B}_1) + \text{span}(\mathcal{B}_2).$$

We shall be using the notation $\mathscr{M}$ to signify a smooth manifold having dimension n, where n is any natural number. Let $y^* \in \mathscr{M}$ be arbitrary. The set that contains every tangent vector at the element $y^* \in \mathscr{M}$ is known as the tangent space at y^*, and is signified by $T_{y^*}\mathscr{M}$. For any element $y^* \in \mathscr{M}$, $T_{y^*}\mathscr{M}$ is a real linear space, having a dimension n, $n \in \mathbb{N}$. In case we are restricted to real manifolds, $T_{y^*}\mathscr{M}$ is isomorphic to the n-dimensional Euclidean space $\mathbb{R}^n$.

A Riemannian metric, denoted by the notation $\mathscr{G}$ on the set $\mathscr{M}$ is a 2-tensor field that is symmetric as well as positive-definite. For every pair of elements $w_1, w_2 \in T_{y^*}\mathscr{M}$, the inner product of w_1 and w_2 is given by:

$$\langle w_1, w_2 \rangle_{y^*} = \mathscr{G}_{y^*}(w_1, w_2),$$

where the symbol $\mathscr{G}_{y^*}$ denotes the Riemannian metric at the element $y^* \in \mathscr{M}$. The norm corresponding to the inner product $\langle w_1, w_2 \rangle_{y^*}$ is denoted by $\| \cdot \|_{y^*}$ (or simply, $\| \cdot \|$, when there is no ambiguity regarding the subscript).

Let $a, b \in \mathbb{R}$, $a < b$ and $\nu : [a, b] \to \mathscr{M}$ be any piecewise differentiable curve that joins the elements y^* and $\hat{z}$ in $\mathscr{M}$. That is, we have:

$$\nu(a) = y^*, \ \nu(b) = \hat{z}.$$

The length of the curve ν is denoted by the notation $l(\nu)$ and is defined in the following manner:

$$l(\nu) := \int_a^b \|\nu'(t)\| dt.$$

For any differentiable curve ν, a vector field Y is referred to be parallel along the curve ν, provided that the following condition holds:

$$\nabla_{\nu'} Y = 0.$$

If $\nabla_{\nu'}\nu' = 0$, then ν is termed as a geodesic. If $\|\nu\| = 1$, then the curve ν is said to be normalised.

For every element $y^* \in \mathscr{M}$, the exponential function $\exp_{y^*} : T_{y^*}\mathscr{M} \to \mathscr{M}$ is given by $\exp_{y^*}(\hat{w}) = \nu(1)$, where ν is a geodesic which satisfies $\nu(0) = y^*$ and $\nu'(0) = \hat{w}$. A Riemannian manifold $\mathscr{M}$ is referred to as geodesic complete, provided that the exponential function $\exp_u(v)$ is defined for every arbitrary $v \in T_{y^*}\mathscr{M}$ and $u \in \mathscr{M}$.

A Riemannian manifold is referred to as a Hadamard manifold (or, Cartan-Hadamard manifold) provided that $\mathscr{M}$ is simply connected, geodesic complete as well as has a nonpositive sectional curvature throughout. Henceforth, in our discussions, the notation $\mathscr{M}$ will always signify a Hadamard manifold of dimension n, unless it is specified otherwise.

Let $y^* \in \mathscr{M}$ be some arbitrary element lying in the Hadamard manifold $\mathscr{M}$. Then, the exponential function on the tangent space $\exp_{y^*} : T_{y^*}\mathscr{M} \to \mathscr{M}$ is a globally diffeomorphic function. Moreover, the inverse of the exponential function $\exp^{-1}_{y^*} : \mathscr{M} \to T_{y^*}\mathscr{M}$ satisfies $\exp^{-1}_{y^*}(y^*) = 0$. Furthermore, for every pair of arbitrary elements $y_1^*, y_2^* \in \mathscr{M}$, there will always exist some unique normalized minimal geodesic $\nu_{y_1^*,y_2^*} : [0,1] \to \mathscr{M}$, such that the geodesic ν satisfies the following:

$$\nu_{y_1^*,y_2^*}(\tau) = \exp_{y_1^*}(\tau \exp{}^{-1}_{y_1^*}(y_2^*)), \quad \forall \tau \in [0,1].$$

Thus, every Hadamard manifold $\mathscr{M}$ of dimension n is diffeomorphic to the corresponding n-dimensional Euclidean space $\mathbb{R}^n$.

The following definition of contingent cone is from [17].

Definition 2.1. *Let $\mathcal{F} \subseteq \mathscr{M}$ and $z \in \mathrm{cl}(\mathcal{F})$. Then the contingent cone (in other terms, Bouligand tangent cone) of $\mathcal{F}$ at z is symbolized by $\mathscr{T}(\mathcal{F}, z)$, and is given by:*

$$\mathscr{T}(\mathcal{F}, z) := \{w \in T_z\mathscr{M} : \exists t_n \downarrow 0, \exists w_n \in T_z\mathscr{M}, w_n \to w, \forall n \in \mathbb{N}, exp_z(t_n w_n) \in \mathcal{F}\}.$$

The following definition is from Udrişte [35].

Definition 2.2. *Any subset $\mathcal{S}$ of a Hadamard manifold $\mathscr{M}$ termed as geodesic convex set, provided that every pair of distinct elements $z_1, z_2 \in \mathcal{S}$ and for the geodesic $\gamma_{z_1,z_2} : [0,1] \to \mathscr{M}$ that connects the elements z_1 and z_2, we have*

$$\gamma_{z_1,z_2}(t) \in \mathcal{S}, \quad \forall t \in [0,1],$$

where, $\gamma_{z_1,z_2}(t) = \exp_{z_1}\left(t \exp^{-1}_{z_1} z_2\right)$.

The following definitions are from Barani [2].

Definition 2.3. *Let $\Theta : \mathscr{M} \to \mathbb{R}$ be any locally Lipschitz function. Let $z_1, z_2 \in \mathscr{M}$ be arbitrary elements. Then, the generalized directional derivative of Θ at z_2 in the direction $w \in T_{z_2}\mathscr{M}$, is denoted by the symbol $\Theta^\circ(z_2; w)$, and is defined as follows*

$$\Theta^\circ(z_2; w) := \limsup_{z_1 \to z_2, t \downarrow 0} \frac{\Theta\left(\exp_{z_1} t \left(d \exp_{z_2}\right)_{\exp^{-1}_{z_2} z_1} w\right) - \Theta(z_1)}{t},$$

where $\left(d \exp_{z_2}\right)_{\exp^{-1}_{z_2} z_1} : T_{\exp^{-1}_{z_2} z_1}(T_{z_2}\mathscr{M}) \simeq T_{z_2}\mathscr{M} \to T_{z_1}\mathscr{M}$ is the differential of the exponential function at $\exp^{-1}_{z_2} z_1$.

Definition 2.4. *Let $\Theta : \mathscr{M} \to \mathbb{R}$ be any locally Lipschitz function. Then, the generalized gradient (in other words, Clarke subdifferential) of Θ at $z_1 \in \mathscr{M}$, denoted by $\partial_c\Theta(z_1)$, is a subset of $T_{z_1}\mathscr{M}$, and is defined by*

$$\partial_c\Theta(z_1) = \{\zeta \in T_{z_1}\mathscr{M} \mid \Theta^\circ(z_1; w) \geq \langle \zeta, w \rangle_{z_1}, \quad \forall w \in T_{z_1}\mathscr{M}\}.$$

The following definitions are from Chen and Fang [5].

Definition 2.5. *Let $D \subseteq \mathscr{M}$ be any geodesic convex set. Let $\Theta : D \to \mathbb{R}$ be any locally Lipschitz function.*
(i) The function $\Theta : D \to \mathbb{R}$ is termed as geodesic pseudoconvex at $z_2 \in D$, provided that for each $z_1 \in D$ and for any $\xi_i \in \partial_c \Theta(z_2)$ we have

$$\Theta(z_1) - \Theta(z_2) < 0 \implies \left\langle \xi_i, \exp_{z_2}^{-1}(z_1) \right\rangle_{z_2} < 0.$$

The function $\Theta : D \to \mathbb{R}$ is termed as geodesic strictly pseudoconvex at $z_2 \in D$, provided that for each $z_1 \in D$ and for any $\xi_i \in \partial_c \Theta(z_2)$ we have

$$\Theta(z_1) - \Theta(z_2) \leq 0 \implies \left\langle \xi_i, \exp_{z_2}^{-1}(z_1) \right\rangle_{z_2} < 0.$$

(ii) The function $\Theta : D \to \mathbb{R}$ is termed as geodesic quasiconvex at $z_2 \in D$, provided that for each $z_1 \in D$ and for any $\xi_i \in \partial_c \Theta(z_2)$ we have

$$\Theta(z_1) - \Theta(z_2) \leq 0 \implies \left\langle \xi_i, \exp_{z_2}^{-1}(z_1) \right\rangle_{z_2} \leq 0.$$

Remark 2.6. *(a) If $\Theta : D \to \mathbb{R}$ be a smooth function, then $\partial_c \Theta(z_2) = \{\operatorname{grad} \Theta(z_2)\}$. In that case, the definitions given above reduce to the corresponding definitions of smooth geodesic pseudoconvex and geodesic quasiconvex functions from [3].*

(b) If $\mathscr{M} = \mathbb{R}^n$, D be some convex subset of $\mathscr{M}$ and $\Theta : D \to \mathbb{R}$ be any smooth function, then, $\partial_c \Theta(z_2) = \{\operatorname{grad} \Theta(z_2)\} = \{\nabla \Theta(z_2)\}$, and $\exp_{z_2}^{-1}(z_1) = z_1 - z_2$. Consequently, the notions of geodesic (strict) pseudoconvexity and quasiconvexity defined above reduce to the corresponding well-known notions of differentiable (strict) pseudoconvexity and quasiconvexity from Mangasarian [18] for $\mathbb{R}^n$.

For further detailed exposition on Hadamard manifolds, we refer the readers to [30, 35, 38–40] and the references cited therein.

3. Problem formulation for (NSIMPEC)

We consider a nonsmooth multiobjective semi-infinite programming problem with equilibrium constraints defined on Hadamard manifolds (abbreviated as, (NSIMPEC)) as follows:

$$
\begin{aligned}
\text{(NSIMPEC)} \quad \text{Minimize} \quad & \Phi(y) := (\Phi_1(y), \ldots, \Phi_m(y)), \\
\text{subject to} \quad & \Psi_t(y) \leq 0, \quad \forall t \in \mathcal{T}, \\
& \vartheta_j(y) = 0, \quad \forall j \in \mathcal{I}^{\vartheta} := \{1, \ldots, q\}, \\
& \mathcal{M}_j(y) \geq 0, \quad \forall j \in \mathcal{S} := \{1, \ldots, p\}, \\
& \mathcal{N}_j(y) \geq 0, \quad \forall j \in \mathcal{S} := \{1, \ldots, p\}, \\
& \mathcal{N}_j(y)\mathcal{M}_j(y) = 0, \quad \forall j \in \mathcal{S} := \{1, \ldots, p\},
\end{aligned}
$$

where all the functions Φ_j ($j \in I := \{1, \ldots, m\}$), Ψ_t ($t \in \mathcal{T}$), ϑ_j ($j \in \mathcal{I}^{\vartheta}$) and $\mathcal{N}_j$, $\mathcal{M}_j$ ($j \in \mathcal{S}$) are locally Lipschitz real-valued functions defined on some finite-dimensional Hadamard manifold $\mathscr{M}$. The index set $\mathcal{T}$ is considered to be arbitrary (possibly infinite). The set containing all feasible elements of the problem (NSIMPEC) is signified by the symbol $\mathcal{F}$.

For any arbitrary $y^* \in \mathcal{F}$, we employ the notation $\mathscr{A}(y^*)$ to signify the set containing each of the active constraint multipliers at y^*, that is:

$$\mathscr{A}(y) := \{\eta \in \mathbb{R}_+^{|\mathcal{T}|} : \eta_t \Psi_t(y) = 0, \ \forall t \in \mathcal{T}\}.$$

The following definitions will be employed in the sequel (see, for instance, [34, 37]).

Definition 3.1. *Let $y^* \in \mathcal{F}$. The element y^* is termed as a Pareto efficient solution of (NSIMPEC), provided that there exists no other element $y \in \mathcal{F}$, satisfying:*

$$\Phi(y) \preceq \Phi(y^*).$$

Definition 3.2. *Let $y^* \in \mathcal{F}$. The element y^* is termed as a weak Pareto efficient solution of (NSIMPEC), provided that there exists no other element $y \in \mathcal{F}$, satisfying:*

$$\Phi(y) \prec \Phi(y^*).$$

Let $y^* \in \mathcal{F}$ be arbitrary. The index sets that are defined below will be crucial in the subsequent discussions of the article.

$$\begin{aligned}
L(y) &:= \{t \in \mathcal{T} \mid \Psi_t(y) = 0\},\\
J_+(y^*) &:= \left\{j \in \mathcal{S} \mid \mathcal{M}_j(y^*) > 0\right\},\\
J_0(y^*) &:= \left\{j \in \mathcal{S} \mid \mathcal{M}_j(y^*) = 0\right\},\\
J_{+0}(y^*) &:= \left\{j \in \mathcal{S} \mid \mathcal{M}_j(y^*) > 0, \mathcal{N}_j(y^*) = 0\right\},\\
J_{0+}(y^*) &:= \left\{j \in \mathcal{S} \mid \mathcal{M}_j(y^*) = 0, \mathcal{N}_j(y^*) > 0\right\},\\
J_{00}(y^*) &:= \left\{j \in \mathcal{S} \mid \mathcal{M}_j(y^*) = 0, \mathcal{N}_j(y^*) = 0\right\}.
\end{aligned}$$

Remark 3.3. *Every index set that is defined above obviously depends on the particular choice of feasible element $y^* \in \mathcal{F}$. Nevertheless, in the remaining portion of the article, we shall not indicate such dependence explicitly when it will be easily perceivable from the context.*

The following definition is an extension of the notion of strong stationary element of (NSIMPEC) given by Tung [34] from the setting of Euclidean space to the setting of Hadamard manifold.

Definition 3.4. *Any arbitrary feasible element $y^* \in \mathcal{F}$ is termed as a strong stationary element of (NSIMPEC), provided that there exist $\alpha \in \mathbb{R}^m_+$, $\sigma^\Psi \in \mathscr{A}(y^*)$, $\sigma^\vartheta \in \mathbb{R}^q$, $\sigma^\mathcal{M} \in \mathbb{R}^p$, $\sigma^\mathcal{N} \in \mathbb{R}^p$, satisfying:*

$$\begin{aligned}
0 \in \sum_{j\in I} \alpha_j \partial_c \Phi_j(y^*) + \sum_{t\in\mathcal{T}} \sigma^\Psi_t \partial_c \Psi_t(y^*) + \sum_{j\in\mathcal{I}^\vartheta} \sigma^\vartheta_j \partial_c \vartheta_j(y^*)&\\
- \sum_{j\in\mathcal{S}} \sigma^\mathcal{M}_j \partial_c \mathcal{M}_j(y^*) - \sum_{j\in\mathcal{S}} \sigma^\mathcal{N}_j \partial_c \mathcal{N}_j(y^*),&
\end{aligned}$$

$$\begin{aligned}
&\sigma^\mathcal{M}_j = 0, \quad \forall j \in J_{+0}(y^*), \quad \sigma^\mathcal{M}_j \geq 0, \quad \forall j \in J_{00}(y^*),\\
&\sigma^\mathcal{N}_j = 0, \quad \forall j \in J_{0+}(y^*), \quad \sigma^\mathcal{N}_j \geq 0, \quad \forall j \in J_{00}(y^*), \quad \textit{and} \sum_{j\in I} \alpha_j = 1.
\end{aligned}$$

Let $y^* \in \mathcal{F}$ be any feasible element of (NSIMPEC). Let $\sigma^\Psi \in \mathbb{R}^{|\mathcal{T}|}_+$, $\sigma^\vartheta \in \mathbb{R}^q$, $\sigma^\mathcal{M} \in \mathbb{R}^p$, $\sigma^\mathcal{N} \in \mathbb{R}^p$. We now define some index sets that will be employed in the rest of the article.

$$\begin{aligned}
&J^+_{\mathcal{I}^\vartheta}(y^*) := \left\{j \in \mathcal{I}^\vartheta(y^*) \mid \sigma^\vartheta_j > 0\right\}, \ J^-_{0+}(y^*) := \left\{j \in J_{0+}(y^*) \mid \sigma^\mathcal{M}_j < 0\right\},\\
&J^-_{\mathcal{I}^\vartheta}(y^*) := \left\{j \in \mathcal{I}^\vartheta(y^*) \mid \sigma^\vartheta_j < 0\right\}, \ \hat{J}^+_{+0}(y^*) := \left\{j \in J_{+0}(y^*) \mid \sigma^\mathcal{N}_j > 0\right\},\\
&J^+_{0+}(y^*) := \left\{j \in J_{0+}(y^*) \mid \sigma^\mathcal{M}_j > 0\right\}, \hat{J}^-_{+0}(y^*) := \left\{j \in J_{+0}(y^*) \mid \sigma^\mathcal{N}_j < 0\right\},
\end{aligned}$$

$$J_{00}^{+}(y^*) := \left\{ j \in J_{00}(y^*) \mid \sigma_j^{\mathcal{M}} > 0,\ \sigma_j^{\mathcal{N}} = 0 \right\},$$

$$J_{00}^{-}(y^*) := \left\{ j \in J_{00}(y^*) \mid \sigma_j^{\mathcal{M}} < 0,\ \sigma_j^{\mathcal{N}} = 0 \right\},$$

$$\hat{J}_{00}^{+}(y^*) := \left\{ j \in J_{00}(y^*) \mid \sigma_j^{\mathcal{M}} = 0,\ \sigma_j^{\mathcal{N}} > 0 \right\},$$

$$\hat{J}_{00}^{-}(y^*) := \left\{ j \in J_{00}(y^*) \mid \sigma_j^{\mathcal{M}} = 0,\ \sigma_j^{\mathcal{N}} < 0 \right\}.$$

The following definitions and theorem are from [33].

Definition 3.5. *Let $y^* \in \mathcal{F}$ be any arbitrary feasible element of (NSIMPEC). Then the linearized cone of (NSIMPEC) at y^*, denoted by $\mathscr{C}^{\mathrm{Lin}}(y^*)$, is defined as follows:*

$$\begin{aligned}
\mathscr{C}^{\mathrm{Lin}}(y^*) := \{ v \in T_{y^*}\mathscr{M} \mid \langle \xi_t^{\Psi}, v \rangle \le 0, &\quad \forall \xi_t^{\Psi} \in \partial_c \Psi_t(y^*),\ \forall t \in L,\\
\langle \xi_j^{\vartheta}, v \rangle = 0, &\quad \forall \xi_t^{\vartheta} \in \partial_c \vartheta_j(y^*),\ \forall j \in \mathcal{I}^{\vartheta},\\
\langle \xi_j^{\mathcal{M}}, v \rangle = 0, &\quad \forall \xi_j^{\mathcal{M}} \in \partial_c \mathcal{M}_j(y^*),\ \forall j \in J_{0+},\\
\langle \xi_j^{\mathcal{M}}, v \rangle \ge 0, &\quad \forall \xi_j^{\mathcal{M}} \in \partial_c \mathcal{M}_j(y^*),\ \forall j \in J_{00},\\
\langle \xi_j^{\mathcal{N}}, v \rangle \ge 0, &\quad \forall \xi_j^{\mathcal{N}} \in \partial_c \mathcal{N}_j(y^*),\ \forall j \in J_{00},\\
\langle \xi_j^{\mathcal{N}}, v \rangle = 0, &\quad \forall \xi_j^{\mathcal{N}} \in \partial_c \mathcal{N}_j(y^*),\ \forall j \in J_{+0} \}.
\end{aligned}$$

For any arbitrary feasible element $y^* \in \mathcal{F}$, we define the following sets for our convenience:

$$\mathcal{G}_{\Psi} := \bigcup_{t \in L} \partial_c \Psi_t(y^*), \quad \mathcal{G}_{\vartheta} := \bigcup_{j \in \mathcal{I}^{\vartheta}} \partial_c \vartheta_j(y^*), \quad \mathcal{G}_{\mathcal{M}_1} := \bigcup_{j \in J_{0+}} \partial_c \mathcal{M}_j(y^*),$$
$$\mathcal{G}_{\mathcal{M}_2} := \bigcup_{j \in J_{00}} -\partial_c \mathcal{M}_j(y^*), \quad \mathcal{G}_{\mathcal{N}_1} := \bigcup_{j \in J_{+0}} \partial_c \mathcal{N}_j(y^*), \quad \mathcal{G}_{\mathcal{N}_2} := \bigcup_{J_{00}} -\partial_c \mathcal{N}_j(y^*).$$

Remark 3.6. *From Definition 3.5, it readily follows that*

$$\mathscr{C}^{\mathrm{Lin}}(y^*) = (\mathcal{G}_{\Psi})^{-} \cap (\mathcal{G}_{\vartheta})^{\perp} \cap (\mathcal{G}_{\mathcal{M}_1})^{\perp} \cap (\mathcal{G}_{\mathcal{M}_2})^{-} \cap (\mathcal{G}_{\mathcal{N}_1}))^{\perp} \cap (\mathcal{G}_{\mathcal{N}_2}))^{-}.$$

Definition 3.7. *Let $y^* \in \mathcal{F}$ be any arbitrary feasible element of (NSIMPEC). The Abadie constraint qualification (abbreviated as, (ACQ)) holds at the point y^*, if the following inclusion is satisfied:*

$$\mathscr{C}^{\mathrm{Lin}}(y^*) \subseteq \mathscr{T}(\mathcal{F}, y^*).$$

Theorem 3.8. *Let $y^* \in \mathcal{F}$ be a weak Pareto efficient solution of (NSIMPEC). If (ACQ) holds at y^*, and the set $\Delta_1 := \mathrm{pos}\left(\mathcal{G}_{\psi} \cup \mathcal{G}_{\mathcal{M}_2} \cup \mathcal{G}_{\mathcal{N}_2}\right) + \mathrm{span}\left(\mathcal{G}_{\vartheta} \cup \mathcal{G}_{\mathcal{M}_1} \cup \mathcal{G}_{\mathcal{N}_1}\right)$ is closed, then y^* is a strong stationary element of (NSIMPEC).*

4. Duality

In the present section, the Wolfe type, as well as the Mond-Weir type dual models related to (NSIMPEC) are formulated. Further, weak, strong as well as strict converse duality results relating (NSIMPEC) and the corresponding dual models are established by invoking generalized geodesic convexity hypothesis.

4.1. Wolfe type duality

Let $y^* \in \mathcal{F}$ be any arbitrary feasible element of (NSIMPEC) and let $w \in \mathscr{M}$. Let $\alpha \in \mathbb{R}^m_+$, $\sigma = (\sigma^\Psi, \sigma^\vartheta, \sigma^{\mathcal{M}}, \sigma^{\mathcal{N}}) \in \mathbb{R}^{|\mathcal{T}|}_+ \times \mathbb{R}^q \times \mathbb{R}^p \times \mathbb{R}^p$ and $e := (1, \ldots, 1) \in \mathbb{R}^m$. The corresponding Wolfe type dual model related to (NSIMPEC) depending on the feasible element $y^* \in \mathcal{F}$ is denoted by WD(y^*) and is formulated as:

$$(WD(y^*)) \quad \text{Maximize } \mathscr{L}(w, \alpha, \sigma) := \Phi(w) + \Big(\sum_{t \in \mathcal{T}} \sigma_t^\Psi \Psi_t(w) + \sum_{j \in \mathcal{I}^\vartheta} \sigma_j^\vartheta \vartheta_j(w) - \sum_{j \in \mathcal{S}} \sigma_j^{\mathcal{M}} \mathcal{M}_j(w) - \sum_{j \in \mathcal{S}} \sigma_j^{\mathcal{N}} \mathcal{N}_j(w) \Big) e,$$

$$\text{subject to } 0 \in \sum_{j \in I} \alpha_j \partial_c \Phi_j(w) + \sum_{t \in \mathcal{T}} \sigma_t^\Psi \partial_c \Psi_t(w) + \sum_{j \in \mathcal{I}^\vartheta} \sigma_j^\vartheta \partial_c \vartheta_j(w) - \sum_{j \in \mathcal{S}} \sigma_j^{\mathcal{M}} \partial_c \mathcal{M}_j(w) - \sum_{j \in \mathcal{S}} \sigma_j^{\mathcal{N}} \partial_c \mathcal{N}_j(w),$$

$$\sigma_j^{\mathcal{M}} \geq 0, \quad \forall j \in J_{00}(y^*), \quad \sigma_j^{\mathcal{N}} \geq 0, \quad \forall j \in J_{00}(y^*),$$
$$\sigma_j^{\mathcal{M}} = 0, \quad \forall j \in J_{+0}(y^*), \quad \sigma_j^{\mathcal{N}} = 0, \quad \forall j \in J_{0+}(y^*), \text{ and } \sum_{j \in I} \alpha_j = 1.$$

The set containing each feasible point of WD(y^*) is denoted by $\mathcal{F}_W(y^*)$. We define an auxiliary function $\mathcal{H} : \mathscr{M} \to \mathbb{R}$ as follows

$$\mathcal{H}(y) := \sum_{j \in I} \alpha_j \Phi_j(y) + \sum_{t \in \mathcal{T}} \sigma_j^\Psi \Psi_t(y) + \sum_{j \in I^\vartheta} \sigma_j^\vartheta \vartheta_j(y) - \sum_{j \in \mathcal{S}} \Big[\sigma_j^{\mathcal{M}} \mathcal{M}_j(y) + \sigma_j^{\mathcal{N}} \mathcal{N}_j(y) \Big],$$

for every $y \in \mathscr{M}$. In the following theorem, we derive weak duality relations that relate (NSIMPEC) and (WD)(y^*).

Theorem 4.1. *Let $y^* \in \mathcal{F}$ and $(w, \alpha, \sigma) \in \mathcal{F}_W(y^*)$ be arbitrary feasible elements of (NSIMPEC) and (WD(y^*)) respectively. Further, let us suppose that $J_{0+}^- \cup \hat{J}_{+0}^- = \emptyset$. Then the following assertions hold true.*

(i) If for every $j \in I$, $\mathcal{H}$ is a geodesic pseudoconvex function at w, then

$$\Phi(y^*) \nless \mathscr{L}(w, \alpha, \sigma).$$

(ii) If for every $j \in I$, Φ_j is a strictly geodesic pseudoconvex function at w, then

$$\Phi(y^*) \nleq \mathscr{L}(w, \alpha, \sigma).$$

Proof. From the given hypothesis, we have $y^* \in \mathcal{F}$. Then it follows that

$$\Psi_t(y^*) \leq 0, \quad \forall t \in \mathcal{T}, \quad \vartheta_j(y^*) = 0, \quad \forall j \in \mathcal{I}^\vartheta,$$
$$\mathcal{M}_j(y^*), \mathcal{N}_j(y^*) \geq 0, \quad \forall j \in \mathcal{S}, \quad \mathcal{N}_j(y^*) \mathcal{M}_j(y^*) = 0, \quad \forall j \in \mathcal{S}.$$

Again, we have $(w, \alpha, \sigma) \in \mathcal{F}_W(y^*)$. Therefore, there exist $\xi_j^\Phi \in \partial_c \Phi_j(w)$ $(j \in I)$, $\xi_t^\Psi \in \partial_c \Psi_t(w)$ $(t \in \mathcal{T})$, $\xi_j^\vartheta \in \partial_c \vartheta_j(w)$ $(j \in \mathcal{I}^\vartheta)$, $\xi_j^{\mathcal{M}} \in \partial_c \mathcal{M}_j(w)$ $(j \in \mathcal{S})$, $\xi_j^{\mathcal{N}} \in \partial_c \mathcal{N}_j(w)$ $(j \in \mathcal{S})$, such that

$$\sum_{j \in I} \alpha_j \xi_j^\Phi + \sum_{t \in \mathcal{T}} \sigma_t^\Psi \xi_t^\Psi + \sum_{j \in \mathcal{I}^\vartheta} \sigma_j^\vartheta \xi_j^\vartheta - \sum_{j \in \mathcal{S}} \sigma_j^{\mathcal{M}} \xi_j^{\mathcal{M}} - \sum_{j \in \mathcal{S}} \sigma_j^{\mathcal{N}} \xi_j^{\mathcal{N}} = 0, \tag{1}$$

$$\sigma_j^{\mathcal{M}} = 0, \quad \forall j \in J_{+0}(y^*), \quad \sigma_j^{\mathcal{M}} \geq 0, \quad \forall j \in J_{00}(y^*),$$
$$\sigma_j^{\mathcal{N}} = 0, \quad \forall j \in J_{0+}(y^*), \quad \sigma_j^{\mathcal{N}} \geq 0, \quad \forall j \in J_{00}(y^*), \quad \text{and } \sum_{j \in I} \alpha_j = 1.$$

(i) On contrary, we suppose that $\Phi(y) < \mathcal{L}(w,\alpha,\sigma)$. Then, we have

$$\Phi_i(y^*) < \Phi_i(w) + \sum_{j\in L}\sigma_j^{\Psi}\Psi_j(w) + \sum_{j\in I^{\vartheta}}\sigma_j^{\vartheta}\vartheta_j(w) - \sum_{j\in\mathcal{S}}\left[\sigma_j^{\mathcal{M}}\mathcal{M}_j(w) + \sigma_j^{\mathcal{N}}\mathcal{N}_j(w)\right],$$

for all $i \in I$. Since $y^* \in \mathcal{F}$, $(w,\alpha,\sigma) \in \mathcal{F}_W(y^*)$, $\alpha \in \mathbb{R}^m_+$, it follows that

$$\begin{aligned}&\sum_{i\in I}\alpha_i\Phi_i(y^*) + \sum_{j\in L}\sigma_j^{\Psi}(y^*) + \sum_{j\in I^{\vartheta}}\sigma_j^{\vartheta}\vartheta_j(y^*) - \sum_{j\in\mathcal{S}}\left[\sigma_j^{\mathcal{M}}\mathcal{M}_j(y^*) + \sigma_j^{\mathcal{N}}\mathcal{N}_j(y^*)\right]\\ &< \sum_{i\in I}\alpha_i\Phi_i(w) + \sum_{j\in L}\sigma_j^{\Psi}(w) + \sum_{j\in I^{\vartheta}}\sigma_j^{\vartheta}\vartheta_j(w) - \sum_{j\in\mathcal{S}}\left[\sigma_j^{\mathcal{M}}\mathcal{M}_j(w) + \sigma_j^{\mathcal{N}}\mathcal{N}_j(w)\right].\end{aligned} \tag{2}$$

From the definition of $\mathcal{H}$, it follows from (2) that $\mathcal{H}(y^*) < \mathcal{H}(w)$. By invoking the geodesic pseudoconvexity restriction on $\mathcal{H}$ at w, we get

$$\langle \xi^{\mathcal{H}}, \exp_w^{-1}(y^*)\rangle_w < 0, \quad \forall \xi^{\mathcal{H}} \in \partial_c\mathcal{H}(w),$$

which is a contradiction to (1). Therefore, the proof is complete.

(ii) On contrary, we suppose that $\Phi(y) \leq \mathcal{L}(w,\alpha,\sigma)$. Then there exists some $k \in I$, such that

$$\Phi_i(y^*) \leq \Phi_i(w) + \sum_{j\in L}\sigma_j^{\Psi}\Psi_j(w) + \sum_{j\in I^{\vartheta}}\sigma_j^{\vartheta}\vartheta_j(w) - \sum_{j\in\mathcal{S}}\left[\sigma_j^{\mathcal{M}}\mathcal{M}_j(w) + \sigma_j^{\mathcal{N}}\mathcal{N}_j(w)\right], \tag{3}$$

for all $i \in I$, $i \neq k$ and the above inequality holds strictly for $i = k$. Since $y \in \mathcal{F}$, $(w,\alpha,\sigma) \in \mathcal{F}_W(y^*)$, $\alpha \in \mathbb{R}^m_+$, it follows from (3) that

$$\begin{aligned}&\sum_{i\in I}\alpha_i\Phi_i(y^*) + \sum_{j\in L}\sigma_j^{\Psi}(y^*) + \sum_{j\in I^{\vartheta}}\sigma_j^{\vartheta}\vartheta_j(y^*) - \sum_{j\in\mathcal{S}}\left[\sigma_j^{\mathcal{M}}\mathcal{M}_j(y^*) + \sigma_j^{\mathcal{N}}\mathcal{N}_j(y^*)\right]\\ &\leq \sum_{i\in I}\alpha_i\Phi_i(w) + \sum_{j\in L}\sigma_j^{\Psi}(w) + \sum_{j\in I^{\vartheta}}\sigma_j^{\vartheta}\vartheta_j(w) - \sum_{j\in\mathcal{S}}\left[\sigma_j^{\mathcal{M}}\mathcal{M}_j(w) + \sigma_j^{\mathcal{N}}\mathcal{N}_j(w)\right].\end{aligned} \tag{4}$$

From the definition of $\mathcal{H}$, it follows from (4) that $\mathcal{H}(y^*) \leq \mathcal{H}(w)$. By invoking the strict geodesic pseudoconvexity restriction on $\mathcal{H}$ at w, we get

$$\langle \xi^{\mathcal{H}}, \exp_w^{-1}(y^*)\rangle_w < 0, \quad \forall \xi^{\mathcal{H}} \in \partial_c\mathcal{H}(w),$$

which is a contradiction to (1). Therefore, the proof is complete. □

Remark 4.2. *(a) If $\mathscr{M} = \mathbb{R}^n$ and each of the objective and constraint functions of (NSIMPEC) is differentiable, then, Theorem 4.1 reduces to Proposition 15 derived by Tung [34].*

(b) Theorem 4.1 extends Theorem 4 of Singh and Mishra [31] for more general category of optimization problems and generalizes it from $\mathbb{R}^n$ to Hadamard manifolds.

In the following theorem, we provide strong duality result relating (NSIMPEC) and (WD(y^*)) by employing certain generalized geodesic convexity assumptions.

Theorem 4.3. *Let us assume that $y^* \in \mathcal{F}$ be a weak Pareto efficient solution of (NSIMPEC) such that (ACQ) holds at y^* and the set $\Delta_1 := \mathrm{pos}\left(\mathcal{G}_{\psi} \cup \mathcal{G}_{\mathcal{M}_2} \cup \mathcal{G}_{\mathcal{N}_2}\right) + \mathrm{span}\left(\mathcal{G}_{\vartheta} \cup \mathcal{G}_{\mathcal{M}_1} \cup \mathcal{G}_{\mathcal{N}_1}\right)$ is closed. Further, let us suppose that $J^-_{0+} \cup \hat{J}^-_{+0} = \emptyset$. Then there exist $\overline{\alpha} \in \mathbb{R}^m_+$, $\overline{\sigma}^{\Psi} \in \mathbb{R}^{|\mathcal{T}|}_+$, $\overline{\sigma}^{\vartheta} \in \mathbb{R}^q$, $\overline{\sigma}^{\mathcal{M}} \in \mathbb{R}^p$, $\overline{\sigma}^{\mathcal{N}} \in \mathbb{R}^p$ such that $(y^*,\overline{\alpha},\overline{\sigma}) \in \mathcal{F}_W(y^*)$ and*

$$\Phi(y^*) = \mathscr{L}(y^*,\overline{\alpha},\overline{\sigma}).$$

Further, the following assertions hold true.

(i) If every assumption of weak duality theorem (Theorem 4.1 (i)) are satisfied, then $(y^*, \overline{\alpha}, \overline{\sigma})$ is a weak Pareto efficient solution of $WD(y^*)$.

(ii) If every assumption of weak duality theorem (Theorem 4.1 (ii)) are satisfied, then $(y^*, \overline{\alpha}, \overline{\sigma})$ is a Pareto efficient solution of $WD(y^*)$.

Proof. According to the given hypothesis, $y^* \in \mathcal{F}$ be a weak Pareto efficient solution of (NSIMPEC) and (ACQ) holds at y^*. Then from Theorem 3.8 it follows that there exist $\overline{\alpha} \in \mathbb{R}^m_+$, $\overline{\sigma}^\Psi \in \mathscr{A}(y^*)$, $\overline{\sigma}^\vartheta \in \mathbb{R}^q$, $\overline{\sigma}^\mathcal{M} \in \mathbb{R}^p$, $\overline{\sigma}^\mathcal{N} \in \mathbb{R}^p$, $\xi^\Phi_j \in \partial_c \Phi_j(y^*)$ $(j \in I)$, $\xi^\Psi_t \in \partial_c \Psi_t(y^*)$ $(t \in \mathcal{T})$, $\xi^\vartheta_j \in \partial_c \vartheta_j(y^*)$ $(j \in \mathcal{I}^\vartheta)$, $\xi^\mathcal{M}_j \in \partial_c \mathcal{M}_j(y^*)$ $(j \in \mathcal{S})$, $\xi^\mathcal{N}_j \in \partial_c \mathcal{N}_j(y^*)$ $(j \in \mathcal{S})$, such that the following conditions are satisfied

$$\sum_{j\in I} \overline{\alpha}_j \xi^\Phi_j + \sum_{t\in\mathcal{T}} \overline{\sigma}^\Psi_t \xi^\Psi_t + \sum_{j\in\mathcal{I}^\vartheta} \overline{\sigma}^\vartheta_j \xi^\vartheta_j - \sum_{j\in\mathcal{S}} \overline{\sigma}^\mathcal{M}_j \xi^\mathcal{M}_j + \sum_{j\in\mathcal{S}} \overline{\sigma}^\mathcal{N}_j \xi^\mathcal{N}_j = 0,$$

$$\overline{\sigma}^\mathcal{M}_j = 0, \quad \forall j \in J_{+0}(y^*), \quad \overline{\sigma}^\mathcal{M}_j \geq 0, \quad \forall j \in J_{00}(y^*),$$
$$\overline{\sigma}^\mathcal{N}_j = 0, \quad \forall j \in J_{0+}(y^*), \quad \overline{\sigma}^\mathcal{N}_j \geq 0, \quad \forall j \in J_{00}(y^*), \quad \text{and} \sum_{j\in I} \overline{\alpha}_j = 1.$$

Since $\overline{\sigma}^\Psi \in \mathscr{A}(y^*)$, we have $\overline{\sigma}^\Psi_t \Psi_t(y^*) = 0,\ \forall t \in \mathcal{T}$. Then, it follows that

$$\sum_{t\in\mathcal{T}} \overline{\sigma}^\Psi_t \Psi_t(y^*) = 0. \tag{5}$$

Again, since $y^* \in \mathcal{F}$ we obtain the following

$$\sum_{j\in\mathcal{I}^\vartheta} \overline{\sigma}^\vartheta_j \vartheta_j(y^*) = 0. \tag{6}$$

Since $\overline{\sigma}^\mathcal{M}_j = 0$, for every $j \in J_{+0}(y^*)$ and $\overline{\sigma}^\mathcal{M}_j \geq 0$, for every $j \in J_{00}(y^*)$, it follows that

$$\sum_{j\in\mathcal{S}} \overline{\sigma}^\mathcal{M}_j \mathcal{M}_j(y^*) = 0. \tag{7}$$

Similarly, as $\overline{\sigma}^\mathcal{N}_j = 0$, for every $j \in \cup J_{0+}(y^*)$, and $\overline{\sigma}^\mathcal{N}_j \geq 0$, for every $j \in J_{00}(y^*)$, it follows that

$$\sum_{j\in\mathcal{S}} \overline{\sigma}^\mathcal{N}_j \mathcal{N}_j(y^*) = 0. \tag{8}$$

Thus, we infer that $(y^*, \overline{\alpha}, \overline{\sigma}) \in \mathcal{F}_W(y^*)$. Moreover, it follows from (5), (6), (7) and (8) that

$$\sum_{t\in\mathcal{T}} \overline{\sigma}^\Psi_t \Psi^\Psi_t(y^*) + \sum_{j\in\mathcal{I}^\vartheta} \overline{\sigma}^\vartheta_j \vartheta_j(y^*) - \sum_{j\in\mathcal{S}} \overline{\sigma}^\mathcal{M}_j \mathcal{M}_j(y^*) - \sum_{j\in\mathcal{S}} \overline{\sigma}^\mathcal{N}_j \mathcal{N}_j(y^*) = 0.$$

Hence we have

$$\Phi(y^*) = \mathscr{L}(y^*, \overline{\alpha}, \overline{\sigma}).$$

(i) On contrary, we suppose that $(y^*, \overline{\alpha}, \overline{\sigma})$ is not a weak Pareto efficient solution of $DW(y^*)$. Then there exists $(w, \alpha', \sigma') \in \mathcal{F}_W(y^*)$ such that

$$\mathscr{L}(y^*, \overline{\alpha}, \overline{\sigma}) \prec \mathscr{L}(w, \alpha', \sigma').$$

Then it follows that

$$\Phi(y^*) < \mathscr{L}(w, \alpha', \sigma').$$

which contradicts Theorem 4.1(i). Therefore, the proof is complete.

(ii) On contrary, we suppose that $(y^*, \overline{\alpha}, \overline{\sigma})$ is not a Pareto efficient solution of $DW(y^*)$. Then there exists $(w, \alpha', \sigma') \in \mathcal{F}_W(y^*)$ such that

$$\mathscr{L}(y^*, \overline{\alpha}, \overline{\sigma}) \leq \mathscr{L}(w, \alpha', \sigma').$$

Then it follows that

$$\Phi(y^*) \leq \mathscr{L}(w, \alpha', \sigma').$$

which contradicts Theorem 4.1(ii). Therefore, the proof is complete. □

Remark 4.4. *(a) If $\mathscr{M} = \mathbb{R}^n$ and each of the objective and constraint functions of (NSIMPEC) is differentiable, then, Theorem 4.1 reduces to Proposition 16 derived by Tung [34].*

(b) Theorem 4.1 is an extension of Theorem 5 from Singh and Mishra [31] for more general category of optimization problems, and further generalizes it from $\mathbb{R}^n$ to Hadamard manifolds.

In the following theorem we provide strict converse duality theorem relating (NSIMPEC) and (WD(y^*)).

Theorem 4.5. *Let us assume that $y^* \in \mathcal{F}$ be a weak Pareto efficient solution of (NSIMPEC). Let $(w, \alpha, \sigma) \in \mathcal{F}_W(y^*)$ be a weak Pareto efficient solution of (WD(y^*)), such that, $\Phi(y^*) \preceq \mathscr{L}(w, \alpha, \sigma)$. If every assumption of the weak duality theorem (Theorem 4.1 (ii)) are satisfied, then $y^* = w$.*

Proof. On contrary, we suppose that $y^* \neq w$. From the given hypothesis, we have $y^* \in \mathcal{F}$. Then it follows that

$$\begin{aligned} \Psi_t(y^*) \leq 0, \quad \forall t \in \mathcal{T}, \quad \vartheta_j(y^*) = 0, \quad \forall j \in \mathcal{I}^{\vartheta}, \\ \mathcal{M}_j(y^*), \mathcal{N}_j(y^*) \geq 0, \quad \forall j \in \mathcal{S}, \quad \mathcal{N}_j(y^*)\mathcal{M}_j(y^*) = 0, \quad \forall j \in \mathcal{S}. \end{aligned}$$

Again, we have $(w, \alpha, \sigma) \in \mathcal{F}_W(y^*)$. Therefore, there exist $\xi_j^{\Phi} \in \partial_c \Phi_j(w)$ $(j \in I)$, $\xi_t^{\Psi} \in \partial_c \Psi_t(w)$ $(t \in \mathcal{T})$, $\xi_j^{\vartheta} \in \partial_c \vartheta_j(w)$ $(j \in \mathcal{I}^{\vartheta})$, $\xi_j^{\mathcal{M}} \in \partial_c \mathcal{M}_j(w)$ $(j \in \mathcal{S})$, $\xi_j^{\mathcal{N}} \in \partial_c \mathcal{N}_j(w)$ $(j \in \mathcal{S})$, such that

$$\sum_{j \in I} \alpha_j \xi_j^{\Phi} + \sum_{t \in \mathcal{T}} \sigma_t^{\Psi} \xi_t^{\Psi} + \sum_{j \in \mathcal{I}^{\vartheta}} \sigma_j^{\vartheta} \xi_j^{\vartheta} - \sum_{j \in \mathcal{S}} \sigma_j^{\mathcal{M}} \xi_j^{\mathcal{M}} - \sum_{j \in \mathcal{S}} \sigma_j^{\mathcal{N}} \xi_j^{\mathcal{N}} = 0, \tag{9}$$

$$\begin{aligned} &\sigma_j^{\mathcal{M}} = 0, \quad \forall j \in J_{+0}(y^*), \quad \sigma_j^{\mathcal{M}} \geq 0, \quad \forall j \in J_{00}(y^*), \\ &\sigma_j^{\mathcal{N}} = 0, \quad \forall j \in J_{0+}(y^*), \quad \sigma_j^{\mathcal{N}} \geq 0, \quad \forall j \in J_{00}(y^*), \quad \text{and} \sum_{j \in I} \alpha_j = 1. \end{aligned}$$

Then there exists some $k \in I$, such that

$$\Phi_i(y^*) \leq \Phi_i(w) + \sum_{j \in L} \sigma_j^{\Psi} \Psi_j(w) + \sum_{j \in I^{\vartheta}} \sigma_j^{\vartheta} \vartheta_j(w) - \sum_{j \in \mathcal{S}} \left[\sigma_j^{\mathcal{M}} \mathcal{M}_j(w) + \sigma_j^{\mathcal{N}} \mathcal{N}_j(w) \right],$$

for all $i \in I$, and the above inequality holds strictly for $i = k$. Since $y^* \in \mathcal{F}$, $(w, \alpha, \sigma) \in \mathcal{F}_W(y^*)$, $\alpha \in \mathbb{R}_+^m$, it follows that

$$\begin{aligned} &\sum_{i \in I} \alpha_i \Phi_i(y^*) + \sum_{j \in L} \sigma_j^{\Psi}(y^*) + \sum_{j \in I^{\vartheta}} \sigma_j^{\vartheta} \vartheta_j(y^*) - \sum_{j \in \mathcal{S}} \left[\sigma_j^{\mathcal{M}} \mathcal{M}_j(y^*) + \sigma_j^{\mathcal{N}} \mathcal{N}_j(y^*) \right] \\ &\leq \sum_{i \in I} \alpha_i \Phi_i(w) + \sum_{j \in L} \sigma_j^{\Psi}(w) + \sum_{j \in I^{\vartheta}} \sigma_j^{\vartheta} \vartheta_j(w) - \sum_{j \in \mathcal{S}} \left[\sigma_j^{\mathcal{M}} \mathcal{M}_j(w) + \sigma_j^{\mathcal{N}} \mathcal{N}_j(w) \right]. \end{aligned} \tag{10}$$

From the definition of $\mathcal{H}$, it follows from (10) that $\mathcal{H}(y^*) < \mathcal{H}(w)$. By invoking the geodesic pseudoconvexity restriction on $\mathcal{H}$ at w, we get

$$\langle \xi^{\mathcal{H}}, \exp_w^{-1}(y^*)\rangle_w < 0, \quad \forall \xi^{\mathcal{H}} \in \partial_c \mathcal{H}(w),$$

which is a contradiction to (9). Therefore, the proof is complete. □

In the following numerical example, we illustrate the significance of Wolfe type dual model related to (NSIMPEC).

Example 4.6. *Consider the set $\mathscr{M} \subset \mathbb{R}$ defined by $\mathscr{M} := \{z \in \mathbb{R} : z > 0\}$. Then the set $\mathscr{M}$ is a Riemannian manifold (see, [26, 30]). At any $\hat{z} \in \mathscr{M}$, the tangent space is given by $T_{\hat{z}}\mathscr{M} = \mathbb{R}$. The corresponding metric on $\mathscr{M}$ is given by:*

$$\langle w_1, w_2\rangle_{\hat{z}} = \langle \mathscr{G}(\hat{z})w_1, w_2\rangle, \quad \forall w_1, w_2 \in T_{\hat{z}}\mathscr{M} = \mathbb{R},$$

where,

$$\mathscr{G}(\hat{z}) = \frac{1}{\hat{z}^2}.$$

It is well-known that $\mathscr{M}$ is also a Hadamard manifold. The inverse of the exponential function $exp_{\hat{z}} : T_{\hat{z}}\mathscr{M} \to \mathscr{M}$ for any $v \in T_{\hat{z}}\mathscr{M}$ is given by

$$exp_{\hat{z}}(v) = (\hat{z}_1 e^{\frac{v_1}{\hat{z}_1}}, \hat{z}_2 e^{\frac{v_2}{\hat{z}_2}}), \quad \forall v = (v_1, v_2) \in \mathscr{M}.$$

Consider the following problem (P1), which is a (NSIMPEC).

$$\begin{aligned} (P1) \quad & Minimize\ \Phi(y) = (\Phi_1(y), \Phi_2(y)) := (|y|, y^2 + 7), \\ & subject\ to\ \ \Psi_t(y) := t(y - e) \leq 0, \quad t \in \mathbb{N}, \\ & \mathcal{M}(y) := y - e \geq 0, \\ & \mathcal{N}(y) := \ln y - 1 \geq 0, \\ & \mathcal{N}(y)\mathcal{M}(y) := (\ln y - 1)(y - e) = 0, \end{aligned}$$

where $\Phi_i : \mathscr{M} \to \mathbb{R}$, $(i = 1, 2)$, $\Psi_t : \mathscr{M} \to \mathbb{R}$, $(t \in \mathbb{N})$ and $\mathcal{M}, \mathcal{N} : \mathscr{M} \to \mathbb{R}$ are locally Lipschitz functions. The feasible set for (P1) is denoted by $\mathcal{F}$. We choose the feasible point $y^* = e \in \mathcal{F}$. Clearly y^* is a Pareto efficient solution of (P1). Then it follows that

$$\begin{aligned} &\partial_c \Phi_1(y) = \text{co}\{-y^2, y^2\}, \\ &\partial_c \Phi_2(y) = \{\mathscr{G}(y)^{-1}(2y)\} = \{2y^3\}, \\ &\partial_c \Psi_t(y) = \{\mathscr{G}(y)^{-1}(t)\} = \{ty^2\}, \quad \forall t \in \mathbb{N}, \\ &\partial_c \mathcal{M}(y) = \{\mathscr{G}(y)^{-1}(1)\} = \{y^2\}, \\ &\partial_c \mathcal{N}(y) = \{\mathscr{G}(y)^{-1}\left(\frac{1}{y}\right)\} = \{y\}. \end{aligned}$$

Using above equations, it can be verified that

$$\mathscr{C}^{\text{Lin}}(e) = \{0\}. \tag{11}$$

Furthermore, for the problem (P1), we can show by simple calculations that $\mathscr{T}(\mathcal{F}, e) = \{0\}$. Then it follows that (ACQ) is satisfied at the point $y^* = e \in \mathcal{F}$. The corresponding Wolfe dual problem related to (P1), denoted by (WD)(y^*), is given by:

$$(WD)(y^*) \quad Maximize\ \mathscr{L}(w, \alpha, \sigma) = \Phi(w) + \Big[\sum_{t \in \mathbb{N}} \sigma_t^{\Psi}\Psi_t(w) - \sigma^{\mathcal{M}}\mathcal{M}(w) - \sigma^{\mathcal{N}}\mathcal{N}(w)\Big]e,$$

$$\text{subject to } 0 \in \sum_{j=1}^{2} \alpha_j \partial_c \Phi_j(w) + \sum_{t\in\mathbb{N}} \sigma_t^{\Psi} \partial_c \Psi_t(w) - \sigma^{\mathcal{M}} \partial_c \mathcal{M}(w) + \sigma^{\mathcal{N}} \partial_c \mathcal{N}(w), \tag{12}$$

$$\sigma^{\mathcal{M}} \geq 0,\ \sigma^{\mathcal{N}} \geq 0, \text{ and } \sum_{j\in I} \alpha_j = 1.$$

where, $\sigma = (\sigma^{\Psi}, \sigma^{\mathcal{M}}, \sigma^{\mathcal{N}}) \in \mathbb{R}_+^{|\mathbb{N}|} \times \mathbb{R} \times \mathbb{R}$, $\alpha_j \in \mathbb{R}$, $\alpha_j \geq 0$, $j = 1, 2$. *Let the feasible set of (WD)(y^*) be denoted by (F_W). We can further verify the fact that* $y^* = (e, e)$ *is a weak Pareto efficient solution of (P1).*
We consider a map $\sigma_t^{\Psi} : \mathbb{N} \to \mathbb{R}$, *such that*

$$\sigma_t^{\Psi} = \begin{cases} e, & \text{if } t = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Then for $\alpha_i = \frac{1}{2}$, $(i = 1, 2)$, $\xi_1^{\Phi} = 0 \in \partial_c \Phi_1(e)$, $\xi_2^{\Phi} = 2e^3 \in \partial_c \Phi_2(e)$, σ_t^{Ψ} *as defined above,* $\xi_t^{\Psi} = te^2 \in \partial_c \Psi_t(e)$, *for all* $t \in \mathbb{N}$, $\xi^{\mathcal{M}} = e^2 \in \partial_c \mathcal{M}(e)$, $\xi^{\mathcal{N}} = e \in \partial_c \mathcal{N}(e)$, $\sigma^{\mathcal{M}} = e$ *and* $\sigma^{\mathcal{N}} = e^2$, *we have*

$$\alpha_1 \xi_1^{\Phi} + \alpha_2 \xi_2^{\Phi} + \sum_{t\in\mathbb{N}} \sigma_t^{\Psi} \xi_t^{\Psi} - \sigma^{\mathcal{M}} \xi^{\mathcal{M}} - \sigma^{\mathcal{N}} \xi^{\mathcal{N}} = 0. \tag{13}$$

This shows that $(y^*, \alpha, \sigma) \in \mathcal{F}_W(y^*)$. *Furthermore*

$$\Phi(y^*) = \mathscr{L}(y^*, \alpha, \sigma).$$

One can very easily verify that every assumption of strong duality theorem (Theorem 4.3) is satisfied. Consequently, it follows that (y^*, α, σ) *is a weak Pareto efficient solution of (WD)(*y^**).*

4.2. Mond-Weir type duality

Let $y^* \in \mathcal{F}$ be any arbitrary feasible element of (NSIMPEC) and let $w \in \mathscr{M}$. Let $\alpha \in \mathbb{R}_+^m$, $\sigma = (\sigma^{\Psi}, \sigma^{\vartheta}, \sigma^{\mathcal{M}}, \sigma^{\mathcal{N}}) \in \mathbb{R}_+^{|\mathcal{T}|} \times \mathbb{R}^q \times \mathbb{R}^p \times \mathbb{R}^p$. The Mond-Weir type dual problem related to (NSIMPEC) depending on the feasible element $y^* \in \mathcal{F}$ is denoted by MWD(y^*) and is formulated as

$$(MWD(y^*)) \quad \text{Maximize } \mathscr{F}(w, \alpha, \sigma) := \Phi(w),$$

$$\text{subject to } 0 \in \sum_{j\in I} \alpha_j \partial_c \Phi_j(w) + \sum_{t\in\mathcal{T}} \sigma_t^{\Psi} \partial_c \Psi_t(w) + \sum_{j\in\mathcal{I}^{\vartheta}} \sigma_j^{\vartheta} \partial_c \vartheta_j(w) - \sum_{j\in\mathcal{S}} \sigma_j^{\mathcal{M}} \partial_c \mathcal{M}_j(w) - \sum_{j\in\mathcal{S}} \sigma_j^{\mathcal{N}} \partial_c \mathcal{N}_j(w),$$

$$\Psi_t(w) \geq 0, \quad \forall t \in L, \quad \vartheta_j(w) = 0, \quad \forall j \in \mathcal{I}^{\vartheta},$$
$$\mathcal{M}_j(w) \geq 0, \quad \forall j \in J_{0+}(y^*) \cup J_{00}(y^*), \quad \mathcal{N}_j(w) \geq 0, \quad \forall j \in J_{+0}(y^*) \cup J_{00}(y^*),$$
$$\sigma_j^{\mathcal{M}} = 0, \quad \forall j \in J_{+0}(y^*), \quad \sigma_j^{\mathcal{M}} \geq 0, \quad \forall j \in J_{00}(y^*),$$
$$\sigma_j^{\mathcal{N}} = 0, \quad \forall j \in J_{0+}(y^*), \quad \sigma_j^{\mathcal{N}} \geq 0, \quad \forall j \in J_{00}(y^*),$$
$$\sigma_t^{\Psi} = 0, \quad \forall t \in \mathcal{T} \setminus L(y^*), \quad \text{and } \sum_{j\in I} \alpha_j = 1.$$

The set containing each feasible point of MWD(y^*) is denoted by $\mathcal{F}_{MW}(y^*)$. In the following theorem, we derive weak duality relations that relate (NSIMPEC) and (MWD).

Theorem 4.7. *Let* $y^* \in \mathcal{F}$ *and* $(w, \alpha, \sigma) \in \mathcal{F}_{MW}$ *be arbitrary feasible elements of (NSIMPEC) and (MWD) respectively. Further, let us suppose that* $J_{0+}^- \cup \hat{J}_{+0}^- = \emptyset$ *and each of the functions* ψ_t $(t \in L)$, $\vartheta_j \left(j \in J_{\mathcal{I}^{\vartheta}}^+\right)$, $-\vartheta_j \left(j \in J_{\mathcal{I}^{\vartheta}}^-\right)$, $-\mathcal{M}_j$ $(j \in J_{0+}^+ \cup J_{00}^+ \cup J_{00}^{++})$, $-\mathcal{N}_j$ $(j \in \hat{J}_{00}^+ \cup \hat{J}_{+0}^+ \cup J_{00}^{++})$ *are all geodesic quasiconvex at w.Then the following assertions hold true.*
(i) If for every $j \in I$, Φ_j *is a geodesic pseudoconvex function at w, then* $\Phi(y^*) \nless \mathscr{F}(w, \alpha, \sigma)$.
(ii) If for every $j \in I$, Φ_j *is a strictly geodesic pseudoconvex function at w, then* $\Phi(y^*) \nleq \mathscr{F}(w, \alpha, \sigma)$.

Proof. From the given hypothesis, we have $y^* \in \mathcal{F}$. Then it follows that

$$\Psi_t(y^*) \leq 0, \quad \forall t \in \mathcal{T}, \quad \vartheta_j(y^*) = 0, \quad \forall j \in \mathcal{I}^{\vartheta},$$
$$\mathcal{M}_j(y^*), \mathcal{N}_j(y^*) \geq 0, \quad \forall j \in \mathcal{S}, \quad \mathcal{N}_j(y^*)\mathcal{M}_j(y^*) = 0, \quad \forall j \in \mathcal{S}.$$

Again, we have $(w, \alpha, \sigma) \in \mathcal{F}_{MW}(y^*)$. Therefore, there exist $\xi_j^{\Phi} \in \partial_c \Phi_j(w)$ $(j \in I)$, $\xi_t^{\Psi} \in \partial_c \Psi_t(w)$ $(t \in \mathcal{T})$, $\xi_j^{\vartheta} \in \partial_c \vartheta_j(w)$ $(j \in \mathcal{I}^{\vartheta})$, $\xi_j^{\mathcal{M}} \in \partial_c \mathcal{M}_j(w)$ $(j \in \mathcal{S})$, $\xi_j^{\mathcal{N}} \in \partial_c \mathcal{N}_j(w)$ $(j \in \mathcal{S})$, such that

$$\sum_{j \in I} \alpha_j \xi_j^{\Phi} + \sum_{t \in \mathcal{T}} \sigma_t^{\Psi} \xi_t^{\Psi} + \sum_{j \in \mathcal{I}^{\vartheta}} \sigma_j^{\vartheta} \xi_j^{\vartheta} - \sum_{j \in \mathcal{S}} \sigma_j^{\mathcal{M}} \xi_j^{\mathcal{M}} - \sum_{j \in \mathcal{S}} \sigma_j^{\mathcal{N}} \xi_j^{\mathcal{N}} = 0, \tag{14}$$

$$\Psi_t(w) \geq 0, \quad \forall t \in \mathcal{T}, \quad \vartheta_j(w) = 0, \quad \forall j \in \mathcal{I}^{\vartheta},$$
$$\mathcal{M}_j(w) \geq 0, \quad \forall j \in J_{0+}(y^*) \cup J_{00}(y^*), \quad \mathcal{N}_j(w) \geq 0, \quad \forall j \in J_{+0}(y^*) \cup J_{00}(y^*),$$
$$\sigma_j^{\mathcal{M}} = 0, \quad \forall j \in J_{+0}(y^*), \quad \sigma_j^{\mathcal{M}} \geq 0, \quad \forall j \in J_{00}(y^*),$$
$$\sigma_j^{\mathcal{N}} = 0, \quad \forall j \in J_{0+}(y^*), \quad \sigma_j^{\mathcal{N}} \geq 0, \quad \forall j \in J_{00}(y^*), \quad \text{and} \sum_{j \in I} \alpha_j = 1.$$

Using the feasibility conditions we have

$$\Psi_t(y^*) = 0 \leq \Psi_t(w), \quad \forall t \in L(y^*).$$

From the geodesic quasiconvexity assumption on Ψ_t for every $t \in L(y^*)$ at w, we yield the following

$$\left\langle \xi_t^{\Psi}, \exp_w^{-1}(y^*) \right\rangle \leq 0, \quad \forall t \in L(y^*). \tag{15}$$

Combining (15) with the fact that $\sigma^{\Psi} \in \mathscr{A}(y^*)$, we obtain the following

$$\left\langle \sum_{t \in \mathcal{T}} \sigma_t^{\Psi} \xi_t^{\Psi}, \exp_w^{-1}(y^*) \right\rangle \leq 0. \tag{16}$$

Using the feasibility conditions we have

$$\vartheta_j(y^*) \leq \vartheta_j(w),\ \forall j \in J^+_{\mathcal{I}^{\vartheta}}, \quad -\vartheta_j(y^*) \leq -\vartheta_j(w),\ \forall j \in J^-_{\mathcal{I}^{\vartheta}}.$$

In view of the geodesic quasiconvexity hypothesis on ϑ_j for every $j \in J^+_{\mathcal{I}^{\vartheta}}$ and on $-\vartheta_j$ for every $j \in J^-_{\mathcal{I}^{\vartheta}}$ at w, we have the following

$$\left\langle \xi_j^{\vartheta}, \exp_w^{-1}(y^*) \right\rangle \leq 0, \quad \forall j \in J^+_{\mathcal{I}^{\vartheta}}, \quad \left\langle -\xi_j^{\vartheta}, \exp_w^{-1}(y^*) \right\rangle \leq 0, \quad \forall j \in J^-_{\mathcal{I}^{\vartheta}}. \tag{17}$$

From the definitions of $J^+_{\mathcal{I}^{\vartheta}}, J^-_{\mathcal{I}^{\vartheta}}$ and above inequalities, we obtain the following

$$\left\langle \sum_{j \in \mathcal{I}^{\vartheta}} \sigma_j^{\vartheta} \xi_j^{\vartheta}, \exp_w^{-1}(y^*) \right\rangle \leq 0. \tag{18}$$

Similarly as before, from the definitions of index sets we have the following

$$-\mathcal{M}_j(y^*) \leq -\mathcal{M}_j(w), \quad \forall j \in J_{0+}^+ \cup J_{00}^+ \cup J_{00}^{++}.$$

From the geodesic quasiconvexity assumptions on $-\mathcal{M}_j$, we have the following

$$-\left\langle \sum_{j \in \mathcal{S}} \sigma_j^{\mathcal{M}} \xi_j^{\mathcal{M}}, \exp_w^{-1}(y^*) \right\rangle \leq 0. \tag{19}$$

Similarly, from the definitions of index sets and geodesic quasiconvexity assumptions on $\mathcal{N}_j$ for every $j \in \hat{J}^+_{00} \cup \hat{J}^+_{+0} \cup J^{++}_{00}$, we have the following

$$\left\langle \sum_{j\in\mathcal{S}} \sigma_j^{\mathcal{N}} \xi_j^{\mathcal{N}}, \exp_w^{-1}(y^*) \right\rangle \leq 0. \tag{20}$$

From the inequalities (16), (18), (19) and (20), we yield the following

$$\begin{aligned} &\left\langle \sum_{j\in I} \alpha_j \xi_j^{\Phi}, \exp_w^{-1}(y^*) \right\rangle \\ &= -\left\langle \sum_{t\in\mathcal{T}} \sigma_t^{\Psi} \xi_t^{\Psi} + \sum_{j\in\mathcal{I}^{\vartheta}} \sigma_j^{\vartheta} \xi_j^{\vartheta} - \sum_{j\in\mathcal{S}} \sigma_j^{\mathcal{M}} \xi_j^{\mathcal{M}} - \sum_{j\in\mathcal{S}} \sigma_j^{\mathcal{N}} \xi_j^{\mathcal{N}}, \exp_w^{-1}(y^*) \right\rangle \geq 0. \end{aligned} \tag{21}$$

(i) On contrary, we suppose that $\Phi(y^*) \prec \mathscr{F}(w,\alpha,\sigma)$. Then, it follows that $\Phi_j(y^*) < \Phi_j(w),\ \forall j \in I$. From the geodesic pseudoconvexity assumption on Φ_j for every $j \in I$ at w, we have the following

$$\left\langle \xi_j^{\Phi}, \exp_w^{-1}(y^*) \right\rangle < 0, \quad \forall j \in I. \tag{22}$$

Since $\alpha \in \mathbb{R}^m_+$ and $\sum_{j\in I} \alpha_j = 1$, we obtain

$$\left\langle \sum_{j=1}^{m} \alpha_j \xi_j^{\Phi}, \exp_w^{-1}(y^*) \right\rangle < 0, \tag{23}$$

which contradicts (21). Therefore, the proof is complete.

(ii) On contrary, we suppose that

$$\Phi(y^*) \preceq \mathscr{F}(w,\alpha,\sigma).$$

From the above inequality, it is clear that $y^* \neq w$. From the strict geodesic pseudoconvexity assumption on Φ_j for every $j \in I$ at w, we have the following

$$\left\langle \xi_j^{\Phi}, \exp_w^{-1}(y^*) \right\rangle < 0, \quad \forall j \in I. \tag{24}$$

Since $\alpha \in \mathbb{R}^m_+$ and $\sum_{j\in I} \alpha_j = 1$ we obtain

$$\left\langle \sum_{j=1}^{m} \alpha_j \xi_j^{\Phi}, \exp_w^{-1}(y^*) \right\rangle < 0, \tag{25}$$

which contradicts (21). Therefore, the proof is complete. □

Remark 4.8. *(a) If $\mathscr{M} = \mathbb{R}^n$ and each of the objective and constraint functions of (NSIMPEC) is differentiable, then, Theorem 4.1 reduces to Proposition 20 derived by Tung [34].*

(b) Theorem 4.1 extends Theorem 6 of Singh and Mishra [31] for more general category of optimization problems and generalizes it from $\mathbb{R}^n$ to Hadamard manifolds.

In the following theorem, we provide strong duality result relating (NSIMPEC) and (MWD(y^*)) by employing geodesic convexity assumptions.

Theorem 4.9. *Let us assume that $y^* \in \mathcal{F}$ such that (ACQ) holds at y^* the set* $\Delta_1 := \text{pos}\left(\mathcal{G}_\psi \cup \mathcal{G}_{\mathcal{M}_2} \cup \mathcal{G}_{\mathcal{N}_2}\right) + \text{span}\left(\mathcal{G}_\vartheta \cup \mathcal{G}_{\mathcal{M}_1} \cup \mathcal{G}_{\mathcal{N}_1}\right)$ *is closed. Further, let us suppose that $\tilde{J}^-_{0+} \cup \hat{J}^-_{+0} = \emptyset$. Then there exist $\overline{\alpha} \in \mathbb{R}^m_+$, $\overline{\sigma}^\Psi \in \mathbb{R}^{|\mathcal{T}|}_+$, $\overline{\sigma}^\vartheta \in \mathbb{R}^q$, $\overline{\sigma}^\mathcal{M} \in \mathbb{R}^p$, $\overline{\sigma}^\mathcal{N} \in \mathbb{R}^p$ such that $(y^*, \overline{\alpha}, \overline{\sigma}) \in \mathcal{F}_{MW}(y^*)$ and*

$$\mathscr{F}(y^*, \overline{\alpha}, \overline{\sigma}) = \Phi(y^*).$$

Further, the following assertions hold true.
(i) If every assumption of weak duality theorem (Theorem 4.7 (i)) are satisfied, then $(y^*, \overline{\alpha}, \overline{\sigma})$ is a weak Pareto efficient solution of $MWD(y^*)$.
(ii) If every assumption of weak duality theorem (Theorem 4.7 (ii)) are satisfied,, then $(y^*, \overline{\alpha}, \overline{\sigma})$ is a Pareto efficient solution of $MWD(y^*)$.

Proof. According to the given hypothesis, $y^* \in \mathcal{F}$ be a weak Pareto efficient solution of (NSIMPEC) at which (NSIMPEC-ACQ) is satisfied. Then from Theorem 3.8 it follows that there exist $\overline{\alpha} \in \mathbb{R}^m_+$, $\overline{\sigma}^\Psi \in \mathscr{A}(y^*)$, $\overline{\sigma}^\vartheta \in \mathbb{R}^q$, $\overline{\sigma}^\mathcal{M} \in \mathbb{R}^p$, $\overline{\sigma}^\mathcal{N} \in \mathbb{R}^p$, $\xi^\Phi_j \in \partial_c \Phi_j(y^*)$ $(j \in I)$, $\xi^\Psi_t \in \partial_c \Psi_t(y^*)$ $(t \in \mathcal{T})$, $\xi^\vartheta_j \in \partial_c \vartheta_j(y^*)$ $(j \in \mathcal{I}^\vartheta)$, $\xi^\mathcal{M}_j \in \partial_c \mathcal{M}_j(y^*)$ $(j \in \mathcal{S})$, $\xi^\mathcal{N}_j \in \partial_c \mathcal{N}_j(y^*)$ $(j \in \mathcal{S})$, such that the following conditions are satisfied

$$\sum_{j\in I} \overline{\alpha}_j \xi^\Phi_j + \sum_{t\in\mathcal{T}} \overline{\sigma}^\Psi_t \xi^\Psi_t + \sum_{j\in\mathcal{I}^\vartheta} \overline{\sigma}^\vartheta_j \xi^\vartheta_j - \sum_{j\in\mathcal{S}} \overline{\sigma}^\mathcal{M}_j \xi^\mathcal{M}_j - \sum_{j\in\mathcal{S}} \overline{\sigma}^\mathcal{N}_j \xi^\mathcal{N}_j = 0,$$

$$\overline{\sigma}^\mathcal{M}_j = 0, \quad \forall j \in J_{+0}(y^*), \quad \overline{\sigma}^\mathcal{M}_j \geq 0, \quad \forall j \in J_{00}(y^*),$$
$$\overline{\sigma}^\mathcal{N}_j = 0, \quad \forall j \in J_{0+}(y^*), \quad \overline{\sigma}^\mathcal{N}_j \geq 0, \quad \forall j \in J_{00}(y^*), \quad \text{and} \sum_{j\in I} \overline{\alpha}_j = 1.$$

Since $\overline{\sigma}^\Psi \in \mathscr{A}(y^*)$, we have $\overline{\sigma}^\Psi_t \Psi_t(y^*) = 0, \quad \forall t \in \mathcal{T}$. Then, it follows that

$$\sum_{t\in\mathcal{T}} \overline{\sigma}^\Psi_t \Psi_t(y^*) = 0. \tag{26}$$

Again, since $y^* \in \mathcal{F}$, we obtain the following

$$\sum_{j\in\mathcal{I}^\vartheta} \overline{\sigma}^\vartheta_j \vartheta_j(y^*) = 0. \tag{27}$$

Since $\overline{\sigma}^\mathcal{M}_j = 0$, for every $j \in J_{+0}(y^*)$ and $\overline{\sigma}^\mathcal{M}_j \geq 0$, for every $j \in J_{00}(y^*)$, it follows that

$$\sum_{j\in\mathcal{S}} \overline{\sigma}^\mathcal{M}_j \mathcal{M}_j(y^*) = 0. \tag{28}$$

Similarly, as $\overline{\sigma}^\mathcal{N}_j = 0$ for every $j \in J_{0+}(y^*)$, and $\overline{\sigma}^\mathcal{N}_j \geq 0$, for every $j \in J_{00}(y^*)$, it follows that

$$\sum_{j\in\mathcal{S}} \overline{\sigma}^\mathcal{N}_j \mathcal{N}_j(y^*) = 0. \tag{29}$$

Thus, we infer that $(y^*, \overline{\alpha}, \overline{\sigma}) \in \mathcal{F}_W(y^*)$.

(i) On contrary, we suppose that $(y^*, \overline{\alpha}, \overline{\sigma})$ is not a weak Pareto efficient solution of $MWD(y^*)$. Consequently, some $(w, \alpha, \sigma) \in \mathcal{F}_{MW}(y^*)$ exists, satisfying:

$$\mathscr{F}(y^*, \overline{\alpha}, \overline{\sigma}) \prec \mathscr{F}(w, \alpha, \sigma).$$

Then it follows that

$$\Phi(y^*) \prec \mathscr{F}(w,\alpha,\sigma),$$

which contradicts Theorem 4.7(i). Therefore, the proof is complete.

(ii) On contrary, we suppose that $(y^*,\overline{\alpha},\overline{\sigma})$ is not a Pareto efficient solution of $MWD(y^*)$. Consequently, some $(w,\alpha,\sigma) \in \mathcal{F}_{MW}(y^*)$ exists, satisfying:

$$\mathscr{F}(y^*,\overline{\alpha},\overline{\sigma}) \leq \mathscr{F}(w,\alpha,\sigma).$$

Then it follows that

$$\Phi(y^*) \leq \mathscr{F}(w,\alpha,\sigma).$$

which contradicts Theorem 4.7(ii). Therefore, the proof is complete. □

Remark 4.10. *(a) If $\mathscr{M} = \mathbb{R}^n$ and each of the objective and constraint functions of (NSIMPEC) is differentiable, then, Theorem 4.1 reduces to Proposition 21 derived by Tung [34].*

(b) Theorem 4.1 is an extension of Theorem 7 from Singh and Mishra [31] for more general category of optimization problems, and also generalizes it from $\mathbb{R}^n$ to Hadamard manifolds.

In the following theorem we provide strict converse duality theorem relating (NSIMPEC) and (MWD(y^*)).

Theorem 4.11. *Let us assume that $y^* \in \mathcal{F}$ be any weak Pareto efficient solution of (NSIMPEC). Let $(w,\alpha,\sigma) \in \mathcal{F}_{MW}(y^*)$ be a weak Pareto efficient solution of (MWD(y^*)), such that $\Phi(y^*) \leq \mathscr{F}(w,\alpha,\sigma)$. If each of the assumptions of the weak duality theorem (Theorem 4.7 (ii)) are satisfied, then $y^* = w$.*

Proof. On contrary, we suppose that $y^* \neq w$. In similar lines of the proof of Theorem 4.7, we yield $\xi_j^{\Phi} \in \partial_c\Phi_j(w)$ $(j \in I)$, $\xi_t^{\Psi} \in \partial_c\Psi_t(w)$ $(t \in \mathcal{T})$, $\xi_j^{\vartheta} \in \partial_c\vartheta_j(w)$ $(j \in \mathcal{I}^{\vartheta})$, $\xi_j^{\mathcal{M}} \in \partial_c\mathcal{M}_j(w)$ $(j \in \mathcal{S})$, $\xi_j^{\mathcal{N}} \in \partial_c\mathcal{N}_j(w)$ $(j \in \mathcal{S})$, satisfying:

$$\begin{aligned}&\left\langle \sum_{j\in I} \alpha_j \xi_j^{\Phi}, \exp_w^{-1}(y^*) \right\rangle \\ &= -\left\langle \sum_{t\in\mathcal{T}} \sigma_t^{\Psi}\xi_t^{\Psi} + \sum_{j\in\mathcal{I}^{\vartheta}} \sigma_j^{\vartheta}\xi_j^{\vartheta} - \sum_{j\in\mathcal{S}} \sigma_j^{\mathcal{M}}\xi_j^{\mathcal{M}} - \sum_{j\in\mathcal{S}} \sigma_j^{\mathcal{N}}\xi_j^{\mathcal{N}}, \exp_w^{-1}(y^*) \right\rangle \geq 0.\end{aligned} \tag{30}$$

In view of the strict geodesic pseudoconvexity hypothesis on Φ_j for every $j \in I$ at w, we have the following

$$\left\langle \xi_j^{\Phi}, \exp_w^{-1}(y^*) \right\rangle < 0, \quad \forall j \in I,\ \xi_j^{\Phi} \in \partial_c\Phi_j(w). \tag{31}$$

Since $\alpha \in \mathbb{R}_+^m$ and $\sum_{j\in I} \alpha_j = 1$, we obtain

$$\left\langle \sum_{j=1}^{m} \alpha_j \xi_j^{\Phi}, \exp_w^{-1}(y^*) \right\rangle < 0, \tag{32}$$

which contradicts (30). Therefore, the proof is complete. □

In the following numerical example, we illustrate the significance of Mond-Weir type dual model related to (NSIMPEC).

Example 4.12. *Consider the problem (P1) as defined in Example 4.6. Let $y^* = e \in \mathcal{F}$. The corresponding Mond-Weir dual problem for (P1), denoted by (MWD)(y^*), may be formulated as:*

(MWD)(y^*) Maximize $\mathscr{F}(w) := \Phi(w)$,

subject to

$$0 \in \sum_{j=1}^{2} \alpha_j \partial_c \Phi_j(w) + \sum_{t \in \mathbb{N}} \sigma_t^{\Psi} \partial_c \Psi_t(w) - \sigma^{\mathcal{M}} \partial_c \mathcal{M}(w) - \sigma^{\mathcal{N}} \partial_c \mathcal{N}(w),$$
$$\Psi_t(w) \geq 0, \quad \forall t \in \mathbb{N}, \quad \mathcal{M}(w) \geq 0, \quad \mathcal{N}(w) \geq 0.$$

where $\sigma = (\sigma^{\Psi}, \sigma^{\mathcal{M}}, \sigma^{\mathcal{N}}) \in \mathbb{R}_+^{|\mathbb{N}|} \times \mathbb{R} \times \mathbb{R}$, $\alpha_j \in \mathbb{R}, \alpha_j \geq 0$, $j = 1, 2$, and $\sum_{j=1}^{2} \alpha_j = 1$. The feasible set of (MWD)(y^*) is denoted by $\mathcal{F}_{MW}(y^*)$. One can very easily verify that (ACQ) holds at y^*. Moreover, we can further verify the fact that $y^* = (e, e)$ is a weak Pareto efficient solution of the problem (P1).
We consider a map $\sigma_t^{\Psi} : \mathbb{N} \to \mathbb{R}$, such that

$$\sigma_t^{\Psi} = \begin{cases} e, & \text{if } t = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Then for $\alpha_i = \frac{1}{2}$, $(i = 1, 2)$, $\xi_1^{\Phi} = 0 \in \partial_c \Phi_1(e)$, $\xi_2^{\Phi} = 2e^3 \in \partial_c \Phi_2(e)$, σ_t^{Ψ} as defined above, $\xi_t^{\Psi} = te^2 \in \partial_c \Psi_t(e)$, for all $t \in \mathbb{N}$, $\xi^{\mathcal{M}} = e^2 \in \partial_c \mathcal{M}(e)$, $\xi^{\mathcal{N}} = e \in \partial_c \mathcal{N}(e)$, $\sigma^{\mathcal{M}} = e$ and $\sigma^{\mathcal{N}} = e^2$, we have

$$\alpha_1 \xi_1^{\Phi} + \alpha_2 \xi_2^{\Phi} + \sum_{t \in \mathbb{N}} \sigma_t^{\Psi} \xi_t^{\Psi} - \sigma^{\mathcal{M}} \xi^{\mathcal{M}} - \sigma^{\mathcal{N}} \xi^{\mathcal{N}} = 0. \tag{33}$$

This shows that $(y^*, \alpha, \sigma) \in \mathcal{F}_{MW}(y^*)$. Furthermore

$$\Phi(y^*) = \mathscr{F}(y^*, \alpha, \sigma).$$

One can very easily verify that all the assumptions of the strong duality theorem (Theorem 4.9) is verified. Consequently, (y^*, α, σ) is a weak Pareto efficient solution of (MWD)(y^*).

5. Conclusions and future directions

In this article, we have explored a class of (NSIMPEC) in the setting of Hadamard manifolds. We have formulated two kinds of dual models related to (NSIMPEC), namely, the Wolfe type and Mond-Weir type dual models. We have derived weak, strong and strict converse duality results that relate (NSIMPEC) and the corresponding dual models. We have also provided non-trivial numerical examples to demonstrate the importance of our derived results.

The results that are established in this article generalize as well as extend several notable results previously existing in literature. For instance, the results that are derived in this article generalize the corresponding results of [34] on a more general and wider space, which is, Hadamard manifolds, as well as for further general category of optimization problems, that is, (NSIMPEC). On the other hand, the results derived in [12, 22, 31] are extended to the setting of semi-infinite optimization problems on Hadamard manifolds by the results derived in this article. To the best of our knowledge, this is for the first time that duality models for (NSIMPEC) have been investigated in the setting of Hadamard manifolds.

The results presented in this article leave several avenues for future research. For instance, it would be an exciting research problem to derive duality results for (NSIMPEC) by using the notion of Mordukhovich limiting subdifferential, which comparatively has a better Lagrange multiplier rule than Clarke subdifferential. We intend to pursue this in our future work.

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Declarations

The authors declare that there is no actual or potential conflict of interest in relation to this article.

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