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Almost convergent Motzkin sequence spaces and core theorems

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Abstract. In this article, we present and investigate new type of sequence spaces called by almost convergent Motzkin sequence spaces. We demonstrate that the newly introduced spaces are linearly isomorphic to the spaces of all almost convergent sequences and compute the $\beta-$dual. Additionally, we characterize $(\mathfrak{M}, Z)$ and $(Z, \mathfrak{M})$ for any given sequence space Z, and also determine the necessary and sufficient condition on a matrix $\mathcal{P}$ such that for every bounded sequence u, $B_{\mathcal{M}}$-core$(\mathcal{P}u) \subseteq K$-core$(u)$ and $B_{\mathcal{M}}$-core$(\mathcal{P}u) \subseteq st$-core$(u)$.

1. Introduction

Sequence spaces have played a vital role across various branches of mathematics, such as functional analysis, operator theory, and approximation theory. The importance of sequence spaces has sparked considerable interest among researchers in summability theory. Many researchers have introduced and investigated different types of sequence spaces to uncover their unique properties. The primary objective of classical theory revolves around the generalization of convergence concepts for both series and sequences. Its main goal is to provide a framework through which limits can be assigned to series and sequences that do not exhibit convergence. This is achieved through the use of transformations defined by infinite matrices. The preference for utilizing matrices, rather than general linear mappings, is based on the fact that a linear mapping between two sequence spaces can be represented by an infinite matrix. This approach offers a powerful framework to analyze and understand the behavior of sequences, enabling researchers to explore and extend the theory of convergence in diverse and meaningful ways.

Throughout the paper, we denote ω, l_∞, l_p, c, c_0, cs as the space of all sequences, the spaces of all bounded, $p-$summable, convergent, null sequences, and convergent series, respectively. We denote $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ as the sets of non-negative integers, real, and complex numbers, respectively.

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The natural density of $G \subseteq \mathbb{N}$ is defined by

$$\delta(G) := \lim_{i\to\infty} \frac{|\{g \in G : g \leq i\}|}{i},$$

where $|\{g \in G : g \leq i\}|$ denotes the cardinality of the set $\{g \in G : g \leq i\}$.

A sequence $u = (u_k)$ is statistically convergent to L, indicated by $st - \lim u_k = L$, if for every $\epsilon > 0$,

$$\lim_{i\to\infty} \frac{1}{i}\Big|\{k \leq i : |u_k - L| \geq \epsilon\}\Big| = 0.$$

We denote st and st_0 as the set of all statistically convergent sequences and statistically null sequences, respectively.

Let Z_1, Z_2 be any two sequence spaces and $\mathcal{P} = (a_{nk})$ be an infinite matrix with $a_{nk} \in \mathbb{R}$. Then we define a matrix mapping $\mathcal{P} : Z_1 \to Z_2$, if $\mathcal{P}u = ((\mathcal{P}u)_n) \in Z_2$, for every sequence $u = (u_k) \in Z_1$, where

$$(\mathcal{P}u)_n = \sum_{k=0}^{\infty} a_{nk}u_k, \quad \text{where } n \in \mathbb{N}. \tag{1}$$

The set of all these matrices mappings from Z_1 to Z_2 is represented by the notation (Z_1, Z_2). A sequence (u_k) is $\mathcal{P}-$summable to L if $\mathcal{P}u$ converges to the limit L. We say that $\mathcal{P}$ maps Z_1 regularly into Z_2 if $\lim_k u_k = \lim \mathcal{P}u$, $\forall\ (u_k) \in Z_1$ and we denote this by $(Z_1, Z_2)_{reg}$.

The concept of almost convergence, which extends the notion of convergence for sequences, was initially introduced by Lorentz [32]. Since then, numerous researchers have developed and examined different forms of generalizations for almost convergence (refer to [5], [7], [19], [29], [30], [31], [34], [41]). The reader can refer to the recent textbooks [38] and [3] for fundamental theorems on functional analysis, summability theory, and the papers [1], [2], [4], [5], [6], [12], [14], [15], [16], [24], [35], [36], [44] and [45] on some developments on the almost convergence and the relevant topics.

A sequence $x = (x_q)$ is almost convergent to L if

$$\lim_{t\to\infty} \frac{1}{t+1} \sum_{j=0}^{t} x_{n+j} = L$$

uniformly in n. The space containing all almost convergent sequences is denoted as $\mathcal{A}$, while the space containing almost null sequences is denoted as $\mathcal{A}_0$, i.e.,

$$\mathcal{A} = \left\{ x = (x_n) \in \omega : \exists\, L \in \mathbb{C} \text{ such that } \lim_{t\to\infty} \frac{1}{t+1} \sum_{j=0}^{t} x_{n+j} = L \text{ uniformly in } n \right\}$$

and

$$\mathcal{A}_0 = \left\{ x = (x_n) \in \omega : \lim_{t\to\infty} \frac{1}{t+1} \sum_{j=0}^{t} x_{n+j} = 0 \text{ uniformly in } n \right\}.$$

Motzkin numbers, named after Theodore Motzkin, are a remarkable sequence of integers. In mathematics, the r^{th} Motzkin number represents the count of distinct chords that can be drawn between r points on a circle without intersecting. It is important to note that the chords do not necessarily have to touch all the points on the circle.

Motzkin numbers, denoted as $M_r (r \in \mathbb{N})$, have diverse applications in various mathematical fields such as geometry, combinatorics, and number theory. They possess a recursive nature and hold significant combinatorial properties, which make them valuable tools in multiple areas of mathematics, algorithmic analysis, and even practical applications like coding theory. The Motzkin numbers have proven to be a rich source of mathematical exploration and have contributed to the understanding of fundamental concepts in different disciplines. They are represented by the following sequence:

$$1, 1, 2, 4, 9, 21, 51, 127, 323, 835, 2188, 5798, 15511, 41835, \ldots$$

The Motzkin numbers satisfy the recurrence relations

$$M_r = M_{r-1} + \sum_{s=0}^{r-2} M_s M_{r-s-2} = \frac{2r+1}{r+2} M_{r-1} + \frac{3r-3}{r+2} M_{r-2}.$$

Another relation provided by the Motzkin numbers is given below:

$$M_{r+2} - M_{r+1} = \sum_{s=0}^{r} M_s M_{r-s}, \text{ for } r \geq 0.$$

Furthermore, there are two another relations between Motzkin and Catalan numbers C_s can be given as

$$M_r = \sum_{s=0}^{\lfloor \frac{r}{2} \rfloor} \binom{r}{2s} C_s \quad \text{and} \quad C_{r+1} = \sum_{s=0}^{r} \binom{r}{s} M_s,$$

where $\lfloor . \rfloor$ is the floor function.

The generating function $m(u) = \sum_{r=0}^{\infty} M_r u^r$ of the Motzkin numbers satisfies

$$u^2 + [m(u)]^2 + (u-1)m(u) + 1 = 0$$

and is described by

$$m(u) = \frac{1 - u - \sqrt{1 - 2u - 3u^2}}{2u^2}.$$

Expression on Motzkin numbers with the help of integral function is as follows:

$$M_r = \frac{2}{\pi} \int_0^{\pi} \sin^2 u (2 \cos u + 1)^r du.$$

They have the asymptotic behaviour

$$M_r \sim \frac{1}{2\sqrt{\pi}} \left(\frac{3}{r}\right)^{\frac{3}{2}} 3^r, \quad r \to \infty.$$

For more detail on Motzkin numbers one can refer to [18].

Using the Motzkin numbers and the form of Schröder matrix [10], the Motzkin matrix $\mathcal{M} = (m_{rs})$ as follows:

$$\mathfrak{m}_{rs} := \begin{cases} \frac{M_s M_{r-s}}{M_{r+2} - M_{r+1}}, & \text{if } 0 \leq s \leq r \\ 0 & \text{if } s > r \end{cases},$$

for all $r, s \in \mathbb{N}$. Note that $\mathcal{M}$ is conservative (see [26]).

The inverse $\mathcal{M}^{-1} = \left(\mathfrak{m}_{rs}^{-1}\right)$ of the Motzkin matrix $\mathcal{M}$ as

$$\mathfrak{m}_{rs}^{-1} := \begin{cases} (-1)^{r-s} \frac{M_{s+2} - M_{s+1}}{M_r} P_{r-s}, & , \text{ if } 0 \leq s \leq r \\ 0, & , \text{ if } s > r \end{cases},$$

where $P_0 = 1$ and

$$P_r = \begin{vmatrix} M_1 & M_0 & 0 & 0 & \cdots & 0 \\ M_2 & M_1 & M_0 & 0 & \cdots & 0 \\ M_3 & M_2 & M_1 & M_0 & \cdots & 0 \\ M_4 & M_3 & M_2 & M_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ M_r & M_{r-1} & M_{r-2} & M_{r-3} & \cdots & M_1 \end{vmatrix}$$

for all $r \in \mathbb{N}\backslash\{0\}$. Erdem et al. [26] introduced Motzkin sequence spaces and explored various core theorems associated with them. Building on this work, Erdem further examined compact operators on Motzkin sequence spaces, specifically focusing on $\ell_p(\mathcal{M})$ in [22] and $c_0(\mathcal{M})$ in [23]. More recently, Erdem et al. [25] extended this line of research by defining Paranormed Motzkin sequence spaces and analyzing their properties.

Lemma 1.1. *[43] An infinite matrix $H = (h_{pq}) \in (c, c)$ iff*

$$\sup_{p\in\mathbb{N}} \sum_q |h_{pq}| < \infty \tag{2}$$

and there are $a_q, a \in \mathbb{C}$ such that

$$\lim_{p\to\infty} h_{pq} = a_q, \text{ for each } q \in \mathbb{N} \tag{3}$$

$$\lim_{p\to\infty} \sum_q |h_{pq}| = a. \tag{4}$$

Additionally, $H = (h_{pq}) \in (c_0, c_0)$ iff the condition (2) and $\lim_{p\to\infty} h_{pq} = 0$ hold.

In a recent study conducted by Jasrotia et al. [28], they examined sequence spaces derived from the Catalan matrix that are associated with almost convergence. They also introduced the concept of the Catalan core for sequences with complex values. Building upon this research and considering the widespread applications of Motzkin numbers and almost convergence in diverse fields of mathematics and computer science, the aim of our research is to extend the notion of almost convergence by employing a matrix transformation characterized by Motzkin numbers. In this paper, we introduce the concept of the almost convergent Motzkin sequence spaces (SSs). Moreover, we establish the β-dual of these sequence spaces and study core theorems for the newly defined sequence spaces.

Motivated by [26], we are exploring almost convergent Motzkin SSs and aim to extend the existing knowledge on convergence and related properties in the context of Motzkin numbers. Through the study of almost convergent Motzkin SSs, we can gain insights into the combinatorial and structural aspects of the underlying Motzkin numbers.

2. Almost convergent Motzkin sequence spaces ($\mathfrak{M}$)

In this section, we define and study the almost convergent Motzkin SSs $\mathfrak{M}$ as a collection of sequences whose $\mathcal{M}$-transforms belong to the space $\mathcal{A}$. We also establish an isomorphism between these spaces. Subsequently, we determine the $\beta-$dual of $\mathfrak{M}$.

Let $v = (v_q)$ be the $\mathcal{M}-$transform of a sequence $u = (u_q)$, which is given by the expression:

$$v_q = (\mathcal{M}u)_q = \frac{1}{M_{q+2} - M_{q+1}} \sum_{s=0}^{q} M_s M_{q-s} u_s \tag{5}$$

for all $q \in \mathbb{N}$. Based on this definition, we define the almost convergent Motzkin SSs as follows:

$$\mathfrak{M} = \left\{ u = (u_s) \in \omega : \exists L \in \mathbb{C} \text{ such that } \lim_{t\to\infty} \frac{1}{t+1} \sum_{j=0}^{t} v_{n+j} = L \text{ uniformly in } n \right\}$$

and

$$\mathfrak{M}_0 = \left\{ u = (u_s) \in \omega : \lim_{t\to\infty} \frac{1}{t+1} \sum_{j=0}^{t} v_{n+j} = 0 \text{ uniformly in } n \right\}.$$

Theorem 2.1. *The almost convergent Motzkin SSs $\mathfrak{M}$ and $\mathfrak{M}_0$ are linearly isomorphic to $\mathcal{A}$ and $\mathcal{A}_0$, respectively.*

Proof. Let us define $\begin{array}{c} S : \mathfrak{M} \to \mathcal{A} \\ u \longmapsto Su = \mathcal{M}(u) \end{array}$. Clearly, S is linear. Also, $S(u) = 0$ implies $u = 0$, which shows that S is injective.

To prove that S is surjective, let $y = (y_q) \in \mathcal{A}$ and define $x = (x_q)$ by

$$x_q = \sum_{i=0}^{q} \left((-1)^{q-i} \frac{M_{i+2} - M_{i+1}}{M_q} P_{q-i} \right) y_i$$

for all $q \in \mathbb{N}$.

Then, we have

$$\begin{aligned} &\frac{1}{M_{n+j+2} - M_{n+j+1}} \sum_{s=0}^{n+j} M_s M_{n+j-s} x_s \\ &\qquad = \frac{1}{M_{n+j+2} - M_{n+j+1}} \sum_{s=0}^{n+j} M_s M_{n+j-s} \sum_{i=0}^{s} \left((-1)^{s-i} \frac{M_{i+2} - M_{i+1}}{M_s} P_{s-i} \right) y_i \\ &\qquad = \frac{1}{M_{n+j+2} - M_{n+j+1}} \sum_{i=0}^{n+j} \left(\sum_{s=0}^{n+j-i} (-1)^s M_{n+j-s-i} P_s \right) (M_{i+2} - M_{i+1}) y_i. \end{aligned} \tag{6}$$

Since $\sum_{s=0}^{n+j-i} (-1)^s M_{n+j-s-i} P_s = 0$ for $n + j \neq i$, so the right hand side of equation (6) reduces to

$$\frac{1}{M_{n+j+2} - M_{n+j+1}} (-1)^0 M_0 P_0 (M_{n+j+2} - M_{n+j+1}) y_{n+j}.$$

Thus, we conclude that

$$\lim_{t\to\infty} \frac{1}{t+1} \sum_{j=0}^{t} \frac{1}{M_{n+j+2} - M_{n+j+1}} \sum_{s=0}^{n+j} M_s M_{n+j-s} x_s = \lim_{t\to\infty} \frac{1}{t+1} \sum_{j=0}^{t} y_{n+j}$$

exists uniformly in n. This shows that $x = (x_q) \in \mathfrak{M}$ and so S is surjective. Hence, $\mathfrak{M}$ is linearly isomorphic to $\mathcal{A}$. The space $\mathfrak{M}_0$ is linearly isomorphic to $\mathcal{A}_0$ can be proved in a similar way. □

For any sequence spaces Z_1 and Z_2, the set $S(Z_1, Z_2)$ is defined by

$$S(Z_1, Z_2) = \left\{ z = (z_q) \in \omega : yz = (y_q z_q) \in Z_2 \text{ for all } y = (y_q) \in Z_1 \right\}.$$

If we take $Z_2 = cs$, the the set $S(Z_1, cs)$ is called the β−dual of Z_1 and is denoted as Z_1^{β}.

Lemma 2.2. *[42] $H = (h_{pq}) \in (\mathcal{A}, c)$ iff*

$$\sup_{p\in\mathbb{N}} \sum_q |h_{pq}| < \infty, \tag{7}$$

and there are $a_q, a \in \mathbb{C}$ such that

$$\lim_{p\to\infty} h_{pq} = a_q, \ \text{for each } q \in \mathbb{N} \tag{8}$$

$$\lim_{p\to\infty} \sum_q |h_{pq}| = a, \tag{9}$$

$$\lim_{p\to\infty} \sum_q |\Delta(h_{pq} - a_q)| = 0. \tag{10}$$

Theorem 2.3. *The β−dual of $\mathfrak{M}$ is defined as*

$$\mathfrak{M}^\beta = \mathfrak{M}_1 \cap \mathfrak{M}_2 \cap \mathfrak{M}_3 \cap \mathfrak{M}_4 \cap \mathfrak{M}_5,$$

where

$$\mathfrak{M}_1 = \left\{ z = (z_q) \in \omega : \sup_{t\to\infty} \sum_{p=0}^{t} \left| \sum_{q=p}^{t} (-1)^{q-p} \frac{M_{p+2} - M_{p+1}}{M_q} P_{q-p} z_q \right| < \infty \right\},$$

$$\mathfrak{M}_2 = \left\{ z = (z_q) \in \omega : \lim_{t\to\infty} \sum_{q=p}^{t} (-1)^{q-p} \frac{M_{p+2} - M_{p+1}}{M_q} P_{q-p} z_q \ \text{ exists} \right\},$$

$$\mathfrak{M}_3 = \left\{ z = (z_q) \in \omega : \lim_{t\to\infty} \sum_{p=0}^{t} \left[\sum_{q=p}^{t} (-1)^{q-p} \frac{M_{p+2} - M_{p+1}}{M_q} P_{q-p} z_q \right] \ \text{ exists} \right\},$$

$$\mathfrak{M}_4 = \left\{ z = (z_q) \in \omega : \lim_{t\to\infty} \sum_{p=0}^{t} \left| \sum_{q=p}^{t} (-1)^{q-p} \frac{M_{p+2} - M_{p+1}}{M_q} P_{q-p} z_q \right| = 0 \right\},$$

$$\mathfrak{M}_5 = \left\{ z = (z_q) \in \omega : \lim_{t\to\infty} \sum_{p} \left| \Delta \left[\sum_{q=p}^{t} (-1)^{q-p} \frac{M_{p+2} - M_{p+1}}{M_q} P_{q-p} z_q - z_p \right] \right| = 0 \right\}.$$

In this case

$$z_p = \lim_{t\to\infty} \sum_{q=p}^{t} (-1)^{q-p} \frac{M_{p+2} - M_{p+1}}{M_q} P_{q-p} z_q.$$

Proof. Let $z = (z_q) \in \omega$ and the equality

$$\begin{aligned} \sum_{p=0}^{t} z_p y_p &= \sum_{p=0}^{t} z_p \left(\sum_{q=0}^{p} (-1)^{p-q} \frac{M_{q+2} - M_{q+1}}{M_p} P_{q-p} x_q \right) \\ &= \sum_{p=0}^{t} \left(\sum_{q=p}^{t} (-1)^{q-p} \frac{M_{p+2} - M_{p+1}}{M_q} P_{q-p} z_q \right) x_p = (Dx)_t \end{aligned}$$

for all $t \in \mathbb{N}$, and the matrix $D = (d_{tp})$ is defined by

$$d_{tp} = \begin{cases} \sum_{q=p}^{t} (-1)^{q-p} \frac{M_{p+2} - M_{p+1}}{M_q} P_{q-p} z_q & \text{if } 0 \le p \le t \\ 0 & \text{otherwise.} \end{cases} \tag{11}$$

Thus, from (11) we get $zy = (z_p y_p) \in cs$ whenever $y = (y_p) \in \mathfrak{M}$ iff $Dx \in c$ for $x = (x_p) \in \mathcal{A}$, where D is defined by (11). Thus, $z = (z_q) \in \mathfrak{M}^\beta$ iff $D \in (\mathcal{A}, c)$. Hence, from Lemma 2.2, we get the result. □

3. Matrix transformations on the spaces $\mathfrak{M}$ and $\mathfrak{M}_0$

Consider the infinite matrices $C = (c_{nk})$ and $E = (e_{nk})$, which map the sequences $u = (u_k)$ and $v = (v_k)$ to the sequences $z = (z_n)$ and $t = (t_n)$, respectively, as defined by the following relations:

$$z_n = (Cu)_n = \sum_k c_{nk} u_k, \; n \in \mathbb{N} \tag{12}$$

and

$$t_n = (Ev)_n = \sum_k e_{nk} v_k, \; n \in \mathbb{N}. \tag{13}$$

Here, the method E is applied to the $\mathcal{M}$-transform of the sequence $v = (v_k)$, while the method C is applied directly to the sequence $u = (u_k)$. Therefore, the methods C and E are fundamentally different.

Now, suppose the matrix product $E\mathcal{M}$ exists, which is a weaker assumption compared to the conditions typically required for the matrix E to belong to any specific matrix class. The methods C and E are called dual summability methods of a new type if t_n reduces to z_n or z_n reduces to t_n through formal summation by parts. This implies that $E\mathcal{M}$ exists and equals C, and the formal relation $(E\mathcal{M})u = E(\mathcal{M}u)$ holds if one side exists. Hence, the entries of $C = (c_{nk})$ and $E = (e_{nk})$ are related by the following expressions:

$$c_{nk} = \sum_{j=k}^{\infty} \frac{M_k M_{j-k}}{M_{j+2} - M_{j+1}} e_{nj} \text{ or } e_{nk} = \frac{M_{k+2} - M_{k+1}}{M_k} (c_{n,k} - c_{n,k+1}) \tag{14}$$

for all $n, k \in \mathbb{N}$.

Lemma 3.1. *$H \in (\mathcal{A}, l_\infty)$ if and only if*

$$\sup_{p \in \mathbb{N}} \sum_q |h_{pq}| < \infty. \tag{15}$$

Theorem 3.2. *Suppose that the infinite matrices $C = (c_{nk})$ and $E = (e_{nk})$ are connected with the relation (14). Then $C \in (\mathfrak{M}, Z)$ if and only if $E \in (\mathcal{A}, Z)$ and*

$$\left(\frac{M_{n+2} - M_{n+1}}{M_n}\right) c_{nk} \in c_0, \tag{16}$$

for every fixed $k \in \mathbb{N}$, where Z is any given sequence space.

Proof. Let $C \in (\mathfrak{M}, Z)$ and take $u \in \mathfrak{M}$ and keep in mind $v = \mathcal{M}u$. Then $(c_{nk})_{k \in \mathbb{N}} \in d_1 \cap cs$, where $d_1 = \left\{u = (u_k) : \left\{\left(\frac{M_{k+2} - M_{k+1}}{M_k}\right) u_k\right\} \in l_\infty\right\}$ and $E\mathcal{M}$ exists which implies that $(e_{nk})_{k \in \mathbb{N}} \in l_1 = \mathcal{A}^\beta$ for each $n \in \mathbb{N}$. Thus, Ev exists for each $v = (v_k) \in \mathcal{A}$.

Now,

$$\begin{aligned}\sum_{k=0}^{m} e_{nk}v_k &= \sum_{k=0}^{m} e_{nk}\left(\sum_{s=0}^{k}\frac{M_sM_{k-s}}{M_{k+2}-M_{k+1}}u_s\right)\\ &= \sum_{k=0}^{m}\sum_{s=0}^{k} e_{nk}\frac{M_sM_{k-s}}{M_{k+2}-M_{k+1}}u_s\\ &= \sum_{s=0}^{m} u_s\sum_{k=s}^{m} e_{nk}\frac{M_sM_{k-s}}{M_{k+2}-M_{k+1}}\\ &= \sum_{k=0}^{m}\sum_{j=k}^{m}\frac{M_kM_{j-k}}{M_{j+2}-M_{j+1}}e_{nj}u_k,\end{aligned} \tag{17}$$

for all $m, n \in \mathbb{N}$. Taking $m \to \infty$ and using the relation (14), we have $Ev = Cu$. This implies $E \in (\mathcal{A}, Z)$.

Conversely, suppose that (16) holds and $E \in (\mathcal{A}, Z)$. Then, $(e_{nk}) \in \ell_1 = \mathcal{A}^\beta$ for all $n \in \mathbb{N}$ which gives together with (16) that Cu exists.

Now,

$$\begin{aligned}(Ev)_n &= \sum_k e_{nk}v_k\\ &= \sum_k e_{nk}\left(\sum_{s=0}^{k}\frac{M_sM_{k-s}}{M_{k+2}-M_{k+1}}u_s\right)\\ &= \sum_k\sum_{j=k}^{\infty}\frac{M_kM_{j-k}}{M_{j+2}-M_{j+1}}e_{nj}u_k\\ &= \sum_k c_{nk}u_k\\ &= (Cu)_n, \quad (n \in \mathbb{N}).\end{aligned} \tag{18}$$

This implies $Cu = Ev$. Hence $Cu \in Z$ for all $u \in \mathfrak{M}$, that is, $C \in (\mathfrak{M}, Z)$. □

Theorem 3.3. *Suppose that the matrices $L = (l_{nk})$ and $V = (v_{nk})$ are connected with the relation*

$$v_{nk} = \sum_{j=0}^{n}\frac{M_jM_{n-j}}{M_{n+2}-M_{n+1}}l_{jk}, \; n, k \in \mathbb{N}.$$

Then $L \in (Z, \mathfrak{M})$ if and only if $V \in (Z, \mathcal{A})$.

Proof. Let $u = (u_k) \in Z$ and consider the following equality

$$\sum_{j=0}^{n}\frac{M_jM_{n-j}}{M_{n+2}-M_{n+1}}\sum_{k=0}^{m} l_{jk}u_k = \sum_{k=0}^{m} v_{nk}u_k, \quad (m, n, k \in \mathbb{N}).$$

Taking $m \to \infty$ implies that $Lu \in \mathfrak{M}$ whenever $u \in Z$ if and only if $Vu \in \mathcal{A}$ whenever $u \in Z$. □

Now, we present the following propositions derived from Lemmas 2.2–3.1 and Theorems 3.2–3.3.

Proposition 3.4. *$\mathcal{P} = (a_{nk}) \in (\mathfrak{M}, l_\infty)$ if and only if*

$$\lim_{n\to\infty}\sum_k\left|\Delta\left(\sum_{j=k}^{\infty}\frac{M_kM_{j-k}}{M_{j+2}-M_{j+1}}a_{nj} - a_k\right)\right| = 0, \tag{19}$$

$$\{a_{nk}\}_{k\in\mathbb{N}} \in \mathfrak{M}^{\beta}, \forall n \in \mathbb{N}. \tag{20}$$

Proposition 3.5. *$\mathcal{P} = (a_{nk}) \in (\mathfrak{M}, c)$ if and only if (19), (20) hold and*

$$\lim_{n\to\infty} \sum_{k} \sum_{j=k}^{\infty} \frac{M_k M_{j-k}}{M_{j+2} - M_{j+1}} a_{nj} = a, \tag{21}$$

$$\lim_{n\to\infty} \sum_{j=k}^{\infty} \frac{M_k M_{j-k}}{M_{j+2} - M_{j+1}} a_{nj} = a_k \text{ for each } k \in \mathbb{N}, \tag{22}$$

also hold.

Proposition 3.6. *$\mathcal{P} = (a_{nk}) \in (l_\infty, \mathfrak{M})$ if and only if*

$$\sup_{n\in\mathbb{N}} \sum_{k} \Big| \sum_{j=k}^{\infty} \frac{M_k M_{j-k}}{M_{j+2} - M_{j+1}} a_{nj} \Big| < \infty, \tag{23}$$

$$\mathcal{A} - \lim_{n\to\infty} \sum_{j=k}^{\infty} \frac{M_k M_{j-k}}{M_{j+2} - M_{j+1}} a_{nj} = \alpha_k \text{ exists for each fixed } k \in \mathbb{N}, \tag{24}$$

$$\lim_{m\to\infty} \sum_{k} \Big| \sum_{i=0}^{m} \frac{1}{m+1} \sum_{j=k}^{\infty} \frac{M_k M_{j-k}}{M_{j+2} - M_{j+1}} a_{n+i,j} - \alpha_k \Big| = 0 \text{ uniformly in } n, \tag{25}$$

also holds.

Proposition 3.7. *$\mathcal{P} = (a_{nk}) \in (c, \mathfrak{M})$ if and only if (23),(24) hold, and*

$$\mathcal{A} - \lim_{n\to\infty} \sum_{k} \sum_{j=k}^{\infty} \frac{M_k M_{j-k}}{M_{j+2} - M_{j+1}} a_{nj} = \alpha \tag{26}$$

hold.

4. Core theorems

Following Knoop, a core theorem is characterized as a class of matrices for which the core of the transformed sequence is included by the core of the original sequence. For instance, Knoop's Core Theorem [8] establishes that K-core($\mathcal{P}u$) $\subseteq$ K-core(u) for any real-valued sequence u, provided that $\mathcal{P}$ is a positive matrix belonging to the class $(c, c)_{reg}$. In this section, we introduce a new type of core, referred to as the $B_{\mathcal{M}}$-core of a bounded sequence and also determine the necessary and sufficient conditions on a matrix $\mathcal{P}$ for which $B_{\mathcal{M}}$-core($\mathcal{P}u$) $\subseteq$ $K-$core(u) and $B_{\mathcal{M}}$-core($\mathcal{P}u$) $\subseteq$ $st-$core(u) for all $u \in \ell_\infty$.

Let us define the following functionals on l_∞:

$$I(u) = \liminf_{k\to\infty} u_k,$$

$$S(u) = \limsup_{k\to\infty} u_k,$$

$$s_\sigma(u) = \limsup_{t\to\infty} \sup_{n\in\mathbb{N}} \frac{1}{t+1} \sum_{i=0}^{t} u_{\sigma^i(n)},$$

$$S^*(u) = \limsup_{t\to\infty} \sup_{n\in\mathbb{N}} \frac{1}{t+1} \sum_{i=0}^{t} u_{n+i}.$$

Let $u \in l_\infty$. Then the σ-core of u is defined as $[-s_\sigma(-u), s_\sigma(u)]$ and $s_\sigma(\mathcal{P}u) \leq S(u)$ (σ-core of $\mathcal{P}u \subseteq K$-core of u) and $s_\sigma(\mathcal{P}u) \leq s_\sigma(u)$ (σ-core of $\mathcal{P}u \subseteq \sigma$-core of u), have been studied in [33]. Here, the K-core (or Knoop-core) of u is $[I(u), S(u)]$ [8]. When $\sigma(n) = n+1$, $s_\sigma(u) = S^*(u)$, that is, σ-core of u reduces to B-core of u. Here, B-core (or Banach-core) of u is $[-S^*(-u), S^*(u)]$ (see [40]). Many authors studied B-core and σ-core (see [9],[17],[33],[37],[40]). Fridy and Orhan [27] introduced the concept of statistical core of a statistically bounded sequence as $[st-\liminf u, st-\limsup u]$, where $st-\liminf u$ and $st-\limsup u$ denote the statistical limit inferior and statistical limit superior of $u \in l_\infty$, respectively and determine the necessary and sufficient conditions on $\mathcal{P}$ such that for every bounded sequence u, K-core$(\mathcal{P}u) \subset st$-core$(u)$. For a more comprehensive understanding of statistical core theorems, (see [13], [20], [21], [39]).

Definition 4.1. *Let $u = (u_k) \in l_\infty$. Then $B_\mathcal{M}$-core of u is defined by $[-U^*(-u), U^*(u)]$, where*

$$U^*(u) = \limsup_{t\to\infty} \sup_{n} \frac{1}{t+1} \sum_{j=0}^{t} \frac{1}{M_{n+j+2} - M_{n+j+1}} \sum_{s=0}^{n+j} M_s M_{n+j-s} u_s$$

$$-U^*(-u) = \liminf_{t\to\infty} \sup_{n} \frac{1}{t+1} \sum_{j=0}^{t} \frac{1}{M_{n+j+2} - M_{n+j+1}} \sum_{s=0}^{n+j} M_s M_{n+j-s} u_s.$$

Lemma 4.2. *[11] Let $\|B\| < \infty$ and $\lim_{t\to\infty} \sup_{n\in\mathbb{N}} |b_{tk}(n)| = 0$. Then there is a bounded sequence $v = (v_k)$ with $\|v\| \leq 1$ and*

$$\limsup_{t\to\infty} \sup_{n\in\mathbb{N}} \sum_k b_{tk}(n) v_k = \limsup_{t\to\infty} \sup_{n\in\mathbb{N}} \sum_k |b_{tk}(n)|. \tag{27}$$

Theorem 4.3. *For every bounded sequence u, $B_\mathcal{M}$-core$(\mathcal{P}u) \subseteq K$-core$(u)$ if and only if $\mathcal{P} \in (c, \mathfrak{M})_{reg}$ and*

$$\lim_{t\to\infty} \sup_{n} \sum_k \frac{1}{t+1} \left| \sum_{j=0}^{t} \frac{1}{M_{n+j+2} - M_{n+j+1}} \sum_{s=0}^{n+j} M_s M_{n+j-s} a_{s,k} \right| = 1. \tag{28}$$

Proof. Suppose $B_\mathcal{M}$-core$(\mathcal{P}u) \subset K$-core(u), for all $u \in l_\infty$. Let $u \in \mathfrak{M}$, then $U^*(\mathcal{P}u) = -U^*(-\mathcal{P}u)$. By the hypothesis, we get

$$-S(-u) \leq -U^*(-\mathcal{P}u) \leq U^*(\mathcal{P}u) \leq S(u).$$

Let $u \in c$, then $S(u) = -S(-u) = \lim u$. So, we have

$$\mathcal{A} - \lim \mathcal{P}u = U^*(\mathcal{P}u) = -U^*(-\mathcal{P}u) = \lim u.$$

This yields that $\mathcal{P} \in (c, \mathfrak{M})_{reg}$.

Now, define the sequence of infinite matrices $B = (b_{tk}(n))$ by

$$b_{tk}(n) = \frac{1}{t+1} \sum_{j=0}^{t} \frac{1}{M_{n+j+2} - M_{n+j+1}} \sum_{s=0}^{n+j} M_s M_{n+j-s} a_{s,k} \text{ for all } t,k,n \in \mathbb{N}.$$

Then, the sequence $B = (b_{tk}(n))$ satisfies the conditions of Lemma 4.2, we have

$$\begin{aligned} 1 \leq \liminf_{t\to\infty} \sup_{n\in\mathbb{N}} \sum_k |b_{tk}(n)| &\leq \limsup_{t\to\infty} \sup_{n\in\mathbb{N}} \sum_k |b_{tk}(n)| \\ &= \limsup_{t\to\infty} \sup_{n\in\mathbb{N}} \sum_k b_{tk}(n) v_k \\ &= U^*(\mathcal{P}v) \leq S(v) \leq \|v\| \leq 1. \end{aligned}$$

Conversely, suppose that $\mathcal{P} \in (c, \mathfrak{M})_{reg}$ and (28) hold for all $u \in l_\infty$. For any real number r, we write $r^+ = \max\{r, 0\}$ and $r^- = \max\{-r, 0\}$ then $|r| = r^+ + r^-$, $r = r^+ - r^-$ and $|r| - r = 2r^-$. Therefore, for given $\epsilon > 0$, there $k_0 \in \mathbb{N}$ such that $u_k < S(u) + \epsilon$ for all $k > k_0$.

Now,

$$\begin{aligned} \sum_k b_{tk}(n) u_k &= \sum_{k<k_0} b_{tk}(n) u_k + \sum_{k\geq k_0} (b_{tk}(n))^+ u_k - \sum_{k\geq k_0} (b_{tk}(n))^- u_k \\ &\leq \|u\| \sum_{k<k_0} |b_{tk}(n)| + [S(u) + \epsilon] \sum_k |b_{tk}(n)| + \|u\| \sum_k [|b_{tk}(n)| - b_{tk}(n)]. \end{aligned}$$

Apply $\limsup_{t\to\infty} \sup_n$ and using hypothesis in the above inequality, we have $U^*(\mathcal{P}u) \leq S(u) + \epsilon$. Since, ϵ is arbitrary, we have $B_{\mathcal{M}}$-core$(\mathcal{P}u) \subseteq K$-core$(u)$ for all $u \in l_\infty$. □

Theorem 4.4. *The necessary and sufficient conditions for a matrix $\mathcal{P} \in (st \cap l_\infty, \mathfrak{M})_{reg}$ is $\mathcal{P} \in (c, \mathfrak{M})_{reg}$ and*

$$\lim_{t\to\infty} \sum_{k\in F, \delta(F)=0} \frac{1}{t+1} \left| \sum_{j=0}^{t} \frac{1}{M_{n+j+2} - M_{n+j+1}} \sum_{s=0}^{n+j} M_s M_{n+j-s} a_{s,k} \right| = 0 \text{ uniformly in } n. \tag{29}$$

Proof. Let $\mathcal{P} \in (st \cap l_\infty, \mathfrak{M})_{reg}$. Then $\mathcal{P} \in (c, \mathfrak{M})_{reg}$, since $c \subset st \cap l_\infty$. For a given sequence $u \in l_\infty$, we construct a new sequence $\hat{u} = (\hat{u}_k)$ such that

$$\hat{u}_k = \begin{cases} u_k & ; \quad \text{for } k \in F \\ 0 & ; \quad \text{for } k \notin F, \end{cases}$$

where $F \subseteq \mathbb{N}$ with zero natural density. Then, $st - \lim \hat{u}_k = 0$ and $\hat{u} \in st_0$, we have $\mathcal{P}\hat{u} \in \mathfrak{M}_0$. Define the matrix $T = (t_{nk})$ as

$$t_{nk} = \begin{cases} a_{nk} & ; \quad \text{for } k \in F \\ 0 & ; \quad \text{for } k \notin F, \end{cases}$$

for all n. Since $(\mathcal{P}\hat{u})_n = \sum_{k\in F} a_{nk} \hat{u}_k$, we have $T = (t_{nk}) \in (l_\infty, \mathfrak{M}_0)$. Hence, by Proposition 3.6, the condition (29) holds.

Conversely, suppose that $\mathcal{P} \in (c, \mathfrak{M})_{reg}$ and (29) holds. Let $u \in st \cap l_\infty$ and $st - \lim u = L$. For any given $\epsilon > 0$, write $F = \{k : |u_k - L| \geq \epsilon\}$ so that $\delta(F) = 0$. Since, $\mathcal{P} \in (c, \mathfrak{M})_{reg}$ and $\mathfrak{M} - \lim \sum_k a_{nk} = 1$, we have

$$\begin{aligned} \mathfrak{M} - \lim(\mathcal{P}u) &= \mathfrak{M} - \lim \left(\sum_k a_{nk}(u_k - L) + L \sum_k a_{nk} \right) \\ &= \mathfrak{M} - \lim \left(\sum_k a_{nk}(u_k - L) + L \right) \\ &= \limsup_{t\to\infty} \sup_n \sum_k \frac{1}{t+1} \sum_{j=0}^{t} \frac{1}{M_{n+j+2} - M_{n+j+1}} \sum_{s=0}^{n+j} M_s M_{n+j-s} a_{s,k}(u_k - L) + L. \end{aligned} \tag{30}$$

Since

$$\left|\sum_k \frac{1}{t+1}\sum_{j=0}^{t}\frac{1}{M_{n+j+2}-M_{n+j+1}}\sum_{s=0}^{n+j} M_s M_{n+j-s} a_{s,k}(u_k - L)\right|$$
$$\leq \|u\|\sum_{k\in F}\frac{1}{t+1}\left|\sum_{j=0}^{t}\frac{1}{M_{n+j+2}-M_{n+j+1}}\sum_{s=0}^{n+j} M_s M_{n+j-s} a_{s,k}\right| + \epsilon\|\mathcal{P}\|. \tag{31}$$

Taking limit $t \to \infty$ in (31) and using (29), we have

$$\lim_{t\to\infty}\sum_k \frac{1}{t+1}\sum_{j=0}^{t}\frac{1}{M_{n+j+2}-M_{n+j+1}}\sum_{s=0}^{n+j} M_s M_{n+j-s} a_{s,k}(u_k - L) = 0 \text{ uniformly in } n.$$

Hence, $\mathfrak{M} - \lim(\mathcal{P}u) = st - \lim u$, that is, $\mathcal{P} \in (st \cap l_\infty, \mathfrak{M})_{reg}$. □

Theorem 4.5. *For every bounded sequence u, $B_{\mathcal{M}}$-core($\mathcal{P}u$) $\subseteq$ st-core(u) if and only if $\mathcal{P} \in (st \cap l_\infty, \mathfrak{M})_{reg}$ and (28) holds.*

Proof. Suppose $B_{\mathcal{M}}$-core($\mathcal{P}u$) $\subseteq$ st-core(u), for a bounded sequence u. Then $U^*(\mathcal{P}u) \leq a(u)$ for all $u \in l_\infty$, where $a(u)$ is the statistical limit superior of u. Since, $a(u) \leq S(u)$ for all $u \in l_\infty$ (see [27]), we have (28) from Theorem 4.3. Also, $-a(-u) \leq -U^*(-\mathcal{P}u) \leq U^*(\mathcal{P}u) \leq a(u)$, that is, $st - \liminf u \leq -U^*(-\mathcal{P}u) \leq U^*(\mathcal{P}u) \leq st - \limsup u$. If $u \in st \cap l_\infty$, then $st - \liminf u = st - \limsup u = st - \lim u$. Thus, $st - \lim u = -U^*(-\mathcal{P}u) = U^*(\mathcal{P}u) = \mathfrak{M} - \lim \mathcal{P}u$, that is, $\mathcal{P} \in (st \cap l_\infty, \mathfrak{M})_{reg}$.

Conversely, suppose that $\mathcal{P} \in (st \cap l_\infty, \mathfrak{M})_{reg}$ and (28) holds. If $u \in l_\infty$, then $a(u) < \infty$. Let $F \subset \mathbb{N}$ defined by $F = \{k : u_k > a(u) + \epsilon\}$ for given $\epsilon > 0$. Then $\delta(F) = 0$ and $u_k \leq a(u) + \epsilon$ if $k \notin F$. For any real number r, we write $r^+ = \max\{r, 0\}$ and $r^- = \max\{-r, 0\}$ then $|r| = r^+ + r^-$, $r = r^+ - r^-$ and $|r| - r = 2r^-$.

Now,

$$\begin{aligned}
\sum_k b_{tk}(n)u_k &= \sum_{k<k_0} b_{tk}(n)u_k + \sum_{k\geq k_0}(b_{tk}(n))u_k \\
&= \sum_{k<k_0} b_{tk}(n)u_k + \sum_{k\geq k_0}(b_{tk}(n))^+u_k - \sum_{k\geq k_0}(b_{tk}(n))^-u_k \\
&\leq \|u\|\sum_{k<k_0}|b_{tk}(n)| + \sum_{k\geq k_0, k\notin F}(b_{tk}(n))^+u_k + \sum_{k\geq k_0, k\in F}(b_{tk}(n))^+u_k \\
&\quad + \|u\|\sum_{k\geq k_0}[|b_{tk}(n)| - b_{tk}(n)] \\
&\leq \|u\|\sum_{k<k_0}|b_{tk}(n)| + [a(u)+\epsilon]\sum_{k\geq k_0, k\notin F}|b_{tk}(n)| \\
&\quad + \|u\|\sum_{k\geq k_0, k\in F}|b_{tk}(n)| + \|u\|\sum_{k\geq k_0}[|b_{tk}(n)| - b_{tk}(n)].
\end{aligned}$$

Thus, by applying $\limsup_{t\to\infty}\sup_n$ and using hypothesis, we have $U^*(\mathcal{P}u) \leq a(u) + \varepsilon$. Hence, we have $B_{\mathcal{M}}$-core($\mathcal{P}u$) $\subseteq$ st-core(u) for all $u \in l_\infty$. □

5. Conclusion

In this article, we focus on modifying the research on Motzkin numbers to introduce a new space termed as the almost convergent Motzkin SS. We demonstrate that this space is linearly isomorphic to the space of all almost convergent sequences. Additionally, we calculate the β-dual of this newly established space. We characterize $(\mathfrak{M}, Z)$ and $(Z, \mathfrak{M})$ and also determine the necessary and sufficient condition on $\mathcal{P}$ such that for every bounded sequence u, $B_{\mathcal{M}}$-core$(\mathcal{P}u) \subseteq K$-core$(u)$ and $B_{\mathcal{M}}$-core$(\mathcal{P}u) \subseteq st$-core$(u)$. In future investigations, it will be possible to obtain results corresponding to those presented in this paper by utilizing the Riesz transform of the Motzkin numbers.

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