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Nonlinear Lie n-centralizers at zero-products on unital algebras

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Abstract. Let $\mathcal{A}$ be a unital algebra with nontrivial idempotents. In this article, it is shown that under certain conditions if a map (not necessarily linear) $\delta : \mathcal{A} \to \mathcal{A}$ satisfies

$$\delta\left(p_n\left(a_1, a_2, \cdots, a_n\right)\right) = p_n\left(\delta(a_1), a_2, \cdots, a_n\right)$$

for all $a_1, a_2, \cdots, a_n \in \mathcal{A}$ with $a_1 a_2 \cdots a_n = 0$, then $\delta(a+b) - \delta(a) - \delta(b) \in \mathcal{Z}(\mathcal{A})$ for all $a, b \in \mathcal{A}$. Moreover, δ is of the form $\delta(a) = \lambda a + \tau(a)$ for all $a \in \mathcal{A}$, where $\lambda \in \mathcal{Z}(\mathcal{A})$ and $\tau : \mathcal{A} \to \mathcal{Z}(\mathcal{A})$ is an almost additive map such that $\tau\left(p_n\left(a_1, a_2, \cdots, a_n\right)\right) = 0$ for all $a_1, a_2, \cdots, a_n \in \mathcal{A}$ with $a_1 a_2 \cdots a_n = 0$. Moreover, this result can also be applied to triangular algebras, von Neumann algebras without central summands of type I_1 and so on.

1. Introduction

Let $\mathcal{A}$ be an algebra. A linear (nonlinear) map $\varphi : \mathcal{A} \to \mathcal{A}$ is called a linear (nonlinear) derivation if $\varphi(ab) = \varphi(a)b + a\varphi(b)$ for all $a, b \in \mathcal{A}$. A linear (nonlinear) map φ of $\mathcal{A}$ is called a linear (nonlinear) Lie derivation if $\varphi([a,b]) = [\varphi(a), b] + [a, \varphi(b)]$ and a linear (nonlinear) Lie triple derivation if $\varphi([[a,b],c]) = [[\varphi(a), b], c] + [[a, \varphi(b)], c] + [[a,b], \varphi(c)]$ for all $a, b, c \in \mathcal{A}$. Obviously, each derivation is a Lie derivation and each Lie derivation is a Lie triple derivation. But the converse statement is not true in general. For this reason, Abdullave [1] extended them in one much more general way. Fix a positive integer $n \geq 2$ and define a sequence of polynomials as follows.

$$\begin{aligned}
p_1\left(x_1\right) &= x_1, \\
p_2\left(x_1, x_2\right) &= \left[p_1\left(x_1\right), x_2\right] = \left[x_1, x_2\right], \\
p_3(x_1, x_2, x_3) &= \left[p_2\left(x_1, x_2\right), x_3\right] = \left[\left[x_1, x_2\right], x_3\right], \\
p_4\left(x_1, x_2, x_3, x_4\right) &= \left[p_3\left(x_1, x_2, x_3\right), x_4\right] = \left[\left[\left[x_1, x_2\right], x_3\right], x_4\right], \\
&\cdots \\
p_n\left(x_1, x_2, \cdots, x_n\right) &= \left[p_{n-1}\left(x_1, x_2, \cdots, x_{n-1}\right), x_n\right].
\end{aligned}$$

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The polynomial $p_n(x_1, x_2, \cdots, x_n)$ is said to be an $(n-1)$-th commutator ($n \geq 2$). Then a map $\varphi : \mathcal{A} \to \mathcal{A}$ is said to be a Lie n-derivation if

$$\varphi(p_n(x_1, x_2, \cdots, x_n)) = p_n(\varphi(x_1), x_2, \cdots, x_n) + p_n(x_1, \varphi(x_2), \cdots, x_n) + \cdots + p_n(x_1, x_2, \cdots, \varphi(x_n)) \tag{1}$$

for all $x_1, x_2, \cdots, x_n \in \mathcal{A}$. There has been a great interest in the characterization of Lie n-derivation on various rings and algebras over the last few decades. Many authors have made essential contributions to related topics (see [6, 16, 17] and references therein).

Recently, more and more authors have paid attention to non-global Lie derivations, non-global Lie triple derivations and non-global Lie n-derivations. One can refer to [2, 4, 12–14, 19, 20] and references therein. Ashraf et al. [2] studied non-global nonlinear Lie n-derivations of unital algebras with nontrivial idempotents. They proved that under some mild assumptions every nonlinear Lie n-derivation is of the form $\delta + \tau$, where $\delta : \mathcal{A} \to \mathcal{A}$ is an additive derivation and $\tau : \mathcal{A} \to \mathcal{Z}(\mathcal{A})$ is an almost additive map vanishing at $p_n(a_1, a_2, \cdots, a_n)$ for all $a_1, a_2, \cdots, a_n \in \mathcal{A}$ with $a_1 a_2 \cdots a_n = 0$.

Based on Lie n-derivations, some authors introduced the definition of generalized Lie n-derivation. A mapping $\Delta : \mathcal{A} \to \mathcal{A}$ is called a multiplicative generalized Lie n-derivation if there exists a multiplicative Lie n-derivation $\varphi : \mathcal{A} \to \mathcal{A}$ such that

$$\Delta(p_n(x_1, x_2, \cdots, x_n)) = p_n(\Delta(x_1), x_2, \cdots, x_n) + p_n(x_1, \varphi(x_2), \cdots, x_n) + \cdots + p_n(x_1, x_2, \cdots, \varphi(x_n)) \tag{2}$$

for all $x_1, x_2, \cdots, x_n \in \mathcal{A}$. Let $\Delta : \mathcal{A} \to \mathcal{A}$ be a multiplicative generalized Lie n-derivation with associated multiplicative Lie n-derivation $\varphi : \mathcal{A} \to \mathcal{A}$. Set $\delta = \Delta - \varphi$. Then it follows from (1) and (2) that δ satisfies

$$\delta(p_n(x_1, x_2, \cdots, x_n)) = p_n(\delta(x_1), x_2, \cdots, x_n) \tag{3}$$

for all $x_1, x_2, \cdots, x_n \in \mathcal{A}$. Any mapping $\delta : \mathcal{A} \to \mathcal{A}$ satisfying (3) is called a Lie n-centralizer. In particular, a linear (nonlinear) map $\delta : \mathcal{A} \to \mathcal{A}$ is called a (nonlinear) centralizer if $\delta(ab) = a\delta(b) = \delta(a)b$; a (nonlinear) Lie centralizer if $\delta([a, b]) = [\delta(a), b]$ and a (nonlinear) Lie triple centralizer if $\delta([[a, b], c]) = [[\delta(a), b], c]$ for all $a, b, c \in \mathcal{A}$. It is easy to prove that δ is a nonlinear Lie centralizer (resp. nonlinear Lie triple centralizer) on $\mathcal{A}$ if and only if $\delta([a, b]) = [a, \delta(b)]$ (resp. $\delta([[a, b], c]) = [[a, \delta(b)], c]$) for all $a, b, c \in \mathcal{A}$. There have been several results on Lie n-centralizers and one can refer to [3, 18] and references therein. Let $\mathcal{A}$ be unital ring with a nontrivial idempotent. Under mild conditions, Ashraf and Ansari [3] showed that a multiplicative Lie n-centralizer $\delta : \mathcal{A} \to \mathcal{A}$ has the form of $\delta(x) = zx + \tau(x)$ for all $x \in \mathcal{A}$, where $z \in \mathcal{Z}(\mathcal{A})$ and $\tau : A \to \mathcal{Z}(\mathcal{A})$ is a mapping vanishes at $p_n(x_1, x_2, ..., x_n)$ for all $x_1, x_2, ..., x_n \in \mathcal{A}$. Let G be a generalized matrix algebra over a commutative ring $\mathcal{R}$. Yuan and Liu [18] showed that, under mild conditions, a $\mathcal{R}$-linear Lie n-centralizer has the similar form as in [3].

Similar to non-global Lie derivations, non-global Lie triple derivations and non-global Lie n-derivations, non-global Lie centralizers, non-global Lie triple centralizers and non-global Lie n-centralizers have attracted more attentions from many authors. One can refer to [9–11] and references therein. However, there has been no concern with non-global Lie n-centralizers so far, including the nonlinear setting.

In this note, we shall investigate non-global nonlinear Lie n-centralizers on unital algebras with nontrivial idempotents and its framework is as follows. In Section 2, we present the preliminaries. In Section 3, we show the almost additivity of the nonlinear non-global Lie n-centralizer δ under certain restrictions. In Section 4, we give the structure of the nonlinear non-global Lie n-centralizer δ under some mild assumptions. It is pointed out that this result can also be applied to triangular algebras, von Neumann algebras without central summands of type I_1 and so on.

Definition 1.1. *A mapping $\delta : \mathcal{A} \to \mathcal{A}$ is called a nonlinear nonglobal Lie n-centralizer if*

$$\delta(p_n(a_1, a_2, \cdots, a_n)) = p_n(\delta(a_1), a_2, \cdots, a_n) = p_n(a_1, \delta(a_2), \cdots, a_n)$$

for all $a_1, a_2, \cdots, a_n \in \mathcal{A}$ with $a_1 a_2 \cdots a_n = 0$.

2. Preliminaries

Let $\mathcal{A}$ be an algebra with a nontrivial idempotent e, and denote the idempotent $f = 1 - e$. In this case, $\mathcal{A}$ can be represented in the so-called Pierce decomposition form as $\mathcal{A} = \mathcal{A}_{11} + \mathcal{A}_{12} + \mathcal{A}_{21} + \mathcal{A}_{22}$, where $\mathcal{A}_{11}$, $\mathcal{A}_{12}$, $\mathcal{A}_{21}$, $\mathcal{A}_{22}$ respectively denote $e\mathcal{A}e$, $e\mathcal{A}f$, $f\mathcal{A}e$, $f\mathcal{A}f$. Then each element $a \in \mathcal{A}$ can be written as $a = a_{11} + a_{12} + a_{21} + a_{22}$, where $a_{ij} \in \mathcal{A}_{ij}, i, j = 1, 2$. Moreover, $e\mathcal{A}f$ is an $(e\mathcal{A}e, f\mathcal{A}f)$-bimodule and $f\mathcal{A}e$ is an $(f\mathcal{A}f, e\mathcal{A}e)$-bimodule. Let us assume that $\mathcal{A}$ satisfies

$$\begin{aligned} &e\mathcal{A}e \cdot e\mathcal{A}f = 0 = f\mathcal{A}e \cdot e\mathcal{A}e \quad \textit{implies} \quad e\mathcal{A}e = 0,\\ &e\mathcal{A}f \cdot f\mathcal{A}f = 0 = f\mathcal{A}f \cdot f\mathcal{A}e \quad \textit{implies} \quad f\mathcal{A}f = 0. \end{aligned} \tag{4}$$

The unital algebras with Condition (4) were proposed by Benkovič and Širovnik in [7]. In view of ([5], Proposition 2.1), the center of $\mathcal{A}$ is given by

$$\mathcal{Z}(\mathcal{A}) = \{a_{11} + a_{22} \in \mathcal{A}_{11} + \mathcal{A}_{22} | a_{11}x_{12} = x_{12}a_{22}, x_{21}a_{11} = a_{22}x_{21} \text{ for all } x_{12} \in \mathcal{A}_{12}, x_{21} \in \mathcal{A}_{21}\}.$$

Furthermore, there exists a unique algebra isomorphism $\theta : \mathcal{Z}(\mathcal{A})e \to \mathcal{Z}(\mathcal{A})f$ such that $a_{11}x_{12} = x_{12}\theta(a_{11})$ and $x_{21}a_{11} = \theta(a_{11})x_{21}$ for all $a_{11} \in \mathcal{Z}(\mathcal{A})e$, $x_{12} \in \mathcal{A}_{12}$, $x_{21} \in \mathcal{A}_{21}$.

In the following, we conclude this section with a fundamental result, which will be frequently referenced throughout the paper without further mention.

Lemma 2.1. *For any $x \in \mathcal{A}$, we have*

(a) $p_n(x, -e, -e, \cdots, -e) = exf + (-1)^{n+1} fxe$,

(b) $p_n(x, -f, -f, \cdots, -f) = fxe + (-1)^{n+1} exf$,

(c) $p_n(x, e, e, \cdots, e) = (-1)^{n-1} exf + fxe$,

(d) $p_n(x, f, f, \cdots, f) = (-1)^{n-1} fxe + exf$.

3. The almost additivity

In this section, we investigate the additivity of non-global nonlinear Lie n-centralizers on unital algebras. The main result presented in this section is as follows.

Theorem 3.1. *Let $\mathcal{A}$ be a unital algebra containing a nontrivial idempotent e satisfying Condition (4). Suppose that $\mathcal{Z}(e\mathcal{A}e) = \mathcal{Z}(\mathcal{A})e$ and $\mathcal{Z}(f\mathcal{A}f) = \mathcal{Z}(\mathcal{A})f$. If a map $\delta : \mathcal{A} \to \mathcal{A}$ satisfies*

$$\delta(p_n(a_1, a_2, \cdots, a_n)) = p_n(\delta(a_1), a_2, \cdots, a_n) = p_n(a_1, \delta(a_2), \cdots, a_n) \, (n \geq 3)$$

for all $a_1, a_2, \cdots, a_n \in \mathcal{A}$ with $a_1a_2\cdots a_n = 0$, then δ is almost additive, that is,

$$\delta(a + b) - \delta(a) - \delta(b) \in \mathcal{Z}(\mathcal{A})$$

for all $a, b \in \mathcal{A}$.

To achieve Theorem 3.1, it is necessary to verify the following lemmas.

Lemma 3.2. $\delta(0) = 0$.

Proof. $\delta(0) = \delta(p_n(0, 0, \cdots, 0)) = p_n(\delta(0), 0, \cdots, 0) = 0$. □

Lemma 3.3. *For any $a_{11} \in \mathcal{A}_{11}$ and $a_{22} \in \mathcal{A}_{22}$, the following statements hold:*

(a) *$\delta(a_{11}) = e\delta(a_{11})e + f\delta(a_{11})f$ and $f\delta(a_{11})f \in \mathcal{Z}(\mathcal{A}_{22})$. Clearly, $\delta(a_{11}) \in \mathcal{A}_{11} + \mathcal{Z}(\mathcal{A})$;*

(b) $\delta(a_{22}) = e\delta(a_{22})e + f\delta(a_{22})f$ *and* $e\delta(a_{22})e \in \mathcal{Z}(\mathcal{A}_{11})$. *Clearly,* $\delta(a_{22}) \in \mathcal{A}_{22} + \mathcal{Z}(\mathcal{A})$.

Proof. (a) Since $a_{11}f\cdots f = 0$ and $p_n(a_{11}, f, \cdots, f) = 0$ for all $a_{11} \in \mathcal{A}_{11}$, we can obtain from Lemmas 2.1 and 3.2 that

$$\begin{aligned}0 =&\delta\left(p_n(a_{11}, f, \cdots, f)\right)\\ =&p_n(\delta(a_{11}), f, \cdots, f)\\ =&e\delta(a_{11})f + (-1)^{n-1}f\delta(a_{11})e.\end{aligned}$$

It follows that $e\delta(a_{11})f = f\delta(a_{11})e = 0$. Consequently,

$$\delta(a_{11}) = e\delta(a_{11})e + f\delta(a_{11})f \in \mathcal{A}_{11} + \mathcal{A}_{22}.$$

In the following, we will prove that $f\delta(a_{11})f \in \mathcal{Z}(\mathcal{A}_{22})$. Since $a_{11}a_{22}a_{12}f\cdots f = 0$ and $p_n(a_{11}, a_{22}, a_{12}, f, \cdots, f) = 0$, by Lemmas 2.1 and 3.2, we have

$$\begin{aligned}0 =&\delta\left(p_n(a_{11}, a_{22}, a_{12}, f, \cdots, f)\right)\\ =&p_n(\delta(a_{11}), a_{22}, a_{12}, f, \cdots, f)\\ =&p_{n-2}([[\delta(a_{11}), a_{22}], a_{12}], f, \cdots, f)\\ =&e[[\delta(a_{11}), a_{22}], a_{12}]f + (-1)^{n-3}f[[\delta(a_{11}), a_{22}], a_{12}]e\\ =&[[\delta(a_{11}), a_{22}], a_{12}]\\ =&[[f\delta(a_{11})f, a_{22}], a_{12}]\\ =&-a_{12}[f\delta(a_{11})f, a_{22}],\end{aligned}$$

which implies that

$$a_{12}[f\delta(a_{11})f, a_{22}] = 0. \tag{5}$$

On the other hand, since $a_{11}a_{22}a_{21}f\cdots f = 0$ and $p_n(a_{11}, a_{22}, a_{21}, f, \cdots, f) = 0$ for all $a_{21} \in \mathcal{A}_{21}$,we can obtain from Lemmas 2.1 and 3.2 that

$$\begin{aligned}0 =&\delta\left(p_n(a_{11}, a_{22}, a_{21}, f, \cdots, f)\right)\\ =&p_n(\delta(a_{11}), a_{22}, a_{21}, f, \cdots, f)\\ =&p_{n-2}([[\delta(a_{11}), a_{22}], a_{21}], f, \cdots, f)\\ =&e[[\delta(a_{11}), a_{22}], a_{21}]f + (-1)^{n-3}f[[\delta(a_{11}), a_{22}], a_{21}]e\\ =&(-1)^{n-3}[[\delta(a_{11}), a_{22}], a_{21}]\\ =&(-1)^{n-3}[[f\delta(a_{11})f, a_{22}], a_{21}]\\ =&(-1)^{n-3}[f\delta(a_{11})f, a_{22}]a_{21},\end{aligned}$$

which implies that

$$[f\delta(a_{11})f, a_{22}]a_{21} = 0. \tag{6}$$

In view of Condition (4), Equations (5) and (6), we have

$$[f\delta(a_{11})f, a_{22}] = 0,$$

which implies that

$$f\delta(a_{11})f \in \mathcal{Z}(\mathcal{A}_{22}).$$

Thus, by hypothesis of Theorem 3.1, there exists some $z \in \mathcal{Z}(\mathcal{A})$ such that $f\delta(a_{11})f = zf$. Therefore,

$$\begin{aligned}\delta(a_{11}) =&e\delta(a_{11})e + f\delta(a_{11})f\\ =&e\delta(a_{11})e + zf = (e\delta(a_{11})e - ze) + z \in \mathcal{A}_{11} + \mathcal{Z}(\mathcal{A}).\end{aligned}$$

(b) Using similar arguments as (a), we can also show (b) holds. □

Lemma 3.4. *$\delta(\mathcal{A}_{ij}) \subseteq \mathcal{A}_{ij}$ for $1 \leq i \neq j \leq 2$.*

Proof. Since $a_{12}e \cdots e = 0$ for each $a_{12} \in \mathcal{A}_{12}$, it follows from Definition 1.1 and Lemma 3.3 that

$$\begin{aligned}\delta((-1)^{n-1}a_{12}) =& \delta\left(p_n(a_{12}, e, \cdots, e)\right)\\ =& p_n(a_{12}, \delta(e), \cdots, e)\\ =& p_{n-1}([a_{12}, \delta(e)], e, \cdots, e)\\ =& (-1)^{n-2}e[a_{12}, \delta(e)]f + f[a_{12}, \delta(e)]e\\ =& (-1)^{n-2}[a_{12}, \delta(e)] \in \mathcal{A}_{12}.\end{aligned}$$

If n is odd then $\delta((-1)^{n-1}a_{12}) = \delta(a_{12}) = -[a_{12}, \delta(e)] \in \mathcal{A}_{12}$. If n is even, then we can also get $\delta(a_{12}) = [-a_{12}, \delta(e)] \in \mathcal{A}_{12}$ by replacing a_{12} with $-a_{12}$ in the equation above.

Since $a_{21}f \cdots f = 0$ for each $a_{21} \in \mathcal{A}_{21}$, with similar discussions as above, we can show $\delta(a_{21}) \in \mathcal{A}_{21}$. □

Lemma 3.5. *For any $a_{11} \in \mathcal{A}_{11}$, $b_{12} \in \mathcal{A}_{12}$, we have*

$$\delta\left(a_{11} + b_{12}\right) - \delta\left(a_{11}\right) - \delta\left(b_{12}\right) \in \mathcal{Z}(\mathcal{A}).$$

Proof. Let $t = \delta\left(a_{11} + b_{12}\right) - \delta\left(a_{11}\right) - \delta\left(b_{12}\right)$. First, we will prove $t_{12} = t_{21} = 0$.

Noticing that $f(a_{11} + b_{12})e \cdots e = 0$ and $p_n(f, a_{11}, e, \cdots, e) = 0$ for all $a_{11} \in \mathcal{A}_{11}$ and $b_{12} \in \mathcal{A}_{12}$, we can obtain from Definition 1.1, Lemmas 2.1 and 3.2 that

$$\begin{aligned}p_n(f, \delta(a_{11} + b_{12}), e, \cdots, e) =& \delta(p_n(f, a_{11} + b_{12}, e, \cdots, e))\\ =& \delta(p_n(f, a_{11}, e, \cdots, e)) + \delta(p_n(f, b_{12}, e, \cdots, e))\\ =& p_n(f, \delta(a_{11}), e, \cdots, e) + p_n(f, \delta(b_{12}), e, \cdots, e)\\ =& p_n(f, \delta(a_{11}) + \delta(b_{12}), e, \cdots, e),\end{aligned}$$

which implies that

$$p_n(f, t, e, \cdots, e) = (-1)^{n-3}t_{12} + t_{21} = 0,$$

and then

$$t_{12} = t_{21} = 0. \tag{7}$$

In the following, we will prove that $t_{11} + t_{22} \in \mathcal{Z}(\mathcal{A})$.

Since $x_{12}(a_{11} + b_{12})e \cdots e = 0$ and $p_n(x_{12}, b_{12}, e, \cdots, e) = 0$ for all $a_{11} \in \mathcal{A}_{11}$ and $b_{12}, x_{12} \in \mathcal{A}_{12}$, we can conclude from Definition 1.1, Lemmas 2.1 and 3.2 that

$$\begin{aligned}p_n(x_{12}, \delta(a_{11} + b_{12}), e, \cdots, e) =& \delta(p_n(x_{12}, a_{11} + b_{12}, e, \cdots, e))\\ =& \delta(p_n(x_{12}, a_{11}, e, \cdots, e)) + \delta(p_n(x_{12}, b_{12}, e, \cdots, e))\\ =& p_n(x_{12}, \delta(a_{11}), e, \cdots, e) + p_n(x_{12}, \delta(b_{12}), e, \cdots, e)\\ =& p_n(x_{12}, \delta(a_{11}) + \delta(b_{12}), e, \cdots, e),\end{aligned}$$

which implies that

$$t_{11}x_{12} = x_{12}t_{22}. \tag{8}$$

Similarly, since $(a_{11} + b_{12})x_{21}f \cdots f = 0$ and $p_n(b_{12}, x_{21}, f, \cdots, f) = 0$ for all $x_{21} \in \mathcal{A}_{21}$, we can obtain from Lemmas 2.1 and 3.2 that

$$x_{21}t_{11} = t_{22}x_{21}. \tag{9}$$

According to Equations (7), (8) and (9), we get

$$t = \delta(a_{11} + b_{12}) - \delta(a_{11}) - \delta(b_{12}) = t_{11} + t_{22} \in \mathcal{Z}(\mathcal{A}).$$

□

Lemma 3.6. (a) $\delta(a_{12}+b_{12})=\delta(a_{12})+\delta(b_{12})$ *for all* $a_{12},b_{12}\in\mathcal{A}_{12}$*;*

(b) $\delta(a_{21}+b_{21})=\delta(a_{21})+\delta(b_{21})$ *for all* $a_{21},b_{21}\in\mathcal{A}_{21}$.

Proof. (a) Note that $(e+a_{12})(f+b_{12})(-e)\cdots(-e)=0$ and $p_n(e+a_{12},f+b_{12},-e,\cdots,-e)=a_{12}+b_{12}$ for all $a_{12},b_{12}\in\mathcal{A}_{12}$. Then we can obtain from Lemmas 3.2 and 3.5 that

$$\begin{aligned}\delta(a_{12}+b_{12})=&\delta(p_n(e+a_{12},f+b_{12},-e,\cdots,-e))\\=&p_n(\delta(e+a_{12}),f+b_{12},-e,\cdots,-e)\\=&p_n(\delta(e)+\delta(a_{12}),f+b_{12},-e,\cdots,-e)\\=&p_n(\delta(e),f,-e,\cdots,-e)+p_n(\delta(a_{12}),f,-e,\cdots,-e)+p_n(\delta(e),b_{12},-e,\cdots,-e)+p_n(\delta(a_{12}),b_{12},-e,\cdots,-e)\\=&\delta(p_n(e,f,-e,\cdots,-e))+\delta(p_n(a_{12},f,-e,\cdots,-e))+\delta(p_n(e,b_{12},-e,\cdots,-e))+\delta(p_n(a_{12},b_{12},-e,\cdots,-e))\\=&\delta(a_{12})+\delta(b_{12}).\end{aligned}$$

(b) Similarly, using the fact that $(f+a_{21})(b_{21}+e)f\cdots f=0$ and $p_n(f+a_{21},b_{21}+e,f,\cdots,f)=a_{21}+b_{21}$, we have from Lemmas 3.2 and 3.5 that δ is additive on $\mathcal{A}_{21}$. □

Lemma 3.7. (a) $\delta(a_{11}+b_{11})-\delta(a_{11})-\delta(b_{11})\in\mathcal{Z}(\mathcal{A})$ *for all* $a_{11},b_{11}\in\mathcal{A}_{11}$*;*

(b) $\delta(a_{22}+b_{22})-\delta(a_{22})-\delta(b_{22})\in\mathcal{Z}(\mathcal{A})$ *for all* $a_{22},b_{22}\in\mathcal{A}_{22}$.

Proof. (a) Let $t=\delta(a_{11}+b_{11})-\delta(a_{11})-\delta(b_{11})$. For all $a_{11},b_{11}\in\mathcal{A}_{11}$, we can easily check that $(a_{11}+b_{11})f\cdots f=0$ and $\delta(p_n(a_{11},f,\cdots,f))=\delta(p_n(b_{11},f,\cdots,f))=0$. Then we have from Lemmas 2.1 and 3.2 that

$$\begin{aligned}p_n(\delta(a_{11}+b_{11}),f,\cdots,f)=&\delta(p_n(a_{11}+b_{11},f,\cdots,f))\\=&\delta(p_n(a_{11},f,\cdots,f))+\delta(p_n(b_{11},f,\cdots,f))\\=&p_n(\delta(a_{11}),f,\cdots,f)+p_n(\delta(b_{11}),f,\cdots,f)\\=&p_n(\delta(a_{11})+\delta(b_{11}),f,\cdots,f),\end{aligned}$$

which implies that

$$p_n(t,f,\cdots,f)=(-1)^{n-1}t_{12}+t_{21}=0,$$

and then

$$t_{12}=t_{21}=0. \tag{10}$$

Since $(a_{11}+b_{11})x_{21}e\cdots e=0$ for any $x_{21}\in\mathcal{A}_{21}$, by Lemmas 2.1 and 3.6, we get

$$\begin{aligned}p_n(\delta(a_{11}+b_{11}),x_{21},e,\cdots,e)=&\delta(p_n(a_{11}+b_{11},x_{21},e,\cdots,e))\\=&\delta(-x_{21}a_{11}-x_{21}b_{11})\\=&\delta(-x_{21}a_{11})+\delta(-x_{21}b_{11})\\=&\delta(p_n(a_{11},x_{21},e,\cdots,e))+\delta(p_n(b_{11},x_{21},e,\cdots,e))\\=&p_n(\delta(a_{11}),x_{21},e,\cdots,e)+p_n(\delta(b_{11}),x_{21},e,\cdots,e)\\=&p_n(\delta(a_{11})+\delta(b_{11}),x_{21},e,\cdots,e),\end{aligned}$$

which implies that

$$x_{21}t_{11}=t_{22}x_{21}. \tag{11}$$

On the other hand, since $(a_{11}+b_{11})x_{12}e\cdots e=0$ for all $x_{12}\in\mathcal{A}_{12}$, we have from Lemmas 2.1 and 3.6 that

$$t_{11}x_{12}=x_{12}t_{22}. \tag{12}$$

According to Equations (10), (11) and (12), we can obtain

$$t = t_{11} + t_{22} = \delta(a_{11} + b_{11}) - \delta(a_{11}) - \delta(b_{11}) \in \mathcal{Z}(\mathcal{A}).$$

(b) Similar to the proof of (a), we can obtain

$$\delta(a_{22} + b_{22}) - \delta(a_{22}) - \delta(b_{22}) \in \mathcal{Z}(\mathcal{A})$$

for all $a_{22}, b_{22} \in \mathcal{A}_{22}$. □

Lemma 3.8. *For any $a_{11} \in \mathcal{A}_{11}$, $b_{12} \in \mathcal{A}_{12}$, $c_{21} \in \mathcal{A}_{21}$ and $d_{22} \in \mathcal{A}_{22}$, we have*

(a) $\delta(a_{11} + b_{12} + d_{22}) - \delta(a_{11}) - \delta(b_{12}) - \delta(d_{22}) \in \mathcal{Z}(\mathcal{A})$;

(b) $\delta(a_{11} + c_{21} + d_{22}) - \delta(a_{11}) - \delta(c_{21}) - \delta(d_{22}) \in \mathcal{Z}(\mathcal{A})$.

Proof. (a) Let $t = \delta(a_{11} + b_{12} + d_{22}) - \delta(a_{11}) - \delta(b_{12}) - \delta(d_{22})$. In the following, we show that $t \in \mathcal{Z}(\mathcal{A})$. For all $a_{11} \in \mathcal{A}_{11}$, $b_{12} \in \mathcal{A}_{12}$, $d_{22} \in \mathcal{A}_{22}$, since $f(a_{11} + b_{12} + d_{22})e \cdots e = 0$, according to the fact that $p_n(f, a_{11}, e, \cdots, e) = p_n(f, d_{22}, e, \cdots, e) = 0$, we have from Definition 1.1 and Lemma 2.1 that

$$\begin{aligned} p_n(f, \delta(a_{11} + b_{12} + d_{22}), e, \cdots, e) =&\delta(p_n(f, a_{11} + b_{12} + d_{22}, e, \cdots, e)) \\ =&\delta(p_n(f, a_{11}, e, \cdots, e)) + \delta(p_n(f, b_{12}, e, \cdots, e)) + \delta(p_n(f, d_{22}, e, \cdots, e)) \\ =&p_n(f, \delta(a_{11}), e, \cdots, e) + p_n(f, \delta(b_{12}), e, \cdots, e) + p_n(f, \delta(d_{22}), e, \cdots, e) \\ =&p_n(f, \delta(a_{11}) + \delta(b_{12}) + \delta(d_{22}), e, \cdots, e), \end{aligned}$$

which implies that

$$p_n(f, t, e, \cdots, e) = (-1)^{n-3} t_{12} + t_{21} = 0,$$

and then

$$t_{12} = t_{21} = 0. \tag{13}$$

Since $x_{12}(a_{11} + b_{12} + d_{22})e \cdots e = 0$ for all $x_{12} \in \mathcal{A}_{12}$, by Definition 1.1, Lemmas 2.1 and 3.6, we have

$$\begin{aligned} p_n(x_{12}, \delta(a_{11} + b_{12} + d_{22}), e, \cdots, e) =&\delta(p_n(x_{12}, a_{11} + b_{12} + d_{22}, e, \cdots, e)) \\ =&\delta((-1)^{n-2}(x_{12}d_{22} - a_{11}x_{12})) \\ =&\delta((-1)^{n-2}(x_{12}d_{22})) + \delta((-1)^{n-3}(a_{11}x_{12})) \\ =&\delta(p_n(x_{12}, a_{11}, e, \cdots, e)) + \delta(p_n(x_{12}, b_{12}, e, \cdots, e)) + \delta(p_n(x_{12}, d_{22}, e, \cdots, e)) \\ =&p_n(x_{12}, \delta(a_{11}), e, \cdots, e) + p_n(x_{12}, \delta(b_{12}), e, \cdots, e) + p_n(x_{12}, \delta(d_{22}), e, \cdots, e) \\ =&p_n(x_{12}, \delta(a_{11}) + \delta(b_{12}) + \delta(d_{22}), e, \cdots, e), \end{aligned}$$

which implies that

$$t_{11}x_{12} = x_{12}t_{22}. \tag{14}$$

It is straight forward to verify that $(a_{11} + b_{12} + d_{22})x_{21} f \cdots f = 0$ for any $x_{21} \in \mathcal{A}_{21}$. By invoking Definition 1.1, Lemmas 2.1 and 3.6, we have

$$x_{21}t_{11} = t_{22}x_{21}. \tag{15}$$

According to Equations (13), (14) and (15), we can obtain

$$t = \delta(a_{11} + b_{12} + d_{22}) - \delta(a_{11}) - \delta(b_{12}) - \delta(d_{22}) = t_{11} + t_{22} \in \mathcal{Z}(\mathcal{A}).$$

(b) With the similar argument as the above, we have

$$\delta(a_{11} + c_{21} + d_{22}) - \delta(a_{11}) - \delta(c_{21}) - \delta(d_{22}) \in \mathcal{Z}(\mathcal{A}).$$

□

Lemma 3.9. *For any $a_{12} \in \mathcal{A}_{12}$, $b_{21} \in \mathcal{A}_{21}$, we have*

$$\delta(a_{12} + b_{21}) = \delta(a_{12}) + \delta(b_{21}).$$

Proof. Since $(e - a_{12})(e - b_{21})f \cdots f = 0$ and $p_n(e - a_{12}, e - b_{21}, f, \cdots, f) = a_{12} + (-1)^{n-2}b_{21}$ for any $a_{12} \in \mathcal{A}_{12}$, $b_{21} \in \mathcal{A}_{21}$, we can show from Lemmas 3.2 and 3.8 that

$$\begin{aligned}
\delta(a_{12} + (-1)^{n-2}b_{21}) =& \delta(p_n(e - a_{12}, e - b_{21}, f, \cdots, f)) \\
& p_n(\delta(e) + \delta(-a_{12}), e - b_{21}, f, \cdots, f) \\
=& p_n(\delta(e), e, f, \cdots, f) + p_n(\delta(-a_{12}), e, f, \cdots, f) \\
& + p_n(\delta(e), -b_{21}, f, \cdots, f) + p_n(\delta(-a_{12}), -b_{21}, f, \cdots, f) \\
=& \delta(p_n(e, e, f, \cdots, f)) + \delta(p_n(-a_{12}, e, f, \cdots, f)) \\
& + \delta(p_n(e, -b_{21}, f, \cdots, f)) + \delta(p_n(-a_{12}, -b_{21}, f, \cdots, f)) \\
=& \delta(a_{12}) + \delta((-1)^{n-2}b_{21}).
\end{aligned}$$

If n is even, then we have $\delta(a_{12}+b_{21}) = \delta(a_{12})+\delta(b_{21})$. If n is odd, we can still obtain $\delta(a_{12}+b_{21}) = \delta(a_{12})+\delta(b_{21})$ by replacing b_{21} with $-b_{21}$. □

Lemma 3.10. *For any $a_{11} \in \mathcal{A}_{11}$, $b_{12} \in \mathcal{A}_{12}$, $c_{21} \in \mathcal{A}_{21}$, $d_{22} \in \mathcal{A}_{22}$, we have*

$$\delta(a_{11} + b_{12} + c_{21} + d_{22}) - \delta(a_{11}) - \delta(b_{12}) - \delta(c_{21}) - \delta(d_{22}) \in \mathcal{Z}(\mathcal{A}).$$

Proof. Let $t = \delta(a_{11} + b_{12} + c_{21} + d_{22}) - \delta(a_{11}) - \delta(b_{12}) - \delta(c_{21}) - \delta(d_{22})$. In the following, we show that $t \in \mathcal{Z}(\mathcal{A})$. We can easily check that $(a_{11} + b_{12} + c_{21} + d_{22})e \cdots f = 0$ and $p_n(a_{11}, e, \cdots, f) = p_n(d_{22}, e, \cdots, f) = 0$. It follows from Lemmas 2.1, 3.2 and 3.9 that

$$\begin{aligned}
p_n(\delta(a_{11} + b_{12} + c_{21} + d_{22}), e, f, \cdots, f) =& \delta(p_n(a_{11} + b_{12} + c_{21} + d_{22}, e, f, \cdots, f) \\
=& \delta(-b_{12} + (-1)^{n-2}c_{21}) \\
=& \delta(-b_{12}) + \delta((-1)^{n-2}c_{21}) \\
=& \delta(p_n(a_{11}, e, f, \cdots, f)) + \delta(p_n(b_{12}, e, f, \cdots, f)) \\
& + \delta(p_n(c_{21}, e, f, \cdots, f)) + \delta(p_n(d_{22}, e, f, \cdots, f)) \\
=& p_n(\delta(a_{11}), e, f, \cdots, f) + p_n(\delta(b_{12}), e, f, \cdots, f) \\
& + p_n(\delta(c_{21}), e, f, \cdots, f) + p_n(\delta(d_{22}), e, f, \cdots, f) \\
=& p_n(\delta(a_{11} + b_{12} + c_{21} + d_{22}), e, f, \cdots, f),
\end{aligned}$$

which implies that

$$p_n(t, e, f, \cdots, f) = (-1)^{n-2}t_{21} - t_{12} = 0,$$

and then

$$t_{12} = t_{21} = 0. \tag{16}$$

Since $(a_{11} + b_{12} + c_{21} + d_{22})x_{12}e \cdots e = 0$ and $p_n(b_{12}, x_{12}, e, \cdots, e) = 0$, we can obtain from Definition 1.1, Lemmas 2.1, 3.2 and 3.8(b) that

$$\begin{aligned}
p_n(\delta(a_{11} + b_{12} + c_{21} + d_{22}), x_{12}, e, \cdots, e) =& \delta(p_n(a_{11} + b_{12} + c_{21} + d_{22}), x_{12}, e, \cdots, e) \\
=& \delta(p_n(a_{11} + c_{21} + d_{22}, x_{12}, e, \cdots, e)) + \delta(p_n(b_{12}, x_{12}, e, \cdots, e)) \\
=& p_n(\delta(a_{11} + c_{21} + d_{22}), x_{12}, e, \cdots, e) + p_n(\delta(b_{12}), x_{12}, e, \cdots, e) \\
=& p_n(\delta(a_{11} + b_{12} + c_{21} + d_{22}), x_{12}, e, \cdots, e),
\end{aligned}$$

which implies that

$$t_{11}x_{12} = x_{12}t_{22}. \tag{17}$$

On the other side, since $(a_{11}+b_{12}+c_{21}+d_{22})x_{21}f\cdots f=0$ and $p_n(c_{21},x_{21},f,\cdots,f)=0$, we have from Lemmas 2.1, 3.2 and 3.8(a) that

$$x_{21}t_{11}=t_{22}x_{21}. \tag{18}$$

According to the Equations (16), (17) and (18), we have

$$t=\delta\left(a_{11}+b_{12}+c_{21}+d_{22}\right)-\delta\left(a_{11}\right)-\delta\left(b_{12}\right)-\delta\left(c_{21}\right)-\delta\left(d_{22}\right)=t_{11}+t_{22}\in\mathcal{Z}(\mathcal{A}).$$

□

Now we are ready to prove Theorem 3.1 as follows.

For two arbitrary elements $a=a_{11}+a_{12}+a_{21}+a_{22}$ and $b=b_{11}+b_{12}+b_{21}+b_{22}$ in $\mathcal{A}$, by Lemmas 3.6, 3.7 and 3.10, there exist some $z_i\in\mathcal{Z}(\mathcal{A})$ $(i=1,\cdots,5)$ such that

$$\begin{aligned}\delta(a+b)=&\delta((a_{11}+b_{11})+(a_{12}+b_{12})+(a_{21}+b_{21})+(a_{22}+b_{22}))\\=&\delta(a_{11}+b_{11})+\delta(a_{12}+b_{12})+\delta(a_{21}+b_{21})+\delta(a_{22}+b_{22})+z_1\\=&\delta(a_{11})+\delta(b_{11})+z_2+\delta(a_{12})+\delta(b_{12})+\delta(a_{21})+\delta(b_{21})+\delta(a_{22})+\delta(b_{22})+z_3+z_1\\=&\delta(a_{11}+a_{12}+a_{21}+a_{22})+z_4+\delta(b_{11}+b_{12}+b_{21}+b_{22})+z_5+z_3+z_2+z_1\\=&\delta(a)+\delta(b)+z_5+z_4+z_3+z_2+z_1,\end{aligned}$$

which implies that δ is almost additive.

4. The structure

In the present section, we consider the question of characterizing non-global nonlinear Lie n-centralizers on unital algebra with a nontrivial idempotent and obtain the following result.

Theorem 4.1. *Let $\mathcal{A}$ be a unital algebra containing a nontrivial idempotent e satisfying Condition (4). Suppose that*

(a) $\mathcal{Z}(e\mathcal{A}e)=\mathcal{Z}(\mathcal{A})e$ *and* $\mathcal{Z}(f\mathcal{A}f)=\mathcal{Z}(\mathcal{A})f$;

(b) *either $\mathcal{A}_{11}$ or $\mathcal{A}_{22}$ does not contain nonzero central ideals.*

If a map $\delta:\mathcal{A}\to\mathcal{A}$ satisfies

$$\delta\left(p_n\left(a_1,a_2,\cdots,a_n\right)\right)=p_n\left(\delta(a_1),a_2,\cdots,,a_n\right)=p_n\left(a_1,\delta(a_2),\cdots,a_n\right)(n\geq 3)$$

for all $a_1,a_2,\cdots,a_n\in\mathcal{A}$ with $a_1a_2\cdots a_n=0$. Then $\delta(a)=\lambda a+\tau(a)$, where $\lambda\in\mathcal{Z}(\mathcal{A})$ and $\tau:\mathcal{A}\to\mathcal{Z}(\mathcal{A})$ is an almost additive map vanishing at $p_n\left(a_1,a_2,\cdots,a_n\right)$ for all $a_1,a_2,\cdots,a_n\in\mathcal{A}$ with $a_1a_2\cdots a_n=0$.

Before the proof, we first give a remark in the following.

Remark 4.2. *By conditions $\mathcal{Z}(\mathcal{A}_{11})=\mathcal{Z}(\mathcal{A})e$, $\mathcal{Z}(\mathcal{A}_{22})=\mathcal{Z}(\mathcal{A})f$ and Lemma 3.3, we have*

$$\delta(a_{11})=e\delta(a_{11})e-\theta^{-1}(f\delta(a_{11})f)+\theta^{-1}(f\delta(a_{11})f)+f\delta(a_{11})f$$

and

$$\delta(a_{22})=e\delta(a_{22})e+\theta(e\delta(a_{22})e)+f\delta(a_{22})f-\theta(e\delta(a_{22})e).$$

Now define two mappings $\eta_1:\mathcal{A}_{11}\to\mathcal{Z}(\mathcal{A})$ and $\eta_2:\mathcal{A}_{22}\to\mathcal{Z}(\mathcal{A})$ as

$$\eta_1(a_{11})=\theta^{-1}(f\delta(a_{11})f)+f\delta(a_{11})f$$

and

$$\eta_2(a_{22})=e\delta(a_{22})e+\theta(e\delta(a_{22})e).$$

And then we define two mappings $\eta:\mathcal{A}\to\mathcal{Z}(\mathcal{A})$ and $\Delta:\mathcal{A}\to\mathcal{A}$ by

$$\eta(a_{11}+b_{12}+c_{21}+d_{22})=\eta_1(a_{11})+\eta_2(d_{22})$$

and

$$\Delta(a_{11}+b_{12}+c_{21}+d_{22})=\delta(a_{11})+\delta(b_{12})+\delta(c_{21})+\delta(d_{22})-\eta(a_{11}+b_{12}+c_{21}+d_{22}).$$

The proof of Theorem 4.1 can be achieved via the following series of lemmas.

First, by the definition of Δ, Lemmas 3.3, 3.4 and 3.6, we can easily check that Δ satisfies the following properties.

Lemma 4.3. (a) $\Delta(a_{11}) = e\delta(a_{11})e - \theta^{-1}(f\delta(a_{11})f) \in \mathcal{Z}(\mathcal{A}_{11})$;

(b) $\Delta(a_{22}) = f\delta(a_{22})f - \theta(e\delta(a_{22})e) \in \mathcal{Z}(\mathcal{A}_{22})$;

(c) $\Delta(a_{12}) = \delta(a_{12}) \in \mathcal{A}_{12}$;

(d) $\Delta(a_{21}) = \delta(a_{21}) \in \mathcal{A}_{21}$;

(e) $\Delta(a_{12} + b_{12}) = \Delta(a_{12}) + \Delta(b_{12})$;

(f) $\Delta(a_{21} + b_{21}) = \Delta(a_{21}) + \Delta(b_{21})$;

(g) $\Delta(a_{11} + b_{12} + c_{21} + d_{22}) = \Delta(a_{11}) + \Delta(b_{12}) + \Delta(c_{21}) + \Delta(d_{22})$.

Lemma 4.4. Δ *is additive.*

Proof. By Lemma 4.3, we only need to show that $\Delta(a_{ii} + b_{ii}) = \Delta(a_{ii}) + \Delta(b_{ii})$ for $i = 1, 2$. For any $a_{11}, b_{11} \in \mathcal{A}_{11}$, by Theorem 3.1 and Remark 4.2, we get

$$\begin{aligned}\Delta(a_{11} + b_{11}) - \Delta(a_{11}) - \Delta(b_{11}) =&\delta(a_{11} + b_{11}) - \delta(a_{11}) - \delta(b_{11})\\ &- \eta(a_{11} + b_{11}) + \eta(a_{11}) + \eta(b_{11}) \in \mathcal{Z}(\mathcal{A}).\end{aligned}$$

On the other side, we have from Lemma 4.3 that

$$\Delta(a_{11} + b_{11}) - \Delta(a_{11}) - \Delta(b_{11}) \in \mathcal{A}_{11}.$$

Since $\mathcal{Z}(\mathcal{A}) \cap \mathcal{A}_{11} = 0$, we have

$$\Delta(a_{11} + b_{11}) = \Delta(a_{11}) + \Delta(b_{11}).$$

Similarly, we can prove that $\Delta(a_{22} + b_{22}) = \Delta(a_{22}) + \Delta(b_{22})$.

Finally, we prove that Δ is additive. Let $a = a_{11} + a_{12} + a_{21} + a_{22}$, $b = b_{11} + b_{12} + b_{21} + b_{22}$ be arbitrary elements in $\mathcal{A}$. Then combining Lemma 4.3, we can obtain

$$\begin{aligned}\Delta(a + b) =&\Delta((a_{11} + b_{11}) + (a_{12} + b_{12}) + (a_{21} + b_{21}) + (a_{22} + b_{22}))\\ =&\Delta(a_{11} + b_{11}) + \Delta(a_{12} + b_{12}) + \Delta(a_{21} + b_{21}) + \Delta(a_{22} + b_{22})\\ =&\Delta(a_{11}) + \Delta(b_{11}) + \Delta(a_{12}) + \Delta(b_{12}) + \Delta(a_{21}) + \Delta(b_{21}) + \Delta(a_{22}) + \Delta(b_{22})\\ =&\Delta(a_{11} + a_{12} + a_{21} + a_{22}) + \Delta(b_{11} + b_{12} + b_{21} + b_{22})\\ =&\Delta(a) + \Delta(b).\end{aligned}$$

This completes the proof. □

Lemma 4.5. (a) $\Delta(a_{11}b_{12}) = \Delta(a_{11})b_{12} = a_{11}\Delta(b_{12})$;

(b) $\Delta(a_{12}b_{22}) = \Delta(a_{12})b_{22} = a_{12}\Delta(b_{22})$;

(c) $\Delta(a_{21}b_{11}) = \Delta(a_{21})b_{11} = a_{21}\Delta(b_{11})$;

(d) $\Delta(a_{22}b_{21}) = \Delta(a_{22})b_{21} = a_{22}\Delta(b_{21})$.

Proof. Here we only consider case (a). The proofs of the rest are similar.

Since $a_{11}b_{12}(-e)\cdots(-e)=0$, by Lemmas 2.1, 3.3 and 4.3(a), we have

$$\begin{aligned}\Delta(a_{11}b_{12})=\delta(a_{11}b_{12}) =&\delta(p_n(a_{11},b_{12},-e,\cdots,-e))\\ =&p_n(\delta(a_{11}),b_{12},-e,\cdots,-e)\\ =&p_{n-1}([\delta(a_{11}),b_{12}],-e,\cdots,-e)\\ =&e[\delta(a_{11}),b_{12}]f+(-1)^n f[\delta(a_{11}),b_{12}]e\\ =&[\delta(a_{11}),b_{12}]\\ =&[e\delta(a_{11})e-\theta^{-1}(f\delta(a_{11})f)+\eta_1(a_{11}),b_{12}]\\ =&[e\delta(a_{11})e-\theta^{-1}(f\delta(a_{11})f),b_{12}]\\ =&[\Delta(a_{11}),b_{12}]\\ =&\Delta(a_{11})b_{12}.\end{aligned}$$

On the other side, by Definition 1.1 and Lemmas 2.1, 3.3 and 4.3(c), we have

$$\begin{aligned}\Delta(a_{11}b_{12})=\delta(a_{11}b_{12}) =&\delta(p_n(a_{11},b_{12},-e,\cdots,-e))\\ =&p_n(a_{11},\delta(b_{12}),-e,\cdots,-e)\\ =&p_{n-1}([a_{11},\delta(b_{12})],-e,\cdots,-e)\\ =&e[a_{11},\delta(b_{12})]f+(-1)^n f[a_{11},\delta(b_{12})]e\\ =&[a_{11},\delta(b_{12})]\\ =&[a_{11},\Delta(b_{12})]\\ =&a_{11}\Delta(b_{12}).\end{aligned}$$

Hence $\Delta(a_{11}b_{12})=\Delta(a_{11})b_{12}=a_{11}\Delta(b_{12})$. □

Lemma 4.6. (a) $\Delta(a_{11}b_{11})=\Delta(a_{11})b_{11}=a_{11}\Delta(b_{11})$;

(b) $\Delta(a_{22}b_{22})=\Delta(a_{22})b_{22}=a_{22}\Delta(b_{22})$.

Proof. (a) For any $x_{12}\in\mathcal{A}_{12}$, by Lemma 4.5(a), we have

$$\Delta(a_{11}b_{11}x_{12})=\Delta(a_{11}b_{11})x_{12};$$

$$\Delta(a_{11}b_{11}x_{12})=\Delta(a_{11})b_{11}x_{12};$$

$$\Delta(a_{11}b_{11}x_{12})=a_{11}\Delta(b_{11}x_{12})=a_{11}\Delta(b_{11})x_{12}.$$

Comparing the above Equations, we conclude

$$\Delta(a_{11}b_{11})x_{12}=\Delta(a_{11})b_{11}x_{12}=a_{11}\Delta(b_{11})x_{12} \tag{19}$$

for all $x_{12}\in\mathcal{A}_{12}$. Now, for any $a_{21}\in\mathcal{A}_{21}$, we have from Lemma 4.5(c) that

$$\Delta(x_{21}a_{11}b_{11})=x_{21}\Delta(a_{11}b_{11});$$

$$\Delta(x_{21}a_{11}b_{11})=\Delta(x_{21}a_{11})b_{11}=x_{21}\Delta(a_{11})b_{11};$$

$$\Delta(x_{21}a_{11}b_{11})=x_{21}a_{11}\Delta(b_{11}).$$

Comparing the above equations, we have

$$x_{21}\Delta(a_{11}b_{11})=x_{21}\Delta(a_{11})b_{11}=x_{21}a_{11}\Delta(b_{11}) \tag{20}$$

for all $x_{21} \in \mathcal{A}_{21}$. Consequently, combining Equations (19) and (20), we have from Condition (4) that

$$\Delta(a_{11}b_{11}) = \Delta(a_{11})b_{11} = a_{11}\Delta(b_{11}).$$

(b) Similarly, using parts (b) and (d) of Lemma 4.5, we can prove that

$$\Delta(a_{22}b_{22}) = \Delta(a_{22})b_{22} = a_{22}\Delta(b_{22}).$$

□

Lemma 4.7. (a) $\Delta(a_{12}b_{21}) = \Delta(a_{12})b_{21} = a_{12}\Delta(b_{21})$;

(b) $\Delta(b_{21}a_{12}) = \Delta(b_{21})a_{12} = b_{21}\Delta(a_{12})$.

Proof. It should be noted that we will simultaneously prove (a) and (b) by two cases.

Case 1 $n = 3$.

Since $a_{12}b_{21}x_{21} = 0$, by Lemmas 2.1 and 4.3(c), we have

$$\begin{aligned}\Delta([[a_{12}, b_{21}], x_{21}]) =&\delta([[a_{12}, b_{21}], x_{21}])\\ =&[[\delta(a_{12}), b_{21}], x_{21}]\\ =&[[\Delta(a_{12}), b_{21}], x_{21}]\\ =&[\Delta(a_{12})b_{21} - b_{21}\Delta(a_{12}), x_{21}];\end{aligned}$$

on the other side, we have from Lemmas 4.3, 4.4 and 4.5 that

$$\begin{aligned}\Delta([[a_{12}, b_{21}], x_{21}]) =&\Delta(-b_{21}a_{12}x_{21} - x_{21}a_{12}b_{21})\\ =&-\Delta(b_{21}a_{12}x_{21}) - \Delta(x_{21}a_{12}b_{21})\\ =&-\Delta(b_{21}a_{12})x_{21} - x_{21}\Delta(a_{12}b_{21})\\ =&[\Delta(a_{12}b_{21}), x_{21}] - [\Delta(b_{21}a_{12}), x_{21}]\\ =&[\Delta(a_{12}b_{21}) - \Delta(b_{21}a_{12}), x_{21}].\end{aligned}$$

Combining the above two equations, we have

$$[\Delta(a_{12}b_{21}) - \Delta(b_{21}a_{12}) - \Delta(a_{12})b_{21} + b_{21}\Delta(a_{12}), x_{21}] = 0. \tag{21}$$

Since $b_{21}a_{12}x_{12} = 0$, by Definition 1.1, Lemmas 2.1 and 4.3(c), we have

$$\begin{aligned}\Delta([[b_{21}, a_{12}], x_{12}]) =&\delta([[b_{12}, a_{12}], x_{12}])\\ =&[[b_{21}, \delta(a_{12})], x_{12}]\\ =&[[b_{21}, \Delta(a_{12})], x_{12}]\\ =&[b_{21}\Delta(a_{12}) - \Delta(a_{12})b_{21}), x_{12}];\end{aligned}$$

on the other side, we have from Lemmas 4.3, 4.4 and 4.5 that

$$\begin{aligned}\Delta([[b_{21}, a_{12}], x_{12}]) =&\Delta(-a_{12}b_{21}x_{12} - x_{12}b_{21}a_{12})\\ =&-\Delta(a_{12}b_{21}x_{12}) - \Delta(x_{12}b_{21}a_{12})\\ =&-\Delta(a_{12}b_{21})x_{12} - x_{12}\Delta(b_{21}a_{12})\\ =&-[\Delta(a_{12}b_{21}), x_{12}] + [\Delta(b_{21}a_{12}), x_{12}]\\ =&[\Delta(b_{21}a_{12}) - \Delta(a_{12}b_{21}), x_{12}].\end{aligned}$$

Combining the above two equations, we have

$$[\Delta(a_{12}b_{21}) - \Delta(b_{21}a_{12}) - \Delta(a_{12})b_{21} + b_{21}\Delta(a_{12}), x_{12}] = 0. \tag{22}$$

According to the Equations (21) and (22), we can get

$$(\Delta(a_{12}b_{21}) - \Delta(a_{12})b_{21}) + (b_{21}\Delta(a_{12}) - \Delta(b_{21}a_{12})) \in \mathcal{Z}(\mathcal{A}). \tag{23}$$

Combining with condition (1) of Theorem 4.1, we have

$$\Delta(a_{12}b_{21}) - \Delta(a_{12})b_{21} \in \mathcal{Z}(\mathcal{A}_{11}) \text{ and } b_{21}\Delta(a_{12}) - \Delta(b_{21}a_{12}) \in \mathcal{Z}(\mathcal{A}_{22})$$

for all $a_{12} \in \mathcal{A}_{12}$ and $b_{21} \in \mathcal{A}_{21}$. Without loss of generality, we assume that $\mathcal{A}_{11}$ does not contain nonzero central ideals. It is easy to see that $\mathcal{A}_{11}(\Delta(a_{12}b_{21}) - \Delta(a_{12})b_{21})$ is a central ideal of $\mathcal{A}_{11}$. Therefore $\Delta(a_{12}b_{21}) - \Delta(a_{12})b_{21} = 0$ and then

$$\Delta(a_{12}b_{21}) = \Delta(a_{12})b_{21}.$$

Combining Equation (23) and Lemma 4.3, we have $b_{21}\Delta(a_{12})-\Delta(b_{21}a_{12}) \in \mathcal{Z}(\mathcal{A})\cap\mathcal{A}_{22}$. Since $\mathcal{Z}(\mathcal{A})\cap\mathcal{A}_{22} = 0$, it follows that

$$\Delta(b_{21}a_{12}) = b_{21}\Delta(a_{12}).$$

Similarly, we can also show that

$$\Delta(a_{12}b_{21}) = a_{12}\Delta(b_{21}) \text{ and } \Delta(b_{21}a_{12}) = \Delta(b_{21})a_{12}.$$

Case 2 $n > 3$.

Since $a_{12}b_{21}x_{21}e\cdots e = 0$, by Lemmas 2.1 and 4.3(c), we have

$$\begin{aligned}
\Delta(p_n(a_{12}, b_{21}, x_{21}, e, \cdots, e)) &= \delta(p_n(a_{12}, b_{21}, x_{21}, e, \cdots, e)) \\
&= p_n(\delta(a_{12}), b_{21}, x_{21}, e, \cdots, e)) \\
&= p_{n-2}([[\delta(a_{12}), b_{21}], x_{21}], e, \cdots, e)) \\
&= (-1)^{n-3}e[[\delta(a_{12}), b_{21}], x_{21}]f + f[[\delta(a_{12}), b_{21}], x_{21}]e \\
&= [[\delta(a_{12}), b_{21}], x_{21}] \\
&= [[\Delta(a_{12}), b_{21}], x_{21}] \\
&= [\Delta(a_{12})b_{21} - b_{21}\Delta(a_{12}), x_{21}];
\end{aligned}$$

on the other side, we have from Lemmas 2.1, 3.6 and 4.4 that

$$\begin{aligned}
\Delta(p_n(a_{12}, b_{21}, x_{21}, e, \cdots, e)) &= \Delta([[a_{12}, b_{21}], x_{21}]) \\
&= \Delta([a_{12}b_{21} - b_{21}a_{12}, x_{21}]) \\
&= \Delta([a_{12}b_{21}, x_{21}] - [b_{21}a_{12}, x_{21}]) \\
&= \Delta([a_{12}b_{21}, x_{21}]) - \Delta([b_{21}a_{12}, x_{21}]) \\
&= \Delta(p_n(a_{12}b_{21}, x_{21}, e, \cdots, e)) - \Delta(p_n(b_{21}a_{12}, x_{21}, e, \cdots, e)) \\
&= \delta(p_n(a_{12}b_{21}, x_{21}, e, \cdots, e)) - \delta(p_n(b_{21}a_{12}, x_{21}, e, \cdots, e)) \\
&= p_n(\delta(a_{12}b_{21}), x_{21}, e, \cdots, e) - p_n(\delta(b_{21}a_{12}), x_{21}, e, \cdots, e) \\
&= p_{n-1}([\delta(a_{12}b_{21}), x_{21}], e, \cdots, e) - p_{n-1}([\delta(b_{21}a_{12}), x_{21}], e, \cdots, e) \\
&= [\delta(a_{12}b_{21}), x_{21}] - [\delta(b_{21}a_{12}), x_{21}] \\
&= [\Delta(a_{12}b_{21}), x_{21}] - [\Delta(b_{21}a_{12}), x_{21}] \\
&= [\Delta(a_{12}b_{21}) - \Delta(b_{21}a_{12}), x_{21}].
\end{aligned}$$

Combining the above two equations, we have

$$[\Delta(a_{12}b_{21}) - \Delta(b_{21}a_{12}) - \Delta(a_{12})b_{21} + b_{21}\Delta(a_{12}), x_{21}] = 0. \tag{24}$$

Since $a_{12}b_{21}x_{12}e\cdots e=0$, by Lemmas 2.1 and 4.3(c), we have

$$\begin{aligned}\Delta(p_n(a_{12},b_{21},x_{12},e,\cdots,e)) =&\delta(p_n(a_{12},b_{21},x_{12},e,\cdots,e))\\ =&p_n(\delta(a_{12}),b_{21},x_{12},e,\cdots,e))\\ =&p_{n-2}([[\delta(a_{12}),b_{21}],x_{12}],e,\cdots,e))\\ =&(-1)^{n-3}e[[\delta(a_{12}),b_{21}],x_{12}]f+f[[\delta(a_{12}),b_{21}],x_{12}]e\\ =&(-1)^{n-3}[[\delta(a_{12}),b_{21}],x_{12}]\\ =&(-1)^{n-3}[[\Delta(a_{12}),b_{21}],x_{12}]\\ =&(-1)^{n-3}[\Delta(a_{12})b_{21}-b_{21}\Delta(a_{12}),x_{12}];\end{aligned}$$

on the other side, we have from Lemmas 2.1, 3.6 and 4.4 that

$$\begin{aligned}\Delta(p_n(a_{12},b_{21},x_{12},e,\cdots,e)) =&\Delta((-1)^{n-3}[[a_{12},b_{21}],x_{12}])\\ =&\Delta((-1)^{n-3}[a_{12}b_{21}-b_{21}a_{12},x_{12}])\\ =&\Delta((-1)^{n-3}[a_{12}b_{21},x_{12}]-(-1)^{n-3}[b_{21}a_{12},x_{12}])\\ =&\Delta((-1)^{n-3}[a_{12}b_{21},x_{12}])-\Delta((-1)^{n-3}[b_{21}a_{12},x_{12}])\\ =&\Delta(p_n(a_{12}b_{21},x_{12},e,\cdots,e))-\Delta(p_n(b_{21}a_{12},x_{12},e,\cdots,e))\\ =&\delta(p_n(a_{12}b_{21},x_{12},e,\cdots,e))-\delta(p_n(b_{21}a_{12},x_{12},e,\cdots,e))\\ =&p_n(\delta(a_{12}b_{21}),x_{12},e,\cdots,e)-p_n(\delta(b_{21}a_{12}),x_{12},e,\cdots,e)\\ =&p_{n-1}([\delta(a_{12}b_{21}),x_{12}],e,\cdots,e)-p_{n-1}([\delta(b_{21}a_{12}),x_{12}],e,\cdots,e)\\ =&(-1)^{n-2}[\delta(a_{12}b_{21}),x_{12}]-(-1)^{n-2}[\delta(b_{21}a_{12}),x_{12}]\\ =&(-1)^{n-2}[\Delta(a_{12}b_{21}),x_{12}]-(-1)^{n-2}[\Delta(b_{21}a_{12}),x_{12}]\\ =&(-1)^{n-2}[\Delta(a_{12}b_{21})-\Delta(b_{21}a_{12}),x_{12}].\end{aligned}$$

Combining the above two equations, we have

$$(-1)^{n-2}[\Delta(a_{12}b_{21})-\Delta(b_{21}a_{12})-\Delta(a_{12})b_{21}+b_{21}\Delta(a_{12}),x_{12}]=0. \tag{25}$$

According to the Equations (24) and (25), we can conclude

$$(\Delta(a_{12}b_{21})-\Delta(a_{12})b_{21})+(b_{21}\Delta(a_{12})-\Delta(b_{21}a_{12}))\in\mathcal{Z}(\mathcal{A}).$$

Up to now, using the same argument as the case of $n=3$, we can also obtain the required assertions. □

Lemma 4.8. $\Delta(ab)=\Delta(a)b=a\Delta(b)$ *and* $\Delta(a)=\lambda a$ *for some* $\lambda\in\mathcal{Z}(\mathcal{A})$.

Proof. By Lemmas 4.4-4.7, we can easily check that $\Delta(ab)=\Delta(a)b=a\Delta(b)$ for all $a,b\in\mathcal{A}$. Then, for any $a\in\mathcal{A}$, we have

$$\Delta(a)=\Delta(1)a=a\Delta(1),$$

which implies that $\Delta(1)\in\mathcal{Z}(\mathcal{A})$. Denote $\lambda=\Delta(1)$ and then $\Delta(a)=\lambda a$. □

Proof of Theorem 4.1.

Proof. Define an additive mapping $\tau:\mathcal{A}\to\mathcal{Z}(\mathcal{A})$ by $\tau(a)=\delta(a)-\Delta(a)$ for all $a\in\mathcal{A}$. Then, by the definition of Δ and Theorem 3.1, it follows that $\tau(\mathcal{A})\subseteq\mathcal{Z}(\mathcal{A})$. By Lemma 4.8, we only need to show that $\tau(p_n(a_1,\cdots,a_n))=0$ with $a_1a_2\cdots a_n=0$. In fact, for all $a_1,a_2,\cdots,a_n\in\mathcal{A}$ with $a_1a_2\cdots a_n=0$, we have from Lemma 4.8 that

$$\begin{aligned}\tau(p_n(a_1,a_2,\cdots,a_n)) =&\delta(p_n(a_1,a_2,\cdots,a_n))-\Delta(p_n(a_1,a_2,\cdots,a_n))\\ =&p_n(\delta(a_1),a_2,\cdots,a_n)-p_n(\Delta(a_1),a_2,\cdots,a_n)\\ =&p_n(\Delta(a_1)+\tau(a_1),a_2,\cdots,a_n)-p_n(\Delta(a_1),a_2,\cdots,a_n)\\ =&p_n(\Delta(a_1),a_2,\cdots,a_n)-p_n(\Delta(a_1),a_2,\cdots,a_n)=0.\end{aligned}$$

□

Finally, we give two applications of Theorems 3.1 and 4.1 in the following.

Let $\mathcal{T}$ be a unital algebra with a nontrivial idempotent e and denote $f = 1 - e$, where 1 is the unit of $\mathcal{T}$. Suppose that $e\mathcal{T}f$ is a faithful $(e\mathcal{T}e, f\mathcal{T}f)$-bimodule, which is faithful as a left $e\mathcal{T}e$-module and also as a right $f\mathcal{T}f$-module, and $f\mathcal{T}e = \{0\}$. Then $\mathcal{T} = e\mathcal{T}e + e\mathcal{T}f + f\mathcal{T}f$ is a triangular algebra. Since $f\mathcal{T}e = \{0\}$, Lemma 4.7 holds trivially in case of triangular algebra. Thus we can omit the assumption (b) of Theorem 4.1. As a consequence of Theorems 3.1 and 4.1, we can obtain the following result:

Corollary 4.9. *Let $\mathcal{T} = e\mathcal{T}e+e\mathcal{T}f+f\mathcal{T}f$ be a triangular algebra as mentioned above. Assume that $\mathcal{Z}(e\mathcal{T}e) = \mathcal{Z}(\mathcal{T})e$ and $\mathcal{Z}(f\mathcal{T}f) = \mathcal{Z}(\mathcal{T})f$. If a map $\delta : \mathcal{T} \to \mathcal{T}$ satisfies*

$$\delta\left(p_n\left(x_1, x_2, \cdots, x_n\right)\right) = p_n\left(\delta(x_1), x_2, \cdots, , x_n\right)$$

for all $x_1, x_2, \cdots, x_n \in \mathcal{T}$ with $x_1x_2\cdots x_n = 0$. Then $\delta(x) = \lambda x + \tau(x)$ for all $x \in \mathcal{T}$, where $\lambda \in \mathcal{Z}(\mathcal{T})$ and $\tau : \mathcal{T} \to \mathcal{Z}(\mathcal{T})$ is an almost additive map vanishing at $p_n\left(x_1, x_2, \cdots, x_n\right)$ for all $x_1, x_2, \cdots, x_n \in \mathcal{T}$ with $x_1x_2\cdots x_n = 0$.

Assume that $\mathcal{M}$ is a von Neumann algebra without central summands of type I_1. Then by [15] there exists a nonzero core-free projection $P \in \mathcal{M}$ with $\overline{P} = I$. Fix such P and note that $\overline{P} = \overline{I-P} = I$. It follows from the definition of the central carrier that both span$\{TP(x) : T \in \mathcal{M}, x \in H\}$ and span$\{T(I-P)(x) : T \in \mathcal{M}, x \in H\}$ are dense in H. So $A\mathcal{M}P = \{0\} \Rightarrow A = 0$ and $A\mathcal{M}(I-P) = \{0\} \Rightarrow A = 0$. Brešar and Miers [8] proved that if $Z \in \mathcal{Z}(\mathcal{M})$ such that $Z\mathcal{M} \subseteq \mathcal{Z}(\mathcal{M})$, then $Z = 0$. This implies that $\mathcal{M}$ has nononzero central ideals. Note that $P\mathcal{M}P$ and $(I-P)\mathcal{M}(I-P)$ are also von Neumann algebras without central summands of type I_1. So both $P\mathcal{M}P$ and $(I-P)\mathcal{M}(I-P)$ have no nonzero central ideals. Moreover, $\mathcal{Z}(P\mathcal{M}P) = P\mathcal{Z}(\mathcal{M})$ and $\mathcal{Z}((I-P)\mathcal{M}(I-P)) = (I-P)\mathcal{Z}(\mathcal{M})$. Therefore, as a consequence of Theorems 3.1 and 4.1, we can obtain the following result:

Corollary 4.10. *Let $\mathcal{M}$ be a von Neumann algebra without central summands of type I_1. If a map $\delta : \mathcal{M} \to \mathcal{M}$ satisfies*

$$\delta\left(p_n\left(x_1, x_2, \cdots, x_n\right)\right) = p_n\left(\delta(x_1), x_2, \cdots, , x_n\right)$$

for all $x_1, x_2, \cdots, x_n \in \mathcal{M}$ with $x_1x_2\cdots x_n = 0$. Then $\delta(x) = \lambda x + \tau(x)$ for all $x \in \mathcal{M}$, where $\lambda \in \mathcal{Z}(\mathcal{M})$ and $\tau : \mathcal{M} \to \mathcal{Z}(\mathcal{M})$ is an almost additive map vanishing at $p_n\left(x_1, x_2, \cdots, x_n\right)$ for all $x_1, x_2, \cdots, x_n \in \mathcal{M}$ with $x_1x_2\cdots x_n = 0$.

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Declarations

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