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$\alpha\beta$-level soft sets of FPFS sets and their application to a decision-making problem

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Abstract. In this study we introduce the notion of $\alpha\beta$-level soft set of a fuzzy parameterized fuzzy soft set. We investigate the related with algebraic properties of this notion. Later, we give a decision problem model based on $\alpha\beta$-level soft sets. Finally, this decision-making method has been tried to be explained on an example.

1. Introduction

The concept of sets is fundamental to classical mathematics, and this requires that all mathematical concepts be exact. Therefore, in classical set theory, our only knowledge about a set is the certainty that the elements of the set belong to the set. In other words, either an element belongs to the set, or it does not. Despite the certainty of the concept of set, the real world is full of uncertain data. For example, linguistic data can vary depending on the person, time and place. For this reason, classical set theory is often insufficient in modelling in many fields of science such as computer science, economics, engineering and similar that use linguistic data. However, in response to this requirement, set theories that include uncertainty based on classical set theory began to emerge in the mid-20th century. The most important of these theories is undoubtedly fuzzy set theory [19]. Recently, many alternative theories that include uncertainty have been proposed as alternatives to fuzzy set theory. One of these theories is the soft set which was revealed in 1999 [16]. To overcome the shortcomings of these theories, researchers have combined them to come up with new hybrid theories. The first of these approaches, which is obtained by hybridizing soft sets and fuzzy sets, is fuzzy soft sets [14]. It has been considered to match them with fuzzy sets instead of matching the parameters with classical sets in fuzzy soft sets. Later fuzzy parameterized soft sets [10] and fuzzy parameterized fuzzy soft sets [9] are defined based on these two theories. Recently, many researchers have worked on hybrid theories in various fields, such as [1–7, 11, 12, 18, 20].

On the other hand, decision-making is a problem that human beings frequently encounter in their daily lives. The right decision is very important in matters such as the correct management of many resources such as economy and time or the correct diagnosis of a disease. In addition, it can be difficult to end this process in the human mind due to the abundance and complexity of data. Therefore, creating a mathematical decision model plays an important role in solving this complexity. Furthermore, data that is either below or

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above a specific threshold is irrelevant for making decisions in many cases. For example, a blood pressure value below a certain level is meaningless. Similarly, using data from economic movements in a crisis can lead to unrealistic decisions. Therefore, it is important to ignore this type of data in decision-making.

In this study, the $\alpha\beta$-level soft sets of fuzzy parameterized fuzzy soft sets are defined, and their basic algebraic properties are examined. In addition to this, a decision-making model that ignores irrelevant data based on the $\alpha\beta$-level soft sets of fuzzy parameterized fuzzy soft sets is created and an example is given.

2. Preliminaries

Definition 2.1. ([19]) Let X be a non empty set and $I = [0,1]$. A fuzzy set $\widetilde{A}$ on X is defined as $\widetilde{A} = \{x^{\mu_A(x)} : x \in X\}$, where $\mu_A : X \to [0,1]$ is membership function of $\widetilde{A}$.

From now on, the set of all fuzzy sets over X will be denoted by I^X.

Definition 2.2. ([19]) Let $\widetilde{A}$ and $\widetilde{B}$ be fuzzy sets on X. Then fundamental operations of fuzzy sets are given as follows:

(1) If $\mu_A(x) \leq \mu_B(x)$ for all $x \in X$, then it is called $\widetilde{A}$ is fuzzy subset of $\widetilde{B}$ and is denoted by $\widetilde{A} \leq \widetilde{B}$.

(2) The fuzzy set $\widetilde{A} \vee \widetilde{B}$ on X, whose membership function is defined by $\mu_{A\vee B}(x) = \max\{\mu_A(x), \mu_B(x)\}$ for all $x \in X$, is called the fuzzy union of $\widetilde{A}$ and $\widetilde{B}$.

(3) The fuzzy set $\widetilde{A} \wedge \widetilde{B}$ on X, whose membership function is defined by $\mu_{A\wedge B}(x) = \min\{\mu_A(x), \mu_B(x)\}$ for all $x \in X$, is called the fuzzy intersection of $\widetilde{A}$ and $\widetilde{B}$.

(4) If $\mu_A(x) = 0$ for all $x \in X$, then $\widetilde{A}$ is called null fuzzy set and is denoted by 0_X.

(5) If $\mu_A(x) = 1$ for all $x \in X$, then $\widetilde{A}$ is called universal fuzzy set and is denoted by 1_X.

(6) The fuzzy set $\widetilde{A}^c$on X, whose membership function $\mu_{A^c}(x) = 1 - \mu_A(x)$ for all $x \in X$, is called the complement of $\widetilde{A}$.

Definition 2.3. ([15]) Let $\widetilde{A}$ and $\widetilde{B}$ be fuzzy set on X. If there exists at least one point $x \in X$ such that $\mu_A(x) + \mu_B(x) > 1$, then it is called $\widetilde{A}$ is quasi-coincident with $\widetilde{B}$ and is denoted by $\widetilde{A}q\widetilde{B}$.

If $\widetilde{A}$ is not quasi-coincident with $\widetilde{B}$, then we write $\widetilde{A}\overline{q}\widetilde{B}$.

Definition 2.4. ([8]) Let X be a non-empty set, E be a set of parameters and $F_A : E \to P(X)$. Then the set $(F, A) = \{(e, F_A(e)) : e \in E, F_A(e) \in P(X)\}$ is called a soft set on X, where the function F_A is called the approximation function.

From now on, the set of all soft sets over X will be denoted by $S(X, E)$ and unless otherwise required, we will represent a soft set with the approximation function.

Definition 2.5. ([8]) Let $F_A, F_B \in S(X, E)$. Then fundamental operations of soft sets are given as follows:
(1) If $F_A(e) \subseteq F_B(e)$ for all $e \in E$, then it is called F_A soft subset F_B and is denoted by $F_A \widetilde{\subseteq} F_B$.

(2) The soft set $F_A \widetilde{\cup} F_B$, whose approximation function is defined by $F_{A\cup B}(e) = F_A(e) \cup G_B(e)$ for all $e \in E$, is called the soft union of F_A and G_B.

(3) The soft set $F_A \widetilde{\cap} G_B$, whose approximation function is defined by $F_{A\cap B}(e) = F_A(e) \cap G_B(e)$ for all $e \in E$, is called the soft intersection of F_A and G_B.

(4) If $F_A(e) = \emptyset$ for all $e \in E$, then F_A is called null soft set and is denoted by $F_\emptyset$.

(5) If $F_A(e) = X$ for all $e \in E$, then F_A is called universal soft set and is denoted by F_E.

(6) The soft set F_A^c, whose approximation function is defined by $F_{A^c}(e) = X \setminus F_A(e)$ for all $e \in E$, is called the soft complement of F_A.

It can easily be seen from the definitions above that $F_A \widetilde{\subseteq} F_B$ if and only if $\{(e, \{x\})\} \widetilde{\subseteq} F_A$ implies $\{(e, \{x\})\} \widetilde{\subseteq} F_B$ for all $e \in E$ and $x \in X$.

Definition 2.6. ([13]) Let $u : X \to Y$ and $p : E \to K$ be functions. Then a function denoted by $u_p : S(X,E) \to S(Y,K)$ called a soft mapping and defined as follows:

(1) If $F_A \in S(X,E)$, then the image of F_A under the u_p is the soft set $u_p(F_A)$, defined by

$$u_p(F_A)(k) = \begin{cases} \bigcup\limits_{e \in p^{-1}(k) \cap A} u(F_A(e)) & ; p^{-1}(k) \cap A \neq \varnothing \\ \varnothing & ; p^{-1}(k) \cap A = \varnothing \end{cases}$$

.

(2) If $G_B \in S(Y,K)$, then the preimage of G_B under the soft mapping u_p is the soft set $u_p^{-1}(G_B)$, defined by

$$u_p^{-1}(G_B)(e) = \begin{cases} u^{-1}(G_S(p(e))) & ; p(e) \in B; \\ \varnothing & ; p(e) \notin B \end{cases}$$

Definition 2.7. ([9]) Let $\widetilde{A}$ be a fuzzy set over E. A fuzzy parameterized fuzzy soft set (FPFS) $f_{\widetilde{A}}$ on the universe X is defined as follows:

$f_{\widetilde{A}} = \{(e^{\mu_A(e)}, f_A(e)) : e \in E, f_A(e) \in I^X, \mu_A(e) \in [0,1]\}$,

where the function $f_A : E \to I^X$ is called approximate function of $f_{\widetilde{A}}$.

From now on, the set of all FPFS sets over X will be denoted by $FPFS(X,E)$.

Definition 2.8. ([9]) Let $f_A, f_B \in FPFS(X,E)$. Then some fundemantal operations for fuzzy parameterized fuzzy soft sets are given as follows:

(1) If $\widetilde{A} \leq \widetilde{B}$ and $f_A(e) \leq g_B(e)$ for all $e \in E$, then it is called $f_{\widetilde{A}}$ is FPFS subset $f_{\widetilde{B}}$ and is denoted by $f_{\widetilde{A}} \widetilde{\sqsubseteq} f_{\widetilde{B}}$.

(2) The FPFS set $f_A \widetilde{\sqcup} f_B$, whose membership function and approximation function are defined by $\mu_{A \vee B}(e) = \max\{\mu_A(e), \mu_B(e)\}$ and $f_{A \cup B}(e) = f_A(e) \vee f_B(e)$ for all $e \in E$, respectively, is called the FPFS union of $f_{\widetilde{A}}$ and $f_{\widetilde{B}}$.

(3) The FPFS set $f_A \widetilde{\sqcap} f_B$, whose membership function and approximation function are defined by $\mu_{A \wedge B}(e) = \min\{\mu_A(e), \mu_B(e)\}$ and $f_{A \cap B}(e) = f_A(e) \wedge f_B(e)$ for all $e \in E$, respectively, is called the FPFS intersection of $f_{\widetilde{A}}$ and $f_{\widetilde{B}}$.

(4) If $\widetilde{A} = 0_E$ and $f_A(e) = 0_X$ for all $e \in E$, then $f_{\widetilde{A}}$ is called null FPFS set and is denoted by $f_{\widetilde{0}}$.

(5) If $\widetilde{A} = 1_E$ and $f_A(e) = 1_X$ for all $e \in E$, then $f_{\widetilde{A}}$ is called universal FPFS set and is denoted by $f_{\widetilde{1}}$.

(6) The FPFS set $f^c_{\widetilde{A}}$, whose membership function and approximation function are defined by $\mu_{A^C}(e) = 1 - \mu_A(e)$ and $f_{A^C}(e) = (f_A(e))^C$ for all $e \in E$, respectively, is called the FPFS complement of $f_{\widetilde{A}}$.

Definition 2.9. ([21]) Let $f_{\widetilde{A}}, f_{\widetilde{B}} \in FPFS(X,E)$. If there exists at least one parameter $e \in E$ such that $f_A(e) q g_B(e)$ or $\widetilde{A} q \widetilde{B}$, then it is called $f_{\widetilde{A}}$ is quasi-coincident with $f_{\widetilde{B}}$ and is denoted by $f_{\widetilde{A}} q f_{\widetilde{B}}$. If $f_{\widetilde{A}}$ is not quasi-coincident with $f_{\widetilde{B}}$, then we write $f_{\widetilde{A}} \overline{q} f_{\widetilde{B}}$.

Definition 2.10. ([21]) Let $FPFS(X,E)$ and $FPFS(Y,K)$ be families of all FPFS sets over X and Y, respectively. Let $u : X \to Y$ and $p : E \to K$ be two functions. Then a FPFS mapping $f_{up} : FPFS(X,E) \to FPFS(Y,K)$ is defined as:

(1) for $f_{\widetilde{A}} \in FPFS(X,E)$, the image of $f_{\widetilde{A}}$ under the f_{up} is the FPFS set $g_{\widetilde{S}}$ over Y defined by the approximate function, $\forall k \in K$ and,

$$g_S(k)(y) = \begin{cases} \bigvee\limits_{x \in u^{-1}(y)} \Big(\bigvee\limits_{e \in p^{-1}(k) \cap \operatorname{supp} \widetilde{A}} f_A(e) \Big)(x), & u^{-1}(y) \neq \varnothing,\ p^{-1}(k) \cap \operatorname{supp} \widetilde{A} \neq \varnothing \\ 0_X, & \text{otherwise.} \end{cases}$$

where $p(\widetilde{A}) = \widetilde{S}$ is fuzzy set in K.

(2) for $g_{\widetilde{S}} \in FPFS(Y,K)$, then the pre-image of $g_{\widetilde{S}}$ under the f_{up} is the FPFS set $f_{\widetilde{A}}$ over X defined by the approximate function, $\forall e \in E$;

$f_A(e)(x) = g_S(p(e))(u(x))$

where $\widetilde{A} = p^{-1}(\widetilde{S})$ is fuzzy set in E.

3. $\alpha\beta$-level soft sets of fuzzy parameterized fuzzy soft sets

For any fuzzy set $A \in I^X$and $\alpha \in (0, 1]$, α-level set is defined by $A_\alpha = \{x : \mu_A(x) \geq \alpha\}$[17]. In [12], α-level set for a fuzzy soft set was chosen as a subset of the universe. Here, we introduce a new approach by defining a soft set as an alpha level set for fuzzy parameterized fuzzy soft set.

Definition 3.1. Let $f_{\widetilde{A}} \in FPFS(X, E)$ and $\alpha, \beta \in (0, 1]$. Then the soft set

$$(f_A)_\alpha^\beta = \{(e, (f_A(e))_\alpha) : e \in \widetilde{A}_\beta\}$$

is called $\alpha\beta$-level soft set of $f_{\widetilde{A}}$.

Example 3.2. Let $X = \{x_1, x_2, x_3, x_4\}$, $E = \{e_1, e_2, e_3\}$, $f_A = \{(e_1^{0.7}, \{x_1^{0.3}, x_2^{0.7}, x_3^{0.4}\}), (e_2^{0.4}, \{x_1^{0,4}, x_2^{0,6}\}), (e_3^{0.8}, (x_1^{1}, x_2^{0,1}, x_3^{0,5})\}$ and $\alpha = 0.5$, $\beta = 0.6$. Then $\alpha\beta$-level soft set of $f_{\widetilde{A}}$ is

$$(f_A)_{0.5}^{0.6} = \{(e_1, \{x_2\}), (e_3, \{x_1, x_3\})\}$$

We will now introduce the fundamental properties of the notion of $\alpha\beta$-level soft set.

Theorem 3.3. *Let $f_{\widetilde{A}}, f_{\widetilde{B}} \in FPFS(X, E)$ and $\alpha, \beta, \theta, \gamma \in (0, 1]$. Then followings are true.*

(1) $f_{\widetilde{A}} \widetilde{\sqsubseteq} f_{\widetilde{B}} \Rightarrow (f_{\widetilde{A}})_\alpha^\beta \widetilde{\subseteq} (f_{\widetilde{B}})_\alpha^\beta$.

(2) $(f_{\widetilde{A}} \widetilde{\sqcap} f_{\widetilde{B}})_\alpha^\beta = (f_A)_\alpha^\beta \widetilde{\cap} (f_B)_\alpha^\beta$.

(3) $(f_{\widetilde{A}} \widetilde{\sqcup} f_{\widetilde{B}})_\alpha^\beta \widetilde{\supseteq} (f_{\widetilde{A}})_\alpha^\beta \widetilde{\cup} (f_{\widetilde{B}})_\alpha^\beta$.

(4) $(f_{\widetilde{0}})_\alpha^\beta = F_\varnothing$.

(5) $(f_{\widetilde{1}})_\alpha^\beta = F_E$.

(6) $\alpha < \theta$ *and* $\beta < \gamma \Rightarrow (f_A)_\theta^\gamma \widetilde{\subseteq} (f_A)_\alpha^\beta$.

Proof. (1) Let $f_{\widetilde{A}} \widetilde{\sqsubseteq} f_{\widetilde{B}}$. Then for all $e \in E$, $f_A(e) \leq g_B(e)$ and $\widetilde{A} \leq \widetilde{B}$. Therefore, we have

$$\begin{aligned}
\{(e, \{x\})\} \widetilde{\subseteq} (f_{\widetilde{A}})_\alpha^\beta \quad &\Rightarrow (f_A)(e)(x) \geq \alpha \text{ and } \mu_A(e) \geq \beta \\
&\Rightarrow (f_B)(e)(x) \geq \alpha \text{ and } \mu_B(e) \geq \beta \\
&\Rightarrow \{(e, \{x\})\} \widetilde{\subseteq} (f_B)_\alpha^\beta
\end{aligned}$$

This shows that $(f_{\widetilde{A}})_\alpha^\beta \widetilde{\subseteq} (f_{\widetilde{B}})_\alpha^\beta$.

(2)

$$\begin{aligned}
\{(e, \{x\})\} \widetilde{\subseteq} (f_{\widetilde{A}} \widetilde{\sqcap} f_{\widetilde{B}})_\alpha^\beta \quad &\Leftrightarrow (f_A \widetilde{\sqcap} f_B)(e)(x) \geq \alpha \text{ and } \mu_{A \wedge B}(e) \geq \beta \\
&\Leftrightarrow \min\{(f_A)(e)(x), (f_B)(e)(x)\} \geq \alpha \\
&\quad \text{ and } \min\{\mu_A(e), \mu_B(e)\} \geq \beta \\
&\Leftrightarrow (f_A)(e)(x) \geq \alpha,\ \mu_A(e) \geq \beta \\
&\quad \text{ and } (f_B)(e)(x) \geq \alpha,\ \mu_B(e) \geq \beta \\
&\Leftrightarrow \{(e, \{x\})\} \widetilde{\subseteq} \left(f_{\widetilde{A}}\right)_\alpha^\beta \text{ and } \{(e, \{x\})\} \widetilde{\subseteq} (f_B)_\alpha^\beta \\
&\Leftrightarrow \{(e, \{x\})\} \widetilde{\subseteq} \left(f_{\widetilde{A}}\right)_\alpha^\beta \widetilde{\cap} (f_{\widetilde{B}})_\alpha^\beta
\end{aligned}$$

These requirements show that $(f_{\widetilde{A}} \widetilde{\sqcap} f_{\widetilde{B}})_\alpha^\beta = \left(f_{\widetilde{A}}\right)_\alpha^\beta \widetilde{\cap} (f_{\widetilde{B}})_\alpha^\beta$.

(3)

$$\{(e,\{x\})\}\widetilde{\subseteq}\left(f_{\widetilde{A}}\right)_{\alpha}^{\beta}\widetilde{\cup}(f_{\widetilde{B}})_{\alpha}^{\beta} \Rightarrow \{(e,\{x\})\}\widetilde{\subseteq}\left(f_{\widetilde{A}}\right)_{\alpha}^{\beta} \text{ or } \{(e,\{x\})\}\widetilde{\subseteq}\left(f_{\widetilde{B}}\right)_{\alpha}^{\beta}$$

$$\Rightarrow (f_A)(e)(x) \geq \alpha,\ \mu_A(e) \geq \beta \text{ or } (f_B)(e)(x) \geq \alpha,\ \mu_B(e) \geq \beta$$

$$\Rightarrow \max\{(f_A)(e)(x), (f_B)(e)(x)\} \geq \alpha \text{ and } \max\{\mu_A(e), \mu_B(e)\} \geq \beta$$

$$\Rightarrow \left(f_A \widetilde{\sqcup} f_B\right)(e)(x) \geq \alpha \text{ and } \mu_{A\vee B}(e) \geq \beta$$

$$\Rightarrow \{(e,\{x\})\}\widetilde{\subseteq}(f_A\widetilde{\sqcup}f_B)_{\alpha}^{\beta}$$

These requirements show that $(f_{\widetilde{A}}\widetilde{\sqcup}f_{\widetilde{B}})_{\alpha}^{\beta}\widetilde{\supseteq}\left(f_{\widetilde{A}}\right)_{\alpha}^{\beta}\widetilde{\cup}(f_{\widetilde{B}})_{\alpha}^{\beta}$.

The equalities (4) and (5) are obvious from definitions.

(6) Let $\alpha < \theta$ and $\beta < \gamma$. For all $e \in E$ and for all $x \in X$,

$$\{(e,\{x\})\}\widetilde{\subseteq}(f_{\widetilde{A}})_{\theta}^{\gamma} \Rightarrow (f_A)(e)(x) \geq \theta \text{ and } \mu_A(e) \geq \gamma$$

$$\Rightarrow (f_A)(e)(x) \geq \alpha \text{ and } \mu_A(e) \geq \beta$$

$$\Rightarrow \{(e,\{x\})\}\widetilde{\subseteq}\left(f_{\widetilde{A}}\right)_{\alpha}^{\beta}$$

This shows that $(f_A)_{\theta}^{\gamma}\widetilde{\subseteq}(f_A)_{\alpha}^{\beta}$. □

Nguyen [17] also introduced to the concept of strongly α-level set. Here we will generalize this concept for fuzzy soft sets.

For a fuzzy set $A \in I^X$and $\alpha \in (0,1]$, strongly α-level set defined as $A_{\alpha}^{*} = \{x : \mu_A(x) > \alpha\}$.

Definition 3.4. Let $f_{\widetilde{A}} \in FPFS(X,E)$ and $\alpha, \beta \in (0,1]$. Then the soft set

$$(f_A)_{\alpha}^{\beta*} = \{(e, (f_A(e))_{\alpha}^{*}) : e \in \widetilde{A}_{\beta}^{*}\}$$

is called strongly α, β-level soft set of $f_{\widetilde{A}}$.

Note that $(f_A)_{\alpha}^{\beta}\widetilde{\subseteq}(f_A)_{\alpha}^{\beta*}$.

Theorem 3.5. *If $f_{\widetilde{A}} \in FPFS(X,E)$ and $\alpha, \beta \in (0,1]$, then $\left((f_A)^c\right)_{\alpha}^{\beta*}\widetilde{\subset}\left((f_A)_{1-\alpha}^{1-\beta}\right)^c$.*

Proof. Let $f_{\widetilde{A}} \in FPFS(X,E)$ and $\alpha, \beta \in (0,1]$. Then

$$\{(e,\{x\})\}\widetilde{\subseteq}(f_{\widetilde{A}}^{c})_{\alpha}^{\beta*} \Rightarrow x \in (f_{\widetilde{A}}^{c})_{\alpha}^{\beta*}(e)$$

$$\Rightarrow (f_A)^c(e)(x) > \alpha \text{ and } \mu_{A^c}(e) > \beta$$

$$\Rightarrow 1 - f_A(e)(x) > \alpha \text{ and } 1 - \mu_A(e) > \beta$$

$$\Rightarrow (f_A)(e)(x) < 1 - \alpha \text{ and } \mu_A(e) < 1 - \beta$$

$$\Rightarrow x \notin \left((f)_{1-\alpha}^{1-\beta}\right)(e)$$

$$\Rightarrow x \in \left((f_A)_{1-\alpha}^{1-\beta}\right)^c(e)$$

$$\Rightarrow \{(e,\{x\})\}\widetilde{\subseteq}\left((f_{\widetilde{A}})_{1-\alpha}^{1-\beta}\right)^c$$

This shows that $\left(\left(f_{\widetilde{A}}\right)^c\right)_{\alpha}^{\beta*}\widetilde{\subset}\left((f_{\widetilde{A}})_{1-\alpha}^{1-\beta}\right)^c$. □

The sufficiency part of this theorem is not true in general. Let's show this on an example.

Example 3.6. Let $X = \{x_1, x_2, x_3, x_4\}$, $E = \{e_1, e_2, e_3\}$ and $f_{\widetilde{A}} = \{(e_1^{0.3}, \{x_1^{0.4}, x_2^{0.8}, x_3^{0.5}\}), (e_2^{0.5}, \{x_1^{0.5}, x_2^{0.7}\}), (e_3^{0.6}, (x_1^{1}, x_2^{0.2}, x_3^{0.6})\}$. If we choose $\alpha = 0.4$ $\beta = 0.5$, then we have

$((f_{\widetilde{A}})_{0,6}^{0.5})^c = \{(e_1, X), (e_2, \{x_1, x_3\}), (e_3, \{x_2\}\}$

$((f_{\widetilde{A}}^c)_{0,4}^{0.5*}) = \{(e_1, \{x_1\}\}$

It is clear that $((f_{\widetilde{A}})_{0,6}^{0.5})^c \widetilde{\not\subseteq} ((f_{\widetilde{A}}^c)_{0,4}^{0.5*})$.

Theorem 3.7. *Let $f_{\widetilde{A}} \in FPFS(X, E)$ and $\alpha, \beta, \theta, \gamma \in (0, 1]$. Then*

(1) $(f_{\widetilde{A}})_\alpha^\beta = \widetilde{\bigcap}_{\theta<\alpha, \gamma<\beta} (f_{\widetilde{A}})_\theta^\gamma$

(2) $(f_{\widetilde{A}})_\alpha^{\beta*} = \widetilde{\bigcup}_{\theta>\alpha, \gamma>\beta} (f_{\widetilde{A}})_\theta^\gamma$.

Proof. (1) By Theorem 3.3 (6), if $\theta < \alpha, \gamma < \beta$, then $(f_{\widetilde{A}})_\alpha^\beta \widetilde{\subseteq} (f_{\widetilde{A}})_\theta^\gamma$ and so we have that $(f_{\widetilde{A}})_\alpha^\beta \widetilde{\subseteq} \widetilde{\bigcap}_{\beta<\alpha} (f_{\widetilde{A}})_\theta^\gamma$.

Conversely, if $\{(e, \{x\})\} \widetilde{\subseteq} \widetilde{\bigcap}_{\theta<\alpha, \gamma<\beta} (f_{\widetilde{A}})_\theta^\gamma$, then for all $\theta < \alpha, \gamma < \beta$, $\{(e, \{x\})\} \widetilde{\subseteq} (f_{\widetilde{A}})_\theta^\gamma$. Therefore, $\{(e, \{x\})\} \widetilde{\subseteq} (f_A)_{\alpha-\varepsilon}^{\beta-\varepsilon}$ for all ε such that $\alpha, \beta > \varepsilon > 0$. So $(f_A)(e)(x) \geq \alpha - \varepsilon$ and $\mu_A(x) \geq \beta - \varepsilon$ for all ε such that $\alpha, \beta > \varepsilon > 0$. Hence, it is clear that $(f_A)(e)(x) \geq \alpha$ and $\mu(x) \geq \beta$. Consequently, $x \in (f_A)_\alpha(e)$, $\mu(x) \geq \beta$ and so $\{(e, \{x\})\} \widetilde{\subseteq} (f_A)_\alpha^\beta$. This completes the proof.

(2) By Theorem 3.3 (6), if $\theta > \alpha, \gamma > \beta$, then $(f_A)_\theta^\gamma \widetilde{\subseteq} (f_A)_\alpha^\beta$ and so we have that $(f_A)_\theta^\gamma \widetilde{\subseteq} (f_A)_\alpha^{\beta*}$. This shows that $\widetilde{\bigcup}_{\beta>\alpha} (f_A)_\theta^\gamma \widetilde{\subseteq} (f_A)_\alpha^{\beta*}$.

Conversely, if $\{(e, \{x\})\} \widetilde{\subseteq} (f_A)_\alpha^{\beta*}$, then $x \in (f_A)_\alpha^{\beta*}(e)$ and so $(f_A)(e)(x) > \alpha$ and $\mu_A(e) > \beta$. Hence $(f_A)(e)(x) \geq \alpha + \varepsilon$ and $\mu_A(e) \geq \beta + \varepsilon$ for any $\varepsilon > 0$. Put $\theta = \alpha + \varepsilon$ and $\gamma = \beta + \varepsilon$. Then $(f_A)(e)(x) \geq \theta$ and $\mu_A(e) \geq \gamma$. Consequently $\{(e, \{x\})\} \widetilde{\subseteq} (f_A)_\theta^\gamma \widetilde{\subseteq} \widetilde{\bigcup}_{\beta>\alpha} (f_A)_\theta^\gamma$. This completes the proof. □

Theorem 3.8. *Let $u : X \to Y$, $p : E \to K$ be two function and $f_{\widetilde{A}} \in FPFS(X, E)$. Then the equality $u_p((f_A)_\alpha^\beta) = (u_p(f_A))_\alpha^\beta$ is true.*

Proof. (Case I) If $u^{-1}(y) = \emptyset$ and $p^{-1}(k) \cap \operatorname{supp} \widetilde{A} = \emptyset$, Then $p(\widetilde{A}) = 0_K$ and $u_p(f_A) = 0_Y$ and so $(u_p(f_A))_\alpha^\beta = \emptyset$. This ends the proof.

(Case II) Let if $u^{-1}(y) \neq \emptyset$ and $p^{-1}(k) \cap \operatorname{supp} \widetilde{A} \neq \emptyset$. Then

$$(k,\{y\})\widetilde{\subseteq}u_p\left((f_A)_\alpha^\beta\right) \Leftrightarrow y\in u_p\left((f_A)_\alpha^\beta\right)(k)$$

$$\Leftrightarrow y\in \bigcup_{e\in p^{-1}(k)\cap \operatorname{supp}\widetilde{A}} u((f_A)_\alpha^\beta(e))$$

$$\Leftrightarrow y\in u((f_A)_\alpha^\beta(e)) \text{ for any } e\in p^{-1}(k)\cap \operatorname{supp}\widetilde{A}$$

$$\Leftrightarrow x\in (f_A)_\alpha^\beta(e)$$
$$\text{for any } e\in p^{-1}(k)\cap \operatorname{supp}\widetilde{A} \text{ and } x\in u^{-1}(y)$$

$$\Leftrightarrow (f_A)(e)(x)\geq\alpha \text{ and } \mu_A(e)\geq\beta$$

$$\text{for any } e\in p^{-1}(k)\cap \operatorname{supp}\widetilde{A} \text{ and } x\in u^{-1}(y)$$
$$\Leftrightarrow \left(\bigvee_{e\in p^{-1}(k)\cap \operatorname{supp}\widetilde{A}} f_A(e)\right)(x)\geq\alpha \text{ and } \mu_A(e)\geq\beta$$
$$\text{for any } e\in p^{-1}(k)\cap \operatorname{supp}\widetilde{A} \text{ and } x\in u^{-1}(y)$$

$$\Leftrightarrow \bigvee_{x\in u^{-1}(y)}\left(\bigvee_{e\in p^{-1}(k)\cap \operatorname{supp}\widetilde{A}} f_A(e)\right)(x)\geq\alpha \text{ and } \mu_A(e)\geq\beta$$
$$\text{for any } e\in p^{-1}(k)\cap \operatorname{supp}\widetilde{A}$$

$$\Leftrightarrow u_p(f_A)(k)(y)\geq\alpha \text{ and } \mu_{p(A)}(e)\geq\beta$$

$$\Leftrightarrow (k,\{y\})\widetilde{\in}(u_p(f_A))_\alpha^\beta$$

□

Theorem 3.9. *Let $u: X\to Y$, $p: E\to K$ be two function and $g_B\in FPFS(Y,K)$. Then the equality $u_p^{-1}((g_B)_\alpha) = (u_p^{-1}(g_B))_\alpha$ is true.*

$$\{(e,\{x\})\}\widetilde{\subseteq}(u_p^{-1}(g_B))_\alpha \Leftrightarrow (u_p^{-1}(g_B))(e)(x)\geq\alpha$$

$$\Leftrightarrow g_B(p(e))(u(x))\geq\alpha$$

$$\Leftrightarrow u(x)\in (g_B)_a(p(e))$$

$$\Leftrightarrow x\in u^{-1}((g_B)_a(p(e)))$$

$$\Leftrightarrow x\in u_p^{-1}((g_B)_\alpha)(e)$$

$$\Leftrightarrow \{(e,\{x\})\}\widetilde{\subseteq}u_p^{-1}((g_B)_\alpha)$$

3.1. Application

The word ”decision” is the process of choosing the best one among various alternatives or possibilities. Decision making is constantly encountered in various areas in our daily lives. However, in some cases, factors such as the multiplicity of alternatives and the complexity of parameters may not allow this process to be completed by the human mind. For this reason, today, it is frequently used to try to solve these difficulties using mathematical models.

In the literature, theories obtained by hybridizing soft sets and these sets with fuzzy sets are used intensively in decision-making methods. In most problems, the weights of the parameters and alternatives are usually taken equally. However, in some problems, it may be desired that the parameters and alternatives below a certain level do not affect the decision. For example, when making a decision with an analysis result in medicine, valuesbelow a certain value may be meaningless. For this reason, reducing some data with level sets will be important in terms of decision-making.

In this study, a decision-making method will be given using $\alpha\beta$ level soft sets. The main purpose of this method is to remove the effectiveness of data below a certain level in the decision. Finally, a decision-making example with unrealistic data will be given, which demonstrates the applicability of this method.

Algorithm

The following steps are followed for decision making.

1. Create the set of jurors selected by the decision maker to evaluate, such as $J = \{A_1, A_2, ..A_n\}$.
2. Determine the E set of parameters that the jury will use in the evaluation.
3. The decision-maker determines the parameter weights according to the jury's areas of expertise and gives in the form of a fuzzy set.
4. Construct fuzzy parameterized fuzzy soft sets that give each juror's evaluations, such as f_{A_i}.
5. The decision maker determines the minimum α and β valuesthat will ensure that the parameters and alternatives participate in the decision.
6. Construct $\alpha\beta$-level soft sets $(f_{A_i})_\alpha^\beta$.
7. Calculate

$$\mu_D(x_j) = \frac{1}{|J|}\sum_{k=1}^{n} \frac{\left|\{e_i : x_j \in f_{A_k}(e_i)\}\right|}{|E|}$$

and find fuzzy decision set D.

Example 3.10. Mr. X will recruit staff for the company. Let's assume that the set of candidates is $\{x_1, x_2, x_3\}$.

Step 1. Let the set of juries that Mr. X has chosen, be $J = \{A_1, A_2, A_3\}$.

Step 2. Juries determine the set of evaluation parameters $E = \{e_1, e_2, e_3\}$; where e_1 is experience, e_2 is foreign language knowledge, e_3 is expertise in this field.

Step 3. The fuzzy set of areas of expertise jurors are $\widetilde{A_1} = \{e_1^{0.2}, e_2^{0.5}, e_3^{0.7}\}$, $\widetilde{A_2} = \{e_1^{0.4}, e_2^{0.5}, e_3^{0.1}\}$, $\widetilde{A_3} = \{e_1^{0.3}, e_2^{0.2}, e_3^{0.8}\}$.

Step 4. The juror's present their evaluations to Mr X in the form of fuzzy soft sets

$f_{\widetilde{A_1}} = \{(e_1^{0.2}, \{x_1^{0.4}, x_2^{0.7}, x_3^{0.6}\}), (e_2^{0.5}, \{x_1^{0.6}, x_2^{0.3}, x_3^{0.9}\}, (e_3^{0.7}, \{x_1^{0.2}, x_2^{0.7}, x_3^{0.8}\})\}$

$f_{\widetilde{A_2}} = \{(e_1^{0.4}, \{x_1^{0.5}, x_2^{0.3}, x_3^{1}\}), (e_2^{0.5}, \{x_1^{0.2}, x_2^{0.6}, x_3^{0.5}\}, (e_3^{0.1}, \{x_1^{0.8}, x_2^{0.5}, x_3^{0.7}\})\}$

$f_{\widetilde{A_3}} = \{(e_1^{0.5}, \{x_1^{0.4}, x_2^{0.7}, x_3^{0.6}\}), (e_2^{0.3}, \{x_1^{0.6}, x_2^{0.3}, x_3^{0.9}\}, (e_3^{0.8}, \{x_1^{1}, x_2^{0.2}, x_3^{0.8}\})\}$

.

Step 5. The decision maker determines the valuethat will ensure the participation of the parameters and alternatives in the decision as $\alpha = 0.4$ and $\beta = 0.3$.

Step 6. The level sets corresponding to the α and β valuesdetermined by Mr. X are as follows;

$(f_{A_1})_{0.4}^{0.3} = \{(e_2, \{x_1, x_3\}), (e_3, \{x_2, x_3\})\}$

$(f_{A_2})_{0.4}^{0.3} = \{(e_1, \{x_1, x_2\}), (e_2, \{x_2, x_3\})\}$

$(f_{A_3})_{0.4}^{0.3} = \{(e_1, X), (e_2, \{x_1, x_3\}), (e_3, \{x_1, x_3\})\}$

Step 7. Lastly, we obtain the set of decision $D = \{x_1^{\frac{5}{9}}, x_2^{\frac{4}{9}}, x_3^{\frac{2}{3}}\}$.

As a result, candidate x_3 is the best candidate for the company.

4. Conclusion

This study introduces the concept of $\alpha\beta$-level sets within the context of fuzzy parameterized fuzzy soft sets and investigates their fundamental properties. Subsequently, a decision-making model is constructed using this concept, and an example is provided to demonstrate the model's efficacy. It is anticipated that the concepts presented in this study will pave the way for further research across various disciplines.

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