Filomat 39:17 (2025), 5915–5928
https://doi.org/10.2298/FIL2517915D

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Some new characterizations of normal matrices

Zeyuan Du[a,*], Junchao Wei[a]

[a] *School of Mathematical Sciences, Yangzhou University, Yangzhou, Jiangsu 225002, P.R.China*

Abstract. In this paper, many interesting properties of normal matrices are given by means of such concepts as the power equivalities, projections, the regularity of vectors, one sided A-equality, A-commutativity and so on.

1. Introduction

Throughout this paper, $C^{n\times n}$ stands for the set of all $n \times n$ complex matrices. A^H denotes the conjugate transpose matrix of $A \in C^{n\times n}$. Let $A \in C^{n\times n}$. Then $B \in C^{n\times n}$ is said to be the Moore-Penrose inverse matrix of A if

$$A = ABA, \qquad B = BAB, \qquad (AB)^H = AB, \qquad (BA)^H = BA.$$

The matrix B always exists by [8, 10] and is uniquely determined by the above equations. We always denote it by $A^\dagger$.

$A \in C^{n\times n}$ is said to be group invertible if there exists $B \in C^{n\times n}$ such that

$$A = ABA, \qquad B = BAB, \qquad AB = BA.$$

The matrix B is called the group inverse matrix of A, which is uniquely by above equations [11], and we denote it by $A^\#$.

Let $A \in C^{n\times n}$ be a group invertible matrix. Then A is called an EP matrix if $A^\# = A^\dagger$. It is known that A is EP if and only if $AA^\dagger = A^\dagger A$. For the study of EP matrices, we can also refer to [1, 5, 7, 10]. A is called a SEP matrix [4, 5] if $A^\# = A^\dagger = A^H$. And A is called a normal matrix if $A^H A = AA^H$. In [5], some properties of normal matrices and the conditions for the establishment of SEP matrices are introduced. The rest study of normal matrix can be found in [10]. The rest study of normal elements over a ring can be found in [6, 9, 12].

In this paper, we continue to study normal matrices. In Section 2, we use power equivalities to characterize normal matrix. In Section 3, with the help of projections, we discuss some new characterizations of normal matrices. In Section 4, we use the regularity of vector to describe normal matrices. In Section 5, we use one sided $A-$equality to characterize the normal matrix. In Section 6, by researching $A-$commutativity matrices, we obtain some interesting results about normal matrix.

2020 *Mathematics Subject Classification*. Primary 16B99; Secondary 16W10, 46L05.
Keywords. *EP* matrix; normal matrix; *SEP* matrix; group invertible matrix; the power equivality; projection; the regularity of vector; one sided A-equality; A-commutativity.
Received: 10 October 2024; Revised: 24 March 2025; Accepted: 03 April 2025
Communicated by Dijana Mosić
Research supported by PPZY2015B109(XCX20240265).
* Corresponding author: Zeyuan Du
Email addresses: 18705271773@163.com (Zeyuan Du), jcweiyz@126.com (Junchao Wei)

2. Using power equivalities to characterize normal matrices

In [3, Theorem 2.1], it is shown that for a group invertible matrix A, A is normal if and only if $AA^H(A^\#)^H = A^HA(A^\dagger)^H$. This inspires us to give the following theorems.

Theorem 2.1. *Let $A \in C^{n\times n}$ be a group invertible matrix. Then A is a normal matrix if and only if $(AA^H(A^\#)^H)^k = (A^HA(A^\dagger)^H)^k$ for $k = 2, 3$.*

Proof. " $\Longrightarrow$ " *If A is a normal matrix. Then, by [3, Theorem 2.1], we have $AA^H(A^\#)^H = A^HA(A^\dagger)^H$. It follows that $(AA^H(A^\#)^H)^k = (A^HA(A^\dagger)^H)^k$ for $k = 2, 3$.*

" $\Longleftarrow$ " *From the hypothesis, one has*

$$(AA^H(A^\#)^H)^2 = (A^HA(A^\dagger)^H)^2.$$

Multiplying the equality on the left by $A^\dagger A$, one gets

$$A^\dagger A^2A^H(A^\#)^HAA^H(A^\#)^H = AA^H(A^\#)^HAA^H(A^\#)^H.$$

Multiplying the last equality on the right by $A^\dagger A^\dagger A$, one obtains

$$A^\dagger A^2 = A.$$

Hence, A is EP. Now, for $k = 2, 3$, one yields

$$A^k = (AA^H(A^\#)^H)^k = (A^HA(A^\dagger)^H)^k = A^HA^k(A^\dagger)^H.$$

This gives

$$A^HA^3(A^+)^H = A^3 = A^2A = A^HA^2(A^+)^HA.$$

Multiplying the equality on the left by $A^\#A^\dagger(A^\#)^H$, we have

$$A(A^\dagger)^H = (A^\dagger)^HA.$$

It follows that

$$A^\#A^H = A^\dagger A^H = (A(A^\dagger)^H)^H = ((A^\dagger)^HA)^H = A^HA^\dagger = A^HA^\#.$$

Hence, A is a normal matrix by [5, Theorem 1.3.2]. □

Let $A, B \in C^{n\times n}$. Then A is called $(3, B)$−regular, if $A^3 = ABA$. It is evident that for any $A \in C^{n\times n}$, A is $(3, A)$−regular. Also, for any $A \in C^{n\times n}$, we have

(1) $A^2 = E_n$ if and only if A is invertible and $(3, A^{-1})$−regular;

(2) $A^2 = A$ if and only if A is group invertible and $(3, E_n)$−regular;

(3) $A^H = A$ if and only if A is EP and $(3, A^H)$−regular.

Where E_n is the n-order identity matrix.

Theorem 2.2. *Let $A \in C^{n\times n}$ be a group invertible matrix. Then A is a normal matrix if and only if A is EP and $AA^H(A^\#)^H$ is $(3, A^HA(A^\dagger)^H)$−regular.*

Proof. " $\Longrightarrow$ " *It is an immediate result of [3, Theorem 2.1].*

" $\Longleftarrow$ " *From the assumption, we have*

$$(AA^H(A^\#)^H)^3 = (AA^H(A^\#)^H)(A^HA(A^\dagger)^H)(AA^H(A^\#)^H).$$

Since A is EP, one gets

$$A^3 = AA^HA(A^\dagger)^HA,$$

This gives

$$A = A^\#A^3A^\# = A^\#AA^HA(A^\dagger)^HAA^\# = A^HA(A^\dagger)^H.$$

Hence, $AA^H(A^\#)^H = A = A^HA(A^\dagger)^H$. By [3, Theorem 2.1], A is normal. □

Let $A,\ B \in C^{n\times n}$. Then A is called left (right) $B-$idempotent, if $A^2 = BA$ ($A^2 = AB$). Clearly, for any $A \in C^{n\times n}$, A is left and right $A-$idempotents. Also, we have

(1) A is an idempotent matrix if and only if A is right or left E_n-idempotent;
(2) if A is left or right $B-$idempotent, then A is $(3,\ B)-$regular;
(3) If A is group invertible, then A is partial isometry if and only if A is left or right $(A^+)^H-$idempotent;
(4) $A^2 = E_n$ if and only if A is invertible and A is left or right $A^{-1}-$idempotent;
(5) A is right $A + E_n - A^+A + U - A^+AU-$idempotent for all $U \in C^{n\times n}$
(6) $A = A^H$ if and only if A is right $A^H + E_n - A^+A + U - A^+AU-$idempotent for all $U \in C^{n\times n}$.

Theorem 2.3. *Let $A \in C^{n\times n}$ be a group invertible matrix. Then A is a normal matrix if and only if $AA^H(A^\#)^H$ is left $A^HA(A^\dagger)^H-$idempotent.*

Proof. *" $\Longrightarrow$ " It is clear by [3, Theorem 2.1].*
" $\Longleftarrow$ "If $AA^H(A^\#)^H$ is left $A^HA(A^\dagger)^H-$idempotent, then

$$(AA^H(A^\#)^H)^2 = (A^HA(A^\dagger)^H)(AA^H(A^\#)^H).$$

Multiplying the equality on the right by $A^\dagger A$, one gets

$$AA^H(A^\#)^HA = A^HA(A^\dagger)^HA.$$

Multiplying the last equality on the left by $A^\dagger A$, one has

$$A^\dagger A^2A^H(A^\#)^HA = AA^H(A^\#)^HA.$$

Noting that

$$AA^\dagger = AA^H(A^\#)^HA^\dagger = A^\dagger A^2A^H(A^\#)^HA^\dagger = A^\dagger A^2A^\dagger.$$

Then A is EP. Since

$$(AA^H(A^\#)^H)^3 = AA^H(A^\#)^H(AA^H(A^\#)^H)^2 = AA^H(A^\#)^H(A^HA(A^\dagger)^H)AA^H(A^\#)^H,$$

$AA^H(A^\#)^H$ is $(3,\ A^HA(A^\dagger)^H)$)-regular. Hence, A is a normal matrix by Theorem 2.2. □

Theorem 2.4. *Let $A \in C^{n\times n}$ be a group invertible matrix. Then A is a normal matrix if and only if $AA^H(A^\#)^H$ is right $A^HA(A^\dagger)^H-$idempotent.*

Proof. *" $\Longrightarrow$ " It is evident by [3, Theorem 2.1].*
" $\Longleftarrow$ " The condition "$AA^H(A^\#)^H$ is right $A^HA(A^\dagger)^H-$idempotent" gives

$$AA^H(A^\#)^HAA^H(A^\#)^H = AA^H(A^\#)^HA^HA(A^\dagger)^H = AA^HA(A^\dagger)^H,$$

and

$$A^HA(A^\dagger)^H = A^\dagger AA^HA(A^\dagger)^H = A^\dagger AA^H(A^\#)^HAA^H(A^\#)^H = A^H(A^\#)^HAA^H(A^\#)^H.$$

Noting thta $(A^\#)^H = (A^\#)^HAA^\dagger$. Then

$$A^HA(A^\dagger)^H = A^HA(A^\dagger)^HAA^\dagger.$$

It follows that

$$(A^\dagger)^H = A^\#(A^\dagger)^HA^HA(A^\dagger)^H = A^\#(A^\dagger)^HA^HA(A^\dagger)^HAA^\dagger = (A^\dagger)^HAA^\dagger.$$

Hence, A is EP and

$$(AA^H(A^\#)^H)^3 = (AA^H(A^\#)^H)^2(AA^H(A^\#)^H) = (AA^H(A^\#)^H)(A^HA(A^\dagger)^H)(AA^H(A^\#)^H).$$

By Theorem 2.2, A is normal. □

Theorem 2.5. *Let $A \in C^{n\times n}$ be a group invertible matrix. Then A is a normal matrix if and only if $A^HA(A^\dagger)^H - AA^H(A^\#)^H$ is right $A^HA(A^\dagger)^H$−idempotent.*

Proof. " $\Longrightarrow$ " *It is an immediate result of [3, Theorem 2.1].*

" $\Longleftarrow$ " *If $A^HA(A^\dagger)^H - AA^H(A^\#)^H$ is right $A^HA(A^\dagger)^H$−idempotent. Then*

$$(A^HA(A^\dagger)^H - AA^H(A^\#)^H)^2 = (A^HA(A^\dagger)^H - AA^H(A^\#)^H)(A^HA(A^\dagger)^H).$$

This gives

$$(AA^H(A^\#)^H)^2 = A^HA(A^\dagger)^HAA^H(A^\#)^H.$$

Hence, $AA^H(A^\#)^H$ is left $A^HA(A^\dagger)^H$-idempotent. By Theorem 2.3, A is normal. □

Theorem 2.6. *Let $A \in C^{n\times n}$ be a group invertible matrix. Then A is a normal matrix if and only if $A^HA(A^\dagger)^H - AA^H(A^\#)^H$ is left $A^HA(A^\dagger)^H$− idempotent.*

Proof. " $\Longrightarrow$ " *It is evident by [3, Theorem 2.1].*

" $\Longleftarrow$ " *The condition "$A^HA(A^\dagger)^H - AA^H(A^\#)^H$ is left $A^HA(A^\dagger)^H$− idempotent" gives*

$$(A^HA(A^\dagger)^H - AA^H(A^\#)^H)^2 = (A^HA(A^\dagger)^H)(A^HA(A^\dagger)^H - AA^H(A^\#)^H).$$

This gives

$$(AA^H(A^\#)^H)^2 = AA^H(A^\#)^HA^HA(A^\dagger)^H.$$

Hence, $AA^H(A^\#)^H$ is right $A^HA(A^\dagger)^H$−idempotent. By Theorem 2.4, A is normal. □

Let $A, B \in C^{n\times n}$. If A is right B−idempotent, then

$$A^3 = A(A^2) = A(AB) = A^2B = AB^2,$$

and

$$A^6 = A^3A^3 = A^3AB^2 = A^2A^2B^2 = A^3B^3.$$

Hence, A^2 is right B^2−idempotent and A^3 is right B^3−idempotent.

Theorem 2.7. *Let $A \in C^{n\times n}$ be a group invertible matrix. Then A is a normal matrix if and only if $(AA^H(A^\#)^H)^k$ is right $(A^HA(A^\dagger)^H)^k$ idempotent for $k = 2,\ 3$.*

Proof. " $\Longrightarrow$ " *It is clear by [3, Theorem 2.1] and Theorem 2.4.*

" $\Longleftarrow$ " *From the hypothesis, one has*

$$(AA^H(A^\#)^H)^4 = (AA^H(A^\#)^H)^2(A^HA(A^\dagger)^H)^2.$$

Multiplying the equality on the right by $AA^\dagger$, one gets

$$(AA^H(A^\#)^H)^2(A^HA(A^\dagger)^H)^2 = (AA^H(A^\#)^H)^2(A^HA(A^\dagger)^H)^2AA^\dagger.$$

Multiplying the last equality on the left by $AA^H(A^\#)^2(A^\dagger)^H(A^\dagger)^2$, we have

$$A = A^2A^\dagger.$$

Hence, A is EP. Now, for $k = 2, 3$, one yields

$$A^k = (AA^H(A^\#)^H)^k\ ;\ (A^HA(A^\dagger)^H)^k = A^HA^k(A^\dagger)^H.$$

From the hypothesis, one has

$$(AA^H(A^\#)^H)^6 = (AA^H(A^\#)^H)^3(A^HA(A^\dagger)^H)^3.$$

This gives

$$A^6 = A^3A^HA^3(A^\dagger)^H = A^4A^2 = A^4A^HA^2(A^\dagger)^H.$$

Multiplying the equality on the left by $(A^\#)^3$ and on the right by $A^H(A^\#)^2$, we have

$$A^HA = AA^H.$$

Hence, A is a normal matrix. □

3. Using projections to characterize normal matrices

Let $A \in C^{n\times n}$. Then A is called projection if $A^2 = A = A^H$. Clearly, A is projection if and only if $A = AA^H$ if and only if $A = A^H A$ if and only if $E_n - A$ is projection. The results of characterizing generalized inverse with projections can be found in [2, 13]

Theorem 3.1. *Let $A \in C^{n\times n}$ be a group invertible matrix. Then A is a normal matrix if and only if $A^H A(A^\dagger)^H A^\dagger$ is projection.*

Proof. " $\Longrightarrow$ " *If A is a normal matrix, then, we have $A^H A(A^\dagger)^H A^\dagger = AA^\dagger$. Clearly, $(AA^\dagger)^2 = AA^\dagger = (AA^\dagger)^H$. Hence, $A^H A(A^\dagger)^H A^\dagger$ is projection.*

" $\Longleftarrow$ " *If $A^H A(A^\dagger)^H A^\dagger$ is projection, then we have*

$$A^H A(A^\dagger)^H A^\dagger (A^H A(A^\dagger)^H A^\dagger)^H = A^H A(A^\dagger)^H A^\dagger.$$

Multiplying the equality on the left by $A^H AA^H A^\#(A^\dagger)^H$, one gets

$$A^\dagger A^H A = A^H.$$

Hence, A is a normal matrix by [5, Theorem 1.3.2]. □

Let $A \in C^{n\times n}$. Then A is called Op-projection if $-A^2 = A = A^H$. Clearly, A is Op-projection if and only if $-A$ is projection.

Corollary 3.2. *Let $A \in C^{n\times n}$ be a group invertible matrix. Then A is a normal matrix if and only if $A^H A(A^\dagger)^H A^\dagger - AA^\dagger$ is Op-projection.*

Proof. " $\Longrightarrow$ " *It is evident by Theorem 3.1.*

" $\Longleftarrow$ " *If $A^H A(A^\dagger)^H A^\dagger - AA^\dagger$ is Op-projection. Then, $AA^\dagger - A^H A(A^\dagger)^H A^\dagger$ is projection. We have*

$$(AA^\dagger - A^H A(A^\dagger)^H A^\dagger)(AA^\dagger - A^H A(A^\dagger)^H A^\dagger)^H = AA^\dagger - A^H A(A^\dagger)^H A^\dagger.$$

This gives

$$(A^H A(A^\dagger)^H A^\dagger)^H = A^H A(A^\dagger)^H A^\dagger (A^H A(A^\dagger)^H A^\dagger)^H.$$

Hence, $A^H A(A^\dagger)^H A^\dagger$ is projection. By Theorem 3.1, A is a normal matrix. □

Theorem 3.3. *Let $A \in C^{n\times n}$ be a group invertible matrix. Then A is a normal matrix if and only if $A^H A(A^\dagger)^H A^\dagger$ and $A^H A(A^\dagger)^H A^\dagger - A^H A(A^\dagger)^H A^\dagger (A^H A(A^\dagger)^H A^\dagger)^H$ are idempotents.*

Proof. " $\Longrightarrow$ " *It is clear by Theorem 3.1.*

" $\Longleftarrow$ " *Since $A^H A(A^\dagger)^H A^\dagger$ and $A^H A(A^\dagger)^H A^\dagger - A^H A(A^\dagger)^H A^\dagger (A^H A(A^\dagger)^H A^\dagger)^H$ are idempotents, by a simple computation, we have*

$$A^H A(A^\dagger)^H A^\dagger (A^H A(A^\dagger)^H A^\dagger)^H A^H A(A^\dagger)^H A^\dagger = (A^H A(A^\dagger)^H A^\dagger (A^H A(A^\dagger)^H A^\dagger)^H)^2.$$

Multiplying the equality on the left by $AA^H A^\#(A^\dagger)^H$ and then applying the involution, one gets

$$(A^H A(A^\dagger)^H A^\dagger)^H A^H A(A^\dagger)^H A^\dagger = A^H A(A^\dagger)^H A^\dagger (A^H A(A^\dagger)^H A^\dagger)^H A^H A(A^\dagger)^H A^\dagger.$$

Multiplying the last equality on the right by $AA^H AA^\# A^\dagger (A^\dagger)^H$, we have

$$(A^H A(A^\dagger)^H A^\dagger)^H = A^H A(A^\dagger)^H A^\dagger (A^H A(A^\dagger)^H A^\dagger)^H.$$

Hence, $A^H A(A^\dagger)^H A^\dagger$ is projection. By Theorem 3.1, A is a normal matrix. □

Theorem 3.4. *Let $A \in C^{n\times n}$ be a group invertible matrix. Then A is a normal matrix if and only if $A^HA(A^\dagger)^HA^\dagger$, $A^HA(A^\dagger)^HA^\dagger(A^HA($ and*

$A^HA(A^\dagger)^HA^\dagger - A^HA(A^\dagger)^HA^\dagger(A^HA(A^\dagger)^HA^\dagger)^H + (A^HA(A^\dagger)^HA^\dagger)^H(A^HA(A^\dagger)^HA^\dagger)$ *are idempotents.*

Proof. " $\Longrightarrow$ " *It is an immediate result of Theorem 3.1.*

" $\Longleftarrow$ " *From the assumption, we have*

$$A^HA(A^\dagger)^HA^\dagger(A^HA(A^\dagger)^HA^\dagger)^H = A^HA(A^\dagger)^HA^\dagger(A^HA(A^\dagger)^HA^\dagger)^HA^HA(A^\dagger)^HA^\dagger(A^HA(A^\dagger)^HA^\dagger)^H.$$

Multiplying the equality on the left by $AA^HA^\#(A^\dagger)^H$, one gets

$$(A^HA(A^\dagger)^HA^\dagger)^H = (A^HA(A^\dagger)^HA^\dagger)^HA^HA(A^\dagger)^HA^\dagger(A^HA(A^\dagger)^HA^\dagger)^H,$$

and

$$A^HA(A^\dagger)^HA^\dagger = A^HA(A^\dagger)^HA^\dagger(A^HA(A^\dagger)^HA^\dagger)^HA^HA(A^\dagger)^HA^\dagger.$$

Since $A^HA(A^\dagger)^HA^\dagger - A^HA(A^\dagger)^HA^\dagger(A^HA(A^\dagger)^HA^\dagger)^H + (A^HA(A^\dagger)^HA^\dagger)^H(A^HA(A^\dagger)^HA^\dagger)$ is idempotent, we have

$$A^HA(A^\dagger)^HA^\dagger + (A^HA(A^\dagger)^HA^\dagger)^H = A^HA(A^\dagger)^HA^\dagger(A^HA(A^\dagger)^HA^\dagger)^H + (A^HA(A^\dagger)^HA^\dagger)^HA^HA(A^\dagger)^HA^\dagger.$$

This gives

$$(A^HA(A^\dagger)^HA^\dagger - (A^HA(A^\dagger)^HA^\dagger)^H)^2 = 0.$$

Noting that

$$(A^HA(A^\dagger)^HA^\dagger - (A^HA(A^\dagger)^HA^\dagger)^H)^H = -(A^HA(A^\dagger)^HA^\dagger - (A^HA(A^\dagger)^HA^\dagger)^H).$$

So

$$(A^HA(A^\dagger)^HA^\dagger - (A^HA(A^\dagger)^HA^\dagger)^H)(A^HA(A^\dagger)^HA^\dagger - (A^HA(A^\dagger)^HA^\dagger)^H)^H = 0.$$

Now we have $A^HA(A^\dagger)^HA^\dagger = (A^HA(A^\dagger)^HA^\dagger)^H$. Noting that $A^HA(A^\dagger)^HA^\dagger$ is idempotent. Then $A^HA(A^\dagger)^HA^\dagger$ is projection. Thus A is a normal matrix by Theorem 3.1. □

Lemma 3.5. *Let $A, B \in C^{n\times n}$ be Hermitians. If AB is projection, then BA is projection.*

Proof. *From the assumption, we have*

$$AB = (AB)^H = B^HA^H = BA.$$

Hence, BA is projection. □

It follows from Theorem 3.1 and Lemma 3.5, we have the following corollary.

Corollary 3.6. *Let $A \in C^{n\times n}$ be a group invertible matrix. Then A is a normal matrix if and only if $(A^+)^HA^+A^HA$ is projection.*

Proof. " $\Longrightarrow$ " *It follows from Theorem 3.1 because $(A^+)^HA^+A^HA = (A^HA(A^+)^HA^+)^H$.*

" $\Longleftarrow$ " *Noting that $(A^\dagger)^HA^\dagger$ and A^HA are Hermitian. Then, by the assumption and Lemma 3.5, we have $A^HA(A^\dagger)^HA^\dagger$ is projection. Hence, A is a normal matrix by Theorem 3.1.* □

Theorem 3.7. *Let $A \in C^{n\times n}$. Then A is a projection if and only if A is right AA^H- and A^HA-idempotents.*

Proof. " $\Longrightarrow$ " *Suppose that A is projection. Then $A = AA^H$ and $A = A^HA$, this deduces*

$$A^2 = A^2A^H = A(AA^H),$$

and

$$A^2 = A(A^HA).$$

Hence, A is both right AA^H- and A^HA-idempotents.

" $\Longleftarrow$ " From the hypothesis, one obtains

$$A^2 = A(AA^H),$$

and

$$A^2 = A(A^H A).$$

Noting that

$$rank(A) = rank(AA^H) = rank((AA^H)(AA^H)^H) = rank((AA^H A)A^H)$$
$$\leq rank(AA^H A) = rank(A^2) \leq rank(A).$$

Then rank(A) = rank(A^2), this induces A is group invertible and

$$A = A^{\#}A^2 = A^{\#}(A^2 A^H) = AA^H.$$

Hence, A is projection.
□

Theorem 3.7 inspires us to give the following result.

Theorem 3.8. *Let $A \in C^{n\times n}$ be a group invertible matrix. Then A is a normal matrix if and only if $A^H A(A^\dagger)^H A^\dagger$ is right $A^H A(A^\dagger)^H A^\dagger(A^H A(A^\dagger)^H A^\dagger)^H$−idempotent.*

Proof. *" $\Longrightarrow$ " It is an immediate result of Theorem 3.1.*
" $\Longleftarrow$ " From the assumption, we have

$$(A^H A(A^\dagger)^H A^\dagger)^2 = (A^H A(A^\dagger)^H A^\dagger)^2 (A^H A(A^\dagger)^H A^\dagger)^H.$$

Multiplying the equality on the left by $(A^{\#})^H A^H AA^H A^{\#}(A^\dagger)^H$, one gets

$$A^H A(A^\dagger)^H A^\dagger = A^H A(A^\dagger)^H A^\dagger(A^H A(A^\dagger)^H A^\dagger)^H.$$

Now $A^H A(A^\dagger)^H A^\dagger$ is projection. Hence, A is a normal matrix by Theorem 3.1. □

4. Using vectors to characterize normal matrices

Theorem 2.1 implies us to give the following result on normal matrix. Let $A \in C^{n\times n}$, A is said to be regular if there exists $B \in C^{n\times n}$ such that $A = ABA$. The matrix B is called an inner inverse of A. The inner inverse matrix of A is not unique , and $A\{1\}$ is used to denote the set of all inner inverses of A.

Theorem 4.1. *Let $A \in C^{n\times n}$ be a group invertible matrix. Then A is a normal matrix if and only if $\begin{pmatrix} A^H, & A - E_n \end{pmatrix} \in \begin{pmatrix} A^{\#}(A^\dagger)^H \\ A^\dagger \end{pmatrix}\{1\}$.*

Proof. *" $\Longrightarrow$ " Assume that A is a normal matrix. Then, A is EP by [5, Lemma 1.3.3], and $A^{\#}(A^\dagger)^H = (A^\dagger)^H A^{\#}$. It follows that*

$$\begin{pmatrix} A^{\#}(A^\dagger)^H \\ A^\dagger \end{pmatrix} \begin{pmatrix} A^H, & A - E_n \end{pmatrix} \begin{pmatrix} A^{\#}(A^\dagger)^H \\ A^\dagger \end{pmatrix} = \begin{pmatrix} A^{\#}(A^\dagger)^H(A^H A^{\#}(A^\dagger)^H + AA^\dagger - A^\dagger) \\ A^\dagger A^H A^{\#}(A^\dagger)^H + A^\dagger AA^\dagger - A^\dagger A^\dagger \end{pmatrix}.$$

Noting that

$$A^{\#}(A^\dagger)^H(A^H A^{\#}(A^\dagger)^H + AA^\dagger - A^\dagger) = A^{\#}(A^\dagger)^H + A^{\#}A^{\#}(A^\dagger)^H - A^{\#}(A^\dagger)^H A^\dagger = A^{\#}(A^\dagger)^H,$$

and

$$A^\dagger A^H A^{\#}(A^\dagger)^H + A^\dagger AA^\dagger - A^\dagger A^\dagger = A^\dagger A^H(A^\dagger)^H A^{\#} + A^\dagger - A^\dagger A^\dagger = A^\dagger.$$

Hence,

$$\begin{pmatrix} A^{\#}(A^{\dagger})^{H} \\ A^{\dagger} \end{pmatrix}\begin{pmatrix} A^{H}, & A-E_n \end{pmatrix}\begin{pmatrix} A^{\#}(A^{\dagger})^{H} \\ A^{\dagger} \end{pmatrix} = \begin{pmatrix} A^{\#}(A^{\dagger})^{H} \\ A^{\dagger} \end{pmatrix}.$$

One gets $\begin{pmatrix} A^{H}, & A-E_n \end{pmatrix} \in \begin{pmatrix} A^{\#}(A^{\dagger})^{H} \\ A^{\dagger} \end{pmatrix}\{1\}$.

" $\Longleftarrow$ " *From the assumption, we have*

$$\begin{pmatrix} A^{\#}(A^{\dagger})^{H} \\ A^{\dagger} \end{pmatrix}\begin{pmatrix} A^{H}, & A-E_n \end{pmatrix}\begin{pmatrix} A^{\#}(A^{\dagger})^{H} \\ A^{\dagger} \end{pmatrix} = \begin{pmatrix} A^{\#}(A^{\dagger})^{H} \\ A^{\dagger} \end{pmatrix}.$$

This gives

$$A^{\#}(A^{\dagger})^{H}(A^{H}A^{\#}(A^{\dagger})^{H} + AA^{\dagger} - A^{\dagger}) = A^{\#}(A^{\dagger})^{H}, \tag{1}$$

$$A^{\dagger}A^{H}A^{\#}(A^{\dagger})^{H} = A^{\dagger}A^{\dagger}. \tag{2}$$

Multiplying (4.2) on the left by $(A^{\#})^{H}A^{H}A$, *we have*

$$A^{H}A^{\#}(A^{\dagger})^{H} = A^{\dagger}.$$

Multiplying the last equality on the right by $A^{\dagger}A$, *one yields* $A^{\dagger} = A^{\dagger}A^{\dagger}A$. *Hence, A is EP. It follows from (4.1) that*

$$A^{\#}(A^{\dagger})^{H} = A^{\#}A^{\#}(A^{\dagger})^{H} + A^{\#}(A^{\dagger})^{H} - A^{\#}(A^{\dagger})^{H}A^{\dagger},$$

this gives

$$A^{\#}A^{\#}(A^{\dagger})^{H} = A^{\#}(A^{\dagger})^{H}A^{\dagger}.$$

Multiplying the above equality on the left by A, one gets

$$A^{\#}(A^{\dagger})^{H} = (A^{\dagger})^{H}A^{\dagger} = (A^{\dagger})^{H}A^{\#}.$$

Hence, A is normal. □

It is well known that A is a normal matrix if and only if $A^{H} = AA^{H}A^{\dagger}$ and $A^{\#}(A^{\dagger})^{H} = (A^{\dagger})^{H}A^{\#}$[5, Theorem 1.3.2]. Hence, Theorem 4.1 leads to the following corollary.

Corollary 4.2. *Let* $A \in C^{n\times n}$ *be a group invertible matrix. Then A is a normal matrix if and only if* $\begin{pmatrix} AA^{H}A^{\dagger}, & A-E_n \end{pmatrix} \in \begin{pmatrix} (A^{\dagger})^{H}A^{\#} \\ A^{\dagger} \end{pmatrix}\{1\}$.

Proof. " $\Longrightarrow$ " *If A is a normal matrix. Then, we have* $A^{H} = AA^{H}A^{\dagger}$ *and* $A^{\#}(A^{\dagger})^{H} = (A^{\dagger})^{H}A^{\#}$. *Hence,* $\begin{pmatrix} AA^{H}A^{\dagger}, & A-E_n \end{pmatrix} \in \begin{pmatrix} (A^{\dagger})^{H}A^{\#} \\ A^{\dagger} \end{pmatrix}\{1\}$ *by Theorem 4.1.*

" $\Longleftarrow$ " *From the assumption, we have*

$$\begin{pmatrix} (A^{\dagger})^{H}A^{\#} \\ A^{\dagger} \end{pmatrix}\begin{pmatrix} AA^{H}A^{\dagger}, & A-E_n \end{pmatrix}\begin{pmatrix} (A^{\dagger})^{H}A^{\#} \\ A^{\dagger} \end{pmatrix} = \begin{pmatrix} (A^{\dagger})^{H}A^{\#} \\ A^{\dagger} \end{pmatrix}.$$

This gives

$$AA^{\dagger}A^{\dagger}(A^{\dagger})^{H}A^{\#} + (A^{\dagger})^{H}A^{\dagger} - (A^{\dagger})^{H}A^{\#}A^{\dagger} = (A^{\dagger})^{H}A^{\#}, \tag{3}$$

$$A^{H}A^{\dagger}(A^{\dagger})^{H}A^{\#} = A^{\dagger}A^{\dagger}. \tag{4}$$

Multiplying (4.4) on the right by $A^{\dagger}A$, *we have*

$$A^{\dagger}A^{\dagger}A^{\dagger}A = A^{\dagger}A^{\dagger}.$$

By [4, Lemma 2.11], $A^\dagger = A^\dagger A^\dagger A$. *Then A is EP. It follows from (4.3) that*

$$A^\dagger (A^\dagger)^H A^\# = (A^\dagger)^H A^\# A^\dagger.$$

This gives

$$A^\# (A^\dagger)^H = A^\dagger (A^\dagger)^H = A^\dagger (A^\dagger)^H A^\# A = (A^\dagger)^H A^\# A^\dagger A = (A^\dagger)^H A^\#.$$

Hence, A is normal. □

Noting that *SEP* matrix is always normal. Hence, Corollary 4.2 implies the following corollary which characterizes *SEP* matrix.

Corollary 4.3. *Let* $A \in C^{n\times n}$ *be a group invertible matrix. Then A is a SEP matrix if and only if* $\begin{pmatrix} AA^H A^H, & A - E_n \end{pmatrix} \in \begin{pmatrix} (A^\dagger)^H A^\# \\ A^\dagger \end{pmatrix}\{1\}$.

Proof. *" ⟹ " It is an immediate result of Corollary 4.2.*
" ⟸ " From the assumption, we have

$$\begin{pmatrix} (A^\dagger)^H A^\# \\ A^\dagger \end{pmatrix} \begin{pmatrix} AA^H A^H, & A - E_n \end{pmatrix} \begin{pmatrix} (A^\dagger)^H A^\# \\ A^\dagger \end{pmatrix}$$
$$= \begin{pmatrix} (A^\dagger)^H A^\# AA^H A^H (A^\dagger)^H A^\# + (A^\dagger)^H A^\dagger - (A^\dagger)^H A^\# A^+ \\ A^H A^H (A^\dagger)^H A^\# + A^\dagger - A^\dagger A^\dagger \end{pmatrix}.$$

Noting that $(A^\dagger)^H A^\# AA^H A^H (A^\dagger)^H A^\# = AA^\dagger A^\dagger AA^\#$. *Then one gets*

$$AA^\dagger A^\dagger AA^\# + (A^\dagger)^H A^\dagger - (A^\dagger)^H A^\# A^\dagger = (A^\dagger)^H A^\#, \tag{5}$$

$$A^H A^H (A^\dagger)^H A^\# = A^\dagger A^\dagger. \tag{6}$$

Multiplying (4.6) on the right by $A^\dagger A$, *we have*

$$A^\dagger A^\dagger A^\dagger A = A^\dagger A^\dagger.$$

A is EP by Corollary 4.2. Hence, the equality (4.5) changes into

$$A^\dagger = (A^\dagger)^H A^\# A^\dagger.$$

So,

$$A = A^\dagger A^2 = (A^\dagger)^H A^\# A^\dagger A^2 = (A^\dagger)^H A^\# A = (A^\dagger)^H.$$

Thus, A is a SEP matrix. □

Since Hermite matrices are normal. Hence, Corollary 4.2 implies the following corollary which characterizes Hermite matrix.

Corollary 4.4. *Let* $A \in C^{n\times n}$ *be a group invertible matrix. Then A is a Hermite matrix if and only if* $\begin{pmatrix} A^2 A^\dagger, & A - E_n \end{pmatrix} \in \begin{pmatrix} (A^\dagger)^H A^\# \\ A^\dagger \end{pmatrix}\{1\}$.

Proof. *" ⟹ " It is an immediate result of Corollary 4.2.*
" ⟸ " From the assumption, we have

$$\begin{pmatrix} (A^\dagger)^H A^\# \\ A^\dagger \end{pmatrix} \begin{pmatrix} A^2 A^\dagger, & A - E_n \end{pmatrix} \begin{pmatrix} (A^\dagger)^H A^\# \\ A^\dagger \end{pmatrix} = \begin{pmatrix} (A^\dagger)^H (A^\dagger)^H A^\# + (A^\dagger)^H A^\dagger - (A^\dagger)^H A^\# A^\dagger \\ A^\dagger A (A^\dagger)^H A^\# + A^\dagger - A^\dagger A^\dagger \end{pmatrix}.$$

This gives

$$(A^{\dagger})^H(A^{\dagger})^H A^{\#} + (A^{\dagger})^H A^{\dagger} - (A^{\dagger})^H A^{\#} A^{\dagger} = (A^{\dagger})^H A^{\#}, \tag{7}$$

$$A^{\dagger}A(A^{\dagger})^H A^{\#} = A^{\dagger}A^{\dagger}. \tag{8}$$

Multiplying (4.8) on the right by $A^{\dagger}A$, we have

$$A^{\dagger}A^{\dagger}A^{\dagger}A = A^{\dagger}A^{\dagger}.$$

A is EP by Corollary 4.2. Multiplying (4.7) on the right by A, we have

$$(A^{\dagger})^H(A^{\dagger})^H = (A^{\dagger})^H A^{\#}.$$

Multiplying the last equality on the left by $A^H A^H$, it gets

$$A^{\dagger}A = A^H A^{\#}.$$

Hence, A is a Hermite matrix. □

5. Using one sided A−equality to characterize normal matrices

Let $A, B, C \in C^{n\times n}$. Then B and C are called left (right) A−equal if $AB = AC$ ($BA = CA$). Clearly, we have
(1) A is right C−idempotent if and only if A and C are left A−equal;
(2) $A^2 = A$ if and only if A and E_n are left A−equal;
(3) A is (3, B)−regular if and only if A^2 and BA are left A−equal;
(4) A is projection if and only if A^H and E_n are left A−equal.

Theorem 5.1. *Let $A \in C^{n\times n}$ be a group invertible matrix. Then A is a normal matrix if and only if $A^H A(A^{\#})^H$ and A are left $AA^H(A^{\#})^H$−equal.*

Proof. *" ⟹ " If A is a normal matrix, then A is EP. We have*

$$A^H A(A^{\#})^H = AA^H(A^{\#})^H = A.$$

Hence, $A^H A(A^{\#})^H$ and A are left $AA^H(A^{\#})^H$−equal.
" ⟸ " The condition "$A^H A(A^{\#})^H$ and A are left $AA^H(A^{\#})^H$−equal" gives

$$AA^H(A^{\#})^H A^H A(A^{\#})^H = AA^H(A^{\#})^H A. \tag{9}$$

Multiplying the equality on the right by $AA^{\dagger}$, we have

$$AA^H(A^{\#})^H A = AA^H(A^{\#})^H A^2 A^{\dagger}.$$

Multiplying the last equality on the left by $(A^{\dagger})^H A^H A^{\dagger}$, it gets

$$A = A^2 A^{\dagger}.$$

So A is EP. Multiplying (5.1) on the left by $A^{\dagger}$ and on the right by A^H, it gets $A^H A = AA^H$. Hence, A is a normal matrix. □

Theorem 5.2. *Let $A \in C^{n\times n}$ be a group invertible matrix. Then A is a normal matrix if and only if $A^H A(A^{+})^H$ and $AA^H(A^{\#})^H$ are left A−equal.*

Proof. ″ $\Longrightarrow$ ″ *It is evident by [3, Theorem 2.1].*

″ $\Longleftarrow$ ″ *From the assumption, we have*

$$AA^HA(A^\dagger)^H = A^2A^H(A^\#)^H. \tag{10}$$

Multiplying (5.2) on the right by $A^\dagger A$*, we have*

$$A^2A^H(A^\#)^H = A^2A^H(A^\#)^HA^\dagger A.$$

Multiplying the last equality on the left by $A^\dagger A^\dagger A^\#$*, it gets*

$$A^\dagger A^\dagger A = A^\dagger.$$

So A is EP. Multiplying (5.2) on the left by $A^\dagger$ *and on the right by* A^H*, it gets* $A^HA = AA^H$*. Hence, A is a normal matrix.* □

Theorem 5.3. *Let* $A \in C^{n\times n}$ *be a group invertible matrix. Then A is a normal matrix if and only if* $A^HA(A^+)^H$ *and* $AA^H(A^\#)^H$ *are left* A^H-*equal.*

Proof. ″ $\Longrightarrow$ ″ *It is evident by [3, Theorem 2.1].*

″ $\Longleftarrow$ ″*From the assumption, we have*

$$A^HA^HA(A^\dagger)^H = A^HAA^H(A^\#)^H. \tag{11}$$

Multiplying (5.3) on the right by $A^\dagger A$*, we have*

$$A^HAA^H(A^\#)^H = A^HAA^H(A^\#)^HA^\dagger A.$$

Multiplying the last equality on the left by $A^\dagger A^\dagger (A^\dagger)^H$*, it gets*

$$A^\dagger = (A^\dagger)^2A.$$

So A is EP. Multiplying (5.3) on the left by $(A^\dagger)^H$ *and on the right by* A^H*, it gets* $A^HA = AA^H$*. Hence, A is a normal matrix.* □

Theorem 5.4. *Let* $A \in C^{n\times n}$ *be a group invertible matrix. Then A is a normal matrix if and only if A and* $(A^\#)^HAA^H$ *are left* $A^HA(A^\#)^H-$*equal.*

Proof. ″ $\Longrightarrow$ ″*If A is a normal matrix, then A is EP. We have*

$$A^HA(A^\#)^H = A = (A^\#)^HAA^H.$$

Hence, A and $(A^\#)^HAA^H$ *are left* $A^HA(A^\#)^H-$*equal.*

″ $\Longleftarrow$ ″ *From the assumption, we have*

$$A^HA(A^\#)^HA = A^HA(A^\#)^H(A^\#)^HAA^H. \tag{12}$$

Multiplying (5.4) on the right by $AA^\dagger$ *and on the left by* $AA^\dagger A^HA^\dagger(A^\dagger)^H$*, we have*

$$A = A^2A^\dagger.$$

So A is EP. Multiplying (5.4) on the left by $A^HA^HA^\dagger(A^\dagger)^H$*, it gets*

$$A^HA = AA^H.$$

Hence, A is a normal matrix. □

Theorem 5.5. *Let $A \in C^{n\times n}$ be a group invertible matrix. Then A is a normal matrix if and only if A^k and $((A^{\#})^H AA^H)^k$ are left $A^H A(A^{\#})^H-$equal for $k = 2,\ 3$.*

Proof. "$\Longrightarrow$"It is evident by Theorem 5.4.
"$\Longleftarrow$" From the hypothesis, one has

$$A^H A(A^{\#})^H A^2 = A^H A(A^{\#})^H((A^{\#})^H AA^H)^2.$$

Multiplying the equality on the right by $AA^{\dagger}$ and on the left by $A^{\#}AA^{\dagger}A^H A^{\dagger}(A^{\dagger})^H$, we have

$$A = A^2 A^{\dagger}.$$

So A is EP. Now, for $k = 2, 3$, one yields

$$((A^{\#})^H AA^H)^k = (A^{\#})^H A^k A^H.$$

This gives

$$A^H A(A^{\#})^H(A^{\#})^H A^3 A^H = A^H A(A^{\#})^H A^2 A = A^H A(A^{\#})^H(A^{\#})^H A^2 A^H A.$$

Multiplying on the left by $A^{\dagger}A^{\#}A^H A^H A^{\dagger}(A^{\dagger})^H$, it gets

$$AA^H = A^H A.$$

Hence, A is a normal matrix. □

6. Constructing $A-$commutativity matrices to characterize normality

Let $A,\ B,\ C \in C^{n\times n}$. Then B and C are called $A-$commuatativity if $AB = CA$. Clearly, we have
(1) A is left C-idempotent if and only if A and C are $A-$commutativity;
(2) A is right $B-$idempotent if and only if B and A are $A-$commutativity;
(3) $A^2 = A$ if and only if A and E_n are $A-$commutativity;
(4) A is normal if and only if A and A are A^H-commutativity;
(5) A is normal if and only if A^H and A^H are $A-$commutativity.

Theorem 6.1. *Let $A \in C^{n\times n}$ be a group invertible matrix. Then A is a normal matrix if and only if $AA^H(A^{\#})^H$ and $A^H A(A^{+})^H$ are $A-$commutativity.*

Proof. "$\Longrightarrow$"It is evident by [3, Theorem 2.1].
"$\Longleftarrow$" From the hypothesis, one has

$$A^2 A^H(A^{\#})^H = A^H A(A^{+})^H A. \tag{13}$$

Multiplying (6.1) on the right by $AA^{\dagger}$, we have

$$A^H A(A^{+})^H A = A^H A(A^{+})^H A^2 A^{\dagger}.$$

Multiplying the equality on the left by $A^{\#}AA^H A^{\#}(A^{\dagger})^H$, it gets

$$A = A^2 A^{\dagger}.$$

So A is EP. Multiplying (6.1) on the right by $A^{\#}A^H$, it gets

$$AA^H = A^H A.$$

Hence, A is a normal matrix. □

Corollary 6.2. *Let $A \in C^{n\times n}$ be a group invertible matrix. Then A is a normal matrix if and only if $AA^H(A^\#)^H$ and $A^HA(A^+)^H$ are $A^\#-$commutativity.*

Proof. It is an immediate result of Theorem 6.1. □

Theorem 6.3. *Let $A \in C^{n\times n}$ be a group invertible matrix. Then A is a normal matrix if and only if $AA^H(A^\#)^H$ and $A^HA(A^\dagger)^H$ are $((A^\dagger)^H)^k-$commutativity for $k = 2,\ 3$.*

Proof. $'' \Longrightarrow ''$ It is evident by [3, Theorem 2.1].
$'' \Longleftarrow ''$From the assumption, we have

$$((A^\dagger)^H)^2AA^H(A^\#)^H = A^HA((A^\dagger)^H)^3.$$

Multiplying the equality on the right by $AA^\dagger$, we have

$$A^HA((A^\dagger)^H)^3 = A^HA((A^\dagger)^H)^3AA^\dagger.$$

Multiplying the last equality on the left by $(A^\#AA^H)^2A^\#(A^\dagger)^H$, it gets

$$(A^\dagger)^H = (A^\dagger)^HAA^\dagger.$$

So A is EP. Now, for $k = 2, 3$, one yields

$$((A^\dagger)^H)^3AA^H(A^\#)^H = A^HA((A^\dagger)^H)^3(A^\dagger)^H = ((A^\dagger)^H)^2AA^H(A^\#)^H(A^\dagger)^H.$$

Multiplying the above equality on the right by A^H and on the left by $(A^H)^3$, it gets

$$AA^H = A^HA.$$

Hence, A is a normal matrix. □

Theorem 6.4. *Let $A \in C^{n\times n}$ be a group invertible matrix. Then A is a normal matrix if and only if $A^HA(A^\dagger)^H$ and $AA^H(A^\#)^H$ are $A^\dagger-$commutativity.*

Proof. $'' \Longrightarrow ''$ It is evident by [3, Theorem 2.1].
$'' \Longleftarrow ''$ From the hypothesis, one has

$$A^\dagger A^HA(A^\dagger)^H = AA^H(A^\#)^HA^\dagger = AA^\dagger. \tag{14}$$

Multiplying (6.2) on the right by $A^\dagger A$, we have

$$AA^\dagger = AA^\dagger A^\dagger A.$$

Hence, A is EP. Multiplying (6.3) on the right by A^H and on the left by A, it gets

$$AA^H = A^HA.$$

Hence, A is a normal matrix. □

Theorem 6.5. *Let $A \in C^{n\times n}$ be a group invertible matrix. Then A is a normal matrix if and only if $A^HA(A^\dagger)^H$ and $AA^H(A^\#)^H$ are $(A^H)^k-$commutativity for $k = 2,\ 3$.*

Proof. $'' \Longrightarrow ''$ It is evident by [3, Theorem 2.1].
$'' \Longleftarrow ''$From the assumption, we have

$$(A^H)^3A(A^\dagger)^H = AA^H(A^\#)^H(A^H)^2 = A(A^H)^2.$$

Multiplying the equality on the right by $A^{\dagger}A$ and on the left by $(A^{\#})^{H}A^{\dagger}$, we have

$$A^{H} = A^{H}A^{\dagger}A.$$

So A is EP. Now, for k = 2, 3, one yields

$$AA^{H}(A^{\#})^{H}(A^{H})^{3} = A^{H}(A^{H})^{3}A(A^{\dagger})^{H} = A^{H}AA^{H}(A^{\#})^{H}(A^{H})^{2},$$

that is,

$$A(A^{H})^{3} = A^{H}A(A^{H})^{2}.$$

Multiplying on the right by $((A^{\#})^{H})^{2}$, it gets

$$AA^{H} = A(A^{H})^{3}((A^{\#})^{H})^{2} = A^{H}A(A^{H})^{H}((A^{\#})^{H})^{2} = A^{H}A.$$

Hence, A is a normal matrix. □

References

[1] A. Ben-Israel, T. N. E. Greville, Generalized inverses: theory and applications, Second Ed., Springer 2003.
[2] B.Gadelseeda, J. C. Wei, One sided x-projection, one sided x-idempotent and strongly *EP* elements in a*-ring[J]. Filomat 39(5)(2025), 1539-1550.
[3] C. M. Peng, H. N. Zhou, J. C. Wei. Some new characterizations of normal matrices. Filomat 38(2)(2024), 393-404.
[4] D. D. Zhao, J. C. Wei, Strongly *EP* elements in rings with involution, J. Algebra Appl. (2022), 2250088, (10 pages), DOI: 10.1142/S0219498822500888.
[5] D. Mosić, Generalized inverses, Faculty of Sciences and Mathematics, University of Niš, Niš 2018.
[6] D. Mosić, D. S. Djordjević, Moore-Penrose invertible normal and Hermitian elements in rings, Linear Algebra Appl. 431 (2009), 732-745.
[7] E. Boasso, Factorizations of *EP* Banach operators and *EP* Banach algebra elements, J. Math. Anal. Appl. 379 (2011), 245-255.
[8] E. Boasso, On the Moore-Penrose inverse in C^*−algebras, Extracta Mathematicae 21(2) (2006), 93-106.
[9] L. Y. Shi, J. C. Wei, Some new characterizations of normal elements, Filomat 33(13) (2019), 4115-4120.
[10] O. M. Baksalary, G. Trenkler, Characterizations of *EP*, normal and Hermitian matrices, Linear Multilinear Algebra 56 (2006), 299-304.
[11] R. Bru, N. Thome, Group inverse and group involutory matrices, Linear Multilinear Algebra 45 (1998), 207-218.
[12] Y. Qu, J. C. Wei, H. Yao, Characterizations of normal elements in rings with involution, Acta. Math. Hungar. 156(2) (2018), 459-464.
[13] Y. Ji, J. C. Wei, *SEP* elements in a ring with involution. Filomat 39(4)(2025), 1119-1133.