Filomat 39:17 (2025), 5871–5882
https://doi.org/10.2298/FIL2517871W

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

The characterizations of partial isometries in a ring with involution

Xinran Wang[a,*], Suting Fan[a], Junchao Wei[a]

[a]*College of mathematical science, Yangzhou University, Yangzhou, Jiangsu 225002, P. R. China*

Abstract. In this paper, we discover some properties of partial isometries by some related idempotent elements, projections, *PE* elements, mainly via constructing some equations to study new characterizations of *PI* elements in a rings with involution. The paper explores the representations of the general solution of univariate and bivariate equations, and then combining *PI* elements, it further optimizes the form of solutions to characterize *PI* elements. This work obtains some necessary and sufficient conditions of *PI* elements, and its new characterizations enrich the understanding of partial isometries.

1. Introduction

In this paper, R is a ring with identity. If a map $*: R \to R$ satisfies for $a, b \in R$,

$$(a^*)^* = a,\ (a+b)^* = a^* + b^*,\ (ab)^* = b^*a^*,$$

then R is said to be an involution ring or a $*$−ring.

Let R be a $*$-ring and $a \in R$. If there exists $a^+ \in R$ such that

$$a = aa^+a,\ a^+ = a^+aa^+,\ (aa^+)^* = aa^+,\ (a^+a)^* = a^+a,$$

then a is called a Moore Penrose invertible element, and a^+ is called the Moore Penrose inverse of a [3, 6]. Let R^+ denote the set of all Moore Penrose invertible elements of R.

If there exists $a^\# \in R$ such that

$$aa^\#a = a,\ a^\#aa^\# = a^\#,\ aa^\# = a^\#a,$$

then a is called a group invertible element and $a^\#$ is called the group inverse of a [4, 7, 8], and if $a^\#$ exists, then it is uniquely determined by these equations. We write $R^\#$ to denote the set of all group invertible elements of R.

If $a = aa^*a$, then a is called a partial isometry of R [3, 6]. Let R^{PI} denote the set of all partial isometries of R. Obviously, if $a \in R^+$, then $a \in R^{PI}$ if and only if $a^* = a^+$.

If $a = a^2$, then a is called an idempotent element. Let $E(R)$ denote the set of all idempotent elements of R.

2020 *Mathematics Subject Classification*. Primary 16B99; Secondary 16W10, 46L05.
Keywords. *PI* element, *EP* element, Solution to equation, Moore-Penrose inverse, Group inverse, Ring with involution.
Received: 12 August 2024; Revised: 10 June 2025; Accepted: 11 June 2025
Communicated by Dijana Mosić
* Corresponding author: Xinran Wang
Email addresses: 1092078512@qq.com (Xinran Wang), jcweiyz@126.com (Junchao Wei)
ORCID iDs: https://orcid.org/0009-0004-4505-1490 (Xinran Wang), https://orcid.org/0009-0003-6731-9219 (Suting Fan), https://orcid.org/0000-0002-7310-1836 (Junchao Wei)

Let R be a $*$-ring and $a \in R$. If $a = a^2 = a^*$, then a is called a projection. Let $PE(R)$ denote the set of all projections of R.

If $a \in R^\# \cap R^+$ and $a^\# = a^+$, then a is called an EP element. On the studies of EP, the readers can refer to [2, 3, 5, 9, 10, 12, 15–17]. We denote the set of all EP elements of R by R^{EP}.

Recent researches in partial isometries have produced some findings. In [12], many characterizations of PI elements are given. In [14], D. Mosić, D. S. Djordjević presented some equivalent conditions for the element a in a ring with involution to be a partial isometry. In [18, 22], some people study the solutions of some related equations in a given set $\chi_a = \{a, a^\#, a^+, a^*, (a^\#)^*, (a^+)^*\}$ to characterize partial isometries. In [1], the paper provided some sufficient and necessary conditions for an element in a ring to be an EP element, a partial isometry element, a normal EP element and a strongly EP element by using solutions of equations. In [20], it characterized partial isometries by using some invertible elements.

Motivated by these results, this paper intends to characterize partial isometries through the close relationships among idempotents, projections, EP elements and PE elements. By constructing, adjusting and revising series of univariate and bivariate equations: $x = ya^*a$ and also giving the solutions of some equations, this paper is intended to provide some new equivalent conditions for an element to be partial isometry in a ring with involution .

2. The idempotency of partial isometries

We begin with the following lemma to characterize partial isometry element by linking it with some projections and idempotent elements of R.

Lemma 2.1. *[12, Theorem 1.5.1] Let $a \in R^+$. Then the followings are equivalent:*
(1) $a \in R^{PI}$; (2) $a^+a = a^*a$; (3) $aa^+ = aa^*$; (4) $a^+ = a^*$; (5) $a = (a^+)^*$.

Theorem 2.2. *Let $a \in R^+$. Then the followings are equivalent:*
(1) $a \in R^{PI}$; (2) $a^*a \in E(R)$; (3) $a^*a \in PE(R)$; (4) $aa^* \in E(R)$; (5) $aa^* \in PE(R)$.

Proof. (1)⇒(2) Assume $a \in R^{PI}$. Then $a^*a = a^+a \in E(R)$ by Lemma 2.1.
(2)⇒(3) Noting that $(a^*a)^* = a^*a$. Then $a^*a \in E(R)$ implies $a^*a \in PE(R)$.
(3)⇒(4) Since $a^*a \in PE(R)$, $a^*a = (a^*a)^2 = a^*aa^*a$. Multiplying the equality on the right by a^+, we get

$$a^* = a^*aa^*.$$

So, $aa^* = aa^*aa^* = (aa^*)^2$ and $aa^* \in E(R)$.
(4)⇒(5) Since $(aa^*)^* = aa^*$, $aa^* \in E(R)$ implies $aa^* \in PE(R)$.
(5)⇒(1) Suppose that $aa^* \in PE(R)$. Then $aa^* = (aa^*)^2 = aa^*aa^*$. Multiplying the equality on the right by $(a^+)^*$, we get

$$a = aa^*(a^+)^* = aa^*aa^*(a^+)^* = aa^*a.$$

Hence, $a \in R^{PI}$. ∎

We generalize Lemma 2.1 as follows.

Theorem 2.3. *Let $a \in R^+$. Then the followings are equivalent:*
(1) $a \in R^{PI}$; (2) $a^+a - a^*a \in E(R)$; (3) $aa^+ - aa^* \in E(R)$.

Proof. (1)⇒(2) Assume that $a \in R^{PI}$. Then, by Lemma 2.1 $a^+a - a^*a = 0 \in E(R)$.
(2)⇒(3) Since $a^+a - a^*a \in E(R)$, $a^+a - a^*a = (a^+a - a^*a)^2 = a^+a - a^*a - a^*a + (a^*a)^2$. So, we get

$$a^*a = (a^*a)^2.$$

By Theorem 2.2, $a \in R^{PI}$. Again, by Lemma 2.1, $aa^+ - aa^* = 0 \in E(R)$.

(3)⇒(1) The condition $aa^+ - aa^* \in E(R)$ implies

$$aa^+ - aa^* = (aa^+ - aa^*)^2 = aa^+ - aa^* - aa^* + (aa^*)^2.$$

It follows that

$$aa^* = (aa^*)^2.$$

By Theorem 2.2, $a \in R^{PI}$. ∎

Theorem 2.4. *Let $a \in R^+$. Then the followings are equivalent:*
(1) $a \in R^{PI}$; (2) $aa^* + (1 - aa^*)xaa^* \in E(R)$ for any $x \in R$; (3) $a^*a + (1 - a^*a)xa^*a \in E(R)$ for any $x \in R$.

Proof. (1)⇒(2) Assume that $a \in R^{PI}$. Then $aa^* \in E(R)$ by Theorem 2.2. Hence,

$$(aa^* + (1 - aa^*)xaa^*)^2 = (aa^*)^2 + aa^*(1 - aa^*)xaa^* + (1 - aa^*)x(aa^*)^2 + (1 - aa^*)xaa^*(1 - aa^*)xaa^*$$

$$= aa^* + 0 + (1 - aa^*)xaa^* + 0 = aa^* + (1 - aa^*)xaa^*.$$

One gets $aa^* + (1 - aa^*)xaa^* \in E(R)$.

(2)⇒(1) Suppose that $aa^* + (1 - aa^*)xaa^* \in E(R)$ for each $x \in R$. Especially, choose $x = 0$, one has $aa^* \in E(R)$. By Theorem 2.2, we get $a \in R^{PI}$.

The proof of (1)⟺(3) is similar. ∎

Let $u \in R$. If $u^2 = 1$, then u is called a square element.

Let $u \in R$. If $u^2 = u + 2$, then u is called a quasi-idempotent element.

Theorem 2.5. *Let R be a ring and $e \in R$. Then e is an idempotent element if and only if $2e - 1$ is a square element and $3e - 1$ is a quasi-idempotent element.*

Proof. " ⟹ " Assume that e is idempotent. Then

$$(2e - 1)^2 = 4e^2 - 4e + 1 = 4e - 4e + 1 = 1,$$

$$(3e - 1)^2 = 9e^2 - 6e + 1 = 9e - 6e + 1 = 3e + 1 = (3e - 1) + 2.$$

" ⟸ " From the assumption, we have

$$1 = (2e - 1)^2 = 4e^2 - 4e + 1,$$

and

$$3e + 1 = (3e - 1) + 2 = (3e - 1)^2 = 9e^2 - 6e + 1.$$

This gives $4e^2 = 4e$, and $9e^2 = 9e$. Hence,

$$e^2 + 4e^2 + 4e^2 = 9e^2 = 9e = e + 4e + 4e.$$

It follows $e^2 = e$. Thus, e is idempotent. ∎

3. Characterize partial isometries by the solution of certain equation in a fixed set

Let $a \in R^\# \cap R^+$. We record $\{a, a^+, a^*, (a^+)^*, a^\#, (a^\#)^*, (a^\#)^+, (a^+)^\#\}$ as ρ_a. Among ρ_a, some equalities have been given in [11].

In [18], it is shown that $a \in R^\# \cap R^+$ is partial isometry if and only if the equation $x = xa^*a$ has at least one solution in $\chi_a = \{a, a^\#, a^+, a^*, (a^+)^*, (a^\#)^*\}$. Inspired by this, we consider the following equation:

$$x + xa^+a = xa^*a + aa^+x. \tag{3.1}$$

Lemma 3.1. *[21, Lemma 2.11] Let $a \in R^{\#} \cap R^{+}$ and $x, y \in R$.*
(1) If $a^+a^+x = 0$, then $a^+x = 0$.
(2) If $ya^+a^+ = 0$, then $ya^+ = 0$.

Lemma 3.2. *[11] Let $a \in R^{\#} \cap R^{+}$. Then $(a^{\#})^+ = a^+a^3a^+$ and $(a^+)^{\#} = (aa^{\#})^*a(aa^{\#})^*$.*

Theorem 3.3. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{PI}$ if and only if Eq.(3.1) has at least one solution in ρ_a.*

Proof. " $\Longrightarrow$ " Assume that $a \in R^{PI}$. Then $a = aa^*a$. It follows

$$a^{\#} = a^{\#}a^{\#}a = a^{\#}a^{\#}aa^*a = a^{\#}a^*a, a = (a^+)^*a^*a.$$

Noting $a^{\#}a^+a = a^{\#} = aa^+a^{\#}$ and $(a^+)^*a^+a = (a^+)^* = aa^+(a^+)^*$. Hence, $a, a^{\#}, (a^+)^*$ are all the solutions to Eq.(3.1).

" $\Longleftarrow$ " From the assumption, we have $x + xa^+a = xa^*a + aa^+x$ for some $x \in \rho_a$.

(1) If $x = a$, then $a + aa^+a = aa^*a + aa^+a$, that is $a = aa^*a$. Thus, $a \in R^{PI}$.

(2) If $x = a^{\#}$, then $a^{\#} + a^{\#}a^+a = a^{\#}a^*a + aa^+a^{\#}$, that is, $a^{\#} = a^{\#}a^*a$. This gives $a \in R^{PI}$ by [12, Theorem 1.5.2].

(3) If $x = a^+$, then $a^+ + a^+a^+a = a^+a^*a + aa^+a^+$. Multipying the equality on the left by a^+a, one gets

$$a^+a^2a^+a^+ = aa^+a^+.$$

By Lemma 3.1, we get

$$a^+a^2a^+ = aa^+.$$

Hence, $a \in R^{EP}$, it follows that $x = a^{\#}$. By (2), $a \in R^{PI}$.

(4) If $x = a^*$, then $a^* + a^*a^+a = a^*a^*a + aa^+a^*$. Multiplying the equality on the left by $(a^{\#})^*$, one yields $a^+a = a^*a$. Hence, $a \in R^{PI}$ by Lemma 2.1.

(5) If $x = (a^+)^*$, then $(a^+)^* + (a^+)^*a^+a = (a^+)^*a^*a + aa^+(a^+)^*$, that is, $(a^+)^* = a$. Hence, $a \in R^{PI}$ by Lemma 2.1.

(6) If $x = (a^{\#})^*$, then

$$(a^{\#})^* + (a^{\#})^*a^+a = (a^{\#})^*a^*a + aa^+(a^{\#})^*.$$

Multiplying the equality on the left by a^*, one obtains $a^+a = a^*a$. Hence, $a \in R^{PI}$ by Lemma 2.1.

(7) If $x = (a^{\#})^+ = a^+a^3a^+$, then

$$a^+a^3a^+ + a^+a^3a^+a^+a = a^+a^3a^+a^*a + aa^+a^+a^3a^+.$$

Multiplying the equality on the left by a^+a, one gets

$$a^+a^2a^+a^+a^3a^+ = aa^+a^+a^3a^+.$$

Noting that $a^3a^+a^{\#} = a$. Then

$$aa^+a^+ = aa^+a^+aa^+ = aa^+a^+a^3a^+a^{\#}a^+ = a^+a^2a^+a^+a^3a^+a^{\#}a^+ = a^+a^2a^+a^+.$$

By Lemma 3.1, $aa^+ = a^+a^2a^+$. Hence, $a \in R^{EP}$, it follows that

$$x = (a^{\#})^+ = (a^+)^+ = a.$$

By (1), $a \in R^{PI}$.

(8) If $x = (a^+)^{\#} = (aa^{\#})^*a(aa^{\#})^*$, then

$$(aa^{\#})^*a(aa^{\#})^* + (aa^{\#})^*a(aa^{\#})^*a^+a = (aa^{\#})^*a(aa^{\#})^*a^*a + aa^+(aa^{\#})^*a(aa^{\#})^*.$$

That is,

$$(aa^{\#})^*a(aa^{\#})^* + (aa^{\#})^*a = (aa^{\#})^*aa^*a + a(aa^{\#})^*.$$

Multiplying the equality on the left by a^+, one gets $a^+a = a^*a$. Hence, $a \in R^{PI}$ by Lemma 2.1. ∎

To further study the characterizations of partial isometries, then we transform univariate equations into bivariate equations. Now we consider the following equation:

$$x = ya^*a. \tag{3.2}$$

Theorem 3.4. *Let $a \in R^+$. Then the general solution to Eq.(3.2) is given by*

$$\begin{cases} x = pa^*a \\ y = p + v - va^+a \end{cases}, \; p,\ u,\ v \in R. \tag{3.3}$$

Proof. First,

$$ya^*a = (p + v - va^+a)a^*a = pa^*a + va^*a - va^+aa^*a = pa^*a = x.$$

So, the formula (3.3) is a solution to Eq.(3.2).

Next, let

$$\begin{cases} x = x_0 \\ y = y_0 \end{cases} \tag{3.4}$$

be any solution to Eq.(3.2). Then

$$x_0 = y_0a^*a.$$

Take $p = x_0a^+(a^+)^*$ and $v = y_0$, then

$$p = x_0a^+(a^+)^* = y_0a^*aa^+(a^+)^* = y_0(a^+aa^+a)^* = y_0a^+a = va^+a.$$

Hence,

$$y_0 = p + y_0 - p = p + v - va^+a,$$

$$x_0 = y_0a^*a = y_0a^+aa^*a = pa^*a.$$

Hence, every solution to Eq.(3.2) has the form of the formula (3.3). Therefore, the general solution to Eq.(3.2) is given by (3.3). ∎

Theorem 3.5. *Let $a \in R^+$. Then $a \in R^{PI}$ if and only if the general solution to Eq.(3.2) is given by*

$$\begin{cases} x = pa^+a \\ y = p + v - va^+a \end{cases}, \; p,\ u,\ v \in R. \tag{3.5}$$

Proof. " $\Longrightarrow$ " Assume that $a \in R^{PI}$. Then $a^+ = a^*$ by Lemma 2.1.

So, the formula (3.5) is the same as the formula (3.3). By Theorem 3.4, we are done.

" $\Longleftarrow$ " From the assumption, for all $p \in R$, we obtain

$$pa^+a = x = ya^*a = (p + v - va^+a)a^*a = pa^*a.$$

Especially, choose $p = a$. Then

$$a = aa^+a = aa^*a.$$

Hence, $a \in R^{PI}$. ∎

Now we consider the following equation:

$$x = ya^+a. \tag{3.6}$$

It is easy to show the following lemma.

Lemma 3.6. *Let $a \in R^+$. Then the general solution to Eq.(3.6) is given by Formula (3.5).*

From Theorem 3.5 and Lemma 3.6, we have the following theorem.

Theorem 3.7. *Let $a \in R^+$. Then $a \in R^{PI}$ if and only if Eq.(3.6) has the same solution as Eq.(3.2).*

We adjust Eq.(3.2) as the following equation:

$$ayx = xy(a^+)^*. \tag{3.7}$$

Theorem 3.8. *Let $a \in R^\# \cap R^+$. Then $a \in R^{PI}$ if and only if Eq.(3.7) has at least one solution in $\rho_a^2 = \{(x, y)|x, y \in \rho_a\}$, where $\rho_a = \{a, (a^+)^*, (a^\#)^*, (a^+)^\#, (a^\#)^+\}$.*

Proof. " $\Longrightarrow$ " Suppose that $a \in R^{PI}$. Then $(a^+)^* = a$ by Lemma 2.1.

Hence,

$$(x, y) = ((a^+)^*,\ a) \tag{3.8}$$

is a solution.

" $\Longleftarrow$ " (1) If $y = a$, then Eq.(3.7) is converted into

$$a^2x = xa(a^+)^*.$$

① If $x = a$, then $a^3 = a^2(a^+)^*$. Thus,

$$a = a^\#a^\#a^3 = a^\#a^\#a^2(a^+)^* = a^\#a(a^+)^* = (a^+)^*.$$

From Lemma 2.1, we get $a \in R^{PI}$.

② If $x = (a^+)^*$, then $a^2(a^+)^* = (a^+)^*a(a^+)^*$.

Multiplying the equality on the right by $a^*a^\#$, one gets

$$a = (a^+)^*.$$

By Lemma 2.1, $a \in R^{PI}$.

③ If $x = (a^\#)^*$, then $a^2(a^\#)^* = (a^\#)^*a(a^+)^* = (a^\#)^*a(a^+)^*aa^\# = a^2(a^\#)^*aa^\#$.

Multiplying the equality on the left by $a^+a^\#$, one gets

$$(a^\#)^* = (a^\#)^*aa^\#.$$

By [12, Theorem 1.1.3], $a \in R^{EP}$. Thus, $x = (a^\#)^* = (a^+)^*$. It follows from ② that $a \in R^{PI}$.

④ If $x = (a^\#)^+ = a^+a^3a^+$, then $a^4a^+ = a^2a^+a^3a^+ = a^+a^3a^+a(a^+)^* = a^+a^3(a^+)^*$.

Multiplying the equality on the left by $a^\#a^\#$, one yields

$$a^2a^+ = (a^+)^* = (a^+)^*aa^\# = a^2a^+aa^\# = a.$$

Hence, $a \in R^{EP}$. It follows that $x = (a^\#)^+ = a$. Thus, $a \in R^{PI}$ by ①.

⑤ If $x = (a^+)^\# = (aa^\#)^*a(aa^\#)^*$, then

$$a^2(aa^\#)^*a(aa^\#)^* = (aa^\#)^*a(aa^\#)^*a(a^+)^* = a^+a(aa^\#)^*a(aa^\#)^*a(a^+)^* = a^+a^3(aa^\#)^*a(aa^\#)^*. \tag{3.9}$$

Noticing that $(aa^\#)^*a(aa^\#)^*a^+a^+ = (aa^\#)^*aa^+a^+ = (aa^\#)^*a^+ = a^+$.

Then we multiply the equality(3.9) on the right by $a^+a^+a^\#$, one yields

$$aa^\# = a^2a^+a^\# = a^+a^3a^+a^\# = a^+a.$$

Hence, $a \in R^{EP}$. This infers $x = (a^+)^\# = a$. Therefore, $a \in R^{PI}$ by ①.

(2) If $y = (a^+)^*$, then Eq.(3.7) is converted into

$$a(a^+)^*x = x(a^+)^*(a^+)^*.$$

① If $x = a$, then $a(a^+)^*a = a(a^+)^*(a^+)^*$.

Multiplying the equality on the left by $a^{\#}$, one gets

$$(a^+)^*a = (a^+)^*(a^+)^*.$$

We take the involution $*$ to bothe sides, one yields $a^+a^+ = a^*a^+$.Hence, $a \in R^{PI}$ by [21, Corollary 2.10].

② If $x = (a^+)^*$, then $a(a^+)^*(a^+)^* = (a^+)^*(a^+)^*(a^+)^*$. Taking $*$ to both sides, one gets

$$a^+a^+a^+ = a^+a^+a^*.$$

By Lemma 3.1, one has $a^+a^+ = a^+a^*$. Hence, $a \in R^{PI}$ by [21, Corollary 2.10].

③ If $x = (a^{\#})^*$, then $a(a^+)^*(a^{\#})^* = (a^{\#})^*(a^+)^*(a^+)^*$. Taking $*$ to both sides, one obtains

$$a^+a^+a^{\#} = a^{\#}a^+a^*.$$

Multiplying the equality on the right by a^+a, one gets $a^{\#}a^+a^* = a^{\#}a^+a^*a^+a$.

Next, multiplying the last equality on the left by a^2, one yields

$$aa^+a^* = aa^+a^*a^+a.$$

Again taking $*$, then we get $a^2a^+ = a^+a^3a^+$. This gives

$$aa^{\#} = a^2a^+a^{\#} = a^+a^3a^+a^{\#} = a^+a.$$

Hence, $a \in R^{EP}$, which induces $x = (a^{\#})^* = (a^+)^*$. Therefore, $a \in R^{PI}$ by ②.

④ If $x = (a^{\#})^+ = a^+a^3a^+$, then $a(a^+)^*a^+a^3a^+ = a^+a^3a^+(a^+)^*(a^+)^*$, that is,

$$a(a^+)^*a^2a^+ = a^+a^2(a^+)^*(a^+)^*.$$

Multiplying the equality on the right by $a^{\#}a$, one yields

$$a(a^+)^*a^2a^+ = a(a^+)^*a^2a^+a^{\#}a = a(a^+)^*a.$$

This leads to

$$(a^+)^*a^2a^+ = a^{\#}a(a^+)^*a^2a^+ = a^{\#}a(a^+)^*a = (a^+)^*a.$$

Taking $*$ to both sides, one gets

$$aa^+a^*aa^+a^+ = aa^+a^*a^+ = a^*a^+ = a^*aa^+a^+.$$

By Lemma 3.1, $aa^+a^* = a^*$. Hence, by [12, Theorem 1.2.1], $a \in R^{EP}$, which infers $x = (a^{\#})^+ = a$. Thus, $a \in R^{PI}$ by ①.

⑤ If $x = (a^+)^{\#} = (aa^{\#})^*a(aa^{\#})^*$, then

$$a(a^+)^*(aa^{\#})^*a(aa^{\#})^* = (aa^{\#})^*a(aa^{\#})^*(a^+)^*(a^+)^*.$$

Multiplying the equality on the left by a^+a, one yields

$$a(a^+)^*(aa^{\#})^*a(aa^{\#})^* = a^+a^2(a^+)^*(aa^{\#})^*a(aa^{\#})^*.$$

Multiplying the last equality on the right by a^+a^+a, one yields

$$a(a^+)^* = a^+a^2(a^+)^*.$$

Taking $*$ to both sides, one gets $a^+a^* = a^+a^*a^+a$. It follows from Lemma 3.1 that $a^* = a^*a^+a$. Hence, $a \in R^{EP}$.

Now $x = (a^+)^{\#} = a$. Thus, $a \in R^{PI}$ by ①.

(3) If $y = (a^{\#})^*$, then Eq.(3.7) is converted into

$$a(a^{\#})^*x = x(a^{\#})^*(a^+)^*.$$

① If $x = a$, then $a(a^\#)^*a = a(a^\#)^*(a^+)^*$. We take $*$ to both sides,

$$a^+a^\#a^* = a^*a^\#a^*.$$

This gives $a^\#a^* = aa^+a^\#a^* = aa^*a^\#a^*$.

Multiplying the last equality on the right by $(a^+)^*a^2$, we get

$$a = aa^*a.$$

Thus, $a \in R^{PI}$.

② If $x = (a^+)^*$, then $a(a^\#)^*(a^+)^* = (a^+)^*(a^\#)^*(a^+)^*$. This gives $a^+a^\#a^* = a^+a^\#a^+$ and

$$aa^* = a^3a^+a^\#a^* = a^3a^+a^\#a^+ = aa^+.$$

Hence, $a \in R^{PI}$.

③ If $x = (a^\#)^*$, then $a(a^\#)^*(a^\#)^* = (a^\#)^*(a^\#)^*(a^+)^*$, so $a^\#a^\#a^* = a^+a^\#a^\#$.

Multiplying the equality on the left by a^3, one gets

$$aa^* = aa^\#.$$

By [12, Theorem 1.5.3], $a \in R^{PI}$.

④ If $x = (a^\#)^+ = a^+a^3a^+$, then $a(a^\#)^*a^+a^3a^+ = a^+a^3a^+(a^\#)^*(a^+)^*$.

Multiplying the equality on the right by $a^\#a$, one yields

$$a(a^\#)^*a^+a^2 = a(a^\#)^*a^+a^3a^+.$$

Multiplying the last equality on the left by $a^\#a^*a^+$, one gets $a^\#a = aa^+$.

Hence, $a \in R^{EP}$. It follows that $x = (a^\#)^+ = a$. Thus, $a \in R^{PI}$ by ①.

⑤ If $x = (a^+)^\# = (aa^\#)^*a(aa^\#)^*$, then $a(a^\#)^*(aa^\#)^*a(aa^\#)^* = (aa^\#)^*a(aa^\#)^*(a^\#)^*(a^+)^*$, that is,

$$a(a^\#)^*a(aa^\#)^* = (aa^\#)^*a(a^\#)^*(a^+)^*.$$

Multiplying the equality on the left by a^+, and on the right by a^*, one yields

$$(a^\#)^*aa^* = (a^\#)^*.$$

This gives $a^\# = aa^*a^\#$. Hence, $a \in R^{PI}$ by [12, Theorem 1.5.2].

(4) If $y = (a^\#)^+ = a^+a^3a^+$, then Eq.(3.7) is converted into

$$a^3a^+x = xa^+a^2(a^+)^*.$$

① If $x = a$, then $a^3a^+a = aa^+a^2(a^+)^*$, that is, $a^3 = a^2(a^+)^*$. Thus, $a \in R^{PI}$.

② If $x = (a^+)^*$, then $a^3a^+(a^+)^* = (a^+)^*a^+a^2(a^+)^*$, that is,

$$a^2(a^+)^* = (a^+)^*a(a^+)^*.$$

Taking $*$ to both sides, one has

$$a^+a^*a^* = a^+a^*a^+.$$

By Lemma 3.1, $a^*a^* = a^*a^+$. Hence, $a \in R^{PI}$ by [21, Corollary 2.10].

③ If $x = (a^\#)^*$, then $a^3a^+(a^\#)^* = (a^\#)^*a^+a^2(a^+)^*$. Taking $*$ to both sides, one gets

$$a^\#aa^+a^*a^* = a^+a^*a^+aa^\#.$$

Multiplying the equality on the left by $(a^\#)^*a$, one gets

$$a^* = a^+aa^\#.$$

This induces

$$a^*a = a^+aa^\#a = a^+a.$$

Hence, $a \in R^{PI}$ by [12, Theorem 1.5.2].

④ If $x = (a^\#)^+ = a^+a^3a^+$, then $a^3a^+a^+a^3a^+ = a^+a^3a^+a^+a^2(a^+)^*$.

Multiplying the equality on the right by $a^\#a$, one yields

$$a^3a^+a^+a^3a^+ = a^3a^+a^+a^2.$$

Multiplying the last equality on the left by $a^+a^\#a^\#$, we get

$$a^+a^+a^3a^+ = a^+a^+a^2.$$

By Lemma 3.1, $a^+a^3a^+ = a^+a^2$.

Hence, $a \in R^{EP}$. It follows that $x = (a^\#)^+ = a$, and so $a \in R^{PI}$ by ①.

⑤ If $x = (a^+)^\# = (aa^\#)^*a(aa^\#)^*$, then $a^3a^+(aa^\#)^*a(aa^\#)^* = (aa^\#)^*a(aa^\#)^*a^+a^2(a^+)^*$, that is,

$$a^3(aa^\#)^* = (aa^\#)^*a^2(a^+)^*.$$

Multiplying the equality on the right by a^+a^2, and on the left by $a^\#a^+$, one yields

$$a^2 = (a^+)^*a.$$

This gives $a^*a^* = a^*a^+$. Hence, $a \in R^{PI}$ by [21, Corollary 2.10].

(5) If $y = (a^+)^\# = (aa^\#)^*a(aa^\#)^*$, then Eq.(3.7) is converted into

$$a(aa^\#)^*a(aa^\#)^*x = x(aa^\#)^*a(aa^\#)^*(a^+)^*.$$

① If $x = a$, then $a(aa^\#)^*a(aa^\#)^*a = a(aa^\#)^*a(aa^\#)^*(a^+)^*$.

Multiplying the equality on the left by $aa^+a^+a^+$, one gets

$$a = (a^+)^*.$$

Thus, $a \in R^{PI}$ by Lemma 2.1.

② If $x = (a^+)^*$, then $a(aa^\#)^*a(aa^\#)^*(a^+)^* = (a^+)^*(aa^\#)^*a(aa^\#)^*(a^+)^*$.

Multiplying the equality on the right by $a^*a^+a^*$, we obtain

$$aa^* = aa^+.$$

Hence, $a \in R^{PI}$ by Lemma 2.1.

③ If $x = (a^\#)^*$, then $a(aa^\#)^*a(aa^\#)^*(a^\#)^* = (a^\#)^*(aa^\#)^*a(aa^\#)^*(a^+)^*$, that is,

$$a(aa^\#)^*a(a^\#)^* = (a^\#)^*a(aa^\#)^*(a^+)^*.$$

Multiplying the equality on the right by $a^*a^+a^*$, one gets

$$aa^* = (aa^\#)^*.$$

It follows that $aa^* = aa^\#$. By [12, Theorem 1.5.3], $a \in R^{PI}$.

④ If $x = (a^\#)^+ = a^+a^3a^+$, then $a(aa^\#)^*a(aa^\#)^*a^+a^3a^+ = a^+a^3a^+(aa^\#)^*a(aa^\#)^*(a^+)^*$, that is,

$$a(aa^\#)^*a^3a^+ = a^+a^3(aa^\#)^*(a^+)^*.$$

Multiplying the equality on the right by $a^\#a$, one yields

$$a(aa^\#)^*a^3a^+ = a(aa^\#)^*a^2.$$

Multiplying the last equality on the left by $a^{\#}a^{+}a^{+}$, we get

$$aa^{+} = aa^{\#}.$$

Hence, $a \in R^{EP}$, which implies $x = (a^{\#})^{+} = a$. Thus, $a \in R^{PI}$ by ①.

⑤ If $x = (a^{+})^{\#} = (aa^{\#})^{*}a(aa^{\#})^{*}$, then

$$a(aa^{\#})^{*}a(aa^{\#})^{*}(aa^{\#})^{*}a(aa^{\#})^{*} = (aa^{\#})^{*}a(aa^{\#})^{*}(aa^{\#})^{*}a(aa^{\#})^{*}(a^{+})^{*},$$

that is,

$$a(aa^{\#})^{*}a(aa^{\#})^{*}a(aa^{\#})^{*} = (aa^{\#})^{*}a(aa^{\#})^{*}a(aa^{\#})^{*}(a^{+})^{*}.$$

Multiplying the equality on the right by $a^{+}a$, one yields

$$a(aa^{\#})^{*}a(aa^{\#})^{*}a(aa^{\#})^{*} = a(aa^{\#})^{*}a(aa^{\#})^{*}a.$$

Multiplying the last equality on the left by $aa^{+}a^{+}a^{+}$, we get

$$a = a(aa^{\#})^{*}.$$

Hence, $a \in R^{EP}$ by [12, Theorem 1.1.3]. Thus, $x = (a^{+})^{\#} = a$, $a \in R^{PI}$ by ①. ∎

Observing the Eq.(3.2), we construct the following equation:

$$a + x = aa^{*}a + ya^{*}a. \tag{3.10}$$

Theorem 3.9. *Let $a \in R^{\#} \cap R^{+}$. Then the general solution to Eq.(3.10) is given by*

$$\begin{cases} x = aa^{*}a + pa^{*}a \\ y = (a^{+})^{*} + p + v - va^{+}a \end{cases},\ p,\ v \in R. \tag{3.11}$$

Proof. First, it can be easily calculated that the formula (3.11) is a solution to Eq.(3.10), as follows

$$a + x = a + aa^{*}a + pa^{*}a,$$

$$aa^{*}a + ya^{*}a = aa^{*}a + ((a^{+})^{*} + p + v - va^{+}a)a^{*}a = aa^{*}a + a + pa^{*}a + va^{*}a - va^{*}a = aa^{*}a + a + pa^{*}a.$$

thus, the formula (3.11) is a solution to Eq.(3.10).

Then, let $x = x_0, y = y_0$ be any solution to Eq.(3.10). It deduces

$$a + x_0 = aa^{*}a + y_0a^{*}a.$$

This infers $x_0 = x_0a^{+}a$. When choosing $p = x_0a^{+}(a^{+})^{*} - a, v = y_0$, it obtains

$$pa^{*}a = (x_0a^{+}(a^{+})^{*} - a)a^{*}a = x_0a^{+}a - aa^{*}a = x_0 - aa^{*}a.$$

Hence, $x_0 = aa^{*}a + pa^{*}a$. Also

$$va^{+}a = y_0a^{+}a = y_0a^{*}(a^{+})^{*} = (y_0a^{*}a)a^{+}(a^{+})^{*} = (a + x_0 - aa^{*}a)a^{+}(a^{+})^{*} = (a^{+})^{*} + x_0a^{+}(a^{+})^{*} - a = (a^{+})^{*} + p.$$

It gives

$$y_0 = (a^{+})^{*} + p + y_0 - ((a^{+})^{*} + p) = (a^{+})^{*} + p + v - va^{+}a.$$

Therefore, the general solution to Eq.(3.10) is given by formula (3.11). ∎

Theorem 3.10. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{PI}$ if and only if the general solution to Eq.(3.10) is given by*

$$\begin{cases} x = a + pa^{*}a \\ y = (a^{+})^{*} + p + v - va^{+}a \end{cases},\ p,\ v \in R. \tag{3.12}$$

Proof. " $\Longrightarrow$ " Suppose that $a \in R^{PI}$, then $a = aa^*a$. This implies the formula (3.11) is the same as formula (3.12). Thus, by Theorem 3.9, we can obtain the general solution to Eq.(3.10) is given by formula (3.12).

" $\Longleftarrow$ " From the assumption, for any $p, v \in R$, we obtain

$$a + (a + pa^*a) = aa^*a + ((a^+)^* + p + v - va^+a)a^*a,$$

that is, $a = aa^*a$. Hence, $a \in R^{PI}$. ∎

Revised Eq.(3.10) as follows

$$a + x = ya^*a. \tag{3.13}$$

Theorem 3.11. *Let $a \in R^\# \cap R^+$. Then the general solution to Eq.(3.13) is given by*

$$\begin{cases} x = pa^*a \\ y = (a^+)^* + p + v - va^+a \end{cases}, \ p,\ v \in R. \tag{3.14}$$

Proof. First, it is obvious that

$$a + x = a + pa^*a = (a^+)^*a^*a + pa^*a + va^*a - va^+aa^*a = ((a^+)^* + p + v - va^+a)a^*a = ya^*a.$$

So, the formula (3.14) is a solution to Eq.(3.13).

Then, let $x = x_0, y = y_0$ be any solution to Eq.(3.13). It gives

$$a + x_0 = y_0a^*a,$$

which infers $x_0 = x_0a^+a$. Choosing $p = x_0a^+(a^+)^*, v = y_0$, one gets

$$pa^*a = (x_0a^+(a^+)^*)a^*a = x_0a^+a = x_0.$$

Therefore, $x_0 = pa^*a$. And

$$va^+a = y_0a^+a = y_0a^*(a^+)^* = (y_0a^*a)a^+(a^+)^* = (a + x_0)a^+(a^+)^* = (a^+)^* + x_0a^+(a^+)^* = (a^+)^* + p.$$

It induces

$$y_0 = (a^+)^* + p + y_0 - ((a^+)^* + p) = (a^+)^* + p + v - va^+a.$$

Hence, the general solution to Eq.(3.13) is given by formula (3.14). ∎

Theorem 3.12. *Let $a \in R^\# \cap R^+$. Then $a \in R^{PI}$ if and only if the general solution to the Eq.(3.13) is given by*

$$\begin{cases} x = pa^*a \\ y = a + p + v - va^+a \end{cases}, \ p,\ v \in R. \tag{3.15}$$

Corollary 3.13. *Let $a \in R^\# \cap R^+$. Then $a \in R^{PI}$ if and only if the general solution to the following equation is provided by formula (3.14)*

$$(a^+)^* + x = ya^*a. \tag{3.16}$$

Proof. " $\Longrightarrow$ " If $a \in R^{PI}$, then $a = (a^+)^*$, it follows that Eq.(3.13) is the same as Eq.(3.16). By Theorem 3.11, we get the general solution to Eq.(3.16) is provided by formula (3.15).

" $\Longleftarrow$ " From the assumption, one has

$$(a^+)^* + pa^*a = ((a^+)^* + p + v - va^+a)a^*a.$$

That is, $(a^+)^* = (a^+)^*a^*a = a$. Hence, $a \in R^{PI}$. ∎

From Theorem 3.11, Theorem 3.12 and Corollary 3.13, we have the following corollary.

Corollary 3.14. *Let $a \in R^\# \cap R^+$. Then $a \in R^{PI}$ if and only if Eq.(3.13) has the same solution to Eq.(3.16).*

References

[1] J. Y. Cai, Z. C. Chen, J. C. Wei, *Partial Isometry and EP Elements*, Filomat. **35:6**(2021), 2121–2128.
[2] D. Drivaliaris, S. Karanasios, D. Pappas, *Factorizations of EP operators*, Linear Algebra Appl. **429**(2008), 1555–1567.
[3] R. E. Hartwig, *Block generalized inverses*, Arch. Retion. Mech. Anal. **61**(1976), 197–251.
[4] R. E. Hartwig, *Generalized inverses, EP elements and associates*, Rev. Roumaine Math. Pures Appl. **23**(1978), 57–60.
[5] R. E. Hartwig, I. J. Katz, *Products of EP elements in reflexive semigroups*, Linear Algebra Appl. **14**(1976),11–19.
[6] R. E. Harte, M. Mbekhta, *On generalized inverses in C^*-algebras*, Studia Math. **103**(1992), 71–77.
[7] J. J. Koliha, *The Drazin and Moore-Penrose inverse in C^*-algebras*, Math. Proc. R. Ir. Acad. **99**(1999), 17–27.
[8] J. J. Koliha, D. Cvetković, D. S. Djordjević, *Moore-Penrose inverse in rings with involution*, Linear Algebra Appl. **426**(2007), 371–381.
[9] J. J. Koliha, P. Patrílcio, *Elements of rings with equal spectral idempotents*, J. Aust. Math. Soc. **72**(2002), 137–152.
[10] S. Karanasios, *EP elements in rings and semigroup with involution and C^*-algebras*, Serdica Math. J. **41**(2015), 83–116.
[11] A. Q. Li, J. C. Wei, *The influence of the expression form of solutions to related equations on SEP elements in a ring with involution*, J. Algebra Appl. **23:6**(2024), 2450113.
[12] D. Mosić, *Generalized inverses*, Faculty of Sciences and Mathematics, University of Niš, Niš, 2018.
[13] D. Mosić, *Polynomially partial isometric operators*, HACET J MATH STAT. **52:1**(2023), 151–62.
[14] D. Mosić, D. S. Djordjević, *Partial isometries and EP elements in rings with involution*, Electron J. Linear Algebra. **18**(2009), 761–722.
[15] D. Mosić, D. S. Djordjević, J. J. Koliha, *EP elements in rings*, Linear Algebra Appl. **431**(2009), 527–535.
[16] D. Mosić, D. S. Djordjević, *New characterizations of EP, generalized normal and generalized Hermitian elements in rings*, Appl. Math. Comput. **218**(2012), 6702–6710.
[17] D. Mosić, D. S. Djordjević, *Further results on partial isometries and EP elements in rings with involution*, Math. Comput. Model. **54**(2011), 460–465.
[18] Y. C. Qu, H. Yao, J. C. Wei, *Some characterizations of partial isometry elements in rings with involutions*, Filomat. **33:19**(2019), 6395–6399.
[19] X. R. Wang, J. C. Wei,*PI elements and solutions of related equations in a ring with involution*, Filomat. **38:24**(2024), 8495–8510.
[20] X. Y. Yang, Z. Y. Fan, J. C. Wei, *Some studies on Partial Isometry in Rings with Involution*, Filomat. **36:3**(2022), 1061–1067.
[21] D. D. Zhao, J. C. Wei,*Strongly EP elements in rings with involution*, J. Algebra Appl. **21:5**(2022), 2250088.
[22] R. Z. Zhao, H. Yao, J. C. Wei,*Characterizations of partial isometries and two kinds of EP elements*, Czecho. Math. J. **70:2**(2020),539–551.