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Optimal control of 2-D wave differential inclusions with state constraints

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Abstract. The paper discusses the optimization of 2-D wave differential inclusions (DFIs) with the Laplacian in a bounded cuboid and the first mixed initial-boundary-value problem. Particular attention is paid to problems with state constraints, for which optimality conditions are formulated in terms of the Euler-Lagrange adjoint inclusions. Next, using the well-known Green's formula, the obtained results are generalized to the multidimensional case. In problems with convex inequalities, dual cones are calculated, which are an integral part of Euler-Lagrange inclusions. The examples show that when constructing an adjoint inclusion, it is very important to consider the phase boundary".

1. Introduction

The problem of optimal control of ordinary [2, 4, 6–8, 11, 14, 16–22, 31] and partial differential equations/inclusions [5, 10, 12–15, 23–25, 27, 28, 30] occurs in many applications, from engineering to science in economic dynamics, classical optimal control problems, differential games, etc. The article [23] is devoted to the optimization of gradient DFIs in a rectangular area, optimality conditions are obtained. The results obtained in terms of the Euler-Lagrange adjoint inclusion divergence operation are extended to the multidimensional case. Such results are based on LAMs, related to Mordukhovich's coderivative concept. The article [15] concerns to the optimal control problem defined by hyperbolic discrete and DFIs. An approximation method is used to formulate sufficient optimality conditions for a continuous problem. To construct associated partial DFIs, equivalence theorems for LAMs are proven. In [25], dynamic optimization problems for hyperbolic systems with boundary controls and pointwise state constraints are considered, and the necessary conditions of optimality are derived. In contrast to parabolic dynamics, such systems have not been sufficiently studied in the literature. In [20], based on the so-called adjoint DFIs of the Euler-Lagrange type of higher order and the Hamiltonian, a sufficient optimality condition is obtained for higher-order DFIs. It is proved that the adjoint inclusion for first-order DFI, defined in terms of LAM, coincides with the classical Euler-Lagrange inclusion. Optimality conditions are formulated for some examples. The work [28]

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is devoted to the optimization of the Mayer problem with Darboux-type DFI and duality. It is shown that each pair of solutions to the primal and dual problems satisfies the duality relations and that the optimal values in the primal convex and dual concave problems are equal. The article [21] is mainly devoted to the theory of duality of boundary value problem for ordinary higher-order DFIs, based on LAM in the form of Euler-Lagrange type inclusions and transversality conditions, sufficient optimality conditions obtained. It is noteworthy that inclusions of Euler-Lagrange type for simple and dual problems are "duality relations". The author's book [14] is a complete study on the approximation and optimization of ordinary and partial DFEs. In [24], the optimization of hyperbolic equations with controls under Neumann boundary conditions was carried out. Focusing on the multidimensional wave equation with a nonlinear term, the necessary conditions for optimality are derived. In [30], a convex optimal control problem is considered, including a class of linear hyperbolic partial differential systems. A computational algorithm has been developed that generates minimizing control sequences.

There are also interesting problems devoted to some qualitative problems of various ordinary/partial differential equations/inclusions [1–3, 5, 9, 10, 29, 32]; the main goal of the paper [2] is to provide new explicit criteria for characterizing weak lower semicontinuous Lyapunov pairs or functions associated with first-order DFIs in Hilbert spaces. These dual criteria are expressed through the proximal and main subdifferentials of the nominal functions. In the paper [1], using the Picard operator technique, some Ulam-type existence and stability results are investigated for the Darboux problem associated with some partial fractional order differential inclusions. The paper [5] proves the existence of global set-valued solutions of the Cauchy problem for partial differential equations and DFIs with both single valued and set-valued initial conditions.

As mentioned above, optimal control problems described by partial DFIs are increasingly encountered in various applications. Hyperbolic differential equations/inclusions arise in many applied problems such as string vibrations, acoustic modelling, supersonic fluid flows, etc.

The novelty of the article is the use of cones of tangent directions and their dual cones in methods for solving problems with 2D wave - hyperbolic DFIs and with phase constraints, which can be useful for problems with parabolic and elliptic DFIs. And when deriving these conditions in problems of this kind, it is inevitable to use the discretized method, equivalence theorems and the passage to the limit to a continuous problem, accompanied by complex cumbersome calculations, and therefore are omitted.

The paper is organized in the following order: In Section 2, for the convenience of readers, some concepts are presented, such as the Hamilton function and the argmaximum sets from the book [14], and the main problem with state constraints are stated. In Section 3, we formulate the main result on sufficient optimality conditions for hyperbolic DFIs with a 2-D self-adjoint Laplacian. Further, by specifying set-valued mapping $\Omega : \mathbb{R}^3 \rightrightarrows \mathbb{R}^n$ nusing the solution set of systems of inequalities, the adjoint differential inclusion is analysed in detail. At the same time, it is interesting to note that the Euler-Lagrange type adjoint DFIs are accompanied by homogeneous boundary conditions and "endpoint conditions" instead of initial conditions. In Section 4, Theorem 3.1 is generalized by analogy to multidimensional hyperbolic DFIs; When proving the optimality condition, Green's formula is used. Some examples are also discussed. Section 5 considers a model with a polyhedral set-valued mapping, using Farkas' Theorem and LAM formulates the optimality condition in terms of "Lagrangian" variables and concludes with a specific example. These examples show that when constructing an adjoint equation, only the "boundary points" of the constraint states are important.

2. Preliminary information and problem statement

Let us give the necessary definitions and concepts from the book [14]; let $\langle v, u\rangle$ is a scalar product of a pair $(v, u) \in \mathbb{R}^n$. A set-valued mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is convex if $\operatorname{gph} F = \{(v, u) : u \in F(v)\}$ is convex. For a set-valued mapping F, we introduce the Hamilton function and the argmaximum set: $H_F(v, u^*) = \sup_u \{\langle u, u^*\rangle : u \in F(v)\}, u^* \in \mathbb{R}^n; F_A(v; u^*) = \{u \in F(v) : \langle u, u^*\rangle = H_F(v, u^*)\}$
For a convex set-valued F we let $H_F(v, u^*) = +\infty$, if $F(v) = \varnothing$.

The cone of tangent directions at a point $(\tilde{v}, \tilde{u}) \in \operatorname{gph} F$ will be denoted by $K_F(\tilde{v}, \tilde{u}) \equiv K_{\mathrm{gph}}(\tilde{v}, \tilde{u})$:

$$K_{\text{ghh}}(\tilde{v},\tilde{u}) = \text{cone}(\text{gph}\,F - (\tilde{v},\tilde{u})) = \{(\bar{v},\bar{u}) :$$
$$\bar{v} = \gamma(v-\tilde{v}), \bar{u} = \gamma(u-\tilde{u}), \gamma > 0, (v,u) \in \text{gph}\,F\}.$$

The cone $K_A(v,u), (v,u) \in A$ of tangent directions of nonconvex set $A \subset \mathbb{R}^{2n}$ is defined as the set of $(\bar{v},\bar{u})$, for which there exists a function $\varphi : (0,+\infty) \to \mathbb{R}^{2n}$ with $\varphi(\gamma) \to 0$ as $\gamma \downarrow 0$ such that $(v,u)+\gamma(\bar{v},\bar{u})+\varphi(\gamma) \in A$ for sufficiently small γ. Moreover, in order to avoid choosing the function φ for each tangent direction, can be used the concepts of a local tent [14].

Recall that a set-valued mapping $F^*(\cdot;(\tilde{v},\tilde{u})) : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ defined by

$$F^*(u^*;(\tilde{v},\tilde{u})) = \{v^* : (v^*,-u^*) \in K_F * (\tilde{v},\tilde{u})\}$$

is called the LAM to F where $K_F{}^*(\tilde{v},\tilde{u})$ is the dual cone. In terms of the Hamiltonian function, LAM in the "non-convex" case is defined as follows:

$$F^*(u^*;(\tilde{v},\tilde{u})) := \{v^* : H_F(v,u^*) - H_F(\tilde{v},u^*) \le \langle v^*, v-\tilde{v}\rangle, \forall v \in \mathbb{R}^n\}, \tilde{u} \in F_A(\tilde{v};u^*).$$

Note that for a convex set-valued mapping $H_F(\cdot,u^*)$ is concave function. Now we present the following important theorem, which appears repeatedly throughout the article.

Theorem 2.1. ([14]) *Let $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a convex mapping. Then*

$$F^*(u^*;(v,u)) = \begin{cases} \partial H_F(v,u^*), & u \in F_A(v;u^*), \\ \varnothing, & u \in F_A(v;u^*), \end{cases} \text{ where } \partial H_F(v,u^*) = -\partial[-H_F(v,u^*)].$$

Note that the concept of LAM given in the article is closely related to the concept of coderivative concept of [24, 25], which is essential in the non-convex case.

The domain and epigraph of a function f are defined as follows

$$\text{dom}\, f = \{x : f(x) < +\infty\}, \textit{epi } f = \left\{\left(x^0,x\right) : x^0 \ge f(x)\right\}.$$

It is important to note that $\left(x^0,x\right) \in \text{epi}\, f$ only when $x \in \text{dom}\, f$, since if $f(x) = +\infty$, then there is no real number x^0 such that $x^0 \ge f(x)$.

In Section 5, we consider the optimal control of 2-D wave DFIs with state constraints. Although the wave equation is a special type of PDE that describes the behaviour of waves such as light, sound or water waves, we also study under this name the optimization of the problem for hyperbolic DFIs in two spatial dimensions, containing a self-adjoint Laplacian in a bounded cuboid:

$$\text{minimize } J[v(\cdot,\cdot,\cdot)] = \int_0^T \iint_Q f(v(x,y,t),x,y,t)dxdydt, \tag{1}$$

$$(WHP) \qquad \frac{\partial^2 v(x,y,t)}{\partial t^2} - \nabla^2 v(x,y,t) \in F(v(x,y,t),x,y,t), \quad (x,y,t) \in Q\times[0,T], \tag{2}$$

$$v(x,y,t) \in \Omega(x,y,t), \quad (x,y,t) \in Q\times[0,T], \quad Q = [0,L]\times[0,S], \tag{3}$$
$$v(x,y,0) = \alpha_1(x,y), v_t(x,y,0) = \alpha_2(x,y), v(x,0,t) = \beta_0(x,t),$$

$$v(x,S,t) = \beta_S(x,t), v(0,y,t) = \gamma_0(y,t), \quad v(L,y,t) = \gamma_L(y,t), \tag{4}$$

where $F(\cdot,x,y,t) : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ and $\Omega : \mathbb{R}^3 \rightrightarrows \mathbb{R}^n$ are a convex set-valued mappings, $f(\cdot,x,y,t)$ is proper convex function, ∇^2 is a 2-D Laplacian: $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ and α_1,α_2 and $\beta_0,\beta_S,\gamma_0,\gamma_L$ are given continuous functions, $\alpha_1,\alpha_2 : Q \to \mathbb{R}^n, \beta_0,\beta_S : [0,L]\times[0,T] \to \mathbb{R}^n; \gamma_0,\gamma_L : [0,S]\times[0,T] \to \mathbb{R}^n; L,S,T$ are real positive numbers. Since hyperbolic DFI (2) is second order in time, values of $v(x,y,t)$ and its partial derivative on t must be specified along the initial time. On the other hand, since DFI (2) is the second order in space, the values of $v(x,y,t)$ must be specified along boundaries of the region $Q\times[0,T]$. A rectangular parallelepiped $Q\times[0,T]$ will also be called a cuboid for brevity. We label this problem (WHP). The problem consists in finding a solution $\tilde{v}(x,y,t)$ of a first mixed initial-boundary value problem (WHP) that minimizes (1). We assume throughout the context that feasible solutions are classical solutions; let $P_T = Q\times(0,T), \Gamma_T = \{(x,y)\in\partial Q, \quad 0<t<T\}, \partial Q$ is the boundary of $Q, M_0 = \{(x,y)\in Q, t=0\}$. Then a function $v(x,y,t)\in C^2(P_T)\cap C^1\left(P_T\cup\Gamma_T\cup\bar{M}_0\right)\cup\Gamma_T\cup\bar{M}_0\big)$ satisfying the DFI (2) and condition $v(x,y,0) = \alpha_1(x,y), v_t(x,y,0) = \alpha_2(x,y)$ on M_0, and the boundary conditions in Γ_T is called the classical solution of the initial-boundary value problem (1)-(4). Note that the definition of a solution in the classical sense in no way prevents the implementation of this method for other (classical, generalized, almost everywhere, etc.) classes of solutions.

3. Optimality conditions for wave DFIS with state constraints

To formulate the theorem on the optimality condition, we first introduce the hyperbolic type DFI with selfadjoint Laplacian, the associated argmaximum conditions and a homogeneous "endpoint" and boundary conditions:

$$(i)\quad \frac{\partial^2 v^*(x,y,t)}{\partial t^2} - \nabla^2 v^*(x,y,t) \in F^*\left(v^*(x,y,t);\left(\tilde{v}(x,y,t),\tilde{v}_{tt}(x,y,t)-\nabla^2\tilde{v}(x,y,t)\right),x,y,t\right)$$
$$-\partial f(\tilde{v}(x,y,t),x,y,t) + K^*_{\Omega}(\tilde{v}(x,y,t)), \quad (x,y,t)\in Q\times[0,T],$$

$$(ii)\qquad v^*(x,y,T)=0, \frac{\partial v^*(x,y,T)}{\partial t}=0, v^*(x,0,t)=v^*(x,S,t)=v^*(0,y,t)=v^*(L,y,t)=0.$$

We formulate conditions guaranteeing the nonemptiness of LAM [14]:

$$(iii)\qquad \frac{\partial^2\tilde{v}(x,y,t)}{\partial t^2} - \nabla^2\tilde{v}(x,y,t)\in F_A\left(\tilde{v}(x,y,t);v^*(x,y,t),x,y,t\right).$$

Here, it is assumed that the feasible solutions $v^*(x,y,t)$ are classical.

Theorem 3.1. *Suppose that $F(\cdot,x,y,t):\mathbb{R}^n\rightrightarrows\mathbb{R}^n$ is a convex mapping and that $f(\cdot,x,y,t)$ is a continuous proper (does not take the value $-\infty$ and identically is not equal to $+\infty$) convex function. Then, for $\tilde{v}(x,y,t)\in\Omega(x,y,t)$ to be optimal in problem (WHP) with initial and boundary conditions and 2-D Laplacian hyperbolic DFI, it is sufficient to have a classical solution $v^*(x,y,t)$ satisfying conditions (i)(iii) with adjoint DFI and homogeneous endpoint and boundary conditions.*

Proof. Using Theorem 2.1, due to the LAM formula, Euler-Lagrange DFI (i) takes the form

$$\begin{aligned}&H_F\left(v(x,y,t);v^*(x,y,t)\right)-H_F\left(\tilde{v}(x,y,t);v^*(x,y,t)\right)\\&\le\left\langle\frac{\partial^2 v^*(x,y,t)}{\partial t^2}-\nabla^2 v^*(x,y,t),v(x,y,t)-\tilde{v}(x,y,t)\right\rangle\\&+f(v(x,y,t),x,y,t)-f(\tilde{v}(x,y,t),x,y,t)-\langle\eta^*(x,y,t),v(x,y,t)-\tilde{v}(x,y,t)\rangle\end{aligned}\tag{5}$$

where $\eta^*(x,y,t)\in K^*_{\Omega}(\tilde{v}(x,y,t))$ or $\langle\eta^*(x,y,t),\tilde{v}(x,y,t)\rangle=\inf_{v(x,y,t)\in\Omega(x,y,t)}\langle\eta^*(x,y,t),v(x,y,t)\rangle$.
Then since, by virtue of the definition of the dual cone $\langle\eta^*(x,y,t),v(x,y,t)-\tilde{v}(x,y,t)\rangle\ge 0$, it follows from inequality (5) and condition (iii) that

$$\begin{aligned}&\left\langle\frac{\partial^2 v(x,y,t)}{\partial t^2}-\nabla^2 v(x,y,t),v^*(x,y,t)\right\rangle-\left\langle\frac{\partial^2\tilde{v}(x,y,t)}{\partial t^2}-\nabla^2\tilde{v}(x,y,t),v^*(x,y,t)\right\rangle\\&\le\left\langle\frac{\partial^2 v^*(x,y,t)}{\partial t^2}-\nabla^2 v^*(x,y,t),v(x,y,t)-\tilde{v}(x,y,t)\right\rangle\\&+f(v(x,y,t),x,y,t)-f(\tilde{v}(x,y,t),x,y,t).\end{aligned}$$

Let us write this inequality in the form

$$\begin{aligned} & f(v(x,y,t),x,y,t) - f(\tilde{v}(x,y,t),x,y,t) \\ \geq & \left\langle (v_{tt}(x,y,t) - \tilde{v}_{tt}(x,y,t)) - \nabla^2(v(x,y,t) - \tilde{v}(x,y,t)), v^*(x,y,t) \right\rangle \\ & - \left\langle v^*_{tt}(x,y,t) - \nabla^2 v^*(x,y,t), v(x,y,t) - \tilde{v}(x,y,t) \right\rangle . \end{aligned} \tag{6}$$

In turn, from (6) we have

$$\begin{aligned} & f(v(x,y,t),x,y,t) - f(\tilde{v}(x,y,t),x,y,t) \\ \geq & \left\langle (v_{tt}(x,y,t) - \tilde{v}_{tt}(x,y,t)), v^*(x,y,t) \right\rangle - \left\langle v^*_{tt}(x,y,t), v(x,y,t) - \tilde{v}(x,y,t) \right\rangle \\ & + \left\langle \nabla^2 v^*(x,y,t), v(x,y,t) - \tilde{v}(x,y,t) \right\rangle - \left\langle \nabla^2(v(x,y,t) - \tilde{v}(x,y,t)), v^*(x,y,t) \right\rangle . \end{aligned} \tag{7}$$

By direct verification it can be established that

$$\begin{aligned} & \left\langle (v_{tt}(x,y,t) - \tilde{v}_{tt}(x,y,t)), v^*(x,y,t) \right\rangle - \left\langle v^*_{tt}(x,y,t), v(x,y,t) - \tilde{v}(x,y,t) \right\rangle \\ = & \frac{\partial}{\partial t}\left[\left\langle (v_t(x,y,t) - \tilde{v}_t(x,y,t)), v^*(x,y,t) \right\rangle - \left\langle v^*_t(x,y,t), v(x,y,t) - \tilde{v}(x,y,t) \right\rangle \right] . \end{aligned} \tag{8}$$

On the other hand, it is not difficult to establish chains of equalities associated with the 2-D Laplacian:

$$\begin{aligned} & \left\langle \nabla^2 v^*(x,y,t), v(x,y,t) - \tilde{v}(x,y,t) \right\rangle - \left\langle \nabla^2(v(x,y,t) - \tilde{v}(x,y,t)), v^*(x,y,t) \right\rangle \\ = & \frac{\partial}{\partial x}\left[\langle v(x,y,t) - \tilde{v}(x,y,t), v^*_x(x,y,t) \rangle - \langle (v_x(x,y,t) - \tilde{v}_x(x,y,t)), v^*(x,y,t) \rangle \right] \\ & + \frac{\partial}{\partial y}\left[\left\langle v(x,y,t) - \tilde{v}(x,y,t), v^*_y(x,y,t) \right\rangle - \left\langle \left(v_y(x,y,t) - \tilde{v}_y(x,y,t) \right), v^*(x,y,t) \right\rangle \right] . \end{aligned} \tag{9}$$

Then, keeping in mind (8) and (9) integrating inequality (7) we obtain

$$\begin{aligned} & \int_0^T \iint_Q [f(v(x,y,t),x,y,t) - f(\tilde{v}(x,y,t),x,y,t)] dxdydt \\ \geq & \int_0^T \iint_Q \frac{\partial}{\partial t}\left[\langle v_t(x,y,t) - \tilde{v}_t(x,y,t), v^*(x,y,t) \rangle - \left\langle v^*_t(x,y,t), v(x,y,t) - \tilde{v}(x,y,t) \right\rangle \right] dxdydt \\ & + \int_0^T \iint_Q \frac{\partial}{\partial x}\left[\langle v(x,y,t) - \tilde{v}(x,y,t), v^*_x(x,y,t) \rangle - \langle v_x(x,y,t) - \tilde{v}_x(x,y,t), v^*(x,y,t) \rangle \right] dxdydt \\ & + \int_0^T \iint_Q \frac{\partial}{\partial y}\left[\left\langle v(x,y,t) - \tilde{v}(x,y,t), v^*_y(x,y,t) \right\rangle - \left\langle v_y(x,y,t) - \tilde{v}_y(x,y,t), v^*(x,y,t) \right\rangle \right] dxdydt. \end{aligned} \tag{10}$$

Now we denote the integrals on the right side of (10) by $I_i, i = 1,2,3$, respectively, and transform them, we will do this for example for I_1:

$$\begin{aligned} I_1 = & \iint_Q \left[\langle v_t(x,y,T) - \tilde{v}_t(x,y,T), v^*(x,y,T) \rangle - \left\langle v^*_t(x,y,T), v(x,y,T) - \tilde{v}(x,y,T) \right\rangle \right] dxdy \\ & - \iint_Q \left[\langle v_t(x,y,0) - \tilde{v}_t(x,y,0), v^*(x,y,0) \rangle - \left\langle v^*_t(x,y,0), v(x,y,0) - \tilde{v}(x,y,0) \right\rangle \right] dxdy. \end{aligned}$$

Then from the feasibility of solutions $v(x,y,t), \tilde{v}(x,y,t)$, and from conditions (4) and (ii) it follows that $I_1 = 0$. The integrals I_2 and I_3 are calculated by analogy. Thus $I_1 = I_2 = I_3 = 0$.

Finally, from inequality (10) we have

$$\int_0^T \iint_Q [f(v(x,y,t),x,y,t) - f(\tilde{v}(x,y,t),x,y,t)] dxdydt \geq 0$$

or $J[v(\cdot,\cdot,\cdot)] \geq J[\tilde{v}(\cdot,\cdot,\cdot)]$ for all admissible solutions. □

The following calculations are useful for computing $K^*_\Omega(\tilde{v}(x,y,t)$ in the case of specifying the set $\Omega(x,y,t)$ by inequalities.

Corollary 3.2. *Let a set $\Omega(x, y, t)$ be given as a solution set to the inequality $\Omega = \{v : \varphi(v) \leq 0\}$, where φ be a convex function continuous in $\tilde{v} \in \Omega = \{v : \varphi(v) \leq 0\}, (x, y, t) \in Q \times [0, T]$, and suppose that there is a point v_1 such that $\varphi(v_1) < 0$. Then for the problem (WHP) with $\Omega = \{v : \varphi(v) \leq 0\}, (x, y, t) \in Q \times [0, T]$ the condition (i) of Theorem 3.1 is replaced as follows:*

$$v^*_{tt}(x, y, t) - \nabla^2 v^*(x, y, t) \in F^*\left(v^*(x, y, t); \left(\tilde{v}(x, y, t), \tilde{v}_{tt}(x, y, t) - \nabla^2 \tilde{v}(x, y, t)\right), x, y, t\right)$$
$$-\partial f(\tilde{v}(x, y, t), x, y, t) - \operatorname{cone} \partial_v \varphi(\tilde{v}(x, y, t)), \varphi(\tilde{v}(x, y, t)) = 0, (x, y, t) \in Q \times [0, T], \tag{11}$$

where cone $\partial_v \varphi(\tilde{v})$ is the cone generated by the subdifferential set $\partial_v \varphi(\tilde{v})$, i.e.

$$cone\ \partial_v \varphi(\tilde{v}) = \{u^* : u^* = \alpha v^*, v^* \in \partial_v \varphi(\tilde{v}), \alpha > 0\}. \tag{12}$$

Proof. By Theorem 1.34 [14] the cone $K^*_\Omega(\tilde{v})$ is the dual cone at $\tilde{v} \in \Omega = \{v : \varphi(v) \leq 0\}$ calculated as follows

$$K^*_\Omega(\tilde{v}) = \begin{cases} \{0\}, & \text{if } \varphi(\tilde{v}) < 0, \\ -\operatorname{cone}\ _v\varphi(\tilde{v}), & \text{if } \varphi(\tilde{v}) = 0. \end{cases}$$

Then, taking into account the last formula in the condition of Theorem 3.1, we immediately obtain the required result. □

Proposition 3.3. *Suppose that the set Ω is defined as follows*

$$\Omega = \{v : \varphi_k(v) \leq 0, k \in I_1; \varphi_k(v) = 0, k \in I_2\},$$

where I_1 and I_2 are finite sets of indices and $\varphi_k, k \in I_1 \cup I_2$ are continuously differentiable convex functions, and the gradient vectors $\varphi'_k(\tilde{v}), k \in I_2, \tilde{v} \in \Omega$ are linearly independent. Then the dual cone $K^*_\Omega(\tilde{v})$ has a form

$$K^*_\Omega(\tilde{v}) = \left\{v^* : v^* = -\sum_{k \in I_1(\tilde{v}) U_2} \gamma_k \varphi'_k(\tilde{v}), \gamma_k \geq 0, k \in I_1(\tilde{v})\right\}, I_1(\tilde{v}) = \{k \in I_1 : \varphi_k(\tilde{v}) = 0\}.$$

Proof. First of all, we show that from the linear independence of the vectors $\varphi'_k(\tilde{v}), k \in I_2$ it follows that the direction $\bar{v}$ satisfying relations

$$\left\langle \bar{v}, \varphi'_k(\tilde{v}) \right\rangle < 0, k \in I_1(\tilde{v}); \left\langle \bar{v}, \varphi'_k(\tilde{v}) \right\rangle = 0, k \in I_2 \tag{13}$$

is a tangent vector. Indeed, on the one hand $\varphi_k(\tilde{v} + \gamma\bar{v}) = 0, k \in I_2$ for small $\gamma > 0$. On the other hand, according to the well-known analysis formula for $k \in I_1$

$$\varphi_k(\tilde{v} + \gamma\bar{v}) = \varphi_k(\tilde{v}) + \gamma \left\langle \bar{v}, \varphi'_k(\xi_k) \right\rangle. \tag{14}$$

Here if $k \in I_1 \backslash I_1(\tilde{v})$, then $\varphi_k(\tilde{v}) < 0$ and $\varphi_k(\tilde{v} + \gamma\bar{v}) < 0$ for sufficiently small $\gamma > 0$. In this case, if $k \in I_1(\tilde{v})$ from (14) we have

$$\varphi_k(\tilde{v} + \gamma\bar{v}) = \gamma \left\langle \bar{v}, \varphi'_k(\tilde{v}) \right\rangle + \gamma \left\langle \bar{v}, \varphi'_k(\xi_k) - \varphi'_k(\tilde{v}) \right\rangle.$$

Hence, since $\xi_k \to 0$ as $\gamma \to 0$ the second term on the right tends to zero faster than γ. Therefore, due to inequality (13) $\varphi_k(\tilde{v} + \gamma\bar{v}) < 0, k \in I_1(\tilde{v})$ for small $\gamma > 0$. So for sufficiently small γ the relations $\varphi_k(\tilde{v} + \gamma\bar{v}) < 0, k \in I_1; \varphi_k(\tilde{v} + \gamma\bar{v}) = 0, k \in I_2$, are satisfied, which means that $\tilde{v} + \gamma\bar{v} \in \Omega$. In turn, this inclusion means that $\bar{v}$ is the tangent direction to the set Ω. Thus, it follows that

$$K_\Omega(\tilde{v}) = \left\{\bar{v} : \left\langle \bar{v}, \varphi'_k(\tilde{v}) \right\rangle < 0, k \in I_1(\tilde{v}); \left\langle \bar{v}, \varphi'_k(\tilde{v}) \right\rangle = 0, k \in I_2\right\}.$$

Consequently, applying the Farkas'Theorem 1.13 [14] we obtain the required form of the dual cone $K^*_\Omega(\tilde{v})$. □

Remark 3.4. If a convex function φ differentiable at a point $\tilde{v}$, then by Theorem 1.28 [14] the subdifferential $\partial_v\varphi(\tilde{v})$ consists of the single gradient vector at this point, i.e. $\partial_v\varphi(\tilde{v}) = \{\varphi'(\tilde{v})\}$. Then by formula (12) cone$^2\,\partial_v(\tilde{v}) = \{\alpha\varphi'(\tilde{v}), \alpha > 0\}$ and therefore in condition (11) of Corollary 3.2 the cone $\partial_v\varphi(\tilde{v}(x,y,t))$ should be replaced with $\alpha\varphi'(\tilde{v}(x,y,t)), \alpha > 0$.

Corollary 3.5. *Let us consider the problem (WHP) without phase constraint, which is equivalent to the fact that* $\Omega(x,y,t) \equiv \mathbb{R}^n, (x,y,t) \in Q\times[0,T]$. *Then in the condition (i) of Theorem 3.1* $K^*_\Omega(\tilde{v}(x,y,t) \equiv \{0\}$.

Proof. Since $\Omega(x,y,t) \equiv \mathbb{R}^n, (x,y,t) \in Q\times[0,T]$ it follows that $K_\Omega(\tilde{v}(x,y,t)) \equiv \mathbb{R}^n$ and $K^*_\Omega(\tilde{v}(x,y,t) \equiv \{0\}$. □

The following theorem shows that all the results presented in this article are also valid in non-convex problems, and we considered the convex case only for the sake of simplicity of presentation.

Theorem 3.6. *Let us assume that a non-convex problem (WHP) with a non-convex Hamilton function* H_F *is given, where LAM* F^* *is determined by the Hamilton function. Then for optimality* $\tilde{v}(x,y,t)$ *in this non-convex problem it is sufficient to have a classical solution* $v^*(x,y,t)$ *satisfying conditions:*

$$
\begin{aligned}
&(1) \quad \frac{\partial^2 v^*(x,y,t)}{\partial t^2} - \nabla^2 v^*(x,y,t) + v^*(x,y,t) \\
&\quad \in F^*\left(v^*(x,y,t); \left(\tilde{v}(x,y,t), \tilde{v}_{tt}(x,y,t) - \nabla^2\tilde{v}(x,y,t)\right), x,y,t\right), \quad (x,y,t) \in Q\times[0,T],\\
&(2) \quad f(v,x,y,t) - f(\tilde{v}(x,y,t),x,y,t) \geq \langle v^*(x,y,t), v - \tilde{v}(x,y,t)\rangle, \forall v \in \mathbb{R}^n,\\
&(3) \quad v^*(x,y,T) = 0, \frac{\partial v^*(x,y,T)}{\partial t} = 0, v^*(x,0,t) = v^*(x,S,t) = v^*(0,y,t) = v^*(L,y,t) = 0,\\
&(4) \quad \frac{\partial^2 \tilde{v}(x,y,t)}{\partial t^2} - \nabla^2\tilde{v}(x,y,t) \in H_F(\tilde{v}(x,y,t); v^*(x,y,t), x,y,t).
\end{aligned}
$$

Proof. Due to the LAM defined by the non-convex Hamilton function, it follows from condition (1) that for any feasible solution $v(,,\cdot,)$, *wehave*

$$
\begin{aligned}
&H_F(v(x,y,t); v^*(x,y,t)) - H_F(\tilde{v}(x,y,t); v^*(x,y,t))\\
&\leq \left\langle v^*_{tt}(x,y,t) - \nabla^2 v^*(x,y,t) + v^*(x,y,t), v(x,y,t) - \tilde{v}(x,y,t)\right\rangle,
\end{aligned}
$$

which by condition (4) implies that

$$
\begin{aligned}
&\left\langle v_{tt}(x,y,t) - \tilde{v}_{tt}(x,y,t) - \nabla^2(v(x,y,t) - \tilde{v}(x,y,t)), v^*(x,y,t)\right\rangle\\
&\leq \left\langle v^*_{tt}(x,y,t) - \nabla^2 v^*(x,y,t) + v^*(x,y,t), v(x,y,t) - \tilde{v}(x,y,t)\right\rangle
\end{aligned}
$$

or by virtue of the condition (2) from the last relation for all feasible solution $v(\cdot,\cdot,\cdot)$ in a more convenient form we have

$$
\begin{aligned}
&f(v(x,y,t),x,y,t) - f(\tilde{v}(x,y,t),x,y,t) \geq \langle v^*(x,y,t), v(x,y,t) - \tilde{v}(x,y,t)\rangle\\
&\geq \left\langle v_{tt}(x,y,t) - \tilde{v}_{tt}(x,y,t) - \nabla^2(v(x,y,t) - \tilde{v}(x,y,t)), v^*(x,y,t)\right\rangle\\
&\quad - \left\langle v(x,y,t) - \tilde{v}(x,y,t), v^*_{tt}(x,y,t) - \nabla^2 v^*(x,y,t)\right\rangle.
\end{aligned}
$$

Now, it is easy to see that the last inequality is nothing more than relation (7). Therefore, integrating this inequality over $Q\times[0,T]$ and using the initial and boundary conditions (3), we obtain the desired result and to avoid repetition we omit further proof. □

4. Optimization of multidimensional problem

The formulated problem (1)-(4) can be generalized to second-order hyperbolic DFIs; Assume that M is a bounded convex closed region of $\mathbb{R}^n$, $P_T \subset \mathbb{R}^{n+1}$ is a bounded cylinder $P_T = \{x \in M, 0 < t < T\}$ and that P_T

is the lateral surface of cylinder, ∂M is the piecewise smooth boundary of M, and $P_0 = \{x \in M, t = 0\}$ is the lower base of P_T. Now let us formulate the first mixed problem:

$$\text{minimize } \int_0^T \int_M f(v(x,t), x, t)dxdt,$$

$$(PGC) \qquad \frac{\partial^2 v(x,t)}{\partial t^2} - \nabla^2 v(x,t) \in G(v(x,t), x, t), \quad (x,t) \in P_T, \tag{15}$$

$$v(x,t) \in V(x,t), \quad \nabla^2 v(x,t) = \frac{\partial^2 v(x,t)}{\partial x_1^2} + \cdots + \frac{\partial^2 v(x,t)}{\partial x_n^2},$$

$$(0) = \varphi(x), \quad v_t(x,0) = \psi(x), x = (x_1, \ldots, x_n) \in M; \quad v(x,t) = \chi(x,t), (x,t) \in \Gamma_T, \tag{16}$$

where $G(v,x,t) : \mathbb{R}^{n+2} \rightrightarrows \mathbb{R}^1$ is a convex multivalued mapping, the function $f(\cdot, x, t)$ is convex φ, ψ, χ are continuous. Then $v(x,t) \in C^2(P_T) \bigcap C^1(P_T \cup \Gamma_T \cup P_0)$, satisfying (15), conditions (16) and conditions $v(x,t) \in V(x,t)$, is called a solution (in the classical sense) of this problem.

Theorem 4.1. *Let $f(\cdot, x, t)$ be a continuous proper convex function and $G(v,x,t) : \mathbb{R}^{n+2} \rightrightarrows \mathbb{R}^1$ be a convex set-valued mapping. Then for $\tilde{v}(x,t) \in V(x,t)$, to be optimal in problem (PGC), it is sufficient that there is a solution $v^*(x,t)$ that satisfies conditions (iv)-(vi):*

$$(iv) \quad \frac{\partial^2 v^*(x,t)}{\partial t^2} - \nabla^2 v^*(x,t) \in G^*\left(v^*(x,t); \left(\tilde{v}(x,t), \tilde{v}_{tt}(x,t) - \nabla^2 \tilde{v}(x,t)\right), x, t\right) - \partial f(\tilde{v}(x,t), x, t) + K_V^*(\tilde{v}(x,t), \quad (x,t) \in M \times [0,T],$$

$$(v) \qquad v^*(x,T) = 0, \frac{\partial v^*(x,T)}{\partial t} = 0, x \in M; v^*(x,t) = 0, (x,t) \in \Gamma_T,$$

$$(vi) \qquad \frac{\partial^2 \tilde{v}(x,t)}{\partial t^2} - \nabla^2 \tilde{v}(x,t) \in G_A(\tilde{v}(x,t); v^*(x,t), x, t)$$

Proof. According to the inequality (5) expressed in terms of the Hamilton function we can write

$$\begin{aligned} &H_G(v(x,t); v^*(x,t)) - H_G(\tilde{v}(x,t); v^*(x,t)) \\ &\leq \left(v_{tt}^*(x,t) - \nabla^2 v^*(x,t)\right)(v(x,t) - \tilde{v}(x,t)) \\ &+ f(v(x,t), x, t) - f(\tilde{v}(x,t), x, t) - \phi^*(x,t)(v(x,t) - \tilde{v}(x,t)), (x,t) \in P_T, \end{aligned} \tag{17}$$

where $\phi^*(x,t) \in K_V^*(\tilde{v}(x,t))$. Then under the condition (vi) of theorem we get from (17)

$$\begin{aligned} &f(v(x,t), x, t) - f(\tilde{v}(x,t), x, t) \\ &\geq \left(v_{tt}(x,t) - \tilde{v}_{tt}(x,t) - \nabla^2(v(x,t) - \tilde{v}(x,t))\right) v^*(x,t) \\ &- \left(v_{tt}^*(x,t) - \nabla^2 v^*(x,t)\right)(v(x,t) - \tilde{v}(x,t)). \end{aligned}$$

Then, bearing in mind (8), if we integrate this inequality, we have

$$\begin{aligned} &\int_0^T \int_M [f(v(x,t), x, t) - f(\tilde{v}(x,t), x, t)]dxdt \\ &\geq \int_0^T \int_M \frac{\partial}{\partial t}\left[v^*(x,t)(v_t(x,t) - \tilde{v}_t(x,t)) - v_t^*(x,t)(v(x,t) - \tilde{v}(x,t))\right] dxdt \\ &+ \int_0^T \int_M^T \left[\nabla^2 v^*(x,t)(v(x,t) - \tilde{v}(x,t)) - v^*(x,t)\nabla^2(v(x,t) - \tilde{v}(x,t))\right] dxdt \end{aligned} \tag{18}$$

where $dx = dx_1 \cdots dx_n$.
We denote the integrals to the right of (18) as L_1 and L_2, respectively. Then

$$\begin{aligned} L_1 &= \int_M \left[v^*(x,T)(v_t(x,T) - \tilde{v}_t(x,T)) - v_t^*(x,T)(v(x,T) - \tilde{v}(x,T)\right]dx \\ &- \int_M \left[v^*(x,0)(v_t(x,0) - \tilde{v}_t(x,0)) - v_t^*(x,0)(v(x,0) - \tilde{v}(x,0))\right] dx. \end{aligned}$$

Since $v(x,t), \tilde{v}(x,t)$ are feasible, by the initial conditions (16) $v(x,0) = \varphi(x), v_t(x,0) = \psi(x)$ and by endpoint conditions $v^*(x,T) = 0, v_t^*(x,T) = 0$ (see (iv)) we obtain that $L_1 = 0$.

Calculate the integral L_2. By the familiar Green's formula, we have

$$\begin{aligned} L_2 &= \int_0^T \int_M \left[\nabla^2 v^*(x,t)(v(x,t)-\tilde{v}(x,t)) - v^*(x,t)\nabla^2(v(x,t)-\tilde{v}(x,t))\right] dxdt \\ &= \int_0^T \int_{\partial M} \left[(v(x,t)-\tilde{v}(x,t))\frac{\partial v^*(x,t)}{\partial n} - v^*(x,t)\frac{\partial(v(x,t)-\tilde{v}(x,t))}{\partial n}\right] dsdt, \end{aligned}$$

where n is the unit normal vector external to the boundary ∂M in $\mathbb{R}^{n-1}$ of region M.
Since by condition $v^*(x,t) = 0, \quad x \in \partial M, \quad 0 < t < T$ of theorem and boundary condition $v(x,t)$ $= \tilde{v}(x,t) = \chi(x,t), (x,t) \in \Gamma_T$ it follows that $L_2 = 0$. Finally, from inequality (18) we have $\int_0^T \int_M [f(v(x,t),x,t) - f(\tilde{v}(x,t),x,t)]dxdt \geq 0$ or $J[v(\cdot,\cdot)] \geq J[\tilde{v}(\cdot,\cdot)]$ for all feasible $v(\cdot,\cdot)$. □

Consider the following problem:

$$\begin{aligned} &\text{minimize} \qquad J[v(\cdot,\cdot,\cdot)] = \int_0^T \iint_Q f(v(x,y,t),x,y,t)dxdydt \\ &\frac{\partial^2 v(x,y,t)}{\partial t^2} - \nabla^2 v(x,y,t) = A(t)v(x,y,t) + w(x,y,t), \qquad (19) \\ &v(x,y,t) \in \Omega(x,y,t), w(x,y,t) \in U, \\ &v(x,y,0) = \alpha_1(x,y), \frac{\partial v(x,y,0)}{\partial t} = \alpha_2(x,y), v(x,0,t) = \beta_0(x,t), v(x,S,t) \\ &= \beta_S(x,t), v(0,y,t) = \gamma_0(y,t), \quad v(L,y,t) = \gamma_L(y,t), Q = [0,L]\times[0,S] \end{aligned}$$

where $A(t), t \in [0,T]$ is a $n \times n$ continuous matrix, $U \subset \mathbb{R}^n$ is a convex closed set, and f is continuously differentiable function of v. We must find the control parameter $\tilde{w}(x,y,t) \in U, (x,y,t) \in Q \times [0,T]$ that minimizes $J[v(\cdot,\cdot,\cdot)]$, where $F(v,x,y,t) = A(t)v + U$. It is easy to see that

$$\begin{aligned} F^*(u^*,(v,u),x,y,t) &= \begin{cases} A^*(t)u^*, -u^* \in K_U^*(w), \\ \varnothing, -u^* \notin K_U*(w), \end{cases} \\ u = A(t)v + w. & \qquad (20) \end{aligned}$$

Therefore, using (20), and Theorem 3.1, we get

$$\begin{aligned} &\frac{\partial^2 v^*(x,y,t)}{\partial t^2} - \nabla^2 v^*(x,y,t) \\ &\in A^*(t)v^*(x,y,t) - f'_v(\tilde{v}(x,y,t),x,y,t) + K_U^*(\tilde{v}(x,y,t), \qquad (21) \\ &v^*(x,y,T) = 0, v_t^*(x,y,T) = 0, v^*(x,0,t) = v^*(x,S,t) \\ &= v^*(0,y,t) = v^*(L,y,t) = 0, \tilde{v}(x,y,t) \in \Omega(x,y,t) \qquad (22) \end{aligned}$$

Besides, we have

$$\langle w - \tilde{w}(x,y,t), v^*(x,y,t)\rangle \leq 0, \quad w \in U$$

where we get Pontryagin's maximum principle [26]:

$$\langle \tilde{w}(x,y,t), v^*(x,y,t)\rangle = \max_{w\in U} \langle w, v^*(x,y,t)\rangle \qquad (23)$$

Let us formulate the result obtained

Theorem 4.2. *Let us assume that $v^*(x,y,t)$ satisfies (21)-(23) for some control function $\tilde{w}(x,y,t)$. Then the solution $\tilde{v}(x,y,t)$ in problem (19) is optimal.*

Corollary 4.3. *Suppose that in the problem (1)-(4) $F(u,x,y,t) \equiv U \subset \mathbb{R}^n$ is a convex compact and there exists a gradient $f'_v(\tilde{v}(x,y,t),x,y,t)$. Then $\tilde{w}(x,y,t) \in U$ is a solution to such a problem if the solution $v^*(x,y,t)$ satisfies the relations*

$$\begin{aligned} &\frac{\partial^2 v^*(x,y,t)}{\partial t^2} - \nabla^2 v^*(x,y,t) \in K_\Omega^*\left(\tilde{v}(x,y,t) - f'_v(\tilde{v}(x,y,t),x,y,t)\right. \\ &\langle \tilde{w}(x,y,t), v^*(x,y,t)\rangle = \max_{w\in U} \langle w, v^*(x,y,t)\rangle \end{aligned}$$

Proof. Since gph $F = \mathbb{R}^n \times U$ it follows that $K_F(v,u) = \mathbb{R}^n \times K_U(w)$. Hence

$$K_F * (v,u) = \{0\} \times K_U^*(w)$$

On the definition of LAM

$$F^*(u^*;(v,u)) = \begin{cases} 0, & \text{if } u^* \in K_U^*(w) \\ \varnothing, & \text{if } u^* \notin K_U * (w) \end{cases}$$

It remains to apply Theorem 3.1:

$$\frac{\partial^2 v^*(x,y,t)}{\partial t^2} - \nabla^2 v^*(x,y,t) \in K_\Omega^*(\tilde{v}(x,y,t)) - f_v'(\tilde{v}(x,y,t),x,y,t);$$
$$\langle w - \tilde{w}(x,y,t), v^*(x,y,t)\rangle \le 0, \quad w \in U.$$

□

5. 2-D wave polyhedral DFIs

Let us write the matrix form of the problem in the form of a hyperbolic polyhedral DFI, defined using a finite number of linear inequalities:

$$\begin{aligned} &\text{minimize} && J[v(\cdot,\cdot,\cdot)] = \int_0^T \iint_Q f(v(x,y,t),x,y,t)dxdydt, \\ &(PHP) && \mathrm{Av}(x,y,t) - B\left(\frac{\partial^2 v(x,y,t)}{\partial t^2} - \nabla^2 v(x,y,t)\right) \le d, (x,y,t) \in Q \times [0,T], \end{aligned}$$
$$v(x,y,t) \in \Omega, \Omega = \{v : Dv \le c\},$$
$$v(x,y,0) = \alpha_1(x,y), v_t(x,y,0) = \alpha_2(x,y), v(x,0,t) = \beta_0(x,t),$$
$$v(x,S,t) = \beta_S(x,t), v(0,y,t) = \gamma_0(y,t), v(L,y,t) = \gamma_L(y,t), Q = [0,L] \times [0,S],$$

where A, B, D are $s \times n$ dimensional matrices, d, c are s-dimensional column-vector, $f(\cdot,x,y,t)$ is a polyhedral function, i.e. epi f is a polyhedral set in $\mathbb{R}^{n+1}$. Here $F(v) = \{u : Av - Bu \le d\}$ is a set-valued mapping, $\Omega = \{v : Dv \le c\}$ is a polyhedral set. Now we will use Theorem 3.1 for problem (PHP). By analogy with the formula for calculating the LAM for a polyhedral mapping [14], we obtain that

$$F^*(u^*;(\tilde{v},\tilde{u})) = \{-A^*q : u^* = -B^*q, q \ge 0, \langle A\tilde{v} - B\tilde{u} - d, q\rangle = 0\}.$$

On the other it is not hard to calculate that $K_\Omega(\tilde{v}) = \{\bar{v} : D\bar{v} \le 0\}$ and then according to the Farkas' Theorem 1.13 [14] we have

$$K_\Omega^*(\tilde{v}) = \{\eta^* : \eta^* = -D^*\gamma, \gamma \ge 0\}.$$

Then applying Theorem 3.1 we get

$$\nabla^2 v^*(x,y,t) - \frac{\partial^2 v^*(x,y,t)}{\partial t^2} - A^*q(x,y,t) \in \partial f(\tilde{v}(x,y,t),x,y,t) - K_\Omega^*(\tilde{v}(x,y,t)), v^*(x,y,t) = -B^*q(x,y,t)$$
$$q(x,y,t) \ge 0$$
$$\left\langle A\tilde{v}(x,y,t) - B\left(\tilde{v}_{tt}(x,y,t) - \nabla^2\tilde{v}(x,y,t)\right) - d, q(x,y,t)\right\rangle = 0$$
$$v^*(x,y,T) = 0, v_t^*(x,y,T) = 0, v^*(x,0,t) = v^*(x,S,t) = v^*(0,y,t) = v^*(L,y,t) = 0 \tag{24}$$

Let's substitute $v^*(x,y,t) = -B^*q(x,y,t)$ and $\eta^*(x,y,t) = -D^*\gamma(x,y,t)$ into relation (24)

$$B^*\left(q_{tt}(x,y,t) - \nabla^2 q(x,y,t)\right) - A^*q(x,y,t) \in \partial f(\tilde{v}(x,y,t),x,y,t) + D^*\gamma(x,y,t),$$
$$\left\langle A\tilde{v}(x,y,t) - B\left(\tilde{v}_{tt}(x,y,t) - \nabla^2\tilde{v}(x,y,t)\right) - d, q(x,y,t)\right\rangle = 0, \tag{25}$$
$$B^*q(x,y,T) = 0, B^*q_t(x,y,T) = 0, \tag{26}$$
$$B^*q(x,0,t) = B^*q(x,S,t) = B^*q(0,y,t) = B^*q(L,y,t) = 0.$$

Thus, we obtain the following theorem.

Theorem 5.1. *Suppose that $f(\cdot,x,y,t):\mathbb{R}^n\to\mathbb{R}^1$ is a polyhedral function and that $F:\mathbb{R}^n\rightrightarrows\mathbb{R}^n$ is a polyhedral mapping. Then, in order for the solution $\tilde{v}(x,y,t)$, to be optimal in the problem (PHP), it is sufficient that there exist functions $q(x,y,t)\geq 0,\gamma(x,y,t)\geq 0$ satisfying (25) and the conjugate condition (26).*

Example 5.2. Consider a problem with 2-D wave DFI for which the exact solution is calculated:

$$\begin{aligned} &\text{minimize} \qquad J[v(\cdot,\cdot,\cdot)]=\int_0^\pi\iint_Q v(x,y,t)dxdydt\\ &\tfrac{\partial^2 v(x,y,t)}{\partial t^2}-\nabla^2 v(x,y,t)\geq 0, Q=[0,\pi]\times[0,\pi],\\ &v(x,y,0)=\sin x\cdot\sin y, v_t(x,y,0)=0,\\ &v(x,0,t)=0, v(0,y,t)=0, v(\pi,y,t)=0, v(x,\pi,t)=0. \end{aligned} \tag{27}$$

Comparing with the problem (WHP) we see that in the problem (27) $F(v)=\{u:u\geq 0\}$, gphF $=\{(v,u):u\geq 0\}$, $\operatorname{dom}F=\mathbb{R}^1$, so that $A=D=[0], B=[1], c=d=0$ and $f(v(x,y,t))=v, F:\mathbb{R}^1\rightrightarrows\mathbb{R}^1, L=S=T=\pi, \alpha_1(x,y)\equiv\sin x\cdot\sin y, \alpha_2(x,y)=\beta_0(x,t)=\gamma_0(y,t)=\beta_\pi(x,t)=\gamma_\pi(y,t)\equiv 0$. Second relation of (25) has a form

$$\left(\frac{\partial^2\tilde{v}(x,y,t)}{\partial t^2}-\nabla^2\tilde{v}(x,y,t)\right)q(x,y,t)=0,$$

which means either $q(x,y,t)=0$ or $\frac{\partial^2\tilde{v}(x,y,t)}{\partial t^2}-\nabla^2\tilde{v}(x,y,t)=0$. On the other hand, taking into account $\partial f(v)=\{1\}$ in the first relation (25) we obtain

$$\frac{\partial^2 q^*(x,y,t)}{\partial t^2}-\nabla^2 q^*(x,y,t)=1,$$

where $q^*(x,y,t)$ is a nontrivial solution (otherwise, we have $0=1$, which is impossible). Finally, it follows that $\tilde{v}(x,y,t)$ is a solution of the following wave equation with inhomogeneous initial-boundary conditions:

$$\begin{aligned} &\tfrac{\partial^2\tilde{v}(x,y,t)}{\partial t^2}-\nabla^2\tilde{v}(x,y,t)=0\\ &v(x,y,0)=\sin x\cdot\sin y,\quad v_t(x,y,0)=0,\\ &v(x,0,t)=0, v(0,y,t)=0, v(\pi,y,t)=0, v(x,\pi,t)=0, \end{aligned}$$

which admits the exact fsolution $\tilde{v}(x,y,t)=\sin x\cdot\sin y\cdot\cos\sqrt{2}t$. It is easy to check that the initial and boundary conditions are met; indeed, $\tilde{v}_t(x,y,t)=-\sqrt{2}\sin x\cdot\sin y\cdot\sin\sqrt{2}t$ and are satisfied:

$$\begin{aligned} &\tilde{v}(x,y,0)=\sin x\cdot\sin y, \tilde{v}_t(x,y,0)=0, \tilde{v}(x,0,t)=\tilde{v}(0,y,t)=0,\\ &\tilde{v}(\pi,y,t)=\sin\pi\sin y\cos\sqrt{2}t=0, \tilde{v}(x,\pi,t)=\sin x\sin\pi\cos\sqrt{2}t=0. \end{aligned}$$

Then, substituting $\tilde{v}(x,y,t)=\sin x\cdot\sin y\cdot\cos\sqrt{2}t$ into $J[v(\cdot,\cdot)]$ we find that the minimum value of (27) is $(4/\sqrt{2})\sin\sqrt{2}\pi$:

$$\begin{aligned} J[\tilde{v}(\cdot,\cdot,\cdot)]&=\int_0^\pi\iint_Q\tilde{v}(x,y,t)dxdydt=\int_0^\pi\int_0^\pi\int_0^\pi(\sin x\cdot\sin y\cdot\cos\sqrt{2}t)dxdydt\\ &=-\int_0^\pi\int_0^\pi[\cos x\cdot\sin y\cdot\cos\sqrt{2}t]_{x=0}^{x=\pi}dydt=2\iint_0^\pi\int_0^\pi\sin y\cdot\cos\sqrt{2}tdydt\\ &=-2\int_0^\pi[\cos y\cdot\cos\sqrt{2}t]_{y=0}^{y=\pi}dt=4\int_0^\pi\cos\sqrt{2}tdt=4\left[\frac{\sin\sqrt{2}t}{\sqrt{2}}\right]_{t=0}^{t=\pi}=\frac{4}{\sqrt{2}}\sin\sqrt{2}\pi. \end{aligned}$$

Remark 5.3. Consider the optimal control of 1-D wave DFIs with state constraints:

$$\begin{aligned} &\text{minimize} \qquad J[v(\cdot\cdot)]=\iint_Q f(v(x,t),x,t)dxdt,\\ &\tfrac{\partial^2 v(x,t)}{\partial t^2}-\nabla^2 v(x,t)\in F(v(x,t),x,t),\\ &v(x,t)\in\Omega(x,t),\quad (x,t)\in Q,\quad Q=[0,L]\times[0,T],\\ &v(x,0)=\alpha_1(x), v_t(x,0)=\alpha_2(x),\quad v(0,t)=\gamma_0(t),\quad v(L,t)=\gamma_L(t), \end{aligned} \tag{28}$$

where $F(\cdot,x,t):\mathbb{R}^n\rightrightarrows\mathbb{R}^n$ and $\Omega:\mathbb{R}^2\rightrightarrows\mathbb{R}^n$ are a convex set-valued mappings, $f(\cdot,x,t)$ is proper convex function, $\nabla^2=\partial^2/\partial x^2$ and α_1,α_2 and γ_0,γ_L are given continuous functions, $\alpha_1,\alpha_2:Q\to\mathbb{R}^n$, $\gamma_0,\gamma_L:[0,L]\times[0,T]\to\mathbb{R}^n;L,T$ are real positive numbers.
The conditions (i)-(iii) of Theorem 3.1 have the form

$$\begin{aligned}&\tfrac{\partial^2 v^*(x,t)}{\partial t^2}-\nabla^2 v^*(x,t)\in F^*\left(v^*(x,t);\left(\tilde{v}(x,t),\tilde{v}_{tt}(x,t)-\nabla^2\tilde{v}(x,t)\right),x,t\right)\\&\qquad-\partial f(\tilde{v}(x,t),x,t)+K^*_{\Omega}(\tilde{v}(x,t);\quad (x,t)\in Q,\\&\qquad v^*(x,T)=0,\tfrac{\partial v^*(x,T)}{\partial t}=0,v^*(0,t)=v^*(L,t)=0,\\&\qquad\tfrac{\partial^2\tilde{v}(x,t)}{\partial t^2}-\nabla^2\tilde{v}(x,t)\in F_A\left(\tilde{v}(x,t);v^*(x,t),x,t\right).\end{aligned}\tag{29}$$

Example 5.4. Let's consider an optimization problem with a one-dimensional wave differential equation of hyperbolic type:

$$\begin{aligned}\text{minimize}\qquad & J[v(\cdot,)]=\iint_Q v(x,t)dxdt,\\&\tfrac{\partial^2 v(x,t)}{\partial t^2}-\nabla^2 v(x,t)\in M,\\&v(x,0)=\sin x,v_t(x,0)=0,\quad v(0,t)=v(\pi,t)=0,\end{aligned}\tag{30}$$

where according to problem (28) $F(\cdot,x,t):\mathbb{R}^1\rightrightarrows\mathbb{R}^1$ is a constant mapping, $F(v)=M=[-1,+1]$, gph $F=\{(v,u):u\in M\},\operatorname{dom}F=\mathbb{R}^1,Q=[0,\pi]\times[0,\pi],\Omega(x,t)\equiv\mathbb{R}^1$. Obviously, according to (29) the LAM $F^*(u^*;(v,u))$ should be calculated. Since $H_F(v,u^*)=\sup_u\{u\cdot u^*:u\in M\},u^*\in\mathbb{R}^1$ it is obvious that $H_F(v,u^*)=\operatorname{sgn}u^*$, i.e.

$$H_F(v,u^*)=\begin{cases}1 & \text{if}\quad u^*>0,\\-1, & \text{if}\quad u^*<0.\end{cases}$$

It follows that $F^*(u^*;(v,u))=\partial H_F(v,u^*)\equiv\{0\}$ and does not depend on v^*. On the other hand, $f'_v(\tilde{v}(x,y,t),x,y,t)=1$ and $K^*_{\Omega}(\tilde{v}(x,t))\equiv\{0\}$. Hence, the adjoint wave equation with a homogeneous endpoint and boundary-value problem according to (29) is

$$\frac{\partial^2 v^*(x,t)}{\partial t^2}-\nabla^2 v^*(x,t)=-1,v^*(x,\pi)=0,\quad v^*_t(x,\pi)=0,\quad v^*(0,t)=v^*(\pi,t)=0.$$

Note that $F_A(v;u^*)=\{-1,+1\}$ and for optimality of $\tilde{v}(x,t)$ on the last condition (29) we should use $\tilde{v}_{tt}(x,t)-\nabla^2\tilde{v}(x,t)=\pm1$ and other values of M are not suitable in advance for optimality. In particular, if in problem (30) $M=\{0\}$, then by direct verification we find that the solution of the problem

$$\begin{gathered}\frac{\partial^2 v(x,t)}{\partial t^2}-\nabla^2 v(x,t)=0,\\v(x,0)=\sin x,v_t(x,0)=0,\quad v(0,t)=v(\pi,t)=0\end{gathered}$$

is $v(x,t)=\sin x\cos t$ and then the minimum value of (30) is equal to zero:

$$\iint_Q\tilde{v}(x,t)dxdt=\int_0^\pi\int_0^\pi(\sin x\cos t)dxdt=-\int_0^\pi[\cos x\cos t]_{x=0}^{x=\pi}dt=2\int_0^\pi\cos tdt=0.$$

6. Conclusion

Another issue of discussion is establishing the necessary conditions for optimality, as well as reducing the "gap" between necessary and sufficient conditions. Examples show that for constructing the Euler-Lagrange DFI, the decisive role is played by the points belonging to the argmaximum set. Based on the polyhedral LAM, optimality conditions for (PHP) are formulated. The study of optimality results for such problems can also make a great contribution to the development of other polyhedral (elliptic, parabolic) optimal control problems. Note that at least the finiteness of the number of discontinuity/switching points is of particular interest in optimal control theory. Thus, in our polyhedral problem, it makes sense to obtain a similar result in a region where the number of vertices is constant.

References

[1] S.Abbas, M. Benchohra, A. Petruşel, *Ulam stability for partial fractional differential inclusions via Picard operators theory*, Electr. J.Qualit. Theory Diff. Equ. **51** (2014), 1–13.
[2] S.Adly, A.Hantoute, M.Théra, *Nonsmooth Lyapunov pairs for infinite-dimensional first- order differential inclusions*, Nonlin. Anal: Theory, Meth. Appl. **75** (2012), 985–1008.
[3] B. Ahmad, S. K. Ntouyas, H. H. Alsulami, *Existence of solutions or nonlinear n-th order differential equations and inclusions with nonlocal and integral boundary conditions via fixed point theory*, Filomat, **28** (2014) 2149–2162.
[4] J. P. Aubin, A. Cellina, *Differential Inclusions, Set-Valued Maps and Viability Theory*, Springer, Berlin Heidelberg, 1984.
[5] J. P.Aubin, H. Frankowska, *Set-valued solutions to the Cauchy problem for hyperbolic system of partial differential inclusion*, NoDEA Nonlin. Diff. Equ. Appl. **4** (1996), 149–168.
[6] D. Azzam-Laouir, S. Lounis, L. Thibault, *Existence solutions for second-order differential inclusions with nonconvex perturbations*, Appl. Anal. **86** (2011), 1199–1210.
[7] G. M. Bahaa, *Fractional optimal control problem for differential system with control constraints*, Filomat, **30** (2016), 2177–2189.
[8] M. Benchohra, S. K. Ntouyas, *Existence results for functional differential inclusions*, Elect. J. Diff. Equ. **2001** (2001), 1–8.
[9] M. Chao, L. Zhi-Bin, Z. Lie-Hui, H. Nan-Jing, *On a system of fuzzy differential inclusions*, Filomat, **29** (2015), 1231–1244.
[10] Yi. Cheng, F. Cong, X. Xue, *Boundary value problems of a class of nonlinear partial differential inclusions*, Nonlin. Anal.: Real World Appl. **12** (2011), 3095–3102.
[11] F. H.Clarke, *Functional Analysis, Calculus of Variations and Optimal Control*, Graduate Texts in Mathematics, 264, Springer, 2013.
[12] M. Kisielewicz, *Some optimal control problems for partial differential inclusions*, Opuscula Math. **28** (2008), 507–516.
[13] E. N. Mahmudov, *Necessary and sufficient conditions for discrete and differential inclusions of elliptic type*, J. Math. Anal. Appl. **323** (2006), 768–789.
[14] E. N. Mahmudov, *Approximation and Optimization of Discrete and Differential Inclusions*, Elsevier, Boston, USA, 2011.
[15] E. N. Mahmudov, *Approximation and optimization of Darboux type differential inclusions with set-valued boundary conditions*, Optim. Lett. **7** (2013), 871–891.
[16] E. N. Mahmudov, S. Demir, Ö. Değer, *Optimization of third-order discrete and differential inclusions described by polyhedral set-valued mappings*, Appl. Anal. **95** (2016), 1831–1844.
[17] E. N. Mahmudov, *Optimal control of evolution differential inclusions with polynomial linear differential operators*, Evol. Equ. Contr. Theory **8** (2019), 603–619.
[18] E. N. Mahmudov, *Infimal convolution and duality in problems with third-order discrete and differential inclusions*, J. Optim. Theory Appl. **184** (2020), 781-809.
[19] E. N. Mahmudov, *Optimal control of higher order differential inclusions with functional constraints*, ESAIM: Contr. Optim. Calc. Variat. **26** (2020), 1–23.
[20] E. N. Mahmudov, D. I. Mastaliyeva, *Optimal control of higher-order retarded differential inclusions*, J. Indus. Manag. Optim. **19** (2022), 6544–6557.
[21] E. N. Mahmudov, *Optimization of boundary value problems for higher order diferential inclusions and duality*, Optim. Lett. **16** (2022), 695–712.
[22] E. N. Mahmudov, M. J. Mardanov, *Optimization of the Nicoletti boundary value problem for second-order differential inclusions*, Proceed.Inst.Math.Mech. **49** (2023), 3–15.
[23] E. N. Mahmudov, D. I. Mastaliyeva, *Optimization of the Dirichlet problem for gradient differential inclusions*, Nonlin. Diff. Equ. Appl. NoDEA, **31** (2024), 1–20.
[24] B. S. Mordukhovich, J-P. Raymond, *Neumann boundary control of hyperbolic equations with pointwise state constraints*, SIAM J. Contr. Optim. **43** (2005), 1354–1372.
[25] B. S. Mordukhovich, J-P. Raymond, *Optimal boundary control of hyperbolic equations with pointwise state constraints*, Nonlin. Anal.: Theory Meth. Appl. **63** (2005), 823–830.
[26] L. S. Pontryagin, V. G. Boltyansii, R. V. Gamkrelidze, E. E. Mishchenko, *The mathematical theory of optimal processes*, John Wiley and Sons, Inc., New York, 1962.
[27] M. A. Ragusa, A. Scapellato, *Mixed Morrey spaces and their applications to partial differential equations*, Nonlin. Anal.: Theory, Meth.Appl. **151** (2017), 51–65.
[28] S. D. Sağlam, *Duality in the problems of optimal control described by Darboux type diferential inclusions*, Optim. Lett.**18** (2024), 1811–1835.
[29] Z. Soltani, *A fixed point theorem for generalized contractive type set-valued mappings with application to nonlinear fractional differential inclusions*, Filomat, **32** (2018), 5361–5370.
[30] Z. S. Wu, K. L. Teo, *A convex optimal control problem involving a class of linear hyperbolic systems*, J. Optim. Theory Appl. **39** (1983), 541–560.
[31] Sh. Sh. Yusubov, *On an Optimality of the Singular with Respect to Components Controls ın the Goursat-Dourboux Systems*, Probl. Uprav. **5** (2014), 2–6.
[32] Sh. Sh. Yusubov, *Nonlocal Problem with Integral Conditions for a High-Order Hyperbolic Equation*, Ukr. Mathem. J. **69** (2017), 121–131.