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Rough intuitionistic fuzzy 2-absorbing primary ideals in semirings

Asma Ali[a], Arshad Zishan[a,*]

[a]*Department of Mathematics, Aligarh Muslim University, Aligarh, India*

Abstract. This study presents the innovative concept of roughness in semirings built on intuitionistic fuzzy frameworks. Specifically, this research introduces the concept of rough intuitionistic fuzzy ideals within the framework of semirings and systematically investigates their properties. The aim of this study is to extend the existing concepts of 2-absorbing, 2-absorbing primary, intuitionistic fuzzy 2-absorbing, and intuitionistic fuzzy 2-absorbing primary ideals of semirings by incorporating the notion of roughness. This extension provides a more comprehensive and refined framework for analyzing ideals and intuitionistic fuzzy ideals in semirings. Furthermore, we explore the conditions that establish a connection between the upper and lower rough 2-absorbing primary ideals and the upper and lower approximations of their homomorphic images.

1. Introduction

Two fundamental concepts in commutative algebra are prime ideals and primary ideals. Several researchers have examined various generalizations of prime ideals. In [7], Badawi introduced the concept of a 2-absorbing ideal (2AI) as another kind of prime ideal, whereas in [8], the authors gave the idea of 2-absorbing primary ideals (2API), which is a generalization of primary ideals. Also, fuzzy prime ideals and fuzzy primary ideals are studied in [37, 38]. Badawi [3] studied n-absorbing ideals of a commutative ring.

The concept of fuzzy sets provides an effective framework for understanding the behavior of systems that are either highly complex or insufficiently defined for precise mathematical analysis using traditional tools and methods. This concept has found extensive applications in fields such as expert systems, pattern recognition, and image processing. Zadeh initially proposed the concept of fuzzy sets in his seminal work[45]. Subsequent studies, including [20, 21, 30, 34, 39, 46], have explored and analyzed concepts such as prime, semiprime, maximal, and radical fuzzy ideals of a ring.

Unfortunately, the failure of the fuzzy set theory was caused by inadequate knowledge regarding the function's negative membership degree. In order to solve this issue, Atanassov [4] gave the concept of intuitionistic fuzzy (IF) set which included the negative membership degree of the function in fuzzy set theory in such a way that sum of the positive membership degree and negative membership degree must not exceed by 1. Furthermore, in [5, 6], Atanassov defined some new operations on the IF set and studied

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* Corresponding author: Arshad Zishan
Email addresses: asma_ali2@rediffmail.com (Asma Ali), arshadzeeshan1@gmail.com (Arshad Zishan)

their properties. Additionally, the work by Hur et al. [23] focused on the concept of IF subrings, and many authors have attempted to extend/generalize this concept. Marashdeh et al. [36] explained how the notion of IF rings is intricately connected to the concept of fuzzy space. Chao et al. [10] introduced interval-valued IF sets to a wider context. Darani [11] investigated L-fuzzy 2-absorbing ideals, while Hashempoor et al. [12] established the concept of L-fuzzy 2-absorbing ideals in semirings.

Pawlak [41] introduced rough set theory as an effective mathematical framework for addressing uncertainty in data analysis. This theory, which extends conventional set theory, characterizes a subset of the universe by employing two classical sets: the lower approximation and the upper approximation. The definitions of these approximations are grounded in equivalence classes [44, 47], which provide the basis for determining the boundaries of the sets. Rough set theory has since become an essential tool for handling incomplete or imprecise information in various domains. Numerous researchers have investigated rough sets in various mathematical structures. Biswas et al. [9] and Kuroki [32] examined the concepts of rough subgroups and rough ideals within semigroups, expanding the application of rough set theory in algebraic structures. Kuroki and Wang [33] also looked at how the lower and upper approximations of rough sets relate to normal subgroups. These works help to understand how rough set theory can be applied to study algebraic systems, especially when there is uncertainty or imprecision in the data. By integrating rough sets with ring theory, Davvaz [13] constructed a new framework where rings functioned as universal sets. he ideas of rough ideals and rough subrings, concerning an ideal in a ring, were proposed by Davvaz[14]. This makes difficult mathematical concepts more understandable by deepening our understanding of rough sets and demonstrating their connections to the larger field of ring theory. In 2008, Kazanci and Davvaz [27] introduced the concepts of rough prime and rough fuzzy prime ideals in commutative rings. These ideas expand on the traditional notion of prime ideals by incorporating roughness and fuzziness, offering a new perspective for studying ideals within commutative rings. Jun [24] explored the concept of roughness in Γ-subsemigroups and ideals within Γ-semigroups. In [25], the notion of rough ideals was introduced as a generalization of ideals in BCK-algebras. Davvaz, in several works [15–17], applied approximation concepts to the theory of algebraic hyperstructures. Additionally, Davvaz and Mahdavipour [18] investigated rough modules.

The concepts of rough prime ideals and rough fuzzy prime ideals in semigroups were introduced by Xiao [43], with further details provided in [19, 26, 28, 47]. Relationships between rough sets, fuzzy sets, and algebraic systems have been explored by many mathematicians. Specifically, in [13], lower and upper approximations were formulated in the context of ring theory. In [1], Ali and Zishan gave the notion of rough 3-prime ideals and rough fuzzy 3-prime ideals in near rings. In 2011, Thomas and Nair investigated the combination of rough set theory, IF set theory, and lattice theory to address uncertainty and vagueness in mathematical structures. They expanded this approach to define Rough IF lattices and Rough IF ideals , establishing rules and conditions for their properties and operations. Ali et al. [2] investigated rough ideals in commutative semirings, demonstrating that the lower and upper approximations of a left (right) ideal of a semiring S are also left (right) ideals of S. In [29], Kumar and Selvan introduced rough fuzzy ideals and rough fuzzy prime ideals in semirings, and later, in [31], the concept of rough IF sets within the context of rings were introduced and established the corresponding algebraic framework. They introduced definitions for rough IF ideals and rough IF prime ideals, focusing on their upper and lower approximations. In 2014, Mandal and Ranadive delved into the concept of rough IF ideals within the framework of IF subrings in commutative rings. Their study integrates rough set theory with IF set theory to define and explore these ideals, offering both theoretical insights and potential applications in algebra and fuzzy systems. This research contributes to the field of mathematics, particularly fuzzy algebraic structures, by extending traditional algebraic concepts through advanced methodologies in fuzzy logic. In 2023, Ozkan et al. [40] studied IF 2-absorbing ideals (IF2AIs) and IF 2-absorbing primary ideals (IF2APIs) in commutative semirings.

2. Motivation

The motivation for this research stems from several key factors:

Table 1: Key motivating factors for rough intuitionistic fuzzy ideals of a semiring, along with corresponding authors

Year	Authors	Motivating Factor
2012	Kumar and Selvan	Introduced rough fuzzy ideals and rough fuzzy prime ideals in semirings ([29]). Later, introduced rough IF sets and their algebraic framework in rings ([31]). Defined rough IF ideals and rough IF prime ideals, focusing on their upper and lower approximations.
2014	Mandal and Ranadive	Explored rough IF ideals within the framework of IF subrings in commutative rings. Integrated rough set theory with IF set theory, offering insights into fuzzy algebraic structures.
2023	Ozkan et al.	Studied IF2AIs and IF2APIs in commutative semirings ([40]).

Motivated by above studies, in this paper, we define rough 2APIs and rough IF2APIs in the framework of a semiring, and obtain some of the special properties of these ideals. We also explore the relationships that exist between the upper and lower rough 2APIs and the upper and lower approximations of their homomorphic images. With more understanding of these mathematical entities' features and relationships inside the semiring framework, this research clarifies the complex interactions between them.

3. Contributions

This study contributes to the field of fuzzy algebra by:

• Introducing the novel concept of rough 2AIs and rough IF2AIs in semirings.

• Establishing fundamental properties and results of rough ideals and rough intuitionistic fuzzy ideals in semirings that is in Lemma 4.2 and Lemma 4.3, we provide sufficient condition for upper and lower approximations of a set of a semirings to become ideals.

• We also provide necessary and sufficient conditions for an upper and a lower approximations of a set $\mathfrak{P}$ of a semiring $\mathscr{R}_1$ to be a 2AI, 2API is an upper and a lower approximations of the homomorphic image of $\mathfrak{P}$ of a semiring $\mathscr{R}_2$ is a 2AI, 2API of $\mathscr{R}_2$ (See Theorem 5.3).

• We also provide necessary and sufficient conditions for an upper and a lower approximations of an IF set $\mathfrak{A}$ of a semiring $\mathscr{R}_1$ to be an IF2AI, IFAPI is an upper and a lower approximations of the homomorphic image of $\mathfrak{A}$ of a semiring $\mathscr{R}_2$ is an IF2AI, IF2API of $\mathscr{R}_2$ (See Theorem 6.3).

• Extending the application of intuitionistic fuzzy set theory in algebraic structures, building upon previous work in related algebraic structures.

4. Preliminaries

This section presents the essential definitions, notations, and foundational results needed for the subsequent sections. We also include short proofs of key resutls to keep the content complete and easy to follow.

Definition 4.1 ([22]). *A semiring $\mathscr{R}$ is a triplet $(\mathscr{R}, +, \cdot)$, where '+' and '$\cdot$' are two binary operations such that*

(*i*) $(\mathscr{R}, +)$ *is a commutative monoid with additive identity 0*

(*ii*) $(\mathscr{R}, \cdot)$ *is a monoid with identity* $1 \neq 0$

(*iii*) $(m_1 + m_2)m_3 = m_1 \cdot m_3 + m_2 \cdot m_3$ *and* $m_1(m_2 + m_3) = m_1 \cdot m_2 + m_1 \cdot m_3$ *for all* m_1, m_2 *and* m_3 *in* $\mathscr{R}$

(*iv*) $m \cdot 0 = 0 = 0 \cdot m$, *for all* $m \in \mathscr{R}$.

If a semiring $\mathscr{R}$ satisfies $mn = nm$ for all $m, n \in \mathscr{R}$, then $\mathscr{R}$ is said to be a commutative semiring. Throughout the paper, $\mathscr{R}$ is a semiring unless and otherwise stated.

Definition 4.2 ([22]). *A subset $\Im$ of a semiring $\mathscr{R}$ is said to be an ideal of $\mathscr{R}$, if $\Im$ satisfies the following conditions:*

(*i*) $m + n \in \Im$, *for all* $m, n \in \Im$

(*ii*) $rm, \;\; mr \in \Im$, *for all* $r \in \mathscr{R}$ *and* $m \in \Im$.

Definition 4.3 ([22]). *An ideal $\Im$ of a semiring $\mathscr{R}$ is said to be subtractive if for $m, n \in \mathscr{R}$, $m + n \in \Im$ and $n \in \Im$ implies that $m \in \Im$.*

Definition 4.4 ([40]). *An IF subset $\mathfrak{A} = (\eta_{\mathfrak{A}}, \delta_{\mathfrak{A}})$ of a semiring $\mathscr{R}$ is said to be an IF ideal of $\mathscr{R}$ if for all x, y in $\mathscr{R}$*

(*i*) $\eta_{\mathfrak{A}}(x + y) \geq \min\{\eta_{\mathfrak{A}}(x), \eta_{\mathfrak{A}}(y)\}$

(*ii*) $\delta_{\mathfrak{A}}(x + y) \leq \max\{\delta_{\mathfrak{A}}(x), \delta_{\mathfrak{A}}(y)\}$

(*iii*) $\eta_{\mathfrak{A}}(xy) \geq \max\{\eta_{\mathfrak{A}}(x), \eta_{\mathfrak{A}}(y)\}$

(*iv*) $\delta_{\mathfrak{A}}(xy) \leq \min\{\delta_{\mathfrak{A}}(x), \delta_{\mathfrak{A}}(y)\}$.

Example 4.5. *Let $\mathscr{R} = \{0, a, b, c\}$ be a semiring defined by the following tables:*

$+$	0	a	b	c
0	0	a	b	c
a	a	a	b	c
b	b	b	b	c
c	c	c	c	b

and

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	a	a	a
b	0	a	a	a
c	0	a	a	a

Then an IF set $\mathfrak{A} = (\eta_{\mathfrak{A}}, \delta_{\mathfrak{A}})$ defined by

$$\eta_{\mathfrak{A}}(0) = 1, \quad \eta_{\mathfrak{A}}(a) = \frac{2}{3}, \quad \eta_{\mathfrak{A}}(b) = \frac{1}{3}, \quad \eta_{\mathfrak{A}}(c) = 0,$$
$$\delta_{\mathfrak{A}}(0) = 0, \quad \delta_{\mathfrak{A}}(a) = \frac{1}{3}, \quad \delta_{\mathfrak{A}}(b) = \frac{1}{3}, \quad \delta_{\mathfrak{A}}(c) = 1$$

is an IF ideal of $\mathscr{R}$.

Definition 4.6 ([42]). *An IF ideal $\mathfrak{A} = (\eta_{\mathfrak{A}}, \delta_{\mathfrak{A}})$ of a semiring $\mathscr{R}$ is said to be subtractive if*

(*i*) $\eta_{\mathfrak{A}}(m) \geq \min\{\eta_{\mathfrak{A}}(m + n), \eta_{\mathfrak{A}}(n)\}$

(*ii*) $\delta_{\mathfrak{A}}(m) \leq \max\{\delta_{\mathfrak{A}}(m + n), \delta_{\mathfrak{A}}(n)\}$, *for all* $m, n \in \mathscr{R}$.

Definition 4.7. *A congruence relation (C.R.) $\mathscr{F}$ on a semiring $\mathscr{R}$ is an equivalence relation that is compatible with the semiring's algebraic operations. Specifically, if $(r_1, r_2) \in \mathscr{F}$, then it follows that $(r_1 + r, r_2 + r), (r_1 r, r_2 r)$, and $(rr_1, rr_2) \in \mathscr{F}$ for all $r \in \mathscr{R}$.*

Definition 4.8 ([2]). *A congruence relation (C.R.) $\mathscr{F}$ on a semiring $\mathscr{R}$ is called a full congruence relation (F.C.R.) on $\mathscr{R}$, if*

(*i*) $[\mathfrak{m}]_{\mathscr{F}} + [\mathfrak{n}]_{\mathscr{F}} = [\mathfrak{m} + \mathfrak{n}]_{\mathscr{F}}$.

(*ii*) $\{ab \mid a \in [\mathfrak{m}]_{\mathscr{F}}, b \in [\mathfrak{n}]_{\mathscr{F}}\} = [\mathfrak{m}\mathfrak{n}]_{\mathscr{F}}$.

Definition 4.9 ([2]). *A congruence relation (C.R.) $\mathscr{F}$ on a semiring $\mathscr{R}$ as $(a, b) \in \mathscr{F}$ is said to be Bourne's congruence relation (B.C.R.) with respect to an ideal I of $\mathscr{R}$ if there exists $i, j \in I$ such that $a + i = b + j$.*

The following result show that if $\mathscr{F}$ is an F.C.R. on a semiring $\mathscr{R}$, then it preserves both addition and multiplication in a component-wise manner, ensuring algebraic consistency.

Theorem 4.10. *If $\mathscr{F}$ is F.C.R. on $\mathscr{R}$, then $(r_1, r_2), (r_3, r_4) \in \mathscr{F}$ implies $(r_1 + r_3, r_2 + r_4), (r_1 r_3, r_2 r_4)$.*

Proof. Consider $\mathscr{F}$ is an F.C.R. on $\mathscr{R}$ and $(r_1, r_2), (r_3, r_4) \in \mathscr{F}$. Then $(r_1 + r, r_2 + r), (r_2 + r, r_2 + s) \in \mathscr{F}$. Hence, $(r_1 + r, r_2 + s) \in \mathscr{F}$. Following analogous reasoning,it is evident that $(r_1r_3, r_2r_4) \in \mathscr{F}$.

In 2012, Ali et al. [2] generalized the concept of the lower and upper approximations of an ideal of a ring to the lower and upper approximations of an ideal of a semiring as follows:

Definition 4.11 ([2]). *Let $\mathscr{F}$ be a F.C.R. on a semiring $\mathscr{R}$ and $\mathfrak{S} \subseteq \mathscr{R}$. Then, the sets $\mathscr{F}_-(\mathfrak{S}) = \{n \in \mathscr{R} \mid [n]_{\mathscr{F}} \subseteq \mathfrak{S}\}$ and $\mathscr{F}^-(\mathfrak{S}) = \{n \in \mathscr{R} \mid [n]_{\mathscr{F}} \cap \mathfrak{S} \neq \phi\}$ are referred to as $\mathscr{F}$-lower and $\mathscr{F}$-upper approximation of the set $\mathfrak{S}$.*

$\mathscr{F}(\mathfrak{S}) = (\mathscr{F}_-(\mathfrak{S}), \mathscr{F}^-(\mathfrak{S}))$ is called a rough set with respect to $\mathscr{F}$ if $\mathscr{F}_-(\mathfrak{S}) \neq \mathscr{F}^-(\mathfrak{S})$ and $\mathfrak{S}$ is said to be an upper rough ideal if $\mathscr{F}^-(\mathfrak{S})$ is an ideal of $\mathscr{R}$.

In [35, Theorem 2.9], Mandal and Ranadive established a relationship between congruence relation on a ring R_1 and congruence relation on a homomorphic image of R_1 which is a subset of another ring R_2. Motivated by the study of Mandal and Ranadive, we establish the following result for semirings (a generalization of rings):

Theorem 4.12. *Let $f : \mathscr{R}_1 \to \mathscr{R}_2$ be an epimorphism from a semiring $\mathscr{R}_1$ to a semiring $\mathscr{R}_2$ and $\mathscr{F}_2$ be a C.R. on $\mathscr{R}_2$. Then;*

(*i*) *$\mathscr{F}_1 = \{(n_1, n_2) \in \mathscr{R}_1 \times \mathscr{R}_2 \mid (f(n_1), f(n_2)) \in \mathscr{F}_2\}$ is C.R. on $\mathscr{R}_1$.*
(*ii*) *If $\mathscr{F}_2$ is F.C.R. on $\mathscr{R}_2$ and f is injective, then $\mathscr{F}_1$ is F.C.R. $\mathscr{R}_1$*
(*iii*) *For any subset $\mathfrak{Y}$ of $\mathscr{R}_1$, $f(\mathscr{F}_1^-(\mathfrak{Y})) = \mathscr{F}_2^-(f(\mathfrak{Y}))$.*
(*iv*) *$f(\mathscr{F}_{1-}(\mathfrak{Y})) \subseteq \mathscr{F}_{2-}(f(\mathfrak{Y}))$. Equality holds if f is injective.*

Proof. Proof runs on the same parallel lines as of [35, Theorem 2.9].

The concept behind the following lemmas is that applying upper approximation $\mathscr{F}^-$ and lower approximation $\mathscr{F}_-$ to an ideal doesn't break its structure. In fact, under certain conditions, lower approximation of the ideal coincide with itself. This concept helps us understand how congruence relations interact with ideals in a semiring.

Lemma 4.13. *Let $\mathscr{F}$ be an F.C.R. on semiring $\mathscr{R}$. Then $\mathscr{F}^-(\mathfrak{I})$ is an ideal of $\mathscr{R}$, if $\mathfrak{I}$ is an ideal of $\mathscr{R}$.*

Proof. Let $x, y \in \mathscr{F}^-(\mathfrak{I})$ and $r \in \mathscr{R}$. Then $[x]_{\mathscr{F}} \cap \mathfrak{I} \neq \phi$ and $[y]_{\mathscr{F}} \cap \mathfrak{I} \neq \phi$. So, there exist $a \in [x]_{\mathscr{F}} \cap \mathfrak{I}$ and $b \in [y]_{\mathscr{F}} \cap \mathfrak{I}$ such that $a + b \in \mathfrak{I}$, $ra \in \mathfrak{I}$, and $a + b \in [x]_{\mathscr{F}} + [y]_{\mathscr{F}} = [x + y]_{\mathscr{F}}$, i.e., $[x + y]_{\mathscr{F}} \cap \mathfrak{I} \neq \phi$ and we have

$$x + y \in \mathscr{F}^-(\mathfrak{I}). \tag{1}$$

Since $(a, x) \in \mathscr{F}$, then we get $(ra, rx) \in \mathscr{F}$. Thus $ra \in [rx]_{\mathscr{F}}$ and $ra \in \mathfrak{I}$, we obtain $[rx]_{\mathscr{F}} \cap \mathfrak{I} \neq \phi$ that is

$$rx \in \mathscr{F}^-(\mathfrak{I}). \tag{2}$$

Thus, $\mathscr{F}^-(\mathfrak{I})$ is an ideal of $\mathscr{R}$.

Lemma 4.14. *Let $\mathscr{F}$ be an F.C.R. on a semiring $\mathscr{R}$ and $\mathfrak{I}$ be an ideal of $\mathscr{R}$. If $\mathscr{F}_-(\mathfrak{I}) \neq \phi$, then $\mathscr{F}_-(\mathfrak{I})$ is an ideal of $\mathscr{R}$. Moreover, if $\mathscr{F}$ is a B.C.R. and $\mathfrak{I}$ is the subtractive ideal of $\mathscr{R}$ containing I, then $\mathscr{F}_-(\mathfrak{I}) = \mathfrak{I}$.*

Proof. Assume that $\mathscr{F}_-(\mathfrak{I})$ is a nonempty set. Then for any $a, b \in \mathscr{F}_-(\mathfrak{I})$, $[a]_{\mathscr{F}} \subseteq \mathfrak{I}$, $[b]_{\mathscr{F}} \subseteq \mathfrak{I}$. Since $\mathscr{F}$ is F.C.R., then $[a + b]_{\mathscr{F}} = [a]_{\mathscr{R}} + [b]_{\mathscr{R}} \subseteq \mathfrak{I} + \mathfrak{I} \subseteq \mathfrak{I}$. Therefore, $a + b \in \mathscr{F}_-(\mathfrak{I})$. Since $\mathfrak{I}$ is an ideal of $\mathscr{R}$, then for $r \in \mathscr{R}$, $[ra]_{\mathscr{F}} = [r]_{\mathscr{F}}[a]_{\mathscr{F}} \subseteq [r]_{\mathscr{F}}\mathfrak{I} \subseteq \mathfrak{I}$. Similarly $[ar]_{\mathscr{F}} = [a]_{\mathscr{F}}[r]_{\mathscr{F}} \subseteq \mathfrak{I}[r]_{\mathscr{F}} \subseteq \mathfrak{I}$.
Moreover, assume that $\mathscr{F}$ is a B.C.R. and $\mathfrak{I}$ is a subtractive ideal of $\mathscr{R}$ containing I. Then the definition yields that $\mathscr{F}_-(\mathfrak{I}) \subseteq \mathfrak{I}$. It remains to demonstrate that $\mathfrak{I} \subseteq \mathscr{F}_-(\mathfrak{I})$. Assume that $j \in \mathfrak{I}$ such that $k \in [j]_{\mathscr{F}}$. Then there exist elements $i_1, i_2 \in I \subseteq \mathfrak{I}$ such that $k + i_1 = j + i_2$. Now, $k + i_1 \in \mathfrak{I}$ and $i_2 \in \mathfrak{I}$. Since $\mathfrak{I}$ is subtractive, then $k \in \mathfrak{I}$. This implies that $[j]_{\mathscr{F}} \subseteq \mathfrak{I}$ that is $j \in \mathscr{F}_-(\mathfrak{I})$. Thus $\mathscr{F}_-(\mathfrak{I}) = \mathfrak{I}$.

5. Rough 2-absorbing primary ideal

From here, we now assume that $\mathscr{R}$ is a commutative semiring.

Definition 5.1 ([7]). *A proper ideal $\Im$ of a semiring $\mathscr{R}$ is called a 2-absorbing ideal (2AI) of $\mathscr{R}$ if for $r, s, t \in \mathscr{R}$ such that $rst \in \Im$ implies that $rs \in \Im$ or $rt \in \Im$ or $st \in \Im$.*

Definition 5.2 ([8]). *A proper ideal $\Im$ of a semiring $\mathscr{R}$ is called a 2-absorbing primary ideal (2API) of $\mathscr{R}$ if $rst \in \Im$ implies $rs \in \Im$ or $rt \in \sqrt{\Im}$ or $st \in \sqrt{\Im}$, for $r, s, t \in \mathscr{R}$.*

Remark 5.3. *Every 2-absorbing ideal of a semiring $\mathscr{R}$ is a 2-absorbing primary ideal of $\mathscr{R}$ but the converse need not be true in general.*

Example 5.4. *Assume that $\mathscr{R} = \mathbb{Z}^+ \cup \{0\}$ is a semiring. Then an ideal $\Im = \langle 8 \rangle$ of $\mathscr{R}$ is a 2API but it is not a 2AI of $\mathscr{R}$, as $2 \cdot 2 \cdot 2 \in \langle 8 \rangle$ but $2 \cdot 2 \notin \langle 8 \rangle$.*

Remark 5.5. *While every primary ideal of a semiring $\mathscr{R}$ is necessarily 2-absorbing primary, the converse fails in general.*

Example 5.6. *In Example 5.4, $\langle 10 \rangle$ is a 2API of $\mathscr{R}$ but it is not a primary ideal of $\mathscr{R}$.*

Lemma 5.7. *Let $\mathscr{F}$ be an F.C.R. on a semiring $\mathscr{R}$. Then $\mathscr{F}^-(\Im)$ is a 2AI of $\mathscr{R}$, if $\Im$ is a 2AI of $\mathscr{R}$.*

Proof. Assume that $\Im$ is a 2AI of a semiring $\mathscr{R}$. Then by Lemma 4.13, $\mathscr{F}^-(\Im)$ is an ideal of $\mathscr{R}$, we only show that $\mathscr{F}^-(\Im)$ is a 2AI. Suppose that $r, s, t \in \mathscr{R}$ such that $rst \in \mathscr{F}^-(\Im)$. Then $[rst]_{\mathscr{F}} \cap \Im \neq \phi$. Since $\mathscr{F}$ is F.C.R., then $xyz \in [rst]_{\mathscr{F}} \cap \Im$, for $x \in [r]_{\mathscr{F}},\ y \in [s]_{\mathscr{F}},\ z \in [t]_{\mathscr{F}}$. Again since $\Im$ is 2-absorbing, then $xy \in \Im$ or $xz \in \Im$ or $yz \in \Im$. Therefore $[rs]_{\mathscr{F}} \cap \Im \neq \phi$ or $[rt]_{\mathscr{F}} \cap \Im \neq \phi$ or $[st]_{\mathscr{F}} \cap \Im \neq \phi$. Thus $rs \in \mathscr{F}^-(\Im)$ or $rt \in \mathscr{F}^-(\Im)$ or $st \in \mathscr{F}^-(\Im)$. This shows that $\mathscr{F}^-(\Im)$ is a 2AI of $\mathscr{R}$.

Theorem 5.8. *Let $\mathscr{F}$ be a F.C.R. on semiring $\mathscr{R}$ and $\mathfrak{P}$ be a 2API of $\mathscr{R}$ such that $\mathscr{F}^-(\mathfrak{P}) \neq \mathscr{R}$. Then $\mathfrak{P}$ is the upper rough 2API of $\mathscr{R}$.*

Proof. Let $\mathfrak{P}$ be a 2API of a semiring $\mathscr{R}$ and $rst \in \mathscr{F}^-(\mathfrak{P})$ for $r, s, t \in \mathscr{R}$. Then $[rst]_{\mathscr{F}} \cap \mathfrak{P} \neq \phi$. Since $\mathscr{F}$ is F.C.R., then $\{xyz \mid x \in [r]_{\mathscr{F}},\ y \in [s]_{\mathscr{F}},\ z \in [t]_{\mathscr{F}}\} \cap \mathfrak{P} \neq \phi$, so it follows that $xyz \in [rst]_{\mathscr{F}} \cap \mathfrak{P}$. Since $\mathfrak{P}$ is a 2API of $\mathscr{R}$, then $xy \in \mathfrak{P}$ or $xz \in \sqrt{\mathfrak{P}}$ or $yz \in \sqrt{\mathfrak{P}}$. Therefore $xy \in [rs]_{\mathscr{F}} \cap \mathfrak{P}$ or $(xz)^n \in [(rt)^n]_{\mathscr{F}} \cap \mathfrak{P}$ or $(yz)^n \in [(st)^n]_{\mathscr{F}} \cap \mathfrak{P}$. Thus $rs \in \mathscr{F}^-(\mathfrak{P})$ or $(rt)^n \in \mathscr{F}^-(\mathfrak{P})$ or $(st)^n \in \mathscr{F}^-(\mathfrak{P})$. This shows that $\mathfrak{P}$ is upper rough 2API of $\mathscr{R}$.

Remark 5.9. *The converse implication fails in general, as shown in following example.*

Example 5.10. *Consider $S = \mathbb{N} \cup \{0\}$ (set of all natural numbers with 0) a semiring with usual addition and multiplication, and $\mathscr{F}$ a congruence relation defined as $\mathscr{F} = \{(a, b) \mid a, b \text{ are even or } a, b \text{ are odd numbers}\} \cup \{(0, 0)\}$. Let $\mathfrak{A} = \{0, 4, 6, 8\}$ be a subset of S. Then $\mathscr{F}^-(\mathfrak{A}) = \{x \in S \mid [x]_{\mathscr{F}} \cap \mathfrak{A} \neq \phi\} = \{0, 2, 4, 6, , , , ,\} = 2\mathbb{N} \cup \{0\}$. We can see that $\mathscr{F}^-(\mathfrak{A})$ is a 2API of S that is $\mathfrak{A}$ is an upper rough 2API of S but $\mathfrak{A}$ is not a 2API since $6 + 4 = 10 \notin \mathfrak{A}$.*

Theorem 5.11. *Let $\mathscr{F}$ be a F.C.R. on a semiring $\mathscr{R}$ and $\mathfrak{P}$ be a subtractive 2API of $\mathscr{R}$. If $\mathscr{F}$ be a B.C.R. and $\mathscr{F}_-(\mathfrak{P}) \neq \phi$, then $\mathfrak{P}$ is a lower rough 2API of $\mathscr{R}$.*

Proof. By Lemma 4.14, $\mathscr{F}_-(\mathfrak{P}) = \mathfrak{P}$ and we get the result.

Remark 5.12. *The converse of the Theorem 5.11 does not hold in general.*

Example 5.13. *Let $S = \{0, 1, 2, 3\}$. Then S is a semiring with the following addition and multiplication*

+	0	1	2	3
0	0	1	2	3
1	1	1	2	3
2	2	2	2	3
3	3	3	3	2

·	0	1	2	3
0	0	0	0	0
1	0	1	1	1
2	0	1	1	1
3	0	1	1	1.

Consider the congruence relation $\mathscr{F} = \{(s_1, s_2) \mid s_1, s_2 \in \{0,1\}$ *or* $s_1, s_2 \in \{2,3\}\}$ *on* $\mathfrak{S}$ *such that* $\mathscr{F}$*-congruence classes are the subsets* $\{0,1\}, \{2,3\}$ *of* $\mathfrak{S}$*. Let* $A = \{0,1,3\}$ *be a subset of S. Then* $\mathscr{F}_-(A) = \{0,1\}$ *is a 2API of S that is A is a lower rough 2API of S but A is not a 2API because* $3+3=2 \notin A$.

If a rough set $\mathfrak{I}$ is a lower rough 2API and an upper rough 2API of the semiring $\mathscr{R}$, then it is called rough 2API of the semiring $\mathscr{R}$.

Theorem 5.14. *Let f be an epimorphism from a semiring* $\mathscr{R}_1$ *to a semiring* $\mathscr{R}_2$ *and* $\mathscr{F}_2$ *be a F.C.R. on* $\mathscr{R}_2$*. Let* $\mathfrak{P}$ *be a subset of* $\mathscr{R}_1$*. If* $\mathscr{F}_1 = \{(s_1, s_2) \in \mathscr{R}_1 \times \mathscr{R}_1 \mid (f(s_1), f(s_2)) \in \mathscr{F}_2\}$*, then*

(*i*) $\mathscr{F}_1^-(\mathfrak{P})$ *is a 2AI of* $\mathscr{R}_1$ *if and only if* $\mathscr{F}_2^-(f(\mathfrak{P}))$ *is a 2AI of* $\mathscr{R}_2$.

(*ii*) *If* $\mathscr{F}_2$ *is F.C.R. on* $\mathscr{R}_2$*, then* $\mathscr{F}_1^-(\mathfrak{P})$ *is a 2API of* $\mathscr{R}_1$ *if and only if* $\mathscr{F}_2^-(f(\mathfrak{P}))$ *is a 2API of* $\mathscr{R}_2$.

Proof. (*i*) Suppose that $\mathscr{F}_1^-(\mathfrak{P})$ is a 2AI of $\mathscr{R}_1$ and $r, s, t \in \mathscr{R}_2$ such that $rst \in \mathscr{F}_2^-(f(\mathfrak{P}))$. Then there exists $u, v, w \in \mathscr{R}_1$ such that $f(u) = r,\ f(v) = s,\ f(w) = t$. Thus $[f(u)f(v)f(w)]_{\mathscr{F}_2} \cap f(\mathfrak{P}) \neq \phi$. Since $\mathscr{F}_2$ is F.C.R., then there exists element $f(x) \in [f(u)]_{\mathscr{F}_2},\ f(y) \in [f(v)]_{\mathscr{F}_2},\ f(z) \in [f(w)]_{\mathscr{F}_2}$ such that $f(x)f(y)f(z) = f(xyz) \in f(\mathfrak{P})$. Then we have $x \in [u]_{\mathscr{F}_1},\ y \in [v]_{\mathscr{F}_1},\ z \in [w]_{\mathscr{F}_1}$ and there exists $a \in \mathfrak{P}$ such that $f(xyz) = f(a)$. Hence $xyz \in [uvw]_{\mathscr{F}_1}$ and $a \in [xyz]_{\mathscr{F}_1}$. Therefore, $[uvw]_{\mathscr{F}_1} \cap \mathfrak{P} \neq \phi$. This implies that $uvw \in \mathscr{F}_1^-(\mathfrak{P})$. Since $\mathscr{F}_1^-(\mathfrak{P})$ is 2-absorbing, then we have $uv \in \mathscr{F}_1^-(\mathfrak{P})$ or $vw \in \mathscr{F}_1^-(\mathfrak{P})$ or $uw \in \mathscr{F}_1^-(\mathfrak{P})$. Then by Theorem 4.12,

$$\begin{aligned} rs &= f(uv) \in f(\mathscr{F}_1^-(\mathfrak{P})) = \mathscr{F}_2^-(f(\mathfrak{P})) \\ \text{or}\quad rt &= f(uw) \in f(\mathscr{F}_1^-(\mathfrak{P})) = \mathscr{F}_2^-(f(\mathfrak{P})) \\ \text{or}\quad st &= f(vw) \in f(\mathscr{F}_1^-(\mathfrak{P})) = \mathscr{F}_2^-(f(\mathfrak{P})). \end{aligned}$$

This shows that $\mathscr{F}_2^-(f(\mathfrak{P}))$ is a 2AI of $\mathscr{R}_2$.

Conversely, assume that $\mathscr{F}_2^-(f(\mathfrak{P}))$ is a 2AI of $\mathscr{R}_2$ and let $r, s, t \in \mathscr{R}_1$ such that $rst \in \mathscr{F}_1^-(\mathfrak{P})$. Then

$$f(rst) = f(r)f(s)f(t) \in f(\mathscr{F}_1^-(\mathfrak{P})) = \mathscr{F}_2^-(f(\mathfrak{P})).$$

This implies that

$$f(r)f(s) \in \mathscr{F}_2^-(f(\mathfrak{P})) \quad \text{or} \quad f(r)f(t) \in \mathscr{F}_2^-(f(\mathfrak{P})) \quad \text{or} \quad f(s)f(t) \in \mathscr{F}_2^-(f(\mathfrak{P}))$$

which is

$$f(rs) \in f(\mathscr{F}_1^-(\mathfrak{P})) \quad \text{or} \quad f(rt) \in f(\mathscr{F}_1^-(\mathfrak{P})) \quad \text{or} \quad f(st) \in f(\mathscr{F}_1^-(\mathfrak{P})).$$

Thus there exists $a, b, c \in \mathscr{F}_1^-(\mathfrak{P})$ such that $f(rs) = f(a)$ or $f(rt) = f(b)$ or $f(st) = f(c)$. Therefore, we have $[a]_{\mathscr{F}_1} \cap \mathfrak{P} \neq \phi$ or $[b]_{\mathscr{F}_1} \cap \mathfrak{P} \neq \phi$ or $[c]_{\mathscr{F}_1} \cap \mathfrak{P} \neq \phi$ and $rs \in [a]_{\mathscr{F}_1}$, $rt \in [b]_{\mathscr{F}_1}$, $st \in [c]_{\mathscr{F}_1}$. This implies that

$$[rs]_{\mathscr{F}_1} \cap \mathfrak{P} \neq \phi \quad \text{or} \quad [rt]_{\mathscr{F}_1} \cap \mathfrak{P} \neq \phi \quad \text{or} \quad [st]_{\mathscr{F}_1} \cap \mathfrak{P} \neq \phi.$$

So, we have $rs \in \mathscr{F}_1^-(\mathfrak{P})$ or $rt \in \mathscr{F}_1^-(\mathfrak{P})$ or $st \in \mathscr{F}_1^-(\mathfrak{P})$.

(*ii*) Suppose that $\mathscr{F}_1^-(\mathfrak{P})$ is a 2API of the semiring $\mathscr{R}_1$ and $a, b, c \in \mathscr{R}_2$ such that $abc \in \mathscr{F}_2^-(f(\mathfrak{P}))$. Then there exists $x, y, z \in \mathscr{R}_1$ such that $f(x) = a,\ f(y) = b$ and $f(z) = c$. Thus $[f(x)f(y)f(z)]_{\mathscr{F}_2} \cap f(\mathfrak{P}) \neq \phi$. Since $\mathscr{F}_2$ is F.C.R., then there exists $f(r) \in [f(x)]_{\mathscr{F}_2},\ f(s) \in [f(y)]_{\mathscr{F}_2}$, and $f(t) \in [f(z)]_{\mathscr{F}_2}$ such that $f(r)f(s)f(t) = f(rst) \in f(\mathfrak{P})$. Using hypothesis, we have $r \in [x]_{\mathscr{F}_1},\ s \in [y]_{\mathscr{F}_1}$, $t \in [z]_{\mathscr{F}_1}$ and there exists $w \in \mathfrak{P}$ such that $f(w) = f(rst)$. Hence $rst \in [xyz]_{\mathscr{F}_1}$. Thus $[xyz]_{\mathscr{F}_1} \cap \mathfrak{P} \neq \phi$. This implies that $xyz \in \mathscr{F}_1^-(\mathfrak{P})$. Since $\mathscr{F}_1^-(\mathfrak{P})$ is 2-absorbing, then $xy \in \mathscr{F}_1^-(\mathfrak{P})$ or $xz \in \sqrt{\mathscr{F}_1^-(\mathfrak{P})}$ or $yz \in \sqrt{\mathscr{F}_1^-(\mathfrak{P})}$. Then we can write

$$\begin{aligned} f(xy) &= ab \in f(\mathscr{F}_1^-(\mathfrak{P})) = \mathscr{F}_2^-(f(\mathfrak{P})) \\ \text{or}\quad (f(xz))^n &= (ac)^n \in \mathscr{F}_2^-(f(\mathfrak{P})) \\ \text{or}\quad (f(yz))^n &= (bc)^n \in \mathscr{F}_2^-(f(\mathfrak{P})). \end{aligned}$$

Therefore, we have

$$ab \in \mathscr{F}_2^-(f(\mathfrak{P})) \quad \text{or} \quad ac \in \sqrt{\mathscr{F}_2^-(f(\mathfrak{P}))} \quad \text{or} \quad bc \in \sqrt{\mathscr{F}_2^-(f(\mathfrak{P}))}.$$

Hence $\mathscr{F}_2^-(f(\mathfrak{P}))$ is a 2API of $\mathscr{R}_2$.
Conversely, suppose that $\mathscr{F}_2^-(f(\mathfrak{P}))$ is a 2API of the semiring $\mathscr{R}_2$. Assume that $a, b, c \in \mathscr{R}_1$ such that $abc \in \mathscr{F}_1^-(\mathfrak{P})$. As a direct consequence of Theorem 4.12(*iii*),

$$f(a)f(b)f(c) = f(abc) \in f(\mathscr{F}_1^-(\mathfrak{P})) = \mathscr{F}_2^-(f(\mathfrak{P})).$$

Since $\mathscr{F}_2^-(f(\mathfrak{P}))$ is a 2API of $\mathscr{R}_2$, then we have

$$\begin{aligned} f(a)f(b) &= f(ab) \in f(\mathscr{F}_1^-(\mathfrak{P})) \\ \text{or } f(a)f(c) &= f(ac) \in \sqrt{f(\mathscr{F}_1^-(\mathfrak{P}))} \\ \text{or } f(b)f(c) &= f(bc) \in \sqrt{f(\mathscr{F}_1^-(\mathfrak{P}))}. \end{aligned}$$

If $f(ab) \in f(\mathscr{F}_1^-(\mathfrak{P}))$, then there exists $x \in \mathscr{F}_1^-(\mathfrak{P})$ such that $f(ab) = f(x)$. Thus $[x]_{\mathscr{F}_1} \cap \mathfrak{P} \neq \phi$ and $x \in [ab]_{\mathscr{F}_1}$. Therefore, $[ab]_{\mathscr{F}_1} \cap \mathfrak{P} \neq \phi$. So, we have $ab \in \mathscr{F}_1^-(\mathfrak{P})$.
If $f(ac) \in \sqrt{f(\mathscr{F}_1^-(\mathfrak{P}))}$, then $(f(ac))^n = f((ac)^n) \in f(\mathscr{F}_1^-(\mathfrak{P}))$. So, there exists $y \in \mathscr{F}_1^-(\mathfrak{P})$ such that $f((ac)^n) = f(y)$. Thus $[y]_{\mathscr{F}_1} \cap \mathfrak{P} \neq \phi$ and $y \in [(ac)^n]_{\mathscr{F}_1}$ and $[(ac)^n]_{\mathscr{F}_1} \cap \mathfrak{P} \neq \phi$. Hence, we have $(ac)^n \in \mathscr{F}_1^-(\mathfrak{P})$ or $ac \in \sqrt{\mathscr{F}_1^-(\mathfrak{P})}$. Similarly if $f(bc) \in f(\mathscr{F}_1^-(\mathfrak{P}))$, then $bc \in \sqrt{\mathscr{F}_1^-(\mathfrak{P})}$. This means $\mathscr{F}_1^-(\mathfrak{P})$ is a 2API of $\mathscr{R}_1$.

Theorem 5.15. *Suppose that f is an isomorphism from $\mathscr{R}_1$ to $\mathscr{R}_2$ and $\mathscr{F}_2$ is a C.R. on $\mathscr{R}_2$. If $\mathfrak{P}$ is a subset of $\mathscr{R}_1$ and $\mathscr{F}_1 = \{(a,b) \in \mathscr{R}_1 \times \mathscr{R}_1 \mid (f(a), f(b)) \in \mathscr{F}_2\}$, then*

- (*i*) *$\mathscr{F}_{1-}(\mathfrak{P})$ is a 2AI of $\mathscr{R}_1$ if and only if $\mathscr{F}_{2-}(f(\mathfrak{P}))$ is a 2AI of $\mathscr{R}_2$.*
- (*ii*) *If $\mathscr{F}_2$ is F.C.R., $\mathscr{F}_{1-}(\mathfrak{P})$ is a 2API of $\mathscr{R}_1$ if and only if $\mathscr{F}_{2-}(f(\mathfrak{P}))$ is a 2API of $\mathscr{R}_2$.*

Proof. Since f is one-one, then by Theorem 4.12(*iv*), $f(\mathscr{F}_{1-}(\mathfrak{P})) = \mathscr{F}_{2-}(f(\mathfrak{P}))$. Now, the result follows by arguments similar to Theorem 5.14.

6. Rough IF 2-absorbing primary ideal

Ozkan et al. [40] studied 2-absorbing ideals (2AIs), 2-absorbing primary ideals (2APIs) using IF set theory. They looked into the properties of IF2AI and also examined how these ideals behave, including their images and inverse images, under semiring homomorphisms.

Definition 6.1 ([40]). *Let $\mathfrak{A}$ be an IF set of a semiring $\mathscr{R}$. An IF ideal $\mathfrak{A} = (\mu_{\mathfrak{A}}, \lambda_{\mathfrak{A}})$ of $\mathscr{R}$ is said to be the IF2AI of $\mathscr{R}$, if*

$$\mu_{\mathfrak{A}}(rst) \geq m \ \textit{implies} \ \mu_{\mathfrak{A}}(rs) \geq m \ \textit{or} \ \mu_{\mathfrak{A}}(st) \geq m \ \textit{or} \ \mu_{\mathfrak{A}}(rt) \geq m$$

and

$$\lambda_{\mathfrak{A}}(rst) \leq n \ \textit{implies} \ \lambda_{\mathfrak{A}}(rs) \leq n \ \textit{or} \ \lambda_{\mathfrak{A}}(st) \leq n \ \textit{or} \ \lambda_{\mathfrak{A}}(rt) \leq n$$

for all $r, s, t \in \mathscr{R}$ and $m, n \in [0, 1]$.

Example 6.2. *Consider $\mathscr{R} = \{0, 1, 2, 3, 4\}$ be a semiring under addition '+' and multiplication '·' defined as*

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

and

·	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

Define IF set $\mathfrak{A} = (\mu_{\mathfrak{A}}, \lambda_{\mathfrak{A}})$ *as* $\mu_{\mathfrak{A}} = \{(0,1),(1,0.4),(2,0.5),(3,0.2),(4,0.3)\}$ *and* $\lambda_{\mathfrak{A}} = \{(0,0),(1,0.3),(2,0.7),(3,0.8),(4,0.5)\}$. *Then* $\mathfrak{A}$ *is an IF 2-absorbing ideal of* $\mathscr{R}$.

Definition 6.3 ([40]). *An IF ideal* $\mathfrak{A} = (\mu_{\mathfrak{A}}, \lambda_{\mathfrak{A}})$ *of a semiring* $\mathscr{R}$ *is said to be IF 2-absorbing primary ideal of* $\mathscr{R}$, *if*

$$\mu_{\mathfrak{A}}(rst) \geq m \text{ implies } \mu_{\mathfrak{A}}(rs) \geq m \text{ or } \mu_{\sqrt{\mathfrak{A}}}(st) \geq m \text{ or } \mu_{\sqrt{\mathfrak{A}}}(rt) \geq m$$

and

$$\lambda_{\mathfrak{A}}(rst) \leq n \text{ implies } \lambda_{\mathfrak{A}}(rs) \leq n \text{ or } \lambda_{\sqrt{\mathfrak{A}}}(st) \leq n \text{ or } \lambda_{\sqrt{\mathfrak{A}}}(rt) \leq n$$

for all $r, s, t \in \mathscr{R}$ *and* $m, n \in [0,1]$.

In 2014, Mandal and Ranadive [35] defined lower and upper approximations of an IF ideal of a ring. Now, we define $\mathscr{F}$-lower and $\mathscr{F}$-upper approximations of an IF ideal of a semiring $\mathscr{R}$ as follows:

Definition 6.4. *Let* $\mathscr{F}$ *be a F.C.R. on semiring* $\mathscr{R}$ *and* $\mathfrak{A} = (\mu_{\mathfrak{A}}, \lambda_{\mathfrak{A}})$ *be an IF set of* $\mathscr{R}$. *Then* $\mathscr{F}$*-lower and* $\mathscr{F}$*-upper approximation of* $\mathfrak{A}$ *are defined as*

$$\mathscr{F}_-(\mathfrak{A}) = (\mathscr{F}_-(\mu_{\mathfrak{A}}), \mathscr{F}_-(\lambda_{\mathfrak{A}})), \qquad \mathscr{F}^-(\mathfrak{A}) = (\mathscr{F}^-(\mu_{\mathfrak{A}}), \mathscr{F}^-(\lambda_{\mathfrak{A}})),$$

where

$$\mathscr{F}_-(\mu_{\mathfrak{A}})(r) = \bigwedge_{t \in [r]_{\mathscr{F}}} \mu_{\mathfrak{A}}(t), \qquad \mathscr{F}_-(\lambda_{\mathfrak{A}})(r) = \bigvee_{t \in [r]_{\mathscr{F}}} \lambda_{\mathfrak{A}}(t)$$
$$\mathscr{F}^-(\mu_{\mathfrak{A}})(r) = \bigvee_{t \in [r]_{\mathscr{F}}} \mu_{\mathfrak{A}}(t), \qquad \mathscr{F}^-(\lambda_{\mathfrak{A}})(r) = \bigwedge_{t \in [r]_{\mathscr{F}}} \lambda_{\mathfrak{A}}(t)$$

where $r, t \in \mathscr{R}$.
$\mathscr{F}(\mathfrak{A}) = (\mathscr{F}_-(\mathfrak{A}), \mathscr{F}^-(\mathfrak{A}))$ *is called a rough IF set with respect to* $\mathscr{F}$ *if* $\mathscr{F}_-(\mathfrak{A}) \neq \mathscr{F}^-(\mathfrak{A})$.

Definition 6.5. *A rough fuzzy set* $\mathscr{F}(\mu) = (\mathscr{F}_-(\mu), \mathscr{F}^-(\mu))$ *of a semiring* $\mathscr{R}$ *is said to be rough fuzzy 2API of* $\mathscr{R}$ *if* $\mathscr{F}_-(\mu)$ *and* $\mathscr{F}^-(\mu)$ *are fuzzy 2APIs of* $\mathscr{R}$.

Definition 6.6. *A rough IF set* $\mathscr{F}(\mathfrak{A}) = (\mathscr{F}_-(\mathfrak{A}), \mathscr{F}^-(\mathfrak{A}))$ *of a semiring* $\mathscr{R}$ *is said to be rough IF2API of* $\mathscr{R}$ *if* $\mathscr{F}_-(\mathfrak{A})$ *and* $\mathscr{F}^-(\mathfrak{A})$ *are IF2APIs of* $\mathscr{R}$.

Remark 6.7. *Every rough fuzzy 2API of a semiring* $\mathscr{R}$ *is a rough IF2API of* $\mathscr{R}$. *But the converse need not be true in general which is shown by following example:*

Example 6.8. *Consider* $S = \mathbb{N} \cup \{0\}$ *(set of all natural numbers with* 0*) a semiring with usual addition and multiplication. Let* $\mathfrak{A} = (\mu_{\mathfrak{A}}, \lambda_{\mathfrak{A}})$ *be an IF ideal of* S *defined by*

$$\mu_{\mathfrak{A}}(x) := \begin{cases} 1 & \text{if } x \in 18\mathbb{N} \cup \{0\} \\ 0.1 & \text{otherwise,} \end{cases} \qquad \lambda_{\mathfrak{A}}(x) := \begin{cases} 0 & \text{if } x \in 18\mathbb{N} \cup \{0\} \\ 0.8 & \text{otherwise.} \end{cases}$$

Here $\mathfrak{A}$ is an IF2API of S. Consider a congruence relation $\mathscr{F}$ on S defined by

$$\mathscr{F} = \{(s_1, s_2) \mid s_1, s_2 \text{ are even numbers or } s_1, s_2 \text{ are odd numbers}\} \cup \{(0,0)\}.$$

Then $\mathscr{F}^-(\mu_{\mathfrak{A}})(r) = 1$, $\mathscr{F}^-(\lambda_{\mathfrak{A}})(r) = 0$ and $\mathscr{F}_-(\mu_{\mathfrak{A}})(r) = 0.1$, $\mathscr{F}_-(\lambda_{\mathfrak{A}})(r) = 0.8$, where $r \in S$. Since $\mathscr{F}^-(\mathfrak{A}) = (\mathscr{F}^-(\mu_{\mathfrak{A}}), \mathscr{F}^-(\lambda_{\mathfrak{A}})) = (1,0) \neq (0.1, 0.8) = (\mathscr{F}_-(\mu_{\mathfrak{A}}), \mathscr{F}_-(\lambda_{\mathfrak{A}})) = \mathscr{F}_-(\mathfrak{A})$ and $\mathscr{F}_-(\mathfrak{A}), \mathscr{F}^-(\mathfrak{A})$ are IF2APIs of S, then $\mathscr{F}(\mathfrak{A}) = (\mathscr{F}_-(\mathfrak{A}), \mathscr{F}^-(\mathfrak{A}))$ is a rough IF2API of S. Since $\mathfrak{A}$ is not an IF ideal of S, then $\mathscr{F}(\mathfrak{A}) = (\mathscr{F}_-(\mathfrak{A}), \mathscr{F}^-(\mathfrak{A}))$ is not a rough fuzzy 2API of S.

Lemma 6.9. *Let $\mathscr{F}$ be a F.C.R. on a semiring $\mathscr{R}$. If $\mathfrak{A} = (\mu_{\mathfrak{A}}, \lambda_{\mathfrak{A}})$ is an IF ideal of $\mathscr{R}$, then $\mathscr{F}^-(\mathfrak{A}) = (\mathscr{F}^-(\mu_{\mathfrak{A}}), \mathscr{F}^-(\lambda_{\mathfrak{A}}))$ is also an IF ideal of $\mathscr{R}$.*

Proof. For $r, s \in \mathscr{R}$,

$$\begin{aligned}
\mathscr{F}^-(\mu_{\mathfrak{A}})(r+s) &= \bigvee_{t\in[r+s]_{\mathscr{F}}} \{\mu_{\mathfrak{A}}(t)\} = \bigvee_{t\in[r]_{\mathscr{F}}+[s]_{\mathscr{F}}} \{\mu_{\mathfrak{A}}(t)\} \\
&= \bigvee_{t_1\in[r]_{\mathscr{F}}, t_2\in[s]_{\mathscr{F}}} \{\mu_{\mathfrak{A}}(t_1+t_2)\} \geq \bigvee_{t_1\in[r]_{\mathscr{F}}, t_2\in[s]_{\mathscr{F}}} \{\mu_{\mathfrak{A}}(t_1) \wedge \mu_{\mathfrak{A}}(t_2)\} \\
&= \bigvee_{t_1\in[r]_{\mathscr{F}}} \{\mu_{\mathfrak{A}}(t_1)\} \wedge \bigvee_{t_2\in[s]_{\mathscr{F}}} \{\mu_{\mathfrak{A}}(t_2)\} \\
&= \mathscr{F}^-(\mu_{\mathfrak{A}})(r) \wedge \mathscr{F}^-(\mu_{\mathfrak{A}})(s).
\end{aligned} \tag{3}$$

$$\begin{aligned}
\mathscr{F}^-(\mu_{\mathfrak{A}})(rs) &= \bigvee_{t\in[rs]_{\mathscr{F}}} \{\mu_{\mathfrak{A}}(t)\} = \bigvee_{t\in[r]_{\mathscr{F}}[s]_{\mathscr{F}}} \{\mu_{\mathfrak{A}}(t)\} \\
&= \bigvee_{t_1\in[r]_{\mathscr{F}}, t_2\in[s]_{\mathscr{F}}} \{\mu_{\mathfrak{A}}(t_1 t_2)\} \geq \bigvee_{t_1\in[r]_{\mathscr{F}}, t_2\in[s]_{\mathscr{F}}} \{\mu_{\mathfrak{A}}(t_1) \wedge \mu_{\mathfrak{A}}(t_2)\} \\
&= \bigvee_{t_1\in[r]_{\mathscr{F}}} \{\mu_{\mathfrak{A}}(t_1)\} \wedge \bigvee_{t_2\in[s]_{\mathscr{F}}} \{\mu_{\mathfrak{A}}(t_2)\} \\
&= \mathscr{F}^-(\mu_{\mathfrak{A}})(r) \wedge \mathscr{F}^-(\mu_{\mathfrak{A}})(s).
\end{aligned} \tag{4}$$

$$\begin{aligned}
\mathscr{F}^-(\lambda_{\mathfrak{A}})(r+s) &= \bigwedge_{t\in[r+s]_{\mathscr{F}}} \{\lambda_{\mathfrak{A}}(t)\} = \bigwedge_{t\in[r]_{\mathscr{F}}+[s]_{\mathscr{F}}} \{\lambda_{\mathfrak{A}}(t)\} \\
&= \bigwedge_{t_1\in[r]_{\mathscr{F}}, t_2\in[s]_{\mathscr{F}}} \{\lambda_{\mathfrak{A}}(t_1+t_2)\} \leq \bigwedge_{t_1\in[r]_{\mathscr{F}}, t_2\in[s]_{\mathscr{F}}} \{\lambda_{\mathfrak{A}}(t_1) \vee \lambda_{\mathfrak{A}}(t_2)\} \\
&= \bigwedge_{t_1\in[r]_{\mathscr{F}}} \{\lambda_{\mathfrak{A}}(t_1)\} \vee \bigwedge_{t_2\in[s]_{\mathscr{F}}} \{\lambda_{\mathfrak{A}}(t_2)\} \\
&= \mathscr{F}^-(\lambda_{\mathfrak{A}})(r) \vee \mathscr{F}^-(\lambda_{\mathfrak{A}})(s).
\end{aligned} \tag{5}$$

$$\begin{aligned}
\mathscr{F}^-(\lambda_{\mathfrak{A}})(rs) &= \bigwedge_{t\in[rs]_{\mathscr{F}}} \{\lambda_{\mathfrak{A}}(t)\} = \bigwedge_{t\in[r]_{\mathscr{F}}[s]_{\mathscr{F}}} \{\lambda_{\mathfrak{A}}(t)\} \\
&= \bigwedge_{t_1\in[r]_{\mathscr{F}}, t_2\in[s]_{\mathscr{F}}} \{\lambda_{\mathfrak{A}}(t_1 t_2)\} \leq \bigwedge_{t_1\in[r]_{\mathscr{F}}, t_2\in[s]_{\mathscr{F}}} \{\lambda_{\mathfrak{A}}(t_1) \vee \lambda_{\mathfrak{A}}(t_2)\} \\
&= \bigwedge_{t_1\in[r]_{\mathscr{F}}} \{\lambda_{\mathfrak{A}}(t_1)\} \vee \bigwedge_{t_2\in[s]_{\mathscr{F}}} \{\lambda_{\mathfrak{A}}(t_2)\} \\
&= \mathscr{F}^-(\lambda_{\mathfrak{A}})(r) \vee \mathscr{F}^-(\lambda_{\mathfrak{A}})(s).
\end{aligned} \tag{6}$$

Thus $\mathscr{F}^-(\mathfrak{A})$ is an IF ideal of the semiring $\mathscr{R}$.

Lemma 6.10. *Let $\mathscr{F}$ be a F.C.R. on a semiring $\mathscr{R}$. If $\mathfrak{A} = (\mu_{\mathfrak{A}}, \lambda_{\mathfrak{A}})$ is an IF 2-absorbing ideal of $\mathscr{R}$, then $\mathscr{F}^{-}(\mathfrak{A}) = (\mathscr{F}^{-}(\mu_{\mathfrak{A}}), \mathscr{F}^{-}(\lambda_{\mathfrak{A}}))$ is also an IF 2-absorbing ideal of $\mathscr{R}$.*

Proof. From Lemma 6.9, $\mathscr{F}^{-}(\mathfrak{A})$ is an IF ideal of a semiring $\mathscr{R}$. Assume that $\mathfrak{A}$ is an IF 2-absorbing ideal and for $r, s, t \in \mathscr{R}$, $rst \in \mathscr{F}^{-}(\mathfrak{A}) = (\mathscr{F}^{-}(\mu_{\mathfrak{A}}), \mathscr{F}^{-}(\lambda_{\mathfrak{A}}))$ such that $\mathscr{F}^{-}(\mu_{\mathfrak{A}})(rst) \geq m$ and $\mathscr{F}^{-}(\lambda_{\mathfrak{A}})(rst) \leq n$ where $m, n \in [0, 1]$.

$$\begin{aligned}\mathscr{F}^{-}(\mu_{\mathfrak{A}})(rst) &= \bigvee_{k\in[rst]_{\mathscr{F}}} \mu_{\mathfrak{A}}(k) = \bigvee_{k\in[r]_{\mathscr{F}}[s]_{\mathscr{F}}[t]_{\mathscr{F}}} \mu_{\mathfrak{A}}(k)\\ &= \bigvee_{abc\in[r]_{\mathscr{F}}[s]_{\mathscr{F}}[t]_{\mathscr{F}}} \mu_{\mathfrak{A}}(abc).\end{aligned}$$

Suppose that $rs, st \notin \mathscr{F}^{-}(\mathfrak{A})$ such that $\mathscr{F}^{-}(\mu_{\mathfrak{A}})(rs) \ngeq m$, $\mathscr{F}^{-}(\mu_{\mathfrak{A}})(st) \ngeq m$, $\mathscr{F}^{-}(\lambda_{\mathfrak{A}})(rs) \nleq n$ and $\mathscr{F}^{-}(\lambda_{\mathfrak{A}})(st) \nleq n$ i.e., $\mathscr{F}^{-}(\mu_{\mathfrak{A}})(rs) = \bigvee_{k\in[rs]_{\mathscr{F}}} \mu_{\mathfrak{A}}(k) = \bigvee_{ab\in[r]_{\mathscr{F}}[s]_{\mathscr{F}}} \mu_{\mathfrak{A}}(ab) \ngeq m$, $\mathscr{F}^{-}(\mu_{\mathfrak{A}})(st) = \bigvee_{k\in[st]_{\mathscr{F}}} \mu_{\mathfrak{A}}(k) = \bigvee_{bc\in[s]_{\mathscr{F}}[t]_{\mathscr{F}}} \mu_{\mathfrak{A}}(ab) \ngeq m$, $\mathscr{F}^{-}(\lambda_{\mathfrak{A}})(rs) = \bigwedge_{k\in[rs]_{\mathscr{F}}} \lambda_{\mathfrak{A}}(k) = \bigwedge_{ab\in[r]_{\mathscr{F}}[s]_{\mathscr{F}}} \lambda_{\mathfrak{A}}(ab) \nleq n$, $\mathscr{F}^{-}(\lambda_{\mathfrak{A}})(st) = \bigwedge_{k\in[st]_{\mathscr{F}}} \lambda_{\mathfrak{A}}(k) = \bigwedge_{bc\in[s]_{\mathscr{F}}[t]_{\mathscr{F}}} \lambda_{\mathfrak{A}}(bc) \nleq n$. Since $\mathfrak{A}$ is an IF2AI, then $\bigvee_{abc\in[r]_{\mathscr{F}}[s]_{\mathscr{F}}[t]_{\mathscr{F}}} \mu_{\mathfrak{A}}(abc) \geq m$ implies that

$$\bigvee_{ac\in[r]_{\mathscr{F}}[t]_{\mathscr{F}}} \mu_{\mathfrak{A}}(ac) \geq m. \tag{7}$$

Also

$$\begin{aligned}\mathscr{F}^{-}(\lambda_{\mathfrak{A}})(rst) &= \bigwedge_{k\in[rst]_{\mathscr{F}}} \lambda_{\mathfrak{A}}(k) = \bigwedge_{k\in[r]_{\mathscr{F}}[s]_{\mathscr{F}}[t]_{\mathscr{F}}} \lambda_{\mathfrak{A}}(k)\\ &= \bigwedge_{abc\in[r]_{\mathscr{F}}[s]_{\mathscr{F}}[t]_{\mathscr{F}}} \lambda_{\mathfrak{A}}(abc).\end{aligned}$$

Since $\mathfrak{A}$ is an IF2AI, then $\bigwedge_{abc\in[r]_{\mathscr{F}}[s]_{\mathscr{F}}[t]_{\mathscr{F}}} \lambda_{\mathfrak{A}}(abc) \leq n$ implies that

$$\bigwedge_{ac\in[r]_{\mathscr{F}}[t]_{\mathscr{F}}} \lambda_{\mathfrak{A}}(ac) \leq n. \tag{8}$$

From equations (7)-(8), $rt \in \mathscr{F}^{-}(\mathfrak{A})$. Thus $\mathscr{F}^{-}(\mathfrak{A})$ is an IF2AI of the semiring $\mathscr{R}$.

Proposition 6.11. *Let $\mathscr{F}$ be a F.C.R. on a semiring $\mathscr{R}$. If $\mathfrak{D} = (\mu_{\mathfrak{D}}, \lambda_{\mathfrak{D}})$ is an IF set of $\mathscr{R}$ and $\alpha, \beta \in [0, 1]$, then*

(*i*) $(\mathscr{F}_{-}(\mathfrak{D}))^{(\alpha,\beta)} = \mathscr{F}_{-}(\mathfrak{D}^{(\alpha,\beta)})$ *and*
(*ii*) $(\mathscr{F}^{-}(\mathfrak{D}))_s^{(\alpha,\beta)} = \mathscr{F}^{-}(\mathfrak{D}_s^{(\alpha,\beta)})$.

Proof. (*i*) Suppose that $x \in (\mathscr{F}_{-}(\mathfrak{D}))^{(\alpha,\beta)}$. Then

$$\begin{aligned}&\bigwedge_{a\in[x]_{\mathscr{F}}} \mu_{\mathfrak{D}}(a) \geq \alpha, \quad \bigvee_{a\in[x]_{\mathscr{F}}} \lambda_{\mathfrak{D}}(a) \leq \beta\\ &\Longleftrightarrow \forall\ a \in [x]_{\mathscr{F}},\ \mu_{\mathfrak{D}}(a) \geq \alpha,\ \lambda_{\mathfrak{D}}(a) \leq \beta\\ &\Longleftrightarrow [x]_{\mathscr{F}} \subseteq \mathfrak{D}^{(\alpha,\beta)}\\ &\Longleftrightarrow x \in \mathscr{F}_{-}(\mathfrak{D}^{(\alpha,\beta)})\\ &\Longleftrightarrow (\mathscr{F}_{-}(\mathfrak{D}))^{(\alpha,\beta)} = \mathscr{F}_{-}(\mathfrak{D}^{(\alpha,\beta)}).\end{aligned}$$

(ii) Let $y \in (\mathscr{F}^{-}(\mathfrak{D}))_s^{(\alpha,\beta)}$. Then

$$\begin{aligned} & \mathscr{F}^{-}(\mu_{\mathfrak{D}})(y) > \alpha, \ \ \mathscr{F}^{-}(\lambda_{\mathfrak{D}})(y) < \beta \\ \Longleftrightarrow & \bigvee_{a \in [y]_{\mathscr{F}}} \mu_{\mathfrak{D}}(a) > \alpha, \ \ \bigwedge_{a \in [y]_{\mathscr{F}}} \lambda_{\mathfrak{D}}(a) < \beta \\ \Longleftrightarrow & \exists a \in [y]_{\mathscr{F}}, \ \ \mu_{\mathfrak{D}}(a) > \alpha, \ \ \lambda_{\mathfrak{D}} < \beta \\ \Longleftrightarrow & [y]_{\mathscr{F}} \cap \mathfrak{D}_s^{(\alpha,\beta)} \neq \phi \\ \Longleftrightarrow & y \in \mathscr{F}^{-}(\mathfrak{D}_s^{\alpha,\beta}) \\ \Longleftrightarrow & (\mathscr{F}^{-}(\mathfrak{D}))_s^{(\alpha,\beta)} = \mathscr{F}^{-}(\mathfrak{D}_s^{(\alpha,\beta)}). \end{aligned}$$

Lemma 6.12. *Let $\mathfrak{A}$ be an IF subset of a semiring $\mathscr{R}$. Then $\mathfrak{A}$ is an IF2AI of $\mathscr{R}$ if and only if $\mathfrak{A}_s^{(\alpha,\beta)}$ is a 2AI of $\mathscr{R}$.*

Proof. Assume that $\mathfrak{A} = (\mu_{\mathfrak{A}}, \lambda_{\mathfrak{A}})$ is an IF2AI of the semiring $\mathscr{R}$ and $r, s, t \in \mathfrak{A}_s^{(\alpha,\beta)}$ such that $rst \in \mathfrak{A}_s^{(\alpha,\beta)}$. Then $\mu_{\mathfrak{A}}(rst) > \alpha$ and $\lambda_{\mathfrak{A}}(rst) < \beta$. Therefore,

$$\mu_{\mathfrak{A}}(rst) > \alpha \ \text{ implies } \ \mu_{\mathfrak{A}}(rs) > \alpha \ \text{ or } \ \mu_{\mathfrak{A}}(st) > \alpha \ \text{ or } \ \mu_{\mathfrak{A}}(rt) > \alpha$$

and

$$\lambda_{\mathfrak{A}}(rst) < \beta \ \text{ implies } \ \lambda_{\mathfrak{A}}(rs) < \beta \ \text{ or } \ \lambda_{\mathfrak{A}}(st) < \beta \ \text{ or } \ \lambda_{\mathfrak{A}}(rt) < \beta.$$

Thus, $rs \in \mathfrak{A}_s^{(\alpha,\beta)}$ or $st \in \mathfrak{A}_s^{(\alpha,\beta)}$ or $rt \in \mathfrak{A}_s^{(\alpha,\beta)}$.
Conversely suppose that $\mathfrak{A}_s^{(\alpha,\beta)}$ is a 2AI of $\mathscr{R}$ and $r, s, t \in \mathscr{R}$ such that $\mu_{\mathfrak{A}}(rst) > \alpha$ and $\lambda_{\mathfrak{A}}(rst) < \beta$. Then $rst \in \mathfrak{A}_s^{(\alpha,\beta)}$. Since $\mathfrak{A}_s^{(\alpha,\beta)}$ is 2-absorbing, then $rs \in \mathfrak{A}_s^{(\alpha,\beta)}$ or $st \in \mathfrak{A}_s^{(\alpha,\beta)}$ or $rt \in \mathfrak{A}_s^{(\alpha,\beta)}$. Thus, $\mu_{\mathfrak{A}}(rs) > \alpha$ or $\mu_{\mathfrak{A}}(st) > \alpha$ or $\mu_{\mathfrak{A}}(rt) > \alpha$ and $\lambda_{\mathfrak{A}}(rs) < \beta$ or $\lambda_{\mathfrak{A}}(st) < \beta$ or $\lambda_{\mathfrak{A}}(rt) < \beta$.

Lemma 6.13. *Let $\mathfrak{A}$ be an IF subset of a semiring $\mathscr{R}$. Then $\mathfrak{A}$ is an IF2API of $\mathscr{R}$ if and only if $\mathfrak{A}_s^{(\alpha,\beta)}$ is a 2API of $\mathscr{R}$.*

Proof. Similar to Lemma 6.12.

Lemma 6.14. *Let $\mathfrak{A} = (\mu_{\mathfrak{A}}, \lambda_{\mathfrak{A}})$ be an IF subset of a semiring $\mathfrak{S}$. Then, $\mathfrak{A}$ is an IF2API of $\mathfrak{S}$ if and only if $\mathfrak{A}^{(\alpha,\beta)}$ is a 2API of $\mathfrak{S}$.*

Proof. Suppose that $\mathfrak{A}$ is an IF2AI of the semiring $\mathfrak{S}$ and for $r, s, t \in \mathfrak{S}$ and $\alpha, \beta \in [0,1]$, $rst \in \mathfrak{A}^{(\alpha,\beta)}$. Then $\mu_{\mathfrak{A}}(rst) \geq \alpha$ implies that $\mu_{\mathfrak{A}}(rs) \geq \alpha$ or $\mu_{\sqrt{\mathfrak{A}}}(st) \geq \alpha$ or $\mu_{\sqrt{\mathfrak{A}}}(rt) \geq \alpha$ and $\lambda_{\mathfrak{A}}(rst) \leq \beta$ implies that $\lambda_{\mathfrak{A}}(rs) \leq \beta$ or $\lambda_{\sqrt{\mathfrak{A}}}(st) \leq \beta$ or $\lambda_{\sqrt{\mathfrak{A}}}(rt) \leq \beta$. Therefore, $rs \in \mathfrak{A}^{(\alpha,\beta)}$ or $(st)^m \in \mathfrak{A}^{(\alpha,\beta)}$ or $(rt)^n \in \mathfrak{A}^{(\alpha,\beta)}$, for positive integers m, n. Hence, $\mathfrak{A}^{(\alpha,\beta)}$ is a 2API of $\mathfrak{S}$.
Conversely, suppose on the contrary that $\mathfrak{A}$ is not an IF2API of $\mathfrak{S}$ with $\mathfrak{A}^{(\alpha,\beta)}$ is a 2API of $\mathfrak{S}$. Then $\mu_{\mathfrak{A}}(rst) \geq \alpha$ doesn't imply that $\mu_{\mathfrak{A}}(rs) \geq \alpha$ or $\mu_{\sqrt{\mathfrak{A}}}(st) \geq \alpha$ or $\mu_{\sqrt{\mathfrak{A}}}(rt) \geq \alpha$, and $\lambda_{\mathfrak{A}}(rst) \leq \beta$ doesn't imply that $\lambda_{\mathfrak{A}}(rs) \leq \beta$ or $\lambda_{\sqrt{\mathfrak{A}}}(st) \leq \beta$ or $\lambda_{\sqrt{\mathfrak{A}}}(rt) \leq \beta$, for $r, s, t, \in \mathfrak{S}$ and $\alpha, \beta \in [0,1]$. Since $rst \in \mathfrak{A}^{(\alpha,\beta)}$, $rs \notin \mathfrak{A}^{(\alpha,\beta)}$, $(st)^m \notin \mathfrak{A}^{(\alpha,\beta)}$ and $(rt)^n \notin \mathfrak{A}^{(\alpha,\beta)}$ for any positive integers m, n. This contradicts the hypothesis $\mathfrak{A}^{(\alpha,\beta)}$ is a 2API of $\mathfrak{S}$. Hence, $\mathfrak{A}$ is an IF2API of $\mathfrak{S}$.

Theorem 6.15. *Let $\mathscr{F}_1$ and $\mathscr{F}_2$ be two F.C.R. on a semiring $\mathscr{R}$. If $\mathfrak{A} = (\mu_{\mathfrak{A}}, \lambda_{\mathfrak{A}})$ and $\mathfrak{B} = (\mu_{\mathfrak{B}}, \lambda_{\mathfrak{B}})$ are any two IF sets of $\mathscr{R}$, then the following hold:*

(i) $\mathscr{F}_{1-}(\mathfrak{A}) \subseteq \mathfrak{A} \subseteq \mathscr{F}_1{}^{-}(\mathfrak{A})$
(ii) $\mathscr{F}_{1-}(\mathscr{F}_{1-}(\mathfrak{A})) = \mathscr{F}_{1-}(\mathfrak{A})$
(iii) $\mathscr{F}_1{}^{-}(\mathscr{F}_1{}^{-}(\mathfrak{A})) = \mathscr{F}_1{}^{-}(\mathfrak{A})$
(iv) $\mathscr{F}_1{}^{-}(\mathscr{F}_{1-}(\mathfrak{A})) = \mathscr{F}_{1-}(\mathfrak{A})$
(v) $\mathscr{F}_{1-}(\mathscr{F}_1{}^{-}(\mathfrak{A})) = \mathscr{F}_1{}^{-}(\mathfrak{A})$
(vi) $(\mathscr{F}_1{}^{-}(\mathfrak{A}^c))^c = \mathscr{F}_{1-}(\mathfrak{A})$

(*vii*) $(\mathscr{F}_{1-}(\mathfrak{A}^c))^c = \mathscr{F}_1{}^-(\mathfrak{A})$
(*viii*) $\mathscr{F}_{1-}(\mathfrak{A} \cap \mathfrak{B}) = \mathscr{F}_{1-}(\mathfrak{A}) \cap \mathscr{F}_{1-}(\mathfrak{B})$
(*ix*) $\mathscr{F}_1{}^-(\mathfrak{A} \cap \mathfrak{B}) \subseteq \mathscr{F}_1{}^-(\mathfrak{A}) \cap \mathscr{F}_1{}^-(\mathfrak{B})$
(*x*) $\mathscr{F}_1{}^-(\mathfrak{A} \cup \mathfrak{B}) = \mathscr{F}_1{}^-(\mathfrak{A}) \cup \mathscr{F}_1{}^-(\mathfrak{B})$
(*xi*) $\mathscr{F}_{1-}(\mathfrak{A} \cup \mathfrak{B}) \supseteq \mathscr{F}_{1-}(\mathfrak{A}) \cup \mathscr{F}_{1-}(\mathfrak{B})$
(*xii*) $\mathfrak{A} \subseteq \mathfrak{B}$ *implies that* $\mathscr{F}_{1-}(\mathfrak{A}) \subseteq \mathscr{F}_{1-}(\mathfrak{B})$
(*xiii*) $\mathfrak{A} \subseteq \mathfrak{B}$ *implies that* $\mathscr{F}_1{}^-(\mathfrak{A}) \subseteq \mathscr{F}_1{}^-(\mathfrak{B})$
(*xiv*) $\mathscr{F}_1 \subseteq \mathscr{F}_2$ *implies that* $\mathscr{F}_{1-}(\mathfrak{A}) \supseteq \mathscr{F}_{2-}(\mathfrak{A})$
(*xv*) $\mathscr{F}_1 \subseteq \mathscr{F}_2$ *implies that* $\mathscr{F}_1{}^-(\mathfrak{A}) \subseteq \mathscr{F}_{2-}(\mathfrak{A})$.

Proof. It is straightforward.

Theorem 6.16. *Let $\mathscr{F}$ be an F.C.R. on a semiring $\mathscr{R}$. If $\mathfrak{A} = (\mu_{\mathfrak{A}}, \lambda_{\mathfrak{A}})$ and $\mathfrak{B} = (\mu_{\mathfrak{B}}, \lambda_{\mathfrak{B}})$ are IF ideals of $\mathscr{R}$, then*

(*i*) $\mathscr{F}^-(\mathfrak{A} + \mathfrak{B}) = \mathscr{F}^-(\mathfrak{A}) + \mathscr{F}^-(\mathfrak{B})$ *and*
(*ii*) $\mathscr{F}_-(\mathfrak{A} + \mathfrak{B}) \supseteq \mathscr{F}_-(\mathfrak{A}) + \mathscr{F}_-(\mathfrak{B})$.

Proof. Proof runs on the same parallel lines as [31, Theorem 3.5 and Theorem 3.6].

Theorem 6.17. *Let $\mathfrak{A}$ be an IF2API of a semiring $\mathscr{R}$ and $\mathscr{F}$ be a F.C.R. Then*

(*i*) *$\mathfrak{A}$ is an upper rough IF2API of $\mathscr{R}$.*
(*ii*) *$\mathfrak{A}$ is a lower rough IF2API of $\mathscr{R}$, provided $\mathscr{F}$ is B.C.R. and $\mathfrak{A}$ is a subtractive ideal of $\mathscr{R}$ such that $\mathscr{F}_-(\mathfrak{A}) \neq \phi$.*

Proof. (*i*) From Lemma 6.9, $\mathscr{F}^-(\mathfrak{A})$ is an IF ideal of the semiring $\mathscr{R}$. Assume that $r, s, t \in \mathscr{R}$ such that $rst \in \mathscr{F}^-(\mathfrak{A}) = (\mathscr{F}^-(\mu_{\mathfrak{A}}), \mathscr{F}^-(\lambda_{\mathfrak{A}}))$ and $\mathscr{F}^-(\mu_{\mathfrak{A}})(rst) \geq m$, and $\mathscr{F}^-(\lambda_{\mathfrak{A}})(rst) \leq n$, where $m, n \in [0,1]$. Then

$$\begin{aligned}\mathscr{F}^-(\mu_{\mathfrak{A}})(rst) &= \bigvee_{k\in[rst]_{\mathscr{F}}} \mu_{\mathfrak{A}}(k) = \bigvee_{k\in[r]_{\mathscr{F}}[s]_{\mathscr{F}}[t]_{\mathscr{F}}} \mu_{\mathfrak{A}}(k)\\ &= \bigvee_{abc\in[r]_{\mathscr{F}}[s]_{\mathscr{F}}[t]_{\mathscr{F}}} \mu_{\mathfrak{A}}(abc).\end{aligned}$$

Suppose that $rs, (st)^i \notin \mathscr{F}^-(\mathfrak{A})$ for some positive integer i such that $\mathscr{F}^-(\mu_{\mathfrak{A}})(rs) \ngeq m$, $\mathscr{F}^-(\mu_{\mathfrak{A}})((st)^i) \ngeq m$, $\mathscr{F}^-(\lambda_{\mathfrak{A}})(rs) \nleq n$ and $\mathscr{F}^-(\lambda_{\mathfrak{A}})((st)^i) \nleq n$. Then $\mathscr{F}^-(\mu_{\mathfrak{A}})(rs) = \bigvee_{k\in[rs]_{\mathscr{F}}} \mu_{\mathfrak{A}}(k) = \bigvee_{ab\in[r]_{\mathscr{F}}[s]_{\mathscr{F}}} \mu_{\mathfrak{A}}(ab) \ngeq m$, $\mathscr{F}^-(\mu_{\mathfrak{A}})((st)^i) = \bigvee_{k\in[(st)^i]_{\mathscr{F}}} \mu_{\mathfrak{A}}(k) = \bigvee_{b^ic^i\in[s^i]_{\mathscr{F}}[t^i]_{\mathscr{F}}} \mu_{\mathfrak{A}}(b^ic^i) \ngeq m$, $\mathscr{F}^-(\lambda_{\mathfrak{A}})(rs) = \bigwedge_{k\in[rs]_{\mathscr{F}}} \lambda_{\mathfrak{A}}(k) = \bigwedge_{ab\in[r]_{\mathscr{F}}[s]_{\mathscr{F}}} \lambda_{\mathfrak{A}}(ab) \nleq n$, $\mathscr{F}^-(\lambda_{\mathfrak{A}})((st)^i) = \bigwedge_{k\in[(st)^i]_{\mathscr{F}}} \lambda_{\mathfrak{A}}(k) = \bigwedge_{b^ic^i\in[s^i]_{\mathscr{F}}[t^i]_{\mathscr{F}}} \lambda_{\mathfrak{A}}(b^ic^i) \nleq n$. Since $\mathfrak{A}$ is an IF2API, then $\bigvee_{abc\in[r]_{\mathscr{F}}[s]_{\mathscr{F}}[t]_{\mathscr{F}}} \mu_{\mathfrak{A}}(abc) \geq m$ implies that

$$\bigvee_{a^ic^i\in[r^i]_{\mathscr{F}}[t^i]_{\mathscr{F}}} \mu_{\mathfrak{A}}(a^ic^i) \geq m. \tag{9}$$

Also

$$\begin{aligned}\mathscr{F}^-(\lambda_{\mathfrak{A}})(rst) &= \bigwedge_{k\in[rst]_{\mathscr{F}}} \lambda_{\mathfrak{A}}(k) = \bigwedge_{k\in[r]_{\mathscr{F}}[s]_{\mathscr{F}}[t]_{\mathscr{F}}} \lambda_{\mathfrak{A}}(k)\\ &= \bigwedge_{abc\in[r]_{\mathscr{F}}[s]_{\mathscr{F}}[t]_{\mathscr{F}}} \lambda_{\mathfrak{A}}(abc).\end{aligned}$$

Since $\mathfrak{A}$ is an IF2API, then $\bigwedge_{abc\in[r]_{\mathscr{F}}[s]_{\mathscr{F}}[t]_{\mathscr{F}}} \lambda_{\mathfrak{A}}(abc) \leq n$ implies that

$$\bigvee_{a^ic^i\in[r^i]_{\mathscr{F}}[t^i]_{\mathscr{F}}} \lambda_{\mathfrak{A}}(ac) \leq n. \tag{10}$$

From equations (9) and (10), $(rt)^i \in \mathscr{F}^-(\mathfrak{A})$. Thus, $\mathfrak{A}$ is an upper rough IF2API of the semiring $\mathscr{R}$.
(ii) Since $\mathfrak{A}$ is an IF2API of semiring $\mathscr{R}$, then by Lemma 6.13 $\mathfrak{A}_s^{(\alpha,\beta)}$ is a 2API of $\mathscr{R}$. Also, by Theorem 5.11, $\mathscr{F}_-(\mathfrak{A}_s^{(\alpha,\beta)})$ is a 2API of $\mathscr{R}$. By Proposition 6.11, $\mathscr{F}_-(\mathfrak{A}_s^{(\alpha,\beta)}) = (\mathscr{F}_-(\mathfrak{A}))_s^{(\alpha,\beta)}$ is a 2API of $\mathscr{R}$. Again by Lemma 6.13, $\mathscr{F}_-(\mathfrak{A})$ is an IF2API of $\mathscr{R}$.

Theorem 6.18. *Let f be an epimorphism from a semiring $\mathscr{R}_1$ to a semiring $\mathscr{R}_2$ and let $\mathscr{F}_2$ be a F.C.R. on $\mathscr{R}_2$. Let $\mathfrak{A}$ be a subset of $\mathscr{R}_1$. If $\mathscr{F}_1 = \{(s_1, s_2) \in \mathscr{R}_1 \times \mathscr{R}_1 \mid (f(s_1), f(s_2)) \in \mathscr{F}_2\}$, then*

(i) *$\mathscr{F}_1^-(\mathfrak{A})$ is an IF2AI of $\mathscr{R}_1$ if and only if $\mathscr{F}_2^-(f(\mathfrak{A}))$ is an IF2AI of $\mathscr{R}_2$.*
(ii) *If $\mathscr{F}_2$ is F.C.R., $\mathscr{F}_1^-(\mathfrak{A})$ is an IF2API of $\mathscr{R}_1$ if and only if $\mathscr{F}_2^-(f(\mathfrak{A}))$ is 2API of $\mathscr{R}_2$.*
Furthermore, if $\mathfrak{A}$ is subtractive IF ideal and f is one-one then
(iii) *$\mathscr{F}_{1-}(\mathfrak{A})$ is an IF2AI of $\mathscr{R}_1$ if and only if $\mathscr{F}_{2-}(f(\mathfrak{A}))$ is 2AI of $\mathscr{R}_2$.*
(iv) *If $\mathscr{F}_2$ is F.C.R., $\mathscr{F}_{1-}(\mathfrak{A})$ is an IF2API of $\mathscr{R}_1$ if and only if $\mathscr{F}_{2-}(f(\mathfrak{A}))$ is an IF2API of $\mathscr{R}_2$.*

Proof. (i) By Lemma 6.12, we get $\mathscr{F}_1^-(\mathfrak{A})$ is an IF2AI of $\mathscr{R}_1$ if and only if $(\mathscr{F}_1^-(\mathfrak{A}))_s^{(\alpha,\beta)}$ is a 2AI of $\mathscr{R}_1$. Using Proposition 6.11 and Theorem 5.14, we have $(\mathscr{F}_1^-(\mathfrak{A}_s^{(\alpha,\beta)}))$ is a 2AI of $\mathscr{R}_1$ if and only if $(\mathscr{F}_2^-(f(\mathfrak{A}_s^{(\alpha,\beta)})))$ is a 2AI of $\mathscr{R}_2$. Again by Lemma 6.12, $(\mathscr{F}_2^-(f(\mathfrak{A}_s^{(\alpha,\beta)}))) = (\mathscr{F}_2^-(f(\mathfrak{A}))_s^{(\alpha,\beta)}) = (\mathscr{F}_2^-(f(\mathfrak{A})))_s^{(\alpha,\beta)}$ is a 2AI of $\mathscr{R}_2$ if and only if $\mathscr{F}_2^-(f(\mathfrak{A}))$ is an IF2AI of $\mathscr{R}_2$.
Other parts' proof are parallels to proof of (i).

7. Conclusion

Rough IF2API of a semiring is a generalization of intuitionistic prime, primary ideal of a semiring. So, we replaced a universe set by a semiring and introduced the notion of rough 2APIs and rough IF2API of semirings. Roughness in semirings is an exciting and mostly unexplored area of research in algebra and fuzzy systems. Working on the suggested future directions and solving the open problems can help us better understand rough IF ideals and find more ways to use them in mathematics. It could also lead to discovering new connections with other fields, opening up fresh opportunities for research. In the future, one may further take this concept to other algebraic structures such as near rings, seminear rings, and near semirings etc. and prove the above results. There is another open problem which leads to neutrosophic fuzzy ideals of near rings. One may try above results for neutrosophic fuzzy ideals.

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