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Optimal control of second order delay-differential inclusions

Dilara I. Mastaliyeva[a]

[a] *Khazar University, Baku, Azerbaijan*

Abstract. The present paper studies a new class of problems of optimal control theory with second order delay-differential inclusions (DFIs). In the forms of Euler-Lagrange and Hamiltonian type inclusions the sufficient conditions of optimality for delay-DFIs, including the peculiar transversality ones, are proved. In particular, applications of these results to the second order semilinear optimal control problem are illustrated as well as the optimality conditions for nondelayed problems are derived.

1. Introduction

Discrete and continuous time processes with first order ordinary discrete-differential and differential inclusions found wide application in the field of mathematical economics and in problems of control dynamic system optimization and differential games (see [3, 6–10, 13–20, 22, 28] and their references). The paper [7] deals with the variational convergence of a sequence of optimal control problems for functional differential state equations with deviating argument. Variational limit problems are found under various conditions of convergence of the input data. It is shown that, upon sufficiently weak assumptions on convergence of the argument deviations, the limit problem can assume a form different from that of the whole sequence. In particular, it can be either an optimal control problem for an integro-differential equation or a purely variational problem. Conditions are found under which the limit problem preserves the form of the original sequence. In the paper [24] are considered evolution inclusions driven by a time dependent subdifferential plus a multivalued perturbation. Are proved existence results for the convex and nonconvex valued perturbations, for extremal trajectories (solutions passing from the extreme points of the multivalued perturbation). Are also proved a strong relaxation theorem showing that each solution of the convex problem can be approximated in the supremum norm by extremal solutions. Finally, are presented some examples illustrating these results. The book [3] is concerned with the optimal convex control problem of Bolza in a Banach space. A distinctive feature is a strong emphasis on the connection between theory and application. The main emphasis is put on the characterization of optimal arcs as well as on the synthesis of optimal controllers. Necessary and sufficient conditions of optimality, generalizing the classical Euler-Lagrange equations, are obtained in Sect. 4.1 in terms of the subdifferential of the convex cost integrand. The abstract cases of distributed and boundary controls are treated separately. The paper [23] concerns constrained dynamic optimization problems governed by delay control systems whose dynamic constraints are described by both DFIs and linear algebraic equations. The authors are

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Communicated by Ljubiša D. R. Kočinac
Email address: dilara.mestelieva@gmail.com (Dilara I. Mastaliyeva)
ORCID iD: https://orcid.org/0000-0002-0384-751X (Dilara I. Mastaliyeva)

not familiar with any results in these directions for such systems even in the delay-free case. In the first part of the paper are established the value convergence of discrete approximations as well as the strong convergence of optimal arcs in the classical Sobolev space $W^{1,2}$. Then using discrete approximations as a vehicle, are derived necessary optimality conditions for the initial continuoustime systems in both Euler-Lagrange and Hamiltonian forms via basic generalized differential constructions of variational analysis. In the paper [27] are examined functional differential inclusions with memory and state constraints. For the case of time-independent state constraints, are shown that the solution set is R_δ under Carathéodory conditions on the orientor field. For the case of time-dependent state constraints are proved two existence theorems. For this second case, the question of whether the solution set is R_δ remains open. In the work [10] dynamic optimization problems for differential inclusions on manifolds are considered. A mathematical framework for derivation of optimality conditions for generalized dynamical systems is proposed. By using metric regularity of terminal and dynamic constraints in form of generalized Euler-Lagrange relations and in form of partially convexified Hamiltonian inclusions are obtained optimality conditions. In [8] are provided intrinsic sufficient conditions on a multifunction F and endpoint data φ so that the value function associated to the Mayer problem is semiconcave. The paper [6] introduces a new class of variational problems for differential inclusions, motivated by the control of forest fires. The area burned by the fire at time $t > 0$ is modelled as the reachable set for a differential inclusion $x' \in F(x)$, starting from an initial set R_0. To block the fire, a wall can be constructed progressively in time, at a given speed. In this paper, is studied the possibility of constructing a wall which completely encircles the fire. Moreover, is derived necessary conditions for an optimal strategy, which minimizes the total area burned by the fire.

ecently, a number of authors have started investigating boundary value problems and controllability problems for second order DFIs. The problems accompanied with the second order discrete and differential inclusions are more complicated due to the second order derivatives and their discrete analogous. Thus, optimal control problems with ordinary DSIs and DFIs are one of the area in mathematical theory of optimal processes being intensively developed. More specifically, we deal with similar problems with delay-differential and state constraints of Bolza type. Observe that such problems arise frequently not only in mechanics, aerospace engineering, management sciences, and economics, but also in problems of automatic control, aviovibration, burning in rocket motors, and biophysics (see [14, 22, 23] and references therein).

For second order differential inclusions, the existence of solutions and other qualitative properties has been intensively analyzed in the recent literature (see [2, 11, 29] and their references). In the paper [2] in a separable Banach space a three point boundary value problem for a second order DFI of the form $x''(t) \in F(t, x(t), x'(t))$, a.e. $t \in [0, 1], x(0) = 0; x(\theta) = x(1)$ are considered. The existence of solutions, when the set-valued mapping F is unbounded-valued and satisfies a pseudo-Lipschitz property are investigated. Then, a Lipschitz case is derived and the associated relaxed problem is studied.

The paper [29] is concerned with the nonlinear boundary value problems for a class of semilinear second order DFIs. Using the tools involving topological transformation and fixed points of the set-valued map, some existence theorems of solutions in the convex case are given.

in the paper [11], second order DFIs with a maximal monotone term and generalized boundary conditions are studied. The nonlinear differential operator need not be necessary homogeneous and incorporates as a special case the one-dimensional p-Laplacian. The generalized boundary conditions incorporate as special cases well-known problems such as the Dirichlet (Picard), Neumann and periodic problems. As application to the proven results existence theorems for both "convex" and "nonconvex" problems are obtained. In the second half of the 20th century, many mathematicians in Russia made great contributions to the field of optimal control theory (see [1, 4, 5, 12] and references therein) In the paper [1] using Linear Lyapunov-Krotov functions are obtained sufficient conditions for strong and global minima for the classical smooth problem of optimal control. In the paper [5] sufficient optimality conditions are proved in the form of a maximum principle for the time-optimal problem of transfer from a set M_0 into a set M_1, where an object's behavior is described by the first order differential inclusion $x' \in F(t, x)$. It is shown that state constraints may be active. This means that the adjoint function may have points of discontinuity or jumps. The paper [12] studies optimization problem described by first order evolution impulsive differential inclusions (DFIs); in terms of locally adjoint mappings in framework of convex and nonsmooth analysis are formulated sufficient conditions of optimality. Then are constructed the dual problems for impulsive DFIs and proved duality results. The article [13] investigates an optimal control problem described by higher order retarded differential inclusions with endpoint constraints. In terms of the Euler-Lagrange type adjoint inclusions and the Hamiltonian, a sufficient optimality condition is derived for higher-order differential inclusions. The problems considered in the paper [9] are described in polyhedral set-valued mappings for higher order discrete and differential inclusions. The paper focuses on the necessary and sufficient conditions of optimality for optimization of these problems.

As is pointed out in [25, 26, 30], boundary value problems (BVPs) for higher order differential equations play a very important role in both theory and applications. In recent years, BVPs for second order differential equations have been extensively studied. In particular, fourth order linear differential equations [26] subject to some boundary conditions arise in the mathematical description of some physical systems. For example, mathematical models of deflection of beams [25, 26].

Optimization of higher order differential inclusions was first developed by Mahmudov in [19, 21]. The paper [19] is mainly concerned with the sufficient conditions of optimality for Cauchy problem (with fixed initial and free endpoint constraints) of third-order DFIs. Some special transversality conditions, which are peculiar to problems including third order derivatives are formulated. It is worthwhile to highlight that optimization problem with higher order (say m thorder) DFIs sometimes has its own importance for every m in the theoretical and practical point of view.

The paper [20] is devoted to a second order polyhedral optimization described by ordinary DSIs and DFIs. The stated second order discrete problem is reduced to the polyhedral minimization problem with polyhedral geometric constraints and in terms of the polyhedral Euler-Lagrange inclusions, necessary and sufficient conditions of optimality are derived. Derivation of the sufficient conditions for the second order polyhedral DFIs is based on the discrete-approximation method.

The paper [21] deals with a Bolza problem of optimal control theory given by second order convex differential inclusions with second order state variable inequality constraints. Necessary and sufficient conditions of optimality including distinctive "transversality" condition are proved in the form of Euler-Lagrange inclusions. Construction of Euler-Lagrange type adjoint inclusions is based on the presence of equivalence relations of locally adjoint mappings.

In our present paper, we discuss a special kind of optimization problem with second order delayDFIs and delay DFIs in which the constraints are defined by set-valed mappings. To the best of our best knowledge, there is no paper which considers an optimality conditions for these problems. We try to fill this gap in the literature in this paper. In fact, the difficulty in the problems with higher order DFIs is rather to construct the Euler-Lagrange type higher order adjoint inclusions and the suitable transversality conditions.

The stated problems and obtained optimality conditions in our paper are new. We pursue a twofold goal: to study optimality conditions for delay-DSIs of control systems with respect to discrete approximations and to derive sufficient optimality conditions for second order delay-DFIs. We are not familiar with any results in these directions for such systems even in the nondelay case. The paper is structured as follows;

In Section 2 for the reader's convenience from the monograph of Mahmudov [14] and papers [16–19] the necessary notions and results such as LAM properties in finite dimensional Euclidean spaces, Hamiltonian functions, argmaximum sets, locally tents, and set-valued mappings are given, etc. Then the problems for second order delay -DFIs are formulated.

In the first part of the paper, an optimal control problem in which the system dynamics are described by a so-called second order delay-DFIs are investigated, optimization of second order delay- DFIs is considered and sufficient conditions of optimality for delay- DFIs are proved.

By using separation theorems of convex analysis it is shown that in terms of Hamiltonian functions these optimality conditions can be rewritten in a more symmetrical form.

In the conclusion of section, is considered an example on the problem of so-called "linear" optimal control problem for the second order delay-differential equations.

2. Preliminary studies and problem statement

In this section we recall the key notions of set-valued mappings from the book [14]; let $\mathbb{R}^n$ be a n-dimensional Euclidean space, $\langle x, v\rangle$ be an inner product of elements $x, v \in \mathbb{R}^n$, (x, v) be a pair of x, v. Let's suppose that $F : \mathbb{R}^{3n} \rightrightarrows \mathbb{R}^n$ is a set-valued mapping from $\mathbb{R}^{3n}$ into the set of subsets of $\mathbb{R}^n$. Therefore $F : \mathbb{R}^{3n} \rightrightarrows \mathbb{R}^n$ is a convex set-valued mapping, if its graph $\text{gph}\, F = \{(x, u_1, u_2, v) : v \in F(x, u_1, u_2)\}$ is a convex subset of $\mathbb{R}^{4n}$. A set-valued mapping F is called closed if its $gphF$ is a closed subset in $\mathbb{R}^{4n}$. The domain of a set-valued mapping F is denoted by $\text{dom}\, F$ and is defined as $\text{dom}\, F = \{(x, u_1, u_2) : F(x, u_1, u_2) \neq \varnothing\}$. A set-valued mapping F is convex-valued if $F(x, u_1, u_2)$ is a convex set for each $(x, u_1, u_2) \in \text{dom}\, F$.

A set-valued mapping $F : \mathbb{R}^{3n} \rightrightarrows \mathbb{R}^n$ is said to be upper semicontinuous at (x^0, u_1^0, u_2^0) if for any neighbourhood U of zero in $\mathbb{R}^n$ there exists a neighborhood V of zero in $\mathbb{R}^{3n}$ such that

$$F(x, u_1, u_2) \subseteq F\left(x^0, u_1^0, u_2^0\right) + U, \forall (x, u_1, u_2) \in \left(x^0, u_1^0, u_2^0\right) + V.$$

The Hamiltonian function and argmaximum set corresponding to a set-valued mapping F are defined by the following relations

$$H_F(x, u_1, u_2, v^*) = \sup_v \{\langle v, v^*\rangle : v \in F(x, u_1, u_2)\}, v^* \in \mathbb{R}^n,$$
$$F_{Arg}(x, u_1, u_2; v^*) = F_A(x, u_1, u_2; v^*) = \{v \in F(x, u_1, u_2) : \langle v, v^*\rangle = H_F(x, u_1, u_2, v^*)\}$$

respectively. For a convex F we put $H_F(x, u_1, u_2, v^*) = -\infty$ if $F(x, u_1, u_2) = \varnothing$. In other terms, $H_F(x, u_1, u_2, v^*)$ is the support function to the set $F(x, u_1, u_2)$, evaluated at v^*.

s usual, $\text{int}\, Q$ denotes the interior of the set $Q \subset \mathbb{R}^{4n}$ and $\text{ri}\, Q$ denotes the relative interior of a set Q, i.e. the set of interior points of Q with respect to its affine hull $\text{Aff}\, Q$. The closure of Q is denoted by $\text{cl}Q$.

A convex cone $K_Q(z_0), z_0 = \left(x^0, u_1^0, u_2^0, v^0\right)$ is called a cone of tangent directions at a point $z_0 \in Q$ to the set Q if from $\bar{z} = (\bar{x}, \bar{u}_1, \bar{u}_2, \bar{v}) \in K_Q(z_0)$ it follows that $\bar{z}$ is a tangent vector to the set Q at a point $z_0 \in Q$, i.e., there exists such function $q(\alpha) \in \mathbb{R}^{4n}$ that $z_0 + \alpha\bar{z} + q(\alpha) \in Q$ for sufficiently small $\alpha > 0$ and $\alpha^{-1}q(\alpha) \to 0$, as $\alpha \downarrow 0$.

We have already seen that the cone of tangent directions involve directions for each of which there exists a function $q(\alpha)$. But in order to predetermine properties of the set, Q, this is not sufficient. Nevertheless, the following notion of a local tent allow us to predetermine mapping in Q for nearest tangent directions among themselves.

Definition 2.1. A cone of tangent directions $K_Q(z_0)$ is called local tent if for any $\bar{z}_0 \in riK_Q(z_0)$ there exists a convex cone $K \subseteq K_Q(z_0)$ and a continuous function $\gamma(\cdot)$ defined in the neighborhood of the origin, such that
(1) $\bar{z}_0 \in \text{ri}K$, $\text{Lin}\, K = \text{Lin}\, K_Q(z_0)$, where $\text{Lin}\, K$ is the linear span of K,
(2) $\gamma(\bar{z}) = \bar{z} + r(\bar{z}), r(\bar{z})\|\bar{z}\|^{-1} \to 0$ as $\bar{z} \to 0$,
(3) $z_0 + \gamma(\bar{z}) \in A, \bar{z} \in K \cap S_\varepsilon(0)$ for some $\varepsilon > 0$, where $S_\varepsilon(0)$ is the ball of radius ε.

Definition 2.2. With respect to [11] $h(\bar{x}, x)$ is called a convex upper approximation (CUA) of the function $g : \mathbb{R}^n \to \mathbb{R}^1\{\pm\infty\}$ at a point $x \in \text{dom}\, g = \{x : |g(x)| < +\infty\}$ if $h(\bar{x}, x) \geq V(\bar{x}, x)$ for all $\bar{x} \neq 0$ and $h(\cdot, x)$ is a convex closed positive homogeneous function, where

$$V(\bar{x}, x) = \sup_{r(\cdot)} \limsup_{\alpha\downarrow 0} \frac{1}{\alpha}[g(x + \alpha\bar{x} + r(\alpha)) - g(x)], \alpha^{-1}r(\alpha) \to 0.$$

Here the exterior supremum is taken on all $r(\alpha)$ such that $\alpha^{-1}r(\alpha) \to 0$ as $\alpha \downarrow 0$.

For a convex set-valued mapping $F : \mathbb{R}^{3n} \rightrightarrows \mathbb{R}^n$ a set-valued mapping defined by $F^* : \mathbb{R}^n \rightrightarrows \mathbb{R}^{3n}$

$$F^*(v^*; (x, u_1, u_2, v)) := \left\{\left(x^*, u_1^*, u_2^*\right) : \left(x^*, u_1^*, u_2^*, -v^*\right) \in K^*_{gphF}(x, u_1, u_2, v)\right\}$$
$$K_{gphF}(x, u_1, u_2, v) = \text{cone}\left[\text{gph}\, F - (x, u_1, u_2, v)\right], \forall \left(x^1, u_1^1, u_2^1, v^1\right) \in \text{gph}\, F$$

is called the LAM to F at a point $(x, u_1, u_2, v) \in gphF$, where $K^* = \{z^* : \langle \bar{z}, z^* \rangle \geq 0, \forall \bar{z} \in K\}$ denotes the dual cone to the cone K, as usual.
Below we will define the LAM to a set-valued mapping F by using the Hamiltonian function, associated to F. Thus, the LAM to nonconvex mapping F is defined as follows

$$F^*(v^*; (x, u_1, u_2, v)) := \left\{ \left(x^*, u_1^*, u_2^*\right) : H_F\left(x^1, u_1^1, u_2^1, v^*\right) - H_F(x, u_1, u_2, v^*) \leq \left\langle x^*, x^1 - x\right\rangle \right.$$
$$\left. + \left\langle u_1^*, u_1^1 - u_1\right\rangle + \left\langle u_2^*, u_2^1 - u_2\right\rangle, \forall \left(x^1, u_1^1, u_2^1\right) \in \mathbb{R}^{3n} \right\}, (x, u_1, u_2, v) \in \operatorname{gph} F, v \in F_A(x, u_1, u_2; v^*).$$

Clearly, for the convex mapping F the Hamiltonian function $H_F(\cdot, \cdot, v^*)$ is concave and the latter definition of LAM coincide with the previous definition of LAM Theorem 2.1 [14].

Note that prior to the LAM the notion of coderivative has been introduced for set-valued mappings in terms of the basic normal cone to their graphs by Mordukhovich [22] (however, for the smooth and convex maps the two notions are equivalent). In the most interesting settings for the theory and applications, coderivatives are nonconvex-valued and hence are not tangentially /derivatively generated. This is the case of the first coderivative for general finite dimensional setvalued mappings for the purpose of applications to optimal control.

For most of this paper we consider optimization problems with second order delay-DFIs and state constraints of the form, labelled as (PCH):

$$\text{infimum} \quad J(x(\cdot)) = \int_0^T g(x(t), t)dt, \tag{1}$$
$$(PCH) \quad \text{subject to} \quad x''(t) \in F(x(t), x'(t), x(t-h), t), \text{ a.e } t \in [0, T], \tag{2}$$
$$x(t) = \xi(t), t \in [-h, 0), x(0) = \theta, x(T) \in P \tag{3}$$

where $F(\cdot, t) : \mathbb{R}^{3n} \rightrightarrows \mathbb{R}^n$ and $g(\cdot, t)$ is time dependent set-valued mapping and continuous proper functions, respectively, $P \subseteq \mathbb{R}^n, \xi(t), t \in [-h, 0)$ is an absolutely continuous initial function, θ is a fixed vector. It is required to find a feasible trajectory (arc) $x(t), t \in [-h, T]$ minimizing the Lagrange type functional $J[x(\cdot)]$ over a set of feasible trajectories. Here, a feasible trajectory $x(t), t \in [-h, T]$ satisfies endpoint constraint $x(T) \in P$, almost everywhere (a.e.) the second order delay-DFI (with a possible jump discontinuity at $t = 0$), whose second order derivative in $[0, T]$ belongs to the standard Lebesgue space $L_1^n([0, T])$. In more detail, a feasible solution $x(\cdot)$ of (PCH) is a mapping $x(\cdot) : [-h, T] \to \mathbb{R}^n$ satisfying $x''(t) \in F(x(t), x'(t), x(t-h), t)$, a.e. $t \in [0, T]$, $x(t) = \xi(t)$ for all $t \in [-h, 0), x(T) \in P$ and $x(0) = \theta$ with $x(\cdot) \in AC([-h, T]) \cap W_{1,2}^n([0, T])$, where $AC([-h, T])$ is a space of absolutely continuous functions from $[-h, T]$ into $\mathbb{R}^n$ and $W_{1,2}^n([0, T])$ is a Banach space of absolutely continuous functions from $[0, T]$ into $\mathbb{R}^n$ together with the first order derivatives for which $x''(\cdot) \in L_1^n([0, T])$. Notice that a Banach space $W_{1,2}^n([0, T])$ can be equipped with the different equivalent norms.

Below we prove that an upper semi-continuous set-valued mapping $F(\cdot, t)$ (not necessarily convex) with closed values is closed ($\operatorname{gph} F(\cdot, t)$ is closed).

Lemma 2.3. *Let $F(\cdot, t) : \mathbb{R}^{3n} \rightrightarrows \mathbb{R}^n$ be an upper semi-continuous set-valued mapping and $F(x, u, t)$ be closed set for each $(x, u) \in \operatorname{dom} F(\cdot, t)$. Then $F(\cdot, t)$ is closed.*

Proof. We proceed by a contradiction argument; suppose that $(x_k, u_k, v_k) \in gphF(\cdot, t)$ is a convergent sequence and $(x_k, u_k, v_k) \to (x_0, u_0, v_0)$, but $v_0 \notin F(x_0, u_0, t)$. Then there exists an open set Λ containing $F(x_0, u_0, t)$ such that $v_0 \notin \operatorname{cl}\Lambda$. Recalling that the set-valued mapping $F(\cdot, t)$ is upper semi-continuous it follows that there exists a positive integer k_0 such that $v_k \in \Lambda$ for $\forall k > k_0$. Therefore, $v_0 \in \operatorname{cl}\Lambda$, a contradiction. □

3. Transversality conditions and optimization of second order delay- DFIs

The construction of sufficient optimality conditions for the problem (1)-(3) with secondorder delay DFI is largely based on the discrete approximation problem; due to the cumbersome nature of the calculations, it is omitted.

At first, let us formulate an adjoint delay-DFIs for convex problem (PCH):

$$(i) \quad \left(\frac{d^2x^*(t)}{dt^2} + \frac{d\psi^*(t)}{dt} - \eta^*(t+h), \psi^*(t), \eta^*(t)\right) \in F^*\left(x^*(t); (\tilde{x}(t), \tilde{x}'(t), \tilde{x}(t-h), \tilde{x}''(t)), t\right)$$
$$-\{\partial g(\tilde{x}(t), t)\} \times \{0\} \times \{0\}, \text{ a.e. } t \in [0, T-h), x^*(0) = 0$$
$$(ii) \quad \left(\frac{d^2x^*(t)}{dt^2} + \frac{d\psi^*(t)}{dt}, \psi^*(t), \eta^*(t)\right) \in F^*\left(x^*(t); (\tilde{x}(t), \tilde{x}'(t), \tilde{x}(t-h), \tilde{x}''(t)), t\right)$$
$$-\{\partial g(\tilde{x}(t), t)\} \times \{0\} \times \{0\}, \text{ a.e. } t \in [T-h, T]$$

and the transversality conditions at point $t = T$:

$$(iii) \quad -\frac{dx^*(T)}{dt} - \psi^*(T) \in K_P^*(\tilde{x}(T)); x^*(T) = 0$$

where $K_P(\tilde{x}(T))$ is a cone of tangent directions at a point $\tilde{x}(T) \in P$.
Here we assume that $x^*(t), t \in [0, T]$ is an absolutely continuous function together with the first order derivatives for which $x^{*\prime\prime}(\cdot) \in L_1^n[0, T]$. Moreover, $\psi^*(t), \eta^*(t), t \in [0, T]$ are absolutely continuous and $\psi^{*\prime}(\cdot) \in L_1^n[0, T]$.

The condition guaranteeing nonemptiness of the LAM F^* at a given point is the following

$$(iv) \quad \frac{d^2\tilde{x}(t)}{dt^2} \in F_A\left(\tilde{x}(t), \tilde{x}'(t), \tilde{x}(t-h); x^*(t), t\right), \text{ a.e. } t \in [0, T].$$

It appears that the following assertion is true.

Theorem 3.1. *Let $g : \mathbb{R}^n \times [0, T] \to \mathbb{R}^1$ be continuous and convex with respect to x function, and $F(\cdot, t) : \mathbb{R}^{3n} \rightrightarrows \Pi^n$ be a convex set-valued mapping. Then for optimality of the arc $\tilde{x}(t)$ to the convex problem (PCH) with second order delay-DFIs it is sufficient that there exist a triple $\{x^*(t), \psi^*(t), \eta^*(t)\}$ of absolutely continuous functions, $x^*(t), \psi^*(t), \eta^*(t), t \in [0, T]$ satisfying a.e. the second order Euler-Lagrange delay-DFIs (i), (ii), inclusion (iv) and transversality condition (iii).*

Proof. By Theorem 2. [14] $F^*\left(v^*, (x, u_1, u_2, v), t\right) = \partial_{(x,u_1,u_2)} H_F\left(x, u_1, u_2, v^*\right), v \in F_A\left(x, u_1, u_2; v^*, t\right)$. Then by using the Moreau-Rockafellar Theorem [8, 11, 22] and the convention that $-\partial_x g(\cdot, t) = \partial_x(-g(\cdot, t))$ from conditions (i), (ii) we obtain the second order adjoint DFIs rewritten in term of Hamiltonian function

$$\left(\frac{d^2x^*(t)}{dt^2} + \frac{d\psi^*(t)}{dt} - \eta^*(t+h), \psi^*(t), \eta^*(t)\right) \in \partial_{(x,u_1,u_2)}\left[H_F\left(\tilde{x}(t), \tilde{x}'(t), \tilde{x}(t-h), x^*(t)\right)\right]$$
$$-\{\partial g(\tilde{x}(t), t)\} \times \{0\} \times \{0\}, \quad \text{a.e. } t \in [0, T-h) \tag{4}$$

$$\left(\frac{d^2x^*(t)}{dt^2} + \frac{d\psi^*(t)}{dt}, \psi^*(t), \eta^*(t)\right) \in \partial_{(x,u_1,u_2)}\left[H_F\left(\tilde{x}(t), \tilde{x}'(t), \tilde{x}(t-h), x^*(t)\right)\right]$$
$$-\{\partial g(\tilde{x}(t), t)\} \times \{0\} \times \{0\}, \text{ a.e. } t \in [T-h, T] \tag{5}$$

Next, by definition of subdifferential of the Hamiltonian function H_F we rewrite relation (4) in the form:

$$H_F\left(x(t), x'(t), x(t-h), x^*(t)\right) - H_F\left(\tilde{x}(t), \tilde{x}'(t), \tilde{x}(t-h), x^*(t)\right) - g(x(t), t) + g(\tilde{x}(t), t)$$
$$\leq \left\langle \frac{d^2x^*(t)}{dt^2} + \frac{d\psi^*(t)}{dt} - \eta^*(t+h), x(t) - \tilde{x}(t)\right\rangle + \left\langle \psi^*(t), x'(t) - \tilde{x}'(t)\right\rangle$$
$$+ \left\langle \eta^*(t), x(t-h) - \tilde{x}(t-h)\right\rangle, \quad t \in [0, T-h) \tag{6}$$

By definition of the Hamiltonian function the inequality (6) can be converted to the relation

$$\left\langle \frac{d^2x(t)}{dt^2}, x^*(t)\right\rangle - \left\langle \frac{d^2\tilde{x}(t)}{dt^2}, x^*(t)\right\rangle - g(x(t), t) + g(\tilde{x}(t), t) \leq \left\langle \frac{d^2x^*(t)}{dt^2}, x(t) - \tilde{x}(t)\right\rangle$$
$$+ \frac{d}{dt}\left\langle \psi^*(t), x(t) - \tilde{x}(t)\right\rangle - \left\langle \eta^*(t+h), x(t) - \tilde{x}(t)\right\rangle + \left\langle \eta^*(t), x(t-h) - \tilde{x}(t-h)\right\rangle, t \in [0, T-h)$$

In turn, the latter inequality can be rewritten as follows

$$g(x(t), t) - g(\tilde{x}(t), t) \geq \left\langle \frac{d^2(x(t) - \tilde{x}(t))}{dt^2}, x^*(t)\right\rangle - \left\langle \frac{d^2x^*(t)}{dt^2}, x(t) - \tilde{x}(t)\right\rangle$$
$$- \frac{d}{dt}\left\langle \psi^*(t), x(t) - \tilde{x}(t)\right\rangle + \left\langle \eta^*(t+h), x(t) - \tilde{x}(t)\right\rangle - \left\langle \eta^*(t), x(t-h) - \tilde{x}(t-h)\right\rangle, t \in [0, T-h)$$

Integrating this inequality over the interval $[0, T-h)$ and taking into account that $x(0) = \tilde{x}(0) = \theta$ we can write

$$\int_0^{T-h}[g(x(t),t) - g(\tilde{x}(t),t)]dt \geq \int_0^{T-h}\left[\left\langle \frac{d^2(x(t)-\tilde{x}(t))}{dt^2}, x^*(t)\right\rangle - \left\langle \frac{d^2x^*(t)}{dt^2}, x(t)-\tilde{x}(t)\right\rangle\right]dt$$
$$+\langle \psi^*(0), x(0)-\tilde{x}(0)\rangle - \langle \psi^*(T-h), x(T-h)-\tilde{x}(T-h)\rangle + \int_0^{T-h}\langle \eta^*(t+h), x(t)-\tilde{x}(t)\rangle dt$$
$$-\int_0^{T-h}\langle \eta^*(t), x(t-h)-\tilde{x}(t-h)\rangle dt = \int_0^{T-h}\left[\left\langle \frac{d^2(x(t)-\tilde{x}(t))}{dt^2}, x^*(t)\right\rangle - \left\langle \frac{d^2x^*(t)}{dt^2}, x(t)-\tilde{x}(t)\right\rangle\right]dt \quad (7)$$
$$-\langle \psi^*(T-h), x(T-h)-\tilde{x}(T-h)\rangle + \int_0^{T-h}\langle \eta^*(t+h), x(t)-\tilde{x}(t)\rangle dt - \int_0^{T-h}\langle \eta^*(t), x(t-h)-\tilde{x}(t-h)\rangle dt$$

By similar way, it follows from second order Euler-Lagrange inclusion (5) that

$$g(x(t),t) - g(\tilde{x}(t),t) \geq \left\langle \frac{d^2(x(t)-\tilde{x}(t))}{dt^2}, x^*(t)\right\rangle - \left\langle \frac{d^2x^*(t)}{dt^2}, x(t)-\tilde{x}(t)\right\rangle$$
$$-\frac{d}{dt}\langle \psi^*(t), x(t)-\tilde{x}(t)\rangle - \langle \eta^*(t), x(t-h)-\tilde{x}(t-h)\rangle, \quad t \in [T-h, T]$$

Then an integration of this inequality over the interval $[T-h, T]$ give us

$$\int_{T-h}^{T}[g(x(t),t) - g(\tilde{x}(t),t)]dt \geq \int_{T-h}^{T}\left[\left\langle \frac{d^2(x(t)-\tilde{x}(t))}{dt^2}, x^*(t)\right\rangle - \left\langle \frac{d^2x^*(t)}{dt^2}, x(t)-\tilde{x}(t)\right\rangle\right]dt \quad (8)$$
$$+\langle \psi^*(T-h), x(T-h)-\tilde{x}(T-h)\rangle - \langle \psi^*(T), x(T)-\tilde{x}(T)\rangle - \int_{T-h}^{T}\langle \eta^*(t), x(t-h)-\tilde{x}(t-h)\rangle dt$$

Hence, summing the inequalities (7) and (8) we have

$$\int_0^{T}[g(x(t),t) - g(\tilde{x}(t),t)]dt \geq \int_0^{T}\left[\left\langle \frac{d^2(x(t)-\tilde{x}(t))}{dt^2}, x^*(t)\right\rangle - \left\langle \frac{d^2x^*(t)}{dt^2}, x(t)-\tilde{x}(t)\right\rangle\right]dt$$
$$-\langle \psi^*(T), x(T)-\tilde{x}(T)\rangle - \int_{T-h}^{T}\langle \eta^*(t), x(t-h)-\tilde{x}(t-h)\rangle dt + \int_0^{T-h}\langle \eta^*(t+h), x(t)-\tilde{x}(t)\rangle dt \quad (9)$$
$$-\int_0^{T-h}\langle \eta^*(t), x(t-h)-\tilde{x}(t-h)\rangle dt$$

On the other hand, because of the initial condition $x(t) = \tilde{x}(t) = \xi(t), t \in [-h, 0)$ we get $\int_{-h}^{0}\langle \eta^*(t+h), x(t)-\tilde{x}(t)\rangle dt = 0$. Then it is easy to compute the sum of the last three integrals on the right hand side of the inequality (9) as follows:

$$\int_0^{T-h}\langle \eta^*(t+h), x(t)-\tilde{x}(t)\rangle dt - \int_0^{T-h}\langle \eta^*(t), x(t-h)-\tilde{x}(t-h)\rangle dt - \int_{T-h}^{T}\langle \eta^*(t), x(t-h)-\tilde{x}(t-h)\rangle dt$$
$$= \int_0^{T-h}\langle \eta^*(t+h), x(t)-\tilde{x}(t)\rangle dt - \int_{-h}^{T-2h}\langle \eta^*(t+h), x(t)-\tilde{x}(t)\rangle dt - \int_{T-2h}^{T-h}\langle \eta^*(t+h), x(t)-\tilde{x}(t)\rangle dt$$
$$= \int_0^{T-h}\langle \eta^*(t+h), x(t)-\tilde{x}(t)\rangle dt - \int_0^{T-h}\langle \eta^*(t+h), x(t)-\tilde{x}(t)\rangle dt = 0$$

Thus, the inequality (9) can be simplified as follows

$$\int_0^{T}[g(x(t),t) - g(\tilde{x}(t),t)]dt \geq \int_0^{T}\left[\left\langle \frac{d^2(x(t)-\tilde{x}(t))}{dt^2}, x^*(t)\right\rangle - \left\langle \frac{d^2x^*(t)}{dt^2}, x(t)-\tilde{x}(t)\right\rangle\right]dt \quad (10)$$
$$-\langle \psi^*(T), x(T)-\tilde{x}(T)\rangle$$

Now we transform the expression in the square parentheses on the right hand side of (10):

$$\left\langle \frac{d^2(x(t)-\tilde{x}(t))}{dt^2}, x^*(t)\right\rangle - \left\langle \frac{d^2x^*(t)}{dt^2}, x(t)-\tilde{x}(t)\right\rangle$$
$$= \frac{d}{dt}\left\langle \frac{d(x(t)-\tilde{x}(t))}{dt}, x^*(t)\right\rangle - \frac{d}{dt}\left\langle \frac{dx^*(t)}{dt}, x(t)-\tilde{x}(t)\right\rangle.$$

Then we use again the simplest and most useful particular case $x(0) = \tilde{x}(0) = \theta$ of feasibility solution of (PCH) and the condition $x^*(0) = x^*(T) = 0$ of theorem. Then the integral on the right hand side of (10) over the interval $[0, T]$ can be computed as follows

$$\int_0^T \left[\left\langle \frac{d^2(x(t)-\tilde{x}(t))}{dt^2}, x^*(t)\right\rangle - \left\langle \frac{d^2x^*(t)}{dt^2}, x(t)-\tilde{x}(t)\right\rangle\right] dt$$
$$= \left\langle \frac{d(x(T)-\tilde{x}(T))}{dt}, x^*(T)\right\rangle - \left\langle \frac{d(x(0)-\tilde{x}(0))}{dt}, x^*(0)\right\rangle \tag{11}$$
$$- \left\langle \frac{dx^*(T)}{dt}, x(T)-\tilde{x}(T)\right\rangle + \left\langle \frac{dx^*(0)}{dt}, x(0)-\tilde{x}(0)\right\rangle = -\left\langle \frac{dx^*(T)}{dt}, x(T)-\tilde{x}(T)\right\rangle.$$

By substituting the expression of the integral (11) into the inequality (10) we deduce that

$$\int_0^T [g(x(t),t) - g(\tilde{x}(t),t)]dt \geq -\left\langle \frac{dx^*(T)}{dt}, x(T)-\tilde{x}(T)\right\rangle \tag{12}$$
$$- \langle \psi^*(T), x(T)-\tilde{x}(T)\rangle = -\left\langle \frac{dx^*(T)}{dt} + \psi^*(T), x(T)-\tilde{x}(T)\right\rangle$$

But since by the definition of the dual cone

$$K_P^*(\tilde{x}(T)) = \{x^* : \langle x^*, x(T)-\tilde{x}(T)\rangle, \forall x(T) \in K_P(\tilde{x}(T))\}$$

from condition (iii) of the theorem that it follows that $-\frac{dx^*(T)}{dt} - \psi^*(T) \geq 0$. Thus, from the inequality (12) we obtain that

$$\int_0^1 [g(x(t),t) - g(\tilde{x}(t),t)]dt \geq 0$$

Finally, we obtain that $J[x(t)] \geq J[\tilde{x}(t)], \forall x(t), t \in [0,T]$, i.e. $\tilde{x}(t), t \in [0,T]$ is optimal. □

Below we prove that if a mapping F depends only on x, then the adjoint inclusion involves only one conjugate variable, that is, there are no auxiliary adjoint variables $\eta^*(t), u^*(t)$ in the conjugate second order delay-DFIs. Apparently, this occurs because a mapping F doesn't depend on derivatives $x'(t), x'(t-h)$.

Corollary 3.2. *Suppose that for the problem (PCH) with second order delay-DFIs a set-valued mapping F depends only on x, that is, $F(\cdot,t) \equiv G(\cdot,t) : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ and that the conditions of Theorem 3.1 are satisfied. Then the second order Euler-Lagrange delay-DFIs and transversality condition (iii) of Theorem 3.1 consist of the following*

$$(i) \quad \frac{d^2x^*(t)}{dt^2} \in G^*\left(x^*(t); (\tilde{x}(t), \tilde{x}''(t)), t\right) - \partial g(\tilde{x}(t),t) \text{ a.e. } t \in [0,T], x^*(0) = 0,$$
$$(ii) \quad -\frac{dx^*(1)}{dt} \in K_p^*(\tilde{x}(1)), x^*(T) = 0,$$
$$(iii) \quad \frac{d^2\tilde{x}(t)}{dt^2} \in G_A\left(\tilde{x}(t); x^*(t), t\right), \text{ a.e. } t \in [0,T].$$

Proof. Indeed, it remains only establish the second order Euler-Lagrange delay-DFIs and transversality condition. This is an immediate consequence of Theorem 3.1; by passing to the formal limit in the conditions of this theorem we have the needed result. □

Corollary 3.3. *In addition, to assumptions of Theorem 3.1 let F be a closed set-valued mapping. Then the conditions (i), (ii), (iv) of Theorem 3.1 can be rewritten in term of subdifferentials of Hamiltonian function in the more symmetric form.*

Proof. Using Lemma 2.3 we should prove only the validity of the inclusion

$$\frac{d^2\tilde{x}(t)}{dt^2} \in \partial_{v^*} H_F\left(\tilde{x}(t), \tilde{x}'(t), \tilde{x}(t-h); x^*(t)\right), \text{ a.e. } t \in [0,T]. \tag{13}$$

Indeed, by Lemma 2.3 the argmaximum set at a given point is the subdifferential of the Hamiltonian function with respect to v^* and the inclusion (iv) of Theorem 3.1 coincides with the inclusion (13). Therefore, the assertions of corollary are equivalent to the conditions (i), (ii), (iv) of Theorem 3.1. □

Corollary 3.4. *For the problem (PCH) with non-delayed second order differential inclusions*

$$\begin{array}{ll} \textit{infimum} & J(x(\cdot)) = \int_0^T g(x(t),t)dt \\ \textit{subject to} & x''(t) \in F_0\left(x(t),x'(t),t\right), \ a.e\ t \in [0,T] \\ & x(0)=\theta, t \in [0,T], x(T) \in P \end{array}$$

the second order Euler-Lagrange inclusion has a form

$$\left(\frac{d^2x^*(t)}{dt^2}+\frac{d\psi^*(t)}{dt},\psi^*(t)\right) \in F_0^*\left(x^*(t);(\tilde{x}(t),\tilde{x}'(t),\tilde{x}''(t)),t\right) - \{\partial g(\tilde{x}(t),t)\} \times \{0\} \ \ a.e.\ t \in [0,T]$$

Proof. Indeed, in this case $F_0(\cdot,t) : \mathbb{R}^{2n} \rightrightarrows \mathbb{R}^n$ and $F(x,u_1,u_2,t) = F_0(x,u_1,t), \forall u_2 \in \mathbb{R}^n$ is defined as $F^*(v^*;(x,u_1,u_2,v),t) = F_0^*(v^*;(x,u_1,v),t) \times \{0\}$. It means that $u_2^* = 0$ and consequently, $\eta^*(t) \equiv 0,\ t \in [0,1]$. Then the proof of the corollary follows immediately from the conditions (i), (ii) of Theorem 3.1. □

Theorem 3.5. *Suppose that $g : \mathbb{R}^n \times [0,T] \to \mathbb{R}^1$ is nonconvex function with respect to x, and F is a nonconvex set-valued mapping such that $K_{gphF(\cdot,t)}(\tilde{x}(t),\tilde{x}'(t),\tilde{x}(t-h),\tilde{x}''(t))$ is a local tent. Besides, suppose that $K_P(\tilde{x}(T)), \tilde{x}(T) \in P$ is a local tent. Then for an optimality of the arc $\tilde{x}(t), t \in [0,T]$ among all feasible solutions in such a nonconvex problem (PCH), it is sufficient that there exist a triple $\{x^*(t),\psi^*(t),\eta^*(t)\}$ of absolutely continuous functions $x^*(t), x^{*\prime}(t), \psi^*(t), \eta^*(t), t \in [0,T]$ satisfying the conditions of Theorem 3.1 in the nonconvex case:*

$$(i)\quad \left(\tfrac{d^2x^*(t)}{dt^2}+\tfrac{d\psi^*(t)}{dt}-\eta^*(t+h)+x^*(t),\psi^*(t),\eta^*(t)\right) \in F^*\left(x^*(t);(\tilde{x}(t),\tilde{x}'(t),\tilde{x}(t-h),\tilde{x}''(t)),t\right), \\ a.e.\ t \in [0,T-h), x^*(0)=0;$$

$$(ii)\quad \left(\tfrac{d^2x^*(t)}{dt^2}+\tfrac{d\psi^*(t)}{dt}+x^*(t),\psi^*(t),\eta^*(t)\right) \in F^*\left(x^*(t);(\tilde{x}(t),\tilde{x}'(t),\tilde{x}(t-h),\tilde{x}''(t)),t\right); \ \ a.e.\ t \in [T-h,T];$$

$$(iii)\quad \tfrac{d^2\tilde{x}(t)}{dt^2} \in F_A\left(\tilde{x}(t),\tilde{x}'(t),\tilde{x}(t-h);x^*(t),t\right), \ a.e. t \in [0,T],$$

$$(iv)\quad g(x,t)-g(\tilde{x}(t),t) \geq \langle x^*(t), x-\tilde{x}(t)\rangle, t \in [0,T], \forall x \in \mathbb{R}^n, x^*(T)=0.$$

Proof. By condition (i) of theorem and definition of LAM in the nonconvex case (see Section 2)

$$\left\langle \tfrac{d^2x(t)}{dt^2},x^*(t)\right\rangle - \left\langle \tfrac{d^2\tilde{x}(t)}{dt^2},x^*(t)\right\rangle \leq \left\langle \tfrac{d^2x^*(t)}{dt^2},x(t)-\tilde{x}(t)\right\rangle + \langle x^*(t),x(t)-\tilde{x}(t)\rangle \\ +\tfrac{d}{dt}\langle \psi^*(t),x(t)-\tilde{x}(t)\rangle - \langle \eta^*(t+h),x(t)-\tilde{x}(t)\rangle + \langle \eta^*(t),x(t-h)-\tilde{x}(t-h)\rangle, t \in [0,T-h)$$

whereas by the condition (iv) of theorem for $x = x(t)$ we can write

$$g(x(t),t)-g(\tilde{x}(t),t) \geq \left\langle \tfrac{d^2(x(t)-\tilde{x}(t))}{dt^2},x^*(t)\right\rangle - \left\langle \tfrac{d^2x^*(t)}{dt^2},x(t)-\tilde{x}(t)\right\rangle \\ -\tfrac{d}{dt}\langle \psi^*(t),x(t)-\tilde{x}(t)\rangle + \langle \eta^*(t+h),x(t)-\tilde{x}(t)\rangle - \langle \eta^*(t),x(t-h)-\tilde{x}(t-h)\rangle, t \in [0,T-h) \tag{14}$$

By a similar way for $x = x(t)$ we obtain

$$g(x(t),t)-g(\tilde{x}(t),t) \geq \left\langle \tfrac{d^2(x(t)-\tilde{x}(t))}{dt^2},x^*(t)\right\rangle - \left\langle \tfrac{d^2x^*(t)}{dt^2},x(t)-\tilde{x}(t)\right\rangle \\ -\tfrac{d}{dt}\langle \psi^*(t),x(t)-\tilde{x}(t)\rangle - \langle \eta^*(t),x(t-h)-\tilde{x}(t-h)\rangle, \quad t \in [T-h,T] \tag{15}$$

In the proof of Theorem 3.1 from the latter inequalities (14), (15) is justified (10) for a nonconvex case. Thus, the furthest proof of theorem is similar to the one for Theorem 3.1. □

We note that in the convex case, the conditions (iv), (v) of Theorem 3.5 are equivalent to the conditions $x^*(t) \in \partial_x g(\tilde{x}(t),t)$ and (iii) of Theorem 3.1, respectively. Then for a convex problem (PCH) it is easy to see that the previous conditions (i) and (ii) of Theorem 3.1 coincide with the conditions (i) and (ii) of Theorem 3.5, respectively.

In the conclusion of this section, let us consider an example on the problem of so-called "linear" optimal control problem for the second order delay-differential equations:

$$
\begin{array}{ll}
\text{minimize} & J[x(\cdot)] = \int_0^T g(x(t),t)dt \\
(PLH) & x''(t) = A_0x(t) + A_1x'(t) + A_2x(t-h) + Bu(t), \text{ a.e. } t \in [0,T] \\
& x(t) = \xi(t), t \in [-h,0), x(0) = \theta, x(T) \in P, u(t) \in U \subseteq \mathbb{R}^r
\end{array}
$$

where g is continuously differentiable function in $x, A_i, i = 0,1,2$ and B are $n \times n$ and $n \times r$ matrices, respectively, $U \subseteq \mathbb{R}^r$ is a convex closed subset. The problem is of finding corresponding to the controlling parameter $\tilde{w}(t) \in U$ an arc $\tilde{x}(t)$, minimizing $J[x(\cdot)]$ over a set of feasible solutions.
We transform this problem to the following problem with second order delay-DFIs of the form:

$$
\begin{array}{ll}
\text{minimize} & J[x(\cdot)] = \int_0^T g(x(t),t)dt \\
& x''(t) \in F(x(t), x'(t), x(t-h)), \text{ a.e. } t \in [0,T] \\
& x(t) = \xi(t), t \in [-h,0), x(0) = \theta, \quad x(T) \in P \\
& F(x,u_1,u_2) = A_0x + A_1u_1 + A_2u_2 + BU
\end{array}
\tag{16}
$$

where an admissible arc $x(\cdot)$ is absolutely continuous function together with the first order derivatives for which $x^{*\prime\prime}(\cdot) \in L_1^n([0,T])$.

Theorem 3.6. *The arc $\tilde{x}(t)$ corresponding to the controlling parameter $\tilde{w}(t)$ minimizes $J(x(\cdot))$ over a set of feasible solutions in the convex second order delay-differential problem (PLH), if there exists an absolutely continuous function $x^*(t)$ together with the first order derivatives, satisfying the second order an adjoint delay-differential equation, the transversality and Weierstrass-Pontryagin conditions :*

$$
\begin{array}{c}
\frac{d^2x^*(t)}{dt^2} = A_0^*x^*(t) - A_1^*\frac{dx^*(t)}{dt} + A_2^*x^*(t+h) - g'(\tilde{x}(t),t), \text{ a.e. } t \in [0,T-h) \\
\frac{d^2x^*(t)}{dt^2} = A_0^*x^*(t) - A_1^*\frac{dx^*(t)}{dt} - g'(\tilde{x}(t),t), \quad \text{a.e. } t \in [T-h,T] \\
-\frac{dx^*(T)}{dt} \in K_P^*(\tilde{x}(T)); \quad x^*(0) = x^*(T) = 0 \\
\langle B\tilde{w}(t), x^*(t)\rangle = \sup_{w\in U} \langle Bw, x^*(t)\rangle, \quad t \in [0,T]
\end{array}
$$

Proof. In this problem we are proceeding on the basic of Theorem 3.1 Thus, taking into account that $F(x,u_1,u_2) \equiv A_0x + A_1u_1 + A_2u_2 + BU$ in the problem (16) it can be easily computed that

$$
\begin{aligned}
H_F(x,u_1,u_2,v^*) &= \sup_v \{\langle v, v^*\rangle : v \in F(x,u_1,u_2)\} \\
&= \sup_v \{\langle A_0x + A_1u_1 + A_2u_2 + Bw, v^*\rangle : v \in F(x,u_1,u_2)\} \\
&= \left\langle x, A_0^*v^*\right\rangle + \left\langle u_1, A_1^*v^*\right\rangle + \left\langle u_2, A_2^*v^*\right\rangle + \sup_w \{\langle Bw, v^*\rangle : w \in U\},
\end{aligned}
$$

where A^* is adjoint (transposed) matrix of A. Then by Theorem 2.1 [14] one has

$$
F^*(v^*; (\tilde{x}, \tilde{u}_1, \tilde{u}_2, \tilde{v})) = \begin{cases} \left(A_0^*v^*, A_1^*v^*, A_2^*v^*\right), & -B^*v^* \in K_U^*(\tilde{w}) \\ \varnothing, & -B^*v^* \notin K_U^*(\tilde{w}) \end{cases} \tag{17}
$$

where $\tilde{v} = A_0\tilde{x} + A_1\tilde{u}_1 + A_2\tilde{u}_2 + B\tilde{w}, \tilde{w} \in U$. Thus, using (17) and the relations (i), (ii) of Theorem 3.1 we have the following system of Euler-Lagrange's type linear adjoint equations:

$$
\begin{array}{l}
\frac{d^2x^*(t)}{dt^2} + \frac{d\psi^*(t)}{dt} - \eta^*(t+h) = A_0^*x^*(t) - g'(\tilde{x}(t),t) \\
\psi^*(t) = A_1^*x^*(t), \eta^*(t) = A_2^*x^*(t), \quad \text{a.e. } t \in [0,T-h)
\end{array}
\tag{18}
$$

$$
\begin{array}{l}
\frac{d^2x^*(t)}{dt^2} + \frac{d\psi^*(t)}{dt} = A_0^*x^*(t) - g'(\tilde{x}(t),t), \psi^*(t) = A_1^*x^*(t) \\
\eta^*(t) = A_2^*x^*(t), \quad \text{a.e. } t \in [T-h,T]
\end{array}
\tag{19}
$$

Substituting the expressions for $\psi^*(t), \eta^*(t)$ into equations (18), (19) we have second order Euler-Lagrange type adjoint DFIs (equations):

$$\frac{d^2x^*(t)}{dt^2} = A_0^*x^*(t) - A_1^*\frac{dx^*(t)}{dt} + A_2^*x^*(t+h) - g'(\tilde{x}(t),t), \text{ a.e. } t \in [0, T-h), \quad x^*(0) = 0 \tag{20}$$

$$\frac{d^2x^*(t)}{dt^2} = A_0^*x^*(t) - A_1^*\frac{dx^*(t)}{dt} - g'(\tilde{x}(t),t), \quad \text{a.e. } t \in [T-h, T]. \tag{21}$$

On the other hand, since $\psi^*(T) = A_1^*x^*(T)$ and $x^*(T) = 0$ it follows that the transversality conditions (iii) of Theorem 3.1 for linear optimal control problem (PLH) consist of the following

$$-\frac{dx^*(1)}{dt} \in K_P^*(\tilde{x}(T)); \quad x^*(T) = 0. \tag{22}$$

Moreover, the Weierstrass-Pontryagin maximum principle [14, 22] of theorem is an immediate consequence of the conditions (iv) of Theorem 3.1 and formula (17). Indeed, the condition $-B^*v^* \in K_U^*(\tilde{w})$ means that $\sup_{w\in U} \langle Bw, v^*\rangle = \langle B\tilde{w}, v^*\rangle$ and finally,

$$\langle B\tilde{w}(t), x^*(t)\rangle = \sup_{w\in U} \langle Bw, x^*(t)\rangle, \quad t \in [0, T]$$

Then by this maximum principle and relations (20)-(22) we have the desired result. The proof is completed. □

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