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Vanishing results of the F-stress energy tensor

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Abstract. In this paper, we study some vanishing results of the F-stress energy tensor S_F associated to the F-energy where the target manifold is equipped with a metric connection having non-vanishing torsion. By estimating the norm of S_F, we introduce a $\Phi_{S,F}$-energy functional for maps. The critical map of this functional is called a $\Phi_{S,F}$-harmonic map. We obtain some vanishing results of S_F by studying Liouville theorems for the $\Phi_{S,F}$-harmonic map. Firstly, we find that the equation of $\Phi_{S,F}$-harmonic map with respect to the metric torsion connection coincides with that of $\Phi_{S,F}$-harmonic map with respect to the Levi-Civita connection. This shows a rigidity signature of $\Phi_{S,F}$-harmonic map being invariant under connection transforms from the Levi-Civita connection to the metric torsion connection. Then, under suitable conditions on the Hessian of the distance function and the degree of $F(t)$, we derive several Liouville theorems for the $\Phi_{S,F}$-harmonic map by assuming either growth condition of the $\Phi_{S,F}$-energy or an asymptotic condition at the infinity for the maps. In the end of paper, we also obtain the unique constant solution of the constant Dirichlet boundary value problemson on starlike domains for the $\Phi_{S,F}$-harmonic map. These vanishing theorems extend some results in [18, 19] where $F(t)$ are given as t and $(2t)^{p/2}/p$ $(p \geq 2)$, respectively, and target manifolds are endowed with Levi-Civita connections.

1. Introduction

M. Ara [4] introduced the F-harmonic map between two Riemannian manifolds and its associated stress energy tensor named as F-stress energy tensor. The concept of F-harmonic maps is a generalization of harmonic maps, p-harmonic maps or exponentially harmonic maps. Let $F : [0, +\infty) \to [0, +\infty)$ be a C^2 function with $F(0) = 0$ such that $F' > 0$ on $(0, +\infty)$. A smooth map $u : (M^m, g) \to (N^n, h)$ between Riemannian manifolds (M^m, g) and (N^n, h) is said to be an F-harmonic map if it is a critical point of the following F-energy functional $E_F(u)$ given by

$$E_F(u) = \int_M F(\frac{|du|^2}{2})dv_g$$

with respect to any compactly supported variation, where $|du|$ is the Hilbert-Schmidt of the differential du of u and dv_g denotes the volume element on M. It is the energy, the p-energy, the α-energy and the exponential energy when $F(t) = t$, $(2t)^{p/2}/p$ $(p \geq 2)$, $(1 + 2t)^{\alpha}$ $(\alpha > 1, dimM = 2)$ and e^t, respectively.

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The F-stress energy tensor associated with the functional $E_F(u)$ is given by

$$S_F(X,Y)=F(\frac{|du|^2}{2})g(X,Y)-F'(\frac{|du|^2}{2})u^*h(X,Y),$$

where X, Y are vectors on M. It is known that the stress energy tensor is a useful tool for studying the energy behavior and vanishing results of related functional (cf. [14]). If M is compact, the set G of the Riemannian metrics on M is an infinite dimensional manifold and its tangent space at g is identified with symmetric $(0,2)$-tensors T_gG. For a deformation $\{g_t\}$ of g, we denote $\omega=\frac{d}{dt}g_t|_{t=0}$. Now we fix $u:M\to(N,h)$ and define the functional $\mathcal{E}_F(g_t)=\int_M F(\frac{|du|^2}{2})dv_{g_t}$. Using lemma 2.1 in [14], we can easily obtain the following result.

Theorem 1.1. *Let $u:M\to(N,h)$ be a smooth map and assume that M is compact. Then $\frac{d}{dt}\mathcal{E}_F(g_t)|_{t=0}=\frac{1}{2}\int_M\langle S_F,\omega\rangle dv_g$.*

From Theorem 1.1, we know that $S_F=0$ is the Euler-Lagrange equation for the functional $\mathcal{E}_F(g_t)$. So it is necessary to get some results of $S_F=0$ under some conditions. In fact, the vanishing results of the stress energy tensor S associated to the energy $E(u)=\int_M\frac{|du|^2}{2}dv_g$ and the p-stress energy tensor S_p associated to the p-energy $E_p(u)=\int_M\frac{|du|^p}{p}dv_g$ have been studied in [17–19]. As natural generalizations of the stress energy tensor S and the p-stress energy tensor S_p, we will obtain the vanishing problem of the F-stress energy tensor S_F.

Similar to [29] (see also [14, 15]), we may define the upper degree d_F and the lower degree l_F of F as follows:

$$d_F=\sup_{t\geq0}\frac{tF'(t)}{F(t)}$$

and

$$l_F=\inf_{t\geq0}\frac{tF'(t)}{F(t)}.$$

In general, we have $l_F\leq d_F$. From now on, we always assume that $d_F<+\infty$, $m>4l_F$ and $F''>0$ on $(0,+\infty)$. Let $\{e_1,\cdots e_m\}$ be a local orthonormal frame field on M. Let $\|S_F\|$ denote the norm of the F-stress energy tensor. Now we first estimate $\|S_F\|^2$ in the following

$$\begin{aligned}
\|S_F\|^2&=\sum_{i,j=1}^m\Big(S_F(e_i,e_j)\Big)^2=\sum_{i,j=1}^m\Big(F(\frac{|du|^2}{2})g(e_i,e_j)-F'(\frac{|du|^2}{2})u^*h(e_i,e_j)\Big)^2\\
&=m\Big(F(\frac{|du|^2}{2})\Big)^2-2F(\frac{|du|^2}{2})F'(\frac{|du|^2}{2})|du|^2+\Big(F'(\frac{|du|^2}{2})\Big)^2\|u^*h\|^2\\
&=\Big[m-4\frac{F'(\frac{|du|^2}{2})\frac{|du|^2}{2}}{F(\frac{|du|^2}{2})}\Big]\Big(F(\frac{|du|^2}{2})\Big)^2+\Big(F'(\frac{|du|^2}{2})\Big)^2\|u^*h\|^2\\
&\leq(m-4l_F)\Big(F(\frac{|du|^2}{2})\Big)^2+\Big(F'(\frac{\sqrt{m}}{2}\|u^*h\|)\Big)^2\|u^*h\|^2\\
&=(m-4l_F)\Big(F(\frac{|du|^2}{2})\Big)^2+\frac{4}{m}\Big[\frac{F'(\frac{\sqrt{m}}{2}\|u^*h\|)\frac{\sqrt{m}}{2}\|u^*h\|}{F(\frac{\sqrt{m}}{2}\|u^*h\|)}\Big]^2F(\frac{\sqrt{m}}{2}\|u^*h\|)^2\\
&\leq(m-4l_F)\Big(F(\frac{|du|^2}{2})\Big)^2+\frac{4d_F^2}{m}\Big(F(\frac{\sqrt{m}}{2}\|u^*h\|)\Big)^2
\end{aligned}\tag{1}$$

where we used $|du|^2\leq\sqrt{m}\|u^*h\|$ and $F''>0$. According to (1), we introduce the definitions of $\Phi_{S,F}$-energy functional $E_{\Phi_{S,F}}$ and the $\Phi_{S,F}$-harmonic map as follows in order to study vanishing problem $S_F=0$.

Definition 1.2. *The $\Phi_{S,F}$-energy density $e_{\Phi_{S,F}}(u)$ of u is given by*

$$e_{\Phi_{S,F}}(u) = (m - 4l_F)\Big(F(\frac{|du|^2}{2})\Big)^2 + \frac{4d_F^2}{m}\Big(F(\frac{\sqrt{m}}{2}\|u^*h\|)\Big)^2.$$

The $\Phi_{S,F}$-energy $E_{\Phi_{S,F}}$ of u is defined by

$$E_{\Phi_{S,F}}(u) = \int_M e_{\Phi_{S,F}}(u)dv_g = \int_M \Big[(m - 4l_F)\Big(F(\frac{|du|^2}{2})\Big)^2 + \frac{4d_F^2}{m}\Big(F(\frac{\sqrt{m}}{2}\|u^*h\|)\Big)^2\Big]dv_g.$$

Definition 1.3. *A smooth map u is said to be a $\Phi_{S,F}$-harmonic map if it is a critical point of the $\Phi_{S,F}$-energy functional $E_{\Phi_{S,F}}$ with respect to any smooth compactly supported variation of u.*

We may study the vanishing results of $S_F = 0$ by investigating the Liouville theorems for $\Phi_{S,F}$-harmonic maps. It is well known that studying Liouville type results is one of important problems for harmonic maps, generalized harmonic maps or generalized harmonic forms (see [5, 12, 14, 16–20, 24, 26–28, 35] and the references therein). Most Liouville results have been established by assuming either the finiteness of the energy of the map or the smallness of whole image of the domain manifold under the map. But Jin [28] proved several interesting Liouville theorems for harmonic maps from complete manifolds, whose assumptions concern the asymptotic behavior of the maps at infinity. Dong, Lin and Yang [16] generalized Jin's method to F-harmonic maps, obtained some Liouville theorems and gave their applications.

In the literature on harmonic maps one usually choose to utilize the Levi-Civita connections. Harmonic maps with a connection different from the Levi-Civita connection on the domain manifold have already been investigated in great generality. Such maps become known as V-harmonic maps and include the classes of Hermitian, affine and Weyl harmonic maps into Riemannian manifolds, see the introduction of [11] for more details. In [8], V. Branding gave the equation for harmonic maps where the target manifold is endowed with a connection with metric torsion, named as harmonic maps with torsion. But the V-harmonic map and the harmonic map with torsion can not be obtained as a critical point of some energy functional in general. They are given only by adding some extra structures to the standard harmonic map equation. In this paper, we are going to study $\Phi_{S,F}$-energy functional $E_{\Phi_{S,F}}$ where the target manifold is equipped with a metric torsion connection.

Firstly, we introduce connections with torsion. Let ∇^N denote the Levi-Civita connection of a Riemannian manifold (N, h). For any affine connection there exists a $(2, 1)$-tensor field A such that

$$\dot{\nabla}^N_U V = \nabla^N_U V + A(U, V)$$

for all vector fields $U, V \in \Gamma(TN)$. The torsion tensor $T(U, V)$ is related to the torsion endomorphism $A(U, V)$ via

$$\dot{\nabla}^N_U V - \dot{\nabla}^N_V U - [U, V] = T(U, V) = A(U, V) - A(V, U).$$

If $\dot{\nabla}^N h = 0$, then $\dot{\nabla}^N$ is called a metric torsion connection on N. Obviously, if $\dot{\nabla}^N$ is a metric torsion, then the endomorphism $A(U, \cdot)$ has to be skew-adjoint

$$h(A(U, V), W) = -h(A(U, W), V).$$

There exist several geometric settings where connections with metric torsion naturally appear. For a compact Hermitian manifold, it has a canonical connection, the so-called Chern connection, which has non-vanishing torsion (see [30]). Another famous connection in Hermitian geometry that carries non-vanishing torsion is the Gauduchon connection (see [21, 22]). In the case of a pseudo-Hermitian manifold, there exists a canonical linear connection which preserves both the CR structure and the Webster metric. This particular connection is called the Tanaka-Webster connection (see [13, 33, 34]), and it also has non-vanishing torsion. Connections with metric torsion have been intensively studied in the physics literature (see [32]). Cartan [10] classified connections with metric torsion. The geodesics of connections with vectorial torsion were investigated in [3], and various geometric aspects of manifolds having a connection with vectorial torsion were studied in [2]. The uniformization theorem on closed surfaces for a metric connection with vectorial torsion was proved via the Ricci flow in [9]. For more details on the geometry of Riemannian having a connection with metric torsion we refer to the lecture notes [1].

In this paper, we derive the first variation formula of the $\Phi_{S,F}$-energy functional $E_{\Phi_{S,F}}$ and obtain the equation of $\Phi_{S,F}$-harmonic maps where the target manifold is equipped with a metric torsion connection. We find that the equations of $\Phi_{S,F}$-harmonic map obtained via the variational principle are same under the Levi-Civita connection and connections with metric torsion, respectively. This shows some rigidity signature of $\Phi_{S,F}$-harmonic map being invariant under connection transforms.

To generalize the Liouville results for harmonic maps to the $\Phi_{S,F}$-harmonic maps, we first introduce the $\Phi_{S,F}$-stress energy tensor $S_{\Phi_{S,F}}$ associated to the functional $\Phi_{S,F}$. We prove that the $\Phi_{S,F}$-harmonic map satisfies the conservation law, that is, $divS_{\Phi_{S,F}} = 0$. Using a basic integral formula linked naturally to the conservation law enables us to establish a monotonicity formula for these $\Phi_{S,F}$-harmonic maps. Consequently, several Liouville type theorems and vanishing results from these monotonicity formulae under suitable growth conditions on the $\Phi_{S,F}$-energy. We also obtain a vanishing result under the condition of slowly divergent energy.

Next we generalize Jin's method and results to the $\Phi_{S,F}$-harmonic maps. The procedure consists of two steps. The first step is to use the monotonicity formula to establish a lower bound for the growth rates of the $\Phi_{S,F}$-energy. The second step is to use the asymptotic assumption of the maps at infinity to obtain the upper functional growth rates of the $\Phi_{S,F}$-harmonic maps. Under suitable conditions of the domain manifolds, one may show that these two growth rates are contradictory unless the $\Phi_{S,F}$-harmonic map is constant. In this way, we establish some Liouville theorems for the $\Phi_{S,F}$-harmonic maps with asymptotic property at infinity from some complete manifolds.

In addition to establishing Liouville type results, we investigate the constant Dirichlet boundary value problem as well. We obtain the unique constant solution of the constant Dirichlet boundary value problem on starlike domains for the $\Phi_{S,F}$-harmonic map.

2. The $\Phi_{S,F}$-harmonic map under metric torsion connections

In this section, we give the first variation formula for the $\Phi_{S,F}$-energy functional $E_{\Phi_{S,F}}$ by using the connection with metric torsion on the target manifold. Then we obtain the equation of the $\Phi_{S,F}$-harmonic map. We may conclude that the equations of the $\Phi_{S,F}$-harmonic maps are invariant under the connection transform by variational principle.

Let ∇, ∇^N and $\dot{\nabla}^N$ always denote the Levi-Civita connections of M and N, and a connection with torsion of N, respectively. Let $\widetilde{\nabla}$ and $\widetilde{\dot{\nabla}}$ be the induced connections by ∇^N and $\dot{\nabla}^N$ on $u^{-1}TN$, which are defined by

$$\widetilde{\nabla}_X W = \nabla^N_{du(X)} W, \qquad \widetilde{\dot{\nabla}}_X W = \dot{\nabla}^N_{du(X)} W,$$

where X is a tangent vector of M and W is a section of $u^{-1}TN$. Now we define the tensor $\sigma_{F,u}$, which plays an important role in our argument, as follows:

$$\begin{aligned}\sigma_{F,u}(X) = &\sum_{j=1}^{m} 8d_F^2(\sqrt{m}\|u^*h\|)^{-1} F(\frac{\sqrt{m}}{2}\|u^*h\|)F'(\frac{\sqrt{m}}{2}\|u^*h\|)h(du(X), du(e_j))du(e_j) \\ &+ 2(m-4l_F)F(\frac{|du|^2}{2})F'(\frac{|du|^2}{2})du(X)\end{aligned} \tag{2}$$

for any vector field X on M, where $\{e_j\}_{j=1}^m$ is a local orthonormal frame. If u is a constant map, we may define $\sigma_{F,u} = 0$.

Denote $div\sigma_{F,u}$ and $\overline{div}\sigma_{F,u}$ the divergence of $\sigma_{F,u}$ with respect to the Levi-Civita connection and the connection with torsion, respectively, that is, $div\sigma_{F,u} = \sum_{i=1}^m(\widetilde{\nabla}_{e_i}\sigma_{F,u})(e_i)$ and $\overline{div}\sigma_{F,u} = \sum_{i=1}^m(\widetilde{\dot{\nabla}}_{e_i}\sigma_{F,u})(e_i)$. We define $\Phi_{S,F}$-tension field $\tau_{\Phi_{S,F}}(u)$ of u by

$$\tau_{\Phi_{S,F}}(u) = \overline{div}\sigma_{F,u} - \sum_{i=1}^{m} A(du(e_i), \sigma_{F,u}(e_i))) = div\sigma_{F,u}.$$

Theorem 2.1. *Let $u : (M^m, g) \to (N^n, h)$ be a smooth map where (N^n, h) is endowed with a metric connection with torsion. Let $u_t : (M^m, g) \times (-\delta, \delta) \to (N^n, h)$, $-\delta < t < \delta$, be a family of compactly supported variations such that $u_0 = u$ and $V = \frac{\partial u_t}{\partial t}|_{t=0}$. Then*

$$\frac{dE_{\Phi_{S,F}(u_t)}}{dt}\Big|_{t=0} = -\int_M h(V, \tau_{\Phi_{S,F}}(u))dv_g. \tag{3}$$

Proof. Let $\Psi : M \times (-\delta, \delta) \to N$ be a smooth map defined by $\Psi(x, t) = u_t(x)$, where $M \times (-\delta, \delta)$ is equipped with the product metric. We extend the vector fields $\frac{\partial}{\partial t}$ on $(-\delta, \delta)$ and X on M naturally to $M \times (-\delta, \delta)$, and denote these vectors also by $\frac{\partial}{\partial t}, X$. We shall use the notations ∇, $\widetilde{\nabla}$ and $\widetilde{\dot{\nabla}}$ for the Levi-Civita connections on $M \times (-\delta, \delta)$ and induced connections on $\Psi^{-1}TN$, respectively.

Now we compute

$$\begin{aligned}
&\frac{\partial}{\partial t}\Big\{(m - 4l_F)[F(\frac{|d\Psi|^2}{2})]^2 + \frac{4d_F^2}{m}[F(\frac{\sqrt{m}}{2}\|\Psi^* h\|)]^2\Big\} \\
&= 2(m - 4l_F)F(\frac{|d\Psi|^2}{2})F'(\frac{|d\Psi|^2}{2})\frac{\partial}{\partial t}(\frac{|d\Psi|^2}{2}) + \frac{4d_F^2}{\sqrt{m}}F(\frac{\sqrt{m}}{2}\|\Psi^* h\|)F'(\frac{\sqrt{m}}{2}\|\Psi^* h\|)\frac{\partial}{\partial t}(\|\Psi^* h\|) \\
&= 2(m - 4l_F)F(\frac{|d\Psi|^2}{2})F'(\frac{|d\Psi|^2}{2})\sum_{i=1}^{m} h(\widetilde{\dot{\nabla}}_{\frac{\partial}{\partial t}} d\Psi(e_i), d\Psi(e_i)) \\
&\quad + \frac{8d_F^2}{\sqrt{m}}\|\Psi^* h\|^{-1}F(\frac{\sqrt{m}}{2}\|\Psi^* h\|)F'(\frac{\sqrt{m}}{2}\|\Psi^* h\|)\sum_{i,j=1}^{m} h(\widetilde{\dot{\nabla}}_{\frac{\partial}{\partial t}} d\Psi(e_i), d\Psi(e_j))h(d\Psi(e_i), d\Psi(e_j)) \\
&= \sum_{i=1}^{m} h(\widetilde{\dot{\nabla}}_{\frac{\partial}{\partial t}} d\Psi(e_i), \sigma_{F,\Psi}(e_i)) \\
&= \sum_{i=1}^{m} h(\widetilde{\dot{\nabla}}_{e_i} d\Psi(\frac{\partial}{\partial t}), \sigma_{F,\Psi}(e_i)) + \sum_{i=1}^{m} h(A(d\Psi(\frac{\partial}{\partial t}), d\Psi(e_i)) - A(d\Psi(e_i), d\Psi(\frac{\partial}{\partial t})), \sigma_{F,\Psi}(e_i)) \\
&= \sum_{i=1}^{m} \Big[e_i h(d\Psi(\frac{\partial}{\partial t}), \sigma_{F,\Psi}(e_i)) - h(d\Psi(\frac{\partial}{\partial t}), \widetilde{\dot{\nabla}}_{e_i}\sigma_{F,\Psi}(e_i))\Big] + \sum_{i=1}^{m} h(A(d\Psi(e_i), \sigma_{F,\Psi}(e_i)), d\Psi(\frac{\partial}{\partial t})).
\end{aligned}$$

Here we used

$$\widetilde{\dot{\nabla}}_{\frac{\partial}{\partial t}} d\Psi(e_i) - \widetilde{\dot{\nabla}}_{e_i} d\Psi(\frac{\partial}{\partial t}) = A(d\Psi(\frac{\partial}{\partial t}), d\Psi(e_i)) - A(d\Psi(e_i), d\Psi(\frac{\partial}{\partial t}))$$

and

$$\sum_{i=1}^{m} h(A(d\Psi(\frac{\partial}{\partial t}), d\Psi(e_i)), \sigma_{F,\Psi}(e_i)) = 0.$$

Let X_t be a compactly supported vector field on M such that $g(X_t, Y) = h(d\Psi(\frac{\partial}{\partial t}), \sigma_{F,u}(Y))$ for any vector

Y on M. Then

$$\begin{aligned}
&\frac{\partial}{\partial t}\Big\{(m-4l_F)[F(\frac{|d\Psi|^2}{2})]^2+\frac{4d_F^2}{m}[F(\frac{\sqrt{m}}{2}\|\Psi^*h\|)]^2\Big\}\\
&=\sum_{i=1}^m\Big[e_ig(X_t,e_i)-h(d\Psi(\frac{\partial}{\partial t}),\tilde{\dot{\nabla}}_{e_i}\sigma_{F,\Psi}(e_i))+h(A(d\Psi(e_i),\sigma_{F,\Psi}(e_i)),d\Psi(\frac{\partial}{\partial t}))\Big]\\
&=\sum_{i=1}^m\Big[g(\nabla_{e_i}X_t,e_i)+g(X_t,\nabla_{e_i}e_i)-h(d\Psi(\frac{\partial}{\partial t}),\tilde{\dot{\nabla}}_{e_i}\sigma_{F,\Psi}(e_i)-A(d\Psi(e_i),\sigma_{F,\Psi}(e_i)))\Big]\\
&=divX_t-\sum_{i=1}^m h(d\Psi(\frac{\partial}{\partial t}),\tilde{\dot{\nabla}}_{e_i}\sigma_{F,\Psi}(e_i)-\sigma_{F,\Psi}(\nabla_{e_i}e_i)-A(d\Psi(e_i),\sigma_{F,\Psi}(e_i)))\\
&=divX_t-h(d\Psi(\frac{\partial}{\partial t}),\overline{div}\sigma_{F,\Psi}-\sum_{i=1}^m A(d\Psi(e_i),\sigma_{F,\Psi}(e_i)).
\end{aligned} \tag{4}$$

From (4) and Green's theorem, we have

$$\frac{d}{dt}\Phi_{S,F}(u_t)\Big|_{t=0}=-\int_M h(V,\overline{div}\sigma_{F,u}-\sum_{i=1}^m A(du(e_i),\sigma_{F,u}(e_i))dv_g=-\int_M h(V,\tau_{\Phi_{S,F}}(u))dv_g.$$

□

Remark 2.2. *If we use the the Levi-Civita connection on the target manifold (N^n,h), we can deduce that the first variation formula for the $\Phi_{S,F}$-energy functional $E_{\Phi_{S,F}}$ is*

$$\frac{d}{dt}\Phi_{S,F}(u_t)\Big|_{t=0}=-\int_M h(V,div\sigma_{F,u})dv_g=-\int_M h(V,\tau_{\Phi_{S,F}}(u))dv_g.$$

This shows that the critical points of the $\Phi_{S,F}$-energy functional $E_{\Phi_{S,F}}$ are the same under the Levi-Civita connection and the connection with metric torsion on the target manifold, respectively.

Proposition 2.3. *A smooth map $u:(M^m,g)\to(N^n,h)$ is $\Phi_{S,F}$-harmonic where N is equipped with metric torsion connections if it is a solution of the Euler-Lagrange equation*

$$\tau_{\Phi_{S,F}}(u)=0. \tag{5}$$

3. $\Phi_{S,F}$-stress energy tensor

In this section, we introduce the definition of the $\Phi_{S,F}$-stress energy tensor $S_{\Phi_{S,F}}$ associated with the $\Phi_{S,F}$-energy functional $E_{\Phi_{S,F}}$ and obtain some properties of the $\Phi_{S,F}$-stress energy tensor $S_{\Phi_{S,F}}$.

Following Baird [7], we define a symmetric 2-tensor $S_{\Phi_{S,F}}$ of u associated to the functional $\Phi_{S,F}$ (which we call, the $\Phi_{S,F}$-stress energy tensor of u, in short) by

$$S_{\Phi_{S,F}}(X,Y)=\Big\{(m-4l_F)[F(\frac{|du|^2}{2})]^2+\frac{4d_F^2}{m}[F(\frac{\sqrt{m}}{2}\|u^*h\|)]^2\Big\}g(X,Y)-h(du(X),\sigma_{F,u}(Y))$$

where X,Y are any smooth vectors on M.

Recall that for a two tensor field $T\in\Gamma(T^*M\otimes T^*M)$, its divergence $divT\in\Gamma(T^*M)$ is given by

$$(divT)(X)=\sum_{i=1}^n(\nabla_{e_i}T)(e_i,X)$$

where $\{e_i\}$ is an orthonormal basis of TM.

Theorem 3.1. *Let $u : (M^m, g) \to (N^n, h)$ be a smooth map where N is endowed with a connection with metric torsion and $S_{\Phi_{S,F}}$ be the associated stress-energy tensor. Then for any vector field X on M, we have*

$$(divS_{\Phi_{S,F}})(X) = -h(\tau_{\Phi_{S,F}}(u), du(X)).$$

Proof. We choose a local orthonormal frame filed $\{e_i\}_{i=1}^m$ at a point x such that $\nabla_{e_i} e_j|_x = 0$. Then for any vector field $X \in \Gamma(TM)$, at x, we have

$$\begin{aligned}
&(divS_{\Phi_{S,F}})(X)\\
&= \sum_{i=1}^m (\nabla_{e_i} S_{\Phi_{S,F}})(e_i, X) = \sum_{i=1}^m \Big[e_i(S_{\Phi_{S,F}}(e_i, X)) - S_{\Phi_{S,F}}(e_i, \nabla_{e_i} X)\Big]\\
&= \sum_{i=1}^m \Big\{e_i\Big[[(m-4l_F)(F(\frac{|du|^2}{2}))^2 + \frac{4d_F^2}{m}(F(\frac{\sqrt{m}}{2}\|u^*h\|))^2]g(e_i, X) - h(\sigma_{F,u}(e_i), du(X))\Big]\\
&\quad - [(m-4l_F)(F(\frac{|du|^2}{2}))^2 + \frac{4d_F^2}{m}(F(\frac{\sqrt{m}}{2}\|u^*h\|))^2]g(e_i, \nabla_{e_i} X) + h(\sigma_{F,u}(e_i), du(\nabla_{e_i} X))\Big\}\\
&= \sum_{i=1}^m e_i[(m-4l_F)(F(\frac{|du|^2}{2}))^2 + \frac{4d_F^2}{m}(F(\frac{\sqrt{m}}{2}\|u^*h\|))^2]g(e_i, X)\\
&\quad - \sum_{i=1}^m e_i(h(\sigma_{F,u}(e_i), du(X))) + \sum_{i=1}^n h(\sigma_{F,u}(e_i), du(\nabla_{e_i} X))\\
&= X((m-4l_F)(F(\frac{|du|^2}{2}))^2 + \frac{4d_F^2}{m}(F(\frac{\sqrt{m}}{2}\|u^*h\|))^2) - \sum_{i=1}^n h(\widetilde{\dot{\nabla}}_{e_i}\sigma_{F,u}(e_i), du(X))\\
&\quad - \sum_{i=1}^m h(\sigma_{F,u}(e_i), \widetilde{\dot{\nabla}}_{e_i} du(X)) + \sum_{i=1}^m h(\sigma_{F,u}(e_i), du(\nabla_{e_i} X))\\
&= \sum_{i=1}^m h(\widetilde{\dot{\nabla}}_X du(e_i), \sigma_{F,u}(e_i)) - \sum_{i=1}^n h(\widetilde{\dot{\nabla}}_{e_i}\sigma_{F,u}(e_i), du(X))\\
&\quad - \sum_{i=1}^m h(\sigma_{F,u}(e_i), \widetilde{\dot{\nabla}}_{e_i} du(X)) + \sum_{i=1}^m h(\sigma_{F,u}(e_i), du(\nabla_{e_i} X))\\
&= \sum_{i=1}^m h((\widetilde{\dot{\nabla}}_X du)(e_i), \sigma_{F,u}(e_i)) - \sum_{i=1}^n h((\widetilde{\dot{\nabla}}_{e_i}\sigma_{F,u})(e_i), du(X)) - \sum_{i=1}^m h(\sigma_{F,u}(e_i), (\widetilde{\dot{\nabla}}_{e_i} du)(X))\\
&= \sum_{i=1}^m h(A(du(X), du(e_i)) - A(du(e_i), du(X)), \sigma_{F,u}(e_i)) - \sum_{i=1}^n h((\widetilde{\dot{\nabla}}_{e_i}\sigma_{F,u})(e_i), du(X))\\
&= \sum_{i=1}^m h(A(du(e_i), \sigma_{F,u}(e_i)), du(X)) - \sum_{i=1}^n h((\widetilde{\dot{\nabla}}_{e_i}\sigma_{F,u})(e_i), du(X))\\
&= -\sum_{i=1}^n h((\widetilde{\dot{\nabla}}_{e_i}\sigma_{F,u})(e_i) - A(du(e_i), \sigma_{F,u}(e_i)), du(X))\\
&= -\sum_{i=1}^n h(\tau_{\Phi_{S,F}}(u), du(X)),
\end{aligned}$$

where we used

$$(\widetilde{\dot{\nabla}}_X du)(e_i) - (\widetilde{\dot{\nabla}}_{e_i} du)(X) = A(du(X), du(e_i)) - A(du(e_i), du(X))$$

and

$$\sum_{i=1}^{m} h(A(du(\frac{\partial}{\partial t}), du(e_i)), \sigma_{F,u}(e_i)) = 0.$$

□

Corollary 3.2. *If* $u : (M^m, g) \to (N^n, h)$ *is a* $\Phi_{S,F}$*-harmonic map. Then* u *satisfies the* $\Phi_{S,F}$*-conservation law, i.e.,* $divS_{\Phi_{S,F}} = 0$.

Recall that for two 2-tensors $T_1, T_2 \in \Gamma(T^*M \otimes T^*M)$, their inner product is defined as follows:

$$\langle T_1, T_2 \rangle = \sum_{i,j=1}^{m} T_1(e_i, e_j) T_2(e_i, e_j)$$

where $\{e_i\}$ is an orthonormal basis with respect to g. For a vector field X on M, its dual one form θ_X is given by

$$\theta_X(Y) = g(X, Y), \forall Y \in TM.$$

The covariant derivative of θ_X is given by

$$(\nabla \theta_X)(Y, Z) = (\nabla_Y \theta_X)(Z) = g(\nabla_Y X, Z), \forall Y, Z \in TM.$$

If $X = \nabla \varphi$ is the gradient of some C^2 function φ on M, then $\theta_X = d\varphi$ and $\nabla_Y \theta_X = Hess_g(\varphi)$.

Lemma 3.3. *([7, 14, 16]) Let* T *be a symmetric* $(0, 2)$*-tensor and Let* X *be a vector field, then*

$$div(i_X T) = (divT)(X) + \langle T, \nabla \theta_X \rangle = (divT)(X) + \frac{1}{2} \langle T, L_X g \rangle, \tag{6}$$

where $i_X T \in A^1(M)$ *denotes the interior product by* X *and* L_X *is the Lie derivative of the metric* g *in the direction of* X.

Let D be any bounded domain of M with C^1 boundary. By applying (6) to $S_{\Phi_{S,F}}$ and using the divergence theorem, we obtain the following integral formula

$$\int_{\partial D} S_{\Phi_{S,F}}(X, \nu) ds_g = \int_D [\langle S_{\Phi_{S,F}}, \frac{1}{2} L_X g \rangle + (divS_{\Phi_{S,F}})(X)] dv_g \tag{7}$$

where ν is the unit outward normal vector field along ∂D. In particular, if u is a $\Phi_{S,F}$-harmonic map, then $divS_{\Phi_{S,F}} = 0$. So we have

$$\int_{\partial D} S_{\Phi_{S,F}}(X, \nu) ds_g = \int_D \langle S_{\Phi_{S,F}}, \frac{1}{2} L_X g \rangle dv_g. \tag{8}$$

4. Monotonicity formulae and vanishing results under the growth condition

In this section we will apply (8) to obtain the monotonicity formula for the $\Phi_{S,F}$-energy of $\Phi_{S,F}$-harmonic map. Furthermore, some Liouville type results will be obtained.

Let (M, g) be a complete Riemannian manifold with a pole x_0. Denote by $r(x)$ be the distance function relative to the pole x_0, that is, $r(x) = dist_g(x, x_0)$. On a complete Riemannian manifold with a pole, one can take r to go to infinity. Set $B(\rho) = \{x \in N : r(x) \leq \rho\}$. Because of $Hess_g(r^2) = 2dr \otimes dr + 2rHess_g(r)$, it is known that $\frac{\partial}{\partial r}$ is always an eigenvector of $Hess_g(r^2)$ associated to eigenvalue 2. Denote by $\lambda_{\max}$ (resp. $\lambda_{\min}$) the maximum (resp. minimal) eigenvalues of $2rHess_g(r)$ at each point of $M \setminus \{x_0\}$.

Theorem 4.1. *Let $u : (M^m, g) \to (N^n, h)$ be a $\Phi_{S,F}$-harmonic map. If there exists the following inequality on (M^m, g),*

$$1 + \frac{m-1}{2}\lambda_{\min} - 2d_F \max\{2, \lambda_{\max}\} \geq \Lambda, \tag{9}$$

where Λ is a positive constant, then we have

$$\frac{\int_{B(\rho_1)} \Big[(m-4l_F)(F(\frac{|du|^2}{2}))^2 + \frac{4d_F^2}{m}(F(\frac{\sqrt{m}}{2}\|u^*h\|))^2\Big]dv_g}{\rho_1^{\Lambda}} \leq \frac{\int_{B(\rho_2)} \Big[(m-4l_F)(F(\frac{|du|^2}{2}))^2 + \frac{4d_F^2}{m}(F(\frac{\sqrt{m}}{2}\|u^*h\|))^2\Big]dv_g}{\rho_2^{\Lambda}}$$

for any $0 < \rho_1 < \rho_2$.

Proof. Taking $D = B(R)$ and $X = \frac{1}{2}\nabla r^2 = r\frac{\partial}{\partial r}$ in (8), we have

$$\int_{\partial B(R)} S_{\Phi_{S,F}}(r\frac{\partial}{\partial r}, \frac{\partial}{\partial r})ds_g = \int_{B(R)} \langle S_{\Phi_{S,F}}, \frac{1}{2}L_{r\frac{\partial}{\partial r}}g\rangle dv_g. \tag{10}$$

Let $\{e_i\}_{i=1}^m$ be an orthonormal basis and $e_m = \frac{\partial}{\partial r}$. We also assume that $Hess_g(r^2)$ becomes a diagonal matric with respect to $\{e_i\}_{i=1}^m$. Then we have

$$\begin{aligned}
&\langle S_{\Phi_{S,F}}, L_{r\frac{\partial}{\partial r}}g\rangle \\
&= \sum_{i,j=1}^m S_{\Phi_{S,F}}(e_i, e_j)(L_{r\frac{\partial}{\partial r}}g)(e_i, e_j) \\
&= \sum_{i,j=1}^m \Big\{\Big[(m-4l_F)(F(\frac{|du|^2}{2}))^2 + \frac{4d_F^2}{m}(F(\frac{\sqrt{m}}{2}\|u^*h\|))^2\Big]g(e_i,e_j)(L_{r\frac{\partial}{\partial r}}g)(e_i,e_j) - h(du(e_i), \sigma_{F,u}(e_j))(L_{r\frac{\partial}{\partial r}}g)(e_i,e_j)\Big\} \\
&= \sum_{i=1}^m \Big[(m-4l_F)(F(\frac{|du|^2}{2}))^2 + \frac{4d_F^2}{m}(F(\frac{\sqrt{m}}{2}\|u^*h\|))^2\Big]Hess_g(r^2)(e_i,e_i) - \sum_{i,j=1}^m h(du(e_i), \sigma_{F,u}(e_j))Hess_g(r^2)(e_i,e_j) \\
&\geq \Big[(m-4l_F)(F(\frac{|du|^2}{2}))^2 + \frac{4d_F^2}{m}(F(\frac{\sqrt{m}}{2}\|u^*h\|))^2\Big][2 + (m-1)\lambda_{\min}] - \max\{2, \lambda_{\max}\}\sum_{i=1}^m h(du(e_i), \sigma_{F,u}(e_i)).
\end{aligned} \tag{11}$$

Here

$$\begin{aligned}
&\sum_{i=1}^m h(du(e_i), \sigma_{F,u}(e_i)) \\
&= \sum_{i=1}^m h(du(e_i), \sum_{j=1}^m \frac{8d_F^2}{\sqrt{m}}(\|u^*h\|)^{-1}F(\frac{\sqrt{m}}{2}\|u^*h\|)F'(\frac{\sqrt{m}}{2}\|u^*h\|)h(du(e_i), du(e_j))du(e_j) \\
&\qquad + 2(m-4l_F)F(\frac{|du|^2}{2})F'(\frac{|du|^2}{2})du(e_i)) \\
&= \frac{8d_F^2}{\sqrt{m}}F(\frac{\sqrt{m}}{2}\|u^*h\|)F'(\frac{\sqrt{m}}{2}\|u^*h\|)\|u^*h\| + 2(m-4l_F)F(\frac{|du|^2}{2})F'(\frac{|du|^2}{2})|du|^2 \\
&= \frac{8d_F^2}{\sqrt{m}}\frac{\frac{\sqrt{m}}{2}\|u^*h\|F'(\frac{\sqrt{m}}{2}\|u^*h\|)}{\frac{\sqrt{m}}{2}F(\frac{\sqrt{m}}{2}\|u^*h\|)}(F(\frac{\sqrt{m}}{2}\|u^*h\|))^2 + 2(m-4l_F)\frac{\frac{|du|^2}{2}F'(\frac{|du|^2}{2})}{\frac{1}{2}F(\frac{|du|^2}{2})}(F(\frac{|du|^2}{2}))^2 \\
&\leq 4d_F\Big[\frac{4d_F^2}{m}(F(\frac{\sqrt{m}}{2}\|u^*h\|))^2 + (m-4l_F)(F(\frac{|du|^2}{2}))^2\Big].
\end{aligned} \tag{12}$$

From (11), (12) and (9), we obtain

$$\begin{aligned}\langle S_{\Phi_{S,F}}, \frac{1}{2}L_{r\frac{\partial}{\partial r}}g\rangle &\geq \Big[1+\frac{m-1}{2}\lambda_{\min}-2d_F\max\{2,\lambda_{\max}\}\Big]\Big[(m-4l_F)(F(\frac{|du|^2}{2}))^2+\frac{4d_F^2}{m}(F(\frac{\sqrt{m}}{2}\|u^*h\|))^2\Big]\\ &\geq \Lambda\Big[(m-4l_F)(F(\frac{|du|^2}{2}))^2+\frac{4d_F^2}{m}(F(\frac{\sqrt{m}}{2}\|u^*h\|))^2\Big].\end{aligned} \tag{13}$$

On the other hand, by the coarea formula, we have

$$\begin{aligned}\int_{\partial B(R)} S_{\Phi_{S,F}}(r\frac{\partial}{\partial r},\frac{\partial}{\partial r})ds_g &= R\int_{\partial B(R)}\Big\{\Big[(m-4l_F)(F(\frac{|du|^2}{2}))^2+\frac{4d_F^2}{m}(F(\frac{\sqrt{m}}{2}\|u^*h\|))^2\Big]-h(du(\frac{\partial}{\partial r}),\sigma_{F,u}(\frac{\partial}{\partial r}))\Big\}ds_g\\ &= R\int_{\partial B(R)}\Big[(m-4l_F)(F(\frac{|du|^2}{2}))^2+\frac{4d_F^2}{m}(F(\frac{\sqrt{m}}{2}\|u^*h\|))^2\Big]ds_g\\ &\quad -R\int_{\partial B(R)}\Big[\sum_{j=1}^m\frac{8d_F^2}{\sqrt{m}}(\|u^*h\|)^{-1}F(\frac{\sqrt{m}}{2}\|u^*h\|)F'(\frac{\sqrt{m}}{2}\|u^*h\|)(h(du(\frac{\partial}{\partial r}),du(e_j)))^2\\ &\qquad +2(m-4l_F)F(\frac{|du|^2}{2})F'(\frac{|du|^2}{2})h(du(\frac{\partial}{\partial r}),du(\frac{\partial}{\partial r}))\Big]ds_g\\ &\leq R\int_{\partial B(R)}\Big[(m-4l_F)(F(\frac{|du|^2}{2}))^2+\frac{4d_F^2}{m}(F(\frac{\sqrt{m}}{2}\|u^*h\|))^2\Big]ds_g\\ &= R\frac{d}{dR}\int_{B(R)}\Big[(m-4l_F)(F(\frac{|du|^2}{2}))^2+\frac{4d_F^2}{m}(F(\frac{\sqrt{m}}{2}\|u^*h\|))^2\Big]dv_g.\end{aligned} \tag{14}$$

Hence by (10), (13) and (14), we obtain

$$R\frac{d}{dR}\int_{B(R)}\Big[(m-4l_F)(F(\frac{|du|^2}{2}))^2+\frac{4d_F^2}{m}(F(\frac{\sqrt{m}}{2}\|u^*h\|))^2\Big]dv_g \geq \Lambda\int_{B(R)}\Big[(m-4l_F)(F(\frac{|du|^2}{2}))^2+\frac{4d_F^2}{m}(F(\frac{\sqrt{m}}{2}\|u^*h\|))^2\Big]dv_g.$$

i.e.,

$$\frac{d}{dR}\frac{\int_{B(R)}[(m-4l_F)(F(\frac{|du|^2}{2}))^2+\frac{4d_F^2}{m}(F(\frac{\sqrt{m}}{2}\|u^*h\|))^2]dv_g}{R^\Lambda}\geq 0.$$

Integrating the above formula on $[\rho_1,\rho_2]$, we can get

$$\frac{\int_{B(\rho_1)}\Big[(m-4l_F)(F(\frac{|du|^2}{2}))^2+\frac{4d_F^2}{m}(F(\frac{\sqrt{m}}{2}\|u^*h\|))^2\Big]dv_g}{\rho_1^\Lambda}\leq\frac{\int_{B(\rho_2)}\Big[(m-4l_F)(F(\frac{|du|^2}{2}))^2+\frac{4d_F^2}{m}(F(\frac{\sqrt{m}}{2}\|u^*h\|))^2\Big]dv_g}{\rho_2^\Lambda}.$$

□

Using the monotonicity formula, we immediately obtain the following vanishing results.

Theorem 4.2. *Under the same conditions of Theorem 4.1 and*

$$\int_{B(r)}\Big[(m-4l_F)(F(\frac{|du|^2}{2}))^2+\frac{4d_F^2}{m}(F(\frac{\sqrt{m}}{2}\|u^*h\|))^2\Big]dv_g=o(r^\Lambda),$$

then u is a constant map. Hence the F-energy stress tensor S_F is vanishing, i.e., $S_F=0$.

Proof. By Theorem 4.1, we have

$$\frac{1}{\rho^\Lambda}\int_{B(\rho)}\Big\{(m-4l_F)[F(\frac{|du|^2}{2})]^2+\frac{4d_F^2}{m}[F(\frac{\sqrt{m}}{2}\|u^*h\|)]^2\Big\}dv_g \le \frac{1}{r^\Lambda}\int_{B(r)}\Big\{(m-4l_F)[F(\frac{|du|^2}{2})]^2+\frac{4d_F^2}{m}[F(\frac{\sqrt{m}}{2}\|u^*h\|)]^2\Big\}dv_g$$

for any $0<\rho<r$.
Letting $r\to+\infty$, under the assumption, we can obtain

$$\int_{B(\rho)}\Big[(m-4l_F)(F(\frac{|du|^2}{2}))^2+\frac{4d_F^2}{m}(F(\frac{\sqrt{m}}{2}\|u^*h\|))^2\Big]dv_g\equiv 0.$$

Since ρ is arbitrary, then

$$(m-4l_F)[F(\frac{|du|^2}{2})]^2+\frac{4d_F^2}{m}[F(\frac{\sqrt{m}}{2}\|u^*h\|)]^2\equiv 0.$$

From the assumption of F, we can obtain

$$du\equiv 0, u^*h\equiv 0.$$

This implies that u is a constant map. So we have the vanishing result of S_F. □

We can apply the above vanishing result to some concrete pinched manifolds. In order to do it, we need the following lemmas.

Lemma 4.3. *(see [14, 23, 31]) Let (M,g) be a complete Riemannian manifold with a pole x_0 and let r be the distance function relative to x_0. Denote by K_r the radial curvature of M.*
(i) If $-\alpha^2\le K_r\le-\beta^2$ with $\alpha>0$, $\beta>0$, then

$$\beta\coth(\beta r)[g-dr\otimes dr]\le Hess_g(r)\le\alpha\coth(\alpha r)[g-dr\otimes dr].$$

(ii) If $-\frac{A}{(1+r^2)^{1+\epsilon}}\le K_r\le\frac{B}{(1+r^2)^{1+\epsilon}}$ with $\epsilon>0$, $A\ge 0$, $0\le B<2\epsilon$, then

$$\frac{1-\frac{B}{2\epsilon}}{r}[g-dr\otimes dr]\le Hess_g(r)\le\frac{e^{\frac{A}{2\epsilon}}}{r}[g-dr\otimes dr].$$

(iii)If $-\frac{A(A-1)}{r^2}\le K_r\le-\frac{B(B-1)}{r^2}$ with $A\ge B\ge 1$, then

$$\frac{A}{r}[g-dr\otimes dr]\le Hess_g(r)\le\frac{B}{r}[g-dr\otimes dr].$$

(iv) If $-\frac{a^2}{1+r^2}\le K_r\le\frac{b^2}{1+r^2}$ with $b^2\in[0,\frac{1}{4}]$, then

$$\frac{1+\sqrt{1-4b^2}}{2r}[g-dr\otimes dr]\le Hess_g(r)\le\frac{1+\sqrt{1+4a^2}}{2r}[g-dr\otimes dr].$$

Lemma 4.4. *Let (M,g) be a complete Riemannian manifold with a pole x_0 and let r be the distance function relative to x_0. Assume that there exist two positive functions $h_1(r)$ and $h_2(r)$ such that*

$$h_1(r)[g-dr\otimes dr]\le Hess_g(r)\le h_2(r)[g-dr\otimes dr]\quad and\quad rh_2(r)\ge 1,$$

then

$$1+\frac{m-1}{2}\lambda_{\min}-2d_F\{2,\lambda_{\max}\}\ge 1+(m-1)rh_1(r)-4d_Frh_2(r).$$

Proof. Applying the Hessian operator to the composed function r^2, we have

$$Hess_g(r^2) = 2rHess_g(r) + 2dr \otimes dr.$$

Let $\{e_i\}_{i=1}^m$ be an orthonormal basis and $e_m = \frac{\partial}{\partial r}$. We also assume that $Hess_g(r^2)$ becomes a diagonal matric with respect to $\{e_i\}_{i=1}^m$. Let e_α and e_β be eigenvectors of $2rHess_g(r)$ associated to eigenvalue $\lambda_{\max}$ and $\lambda_{\min}$, respectively. Then

$$\begin{aligned}
&1 + \frac{m-1}{2}\lambda_{\min} - 2d_F \max\{2, \lambda_{\max}\} \\
&= 1 + \frac{m-1}{2} 2rHess_g(r)(e_\alpha, e_\alpha) - 2d_F \max\{2, 2rHess_g(r)(e_\beta, e_\beta)\} \\
&\geq 1 + \frac{m-1}{2} 2rh_1 g(e_\alpha, e_\alpha) - 2d_F \max\{2, 2rh_2 g(e_\beta, e_\beta)\} \\
&= 1 + (m-1)rh_1(r) - 2d_F \max\{2, 2rh_2(r)\} \\
&= 1 + (m-1)rh_1(r) - 4d_F rh_2(r).
\end{aligned}$$

□

From Lemma 4.3 and Lemma 4.4, we have the following lemma.

Lemma 4.5. *Let (M, g) be a complete Riemannian manifold with a pole x_0 and let r be the distance function relative to x_0. Denote by K_r the radial curvature of M.*
(i) If $-\alpha^2 \leq K_r \leq -\beta^2$ with $\alpha > 0$, $\beta > 0$, then

$$1 + \frac{m-1}{2}\lambda_{\min} - 2d_F \max\{2, \lambda_{\max}\} \geq m - 4d_F \frac{\alpha}{\beta}.$$

(ii) If $-\frac{A}{(1+r^2)^{1+\epsilon}} \leq K_r \leq \frac{B}{(1+r^2)^{1+\epsilon}}$ with $\epsilon > 0$, $A \geq 0$, $0 \leq B < 2\epsilon$, then

$$1 + \frac{m-1}{2}\lambda_{\min} - 2d_F \max\{2, \lambda_{\max}\} \geq 1 + (m-1)(1 - \frac{B}{2\epsilon}) - 4d_F e^{\frac{A}{2\epsilon}}.$$

(iii)If $-\frac{A(A-1)}{r^2} \leq K_r \leq -\frac{B(B-1)}{r^2}$ with $A \geq B \geq 1$, then

$$1 + \frac{m-1}{2}\lambda_{\min} - 2d_F \max\{2, \lambda_{\max}\} \geq 1 + (m-1)A - 4d_F B.$$

(iv) If $-\frac{a^2}{1+r^2} \leq K_r \leq \frac{b^2}{1+r^2}$ with $b^2 \in [0, \frac{1}{4}]$, then

$$1 + \frac{m-1}{2}\lambda_{\min} - 2d_F \max\{2, \lambda_{\max}\} \geq 1 + (m-1)\frac{1 + \sqrt{1-4b^2}}{2} - 2d_F(1 + \sqrt{1+4a^2}).$$

Using Theorem 4.2 and Lemma 4.5, we can obtain the following vanishing result.

Corollary 4.6. *Let $u : (M^m, g) \to (N^n, h)$ be a $\Phi_{S,F}$-harmonic map from a complete Riemannian manifold with a pole x_0. Assume that the radial curvature K_r of M satisfies one of the following conditions:*
(i) $-\alpha^2 \leq K_r \leq -\beta^2$ with $\alpha > 0$, $\beta > 0$ and $(m-1)\beta - 4d_F\alpha > 0$;
(ii) $-\frac{A}{(1+r^2)^{1+\epsilon}} \leq K_r \leq \frac{B}{(1+r^2)^{1+\epsilon}}$ with $\epsilon > 0$, $A \geq 0$, $0 \leq B < 2\epsilon$ and $1 + (m-1)(1 - \frac{B}{2\epsilon}) - 4d_F e^{\frac{A}{2\epsilon}} > 0$;
(iii) $-\frac{A(A-1)}{r^2} \leq K_r \leq -\frac{B(B-1)}{r^2}$ with $A \geq B \geq 1$ and $1 + (m-1)A - 4d_F B > 0$;
(iv) $-\frac{a^2}{1+r^2} \leq K_r \leq \frac{b^2}{1+r^2}$ with $b^2 \in [0, \frac{1}{4}]$ and $1 + (m-1)\frac{1+\sqrt{1-4b^2}}{2} - 2d_F(1 + \sqrt{1+4a^2}) > 0$.
If

$$\int_{B(r)} \Big\{(m - 4l_F)[F(\frac{|du|^2}{2})]^2 + \frac{4d_F^2}{m}[F(\frac{\sqrt{m}}{2}\|u^*h\|)]^2\Big\} dv_g = o(r^\Lambda),$$

where

$$\Lambda = \begin{cases} m - 4d_F\frac{\alpha}{\beta}, & if \quad K_r \quad satisfies \quad (i) \\ 1 + (m-1)(1 - \frac{B}{2\epsilon}) - 4d_F e^{\frac{A}{2\epsilon}}, & if \quad K_r \quad satisfies \quad (ii) \\ 1 + (m-1)A - 4d_F B, & if \quad K_r \quad satisfies \quad (iii) \\ 1 + (m-1)\frac{1+\sqrt{1-4b^2}}{2} - 2d_F(1 + \sqrt{1+4a^2}), & if \quad K_r \quad satisfies \quad (iv) \end{cases}$$

then u is a constant map. So we know $S_F = 0$.

The functional $\Phi_{S,F}$ of u is said to be slowly divergent, if there exists a positive function $\psi(r)$ such that

$$\int_{R_0}^{+\infty} \frac{dr}{r\psi(r)} = +\infty$$

for some R_0, and

$$\lim_{R\to+\infty} \int_{B(R)} \frac{(m-4l_F)[F(\frac{|du|^2}{2})]^2 + \frac{4d_F^2}{m}[F(\frac{\sqrt{m}}{2}\|u^*h\|)]^2}{\psi(r)} dv_g < +\infty. \tag{15}$$

Theorem 4.7. *Let $u : (M^m, g) \to (N^n, h)$ be a $\Phi_{S,F}$-harmonic map. If $r(x)$ satisfies the condition (9) and $\Phi_{S,F}$ is slowly divergent, then u is a constant map and $S_F = 0$.*

Proof. From the proof of Theorem 4.1, we have

$$R\frac{d}{dR}\int_{B(R)} \Big[(m-4l_F)(F(\frac{|du|^2}{2}))^2 + \frac{4d_F^2}{m}(F(\frac{\sqrt{m}}{2}\|u^*h\|))^2\Big]dv_g \geq \Lambda \int_{B(R)} \Big[(m-4l_F)(F(\frac{|du|^2}{2}))^2 + \frac{4d_F^2}{m}(F(\frac{\sqrt{m}}{2}\|u^*h\|))^2\Big]dv_g.$$

If u is not a constant map, there exists constants $R_1 > 0$ and $c_0 > 0$ such that

$$\int_{B(R)} \Big[(m-4l_F)(F(\frac{|du|^2}{2}))^2 + \frac{4d_F^2}{m}(F(\frac{\sqrt{m}}{2}\|u^*h\|))^2\Big]dv_g \geq c_0$$

for any $R \geq R_1$. Thus

$$\int_{\partial B(R)} \Big[(m-4l_F)(F(\frac{|du|^2}{2}))^2 + \frac{4d_F^2}{m}(F(\frac{\sqrt{m}}{2}\|u^*h\|))^2\Big]ds_g \geq \frac{\Lambda c_0}{R}, \forall R \geq R_1.$$

Since $\Phi_{S,F}$ is slowly divergent, then

$$\begin{aligned} &\lim_{R\to+\infty} \int_{B(R)} \frac{(m-4l_F)[F(\frac{|du|^2}{2})]^2 + \frac{4d_F^2}{m}[F(\frac{\sqrt{m}}{2}\|u^*h\|)]^2}{\psi(r)} dv_g \\ &= \int_0^{+\infty} \frac{dR}{\psi(R)} \int_{\partial B(R)} \Big[(m-4l_F)(F(\frac{|du|^2}{2}))^2 + \frac{4d_F^2}{m}(F(\frac{\sqrt{m}}{2}\|u^*h\|))^2\Big]ds_g \\ &\geq \int_{R_1}^{+\infty} \frac{dR}{\psi(R)} \int_{\partial B(R)} \Big[(m-4l_F)(F(\frac{|du|^2}{2}))^2 + \frac{4d_F^2}{m}(F(\frac{\sqrt{m}}{2}\|u^*h\|))^2\Big]ds_g \\ &\geq \int_{R_1}^{+\infty} \frac{dR}{\psi(R)} \frac{c_0\Lambda}{R} \\ &= c_0\Lambda \int_{R_1}^{+\infty} \frac{dR}{R\psi(R)} \\ &= +\infty. \end{aligned}$$

This contracts with (15). Therefore, u is a constant map and $S_F = 0$. □

5. Vanishing results under the asymptotic condition

In this section, using a similar technique in [16], we can derive a Liouville theorem for the $\Phi_{S,F}$-harmonic map. Furthermore we can obtain the vanishing result of F-stress energy tensor S_F. First we give the lower $\Phi_{S,F}$ functional growth rate for the $\Phi_{S,F}$-harmonic map by using the monotonicity formula.

Proposition 5.1. *Let $u : (M^m, g) \to (N^n, h)$ be a $\Phi_{S,F}$-harmonic map from a complete Riemannian manifold with a pole x_0. $r(x)$ is the distance function relative to the pole x_0. If $r(x)$ satisfies (9) and u is not constant, then*

$$\int_{B(R)} \Big[(m-4l_F)(F(\frac{|du|^2}{2}))^2 + \frac{4d_F^2}{m}(F(\frac{\sqrt{m}}{2}\|u^*h\|))^2\Big]dv_g \geq C(u)R^{\Lambda} \quad as \quad R \to +\infty,$$

where $C(u)$ is a positive constant only depending on u.

Proof. Since u satisfies the condition in Theorem 4.1, we have

$$\frac{\int_{B(\rho)} \Big[(m-4l_F)(F(\frac{|du|^2}{2}))^2 + \frac{4d_F^2}{m}(F(\frac{\sqrt{m}}{2}\|u^*h\|))^2\Big]dv_g}{\rho^{\Lambda}} \leq \frac{\int_{B(R)} \Big[(m-4l_F)(F(\frac{|du|^2}{2}))^2 + \frac{4d_F^2}{m}(F(\frac{\sqrt{m}}{2}\|u^*h\|))^2\Big]dv_g}{R^{\Lambda}}$$

for any $0 < \rho < R$. Note that u is not a constant map, there exists some $\rho > 0$ such that

$$\int_{B(\rho)} \Big[(m-4l_F)(F(\frac{|du|^2}{2}))^2 + \frac{4d_F^2}{m}(F(\frac{\sqrt{m}}{2}\|u^*h\|))^2\Big]dv_g > 0.$$

Set $C(u) = \frac{\int_{B(\rho)} \Big[(m-4l_F)(F(\frac{|du|^2}{2}))^2 + \frac{4d_F^2}{m}(F(\frac{\sqrt{m}}{2}\|u^*h\|))^2\Big]dv_g}{\rho^{\Lambda}}$, then

$$\int_{B(R)} \Big[(m-4l_F)(F(\frac{|du|^2}{2}))^2 + \frac{4d_F^2}{m}(F(\frac{\sqrt{m}}{2}\|u^*h\|))^2\Big]dv_g \geq C(u)R^{\Lambda}.$$

This completes the proof of the proposition. □

Set

$$E^R_{\Phi_{S,F}}(u) = \int_{B(R)} \Big[(m-4l_F)\Big(F(\frac{|du|^2}{2})\Big)^2 + \frac{4d_F^2}{m}\Big(F(\frac{\sqrt{m}}{2}\|u^*h\|)\Big)^2\Big]dv_g.$$

Next we will use the assumption for the map at infinity to derive an upper bound for the growth rate of $E^R_{\Phi_{S,F}}(u)$ as $R \to +\infty$.

Proposition 5.2. *Let $u : (M^m, g) \to (N^n, h)$ be a $\Phi_{S,F}$-harmonic map from a Riemannian manifold M with a pole x_0. Suppose $l_F > 0$ and $F(\frac{\sqrt{m}}{2}\|u^*h\|)F'(\frac{\sqrt{m}}{2}\|u^*h\|) < +\infty$. Assume that $r(x)$ satisfies (9) and $(\int_R^{+\infty} \frac{1}{vol(\partial B_r)}dr)^{-1} \leq C_0R^{\Lambda}$ for R large enough. If $u(x) \to p_0 \in N$ as $r(x) \to +\infty$, then u must be a constant map, or there exists positive constants R_0, C, $c(u)$ and $\eta(R) \to 0$ as $R \to +\infty$, such that*

$$E^R_{\Phi_{S,F}}(u) \leq C(\frac{\eta(R)}{2l_F} + \frac{c(u)}{R^{\sigma}})R^{\sigma} \quad for \quad R \geq R_0.$$

Proof. Suppose the $\Phi_{S,F}$-harmonic map u is not constant, then by Proposition 5.1, the $\Phi_{S,F}$-energy of u must be infinite. That is, $E^R_{\Phi_{S,F}}(u) \to +\infty$ as $R \to +\infty$.

Choose a local coordinate neighborhood (U, ϕ) of p_0 in N^n, such that $\phi(p_0) = 0$, and $h = \sum_{\alpha,\beta=1}^n h_{\alpha\beta}dy^{\alpha} \otimes dy^{\beta}$, $y \in U$ satisfies

$$(\frac{\partial h_{\alpha\beta}(y)}{\partial y^{\gamma}}y^{\gamma} + 2h_{\alpha\beta}(y)) \geq (h_{\alpha\beta}(y)) \quad on \quad U$$

in the matrices sense (that is , for two $n \times n$ matrices A, B, by $A \geq B$, we mean that $A - B$ is a positive semi-definite matrix).

Now the assumption that $u(x) \to 0$ as $r(x) \to +\infty$ implies that there exists an R_1 such that $u(x) \in U$ for $r(x) > R_1$ and

$$(\frac{\partial h_{\alpha\beta}(u)}{\partial u^\gamma} u^\gamma + 2h_{\alpha\beta}(u)) \geq (h_{\alpha\beta}(u)) \quad for \quad r(x) > R_1. \tag{16}$$

For $\omega \in C_0^2(M^m \setminus B(R_1), \phi(U))$, we consider the variation $u + t\omega : M^m \to N^n$ defined as follows:

$$(u + t\omega)(q) = \begin{cases} u(q), & if \quad q \in B(R_1) \\ \phi^{-1}[(\phi(u) + t\omega)(q)], & if \quad q \in M \setminus B(R_1) \end{cases}$$

for sufficient small $|t|$. By the definition of $\Phi_{S,F}$-harmonic maps, we have

$$\frac{d}{dt}\Big|_{t=0} E_{\Phi_{S,F}}(u + t\omega) = 0,$$

that is,

$$\begin{aligned}&\int_{M\setminus B(R_1)} (m - 4l_F)F(\frac{|du|^2}{2})F'(\frac{|du|^2}{2}) g^{ij}\Big[2h_{\alpha\beta}\frac{\partial u^\alpha}{\partial x_i}\frac{\partial \omega^\beta}{\partial x_j} + \frac{\partial h_{\alpha\beta}}{\partial u^\zeta}\omega^\zeta \frac{\partial u^\alpha}{\partial x_i}\frac{\partial u^\beta}{\partial x_j}\Big] dv_g \\ &+ \int_{M\setminus B(R_1)} \frac{4d_F^2}{\sqrt{m}} \|u^*h\|^{-1} F(\frac{\sqrt{m}}{2}\|u^*h\|)F'(\frac{\sqrt{m}}{2}\|u^*h\|) g^{ik}g^{jl}\Big[2h_{\alpha\beta}\frac{\partial u^\alpha}{\partial x_i}\frac{\partial \omega^\beta}{\partial x_j} + \frac{\partial h_{\alpha\beta}}{\partial u^\zeta}\omega^\zeta \frac{\partial u^\alpha}{\partial x_i}\frac{\partial u^\beta}{\partial x_j}\Big] h_{\gamma\delta}\frac{\partial u^\gamma}{\partial x_k}\frac{\partial u^\delta}{\partial x_l} dv_g = 0\end{aligned} \tag{17}$$

Now taking $\omega(x) = \varphi(r(x))u(x)$ in (17) for $\varphi(r) \in C_0^\infty(R_1, +\infty)$, we obtain

$$\begin{aligned}&\int_{M\setminus B(R_1)} \Big\{\frac{4d_F^2}{\sqrt{m}} \|u^*h\|^{-1} F(\frac{\sqrt{m}}{2}\|u^*h\|)F'(\frac{\sqrt{m}}{2}\|u^*h\|) g^{ik}g^{jl}\Big[2h_{\alpha\beta} + \frac{\partial h_{\alpha\beta}}{\partial u^\zeta}u^\zeta\Big] \times \varphi(r(x))\frac{\partial u^\alpha}{\partial x_i}\frac{\partial u^\beta}{\partial x_j} h_{\gamma\delta}\frac{\partial u^\gamma}{\partial x_k}\frac{\partial u^\delta}{\partial x_l}\Big\} dv_g \\ &+ \int_{M\setminus B(R_1)} (m - 4l_F)F(\frac{|du|^2}{2})F'(\frac{|du|^2}{2}) g^{ij}\Big[2h_{\alpha\beta} + \frac{\partial h_{\alpha\beta}}{\partial u^\zeta}u^\zeta\Big]\varphi(r(x))\frac{\partial u^\alpha}{\partial x_i}\frac{\partial u^\beta}{\partial x_j} dv_g \\ &+ 2\int_{M\setminus B(R_1)} \Big\{\frac{4d_F^2}{\sqrt{m}} \|u^*h\|^{-1} F(\frac{\sqrt{m}}{2}\|u^*h\|)F'(\frac{\sqrt{m}}{2}\|u^*h\|) g^{ik}g^{jl} h_{\alpha\beta}\frac{\partial u^\alpha}{\partial x_i}\frac{\partial \varphi(r(x))}{\partial x_j} \times u^\beta h_{\gamma\delta}\frac{\partial u^\gamma}{\partial x_k}\frac{\partial u^\delta}{\partial x_l}\Big\} dv_g \\ &+ 2\int_{M\setminus B(R_1)} (m - 4l_F)F(\frac{|du|^2}{2})F'(\frac{|du|^2}{2}) g^{ij} h_{\alpha\beta}\frac{\partial u^\alpha}{\partial x_i}\frac{\partial \varphi(r(x))}{\partial x_j} u^\beta dv_g = 0\end{aligned} \tag{18}$$

By the standard approximation argument, (18) holds for any Lipschitz function φ with compact support. For $0 \leq \varepsilon \leq 1$, we define

$$\psi_\varepsilon(t) = \begin{cases} 1, & t \leq 1; \\ 1 + \frac{1-t}{\varepsilon}, & 1 < t < 1 + \varepsilon; \\ 0, & t \geq 1 + \varepsilon. \end{cases}$$

and choose the Lipschitz function $\varphi(r(x))$ to be

$$\varphi(r(x)) = \psi_\varepsilon(\frac{r(x)}{R})(1 - \psi_1(\frac{r(x)}{R_1})), \quad R > R_2 = 2R_1. \tag{19}$$

By (19), we have

$$\begin{aligned}
&\int_{M\backslash B(R_1)}\Big\{\frac{4d_F^2}{\sqrt{m}}\|u^*h\|^{-1}F(\frac{\sqrt{m}}{2}\|u^*h\|)F'(\frac{\sqrt{m}}{2}\|u^*h\|)g^{ik}g^{jl}\Big[2h_{\alpha\beta}+\frac{\partial h_{\alpha\beta}}{\partial u^\zeta}u^\zeta\Big]\times\varphi(r(x))\frac{\partial u^\alpha}{\partial x_i}\frac{\partial u^\beta}{\partial x_j}h_{\gamma\delta}\frac{\partial u^\gamma}{\partial x_k}\frac{\partial u^\delta}{\partial x_l}\Big\}dv_g\\
&=\int_{B(R_2)\backslash B(R_1)}\Big\{\frac{4d_F^2}{\sqrt{m}}\|u^*h\|^{-1}F(\frac{\sqrt{m}}{2}\|u^*h\|)F'(\frac{\sqrt{m}}{2}\|u^*h\|)g^{ik}g^{jl}\Big[2h_{\alpha\beta}+\frac{\partial h_{\alpha\beta}}{\partial u^\zeta}u^\zeta\Big]\\
&\qquad\times(1-\psi_1(\frac{r(x)}{R_1}))\frac{\partial u^\alpha}{\partial x_i}\frac{\partial u^\beta}{\partial x_j}h_{\gamma\delta}\frac{\partial u^\gamma}{\partial x_k}\frac{\partial u^\delta}{\partial x_l}\Big\}dv_g\\
&\quad+\int_{B(R)\backslash B(R_2)}\Big\{\frac{4d_F^2}{\sqrt{m}}\|u^*h\|^{-1}F(\frac{\sqrt{m}}{2}\|u^*h\|)F'(\frac{\sqrt{m}}{2}\|u^*h\|)g^{ik}g^{jl}\Big[2h_{\alpha\beta}+\frac{\partial h_{\alpha\beta}}{\partial u^\zeta}u^\zeta\Big]\times\frac{\partial u^\alpha}{\partial x_i}\frac{\partial u^\beta}{\partial x_j}h_{\gamma\delta}\frac{\partial u^\gamma}{\partial x_k}\frac{\partial u^\delta}{\partial x_l}\Big\}dv_g\\
&\quad+\int_{B((1+\varepsilon)R)\backslash B(R)}\Big\{\frac{4d_F^2}{\sqrt{m}}\|u^*h\|^{-1}F(\frac{\sqrt{m}}{2}\|u^*h\|)F'(\frac{\sqrt{m}}{2}\|u^*h\|)g^{ik}g^{jl}\Big[2h_{\alpha\beta}+\frac{\partial h_{\alpha\beta}}{\partial u^\zeta}u^\zeta\Big]\\
&\qquad\times\psi_\varepsilon(\frac{r(x)}{R_1})\frac{\partial u^\alpha}{\partial x_i}\frac{\partial u^\beta}{\partial x_j}h_{\gamma\delta}\frac{\partial u^\gamma}{\partial x_k}\frac{\partial u^\delta}{\partial x_l}\Big\}dv_g
\end{aligned}\tag{20}$$

and

$$\begin{aligned}
&\int_{M\backslash B(R_1)}(m-4l_F)F(\frac{|du|^2}{2})F'(\frac{|du|^2}{2})g^{ij}\Big[2h_{\alpha\beta}+\frac{\partial h_{\alpha\beta}}{\partial u^\zeta}u^\zeta\Big]\varphi(r(x))\frac{\partial u^\alpha}{\partial x_i}\frac{\partial u^\beta}{\partial x_j}dv_g\\
&=\int_{B(R_2)\backslash B(R_1)}(m-4l_F)F(\frac{|du|^2}{2})F'(\frac{|du|^2}{2})g^{ij}\Big[2h_{\alpha\beta}+\frac{\partial h_{\alpha\beta}}{\partial u^\zeta}u^\zeta\Big](1-\psi_1(\frac{r(x)}{R_1}))\frac{\partial u^\alpha}{\partial x_i}\frac{\partial u^\beta}{\partial x_j}dv_g\\
&\quad+\int_{B(R)\backslash B(R_2)}(m-4l_F)F(\frac{|du|^2}{2})F'(\frac{|du|^2}{2})g^{ij}\Big[2h_{\alpha\beta}+\frac{\partial h_{\alpha\beta}}{\partial u^\zeta}u^\zeta\Big]\frac{\partial u^\alpha}{\partial x_i}\frac{\partial u^\beta}{\partial x_j}dv_g\\
&\quad+\int_{B((1+\varepsilon)R)\backslash B(R)}(m-4l_F)F(\frac{|du|^2}{2})F'(\frac{|du|^2}{2})g^{ij}\Big[2h_{\alpha\beta}+\frac{\partial h_{\alpha\beta}}{\partial u^\zeta}u^\zeta\Big]\psi_\varepsilon(\frac{r(x)}{R_1})\frac{\partial u^\alpha}{\partial x_i}\frac{\partial u^\beta}{\partial x_j}dv_g
\end{aligned}\tag{21}$$

and

$$\begin{aligned}
&2\int_{M\backslash B(R_1)}\Big\{\frac{4d_F^2}{\sqrt{m}}\|u^*h\|^{-1}F(\frac{\sqrt{m}}{2}\|u^*h\|)F'(\frac{\sqrt{m}}{2}\|u^*h\|)g^{ik}g^{jl}h_{\alpha\beta}\frac{\partial u^\alpha}{\partial x_i}\frac{\partial\varphi(r(x))}{\partial x_j}\times u^\beta h_{\gamma\delta}\frac{\partial u^\gamma}{\partial x_k}\frac{\partial u^\delta}{\partial x_l}\Big\}dv_g\\
&=-2\int_{B(R_2)\backslash B(R_1)}\Big\{\frac{4d_F^2}{\sqrt{m}}\|u^*h\|^{-1}F(\frac{\sqrt{m}}{2}\|u^*h\|)F'(\frac{\sqrt{m}}{2}\|u^*h\|)g^{ik}g^{jl}h_{\alpha\beta}\frac{\partial u^\alpha}{\partial x_i}\times\frac{\partial\psi_1(\frac{r(x)}{R_1})}{\partial x_j}u^\beta h_{\gamma\delta}\frac{\partial u^\gamma}{\partial x_k}\frac{\partial u^\delta}{\partial x_l}\Big\}dv_g\\
&=-2\frac{1}{R\varepsilon}\int_{B((1+\varepsilon)R)\backslash B(R)}\Big\{\frac{4d_F^2}{\sqrt{m}}\|u^*h\|^{-1}F(\frac{\sqrt{m}}{2}\|u^*h\|)F'(\frac{\sqrt{m}}{2}\|u^*h\|)g^{ik}g^{jl}h_{\alpha\beta}\times\frac{\partial u^\alpha}{\partial x_i}\frac{\partial r(x)}{\partial x_j}u^\beta h_{\gamma\delta}\frac{\partial u^\gamma}{\partial x_k}\frac{\partial u^\delta}{\partial x_l}\Big\}dv_g
\end{aligned}\tag{22}$$

and

$$\begin{aligned}
&2\int_{M\backslash B(R_1)}(m-4l_F)F(\frac{|du|^2}{2})F'(\frac{|du|^2}{2})g^{ij}h_{\alpha\beta}\frac{\partial u^\alpha}{\partial x_i}\frac{\partial\varphi(r(x))}{\partial x_j}u^\beta dv_g\\
&=-2\int_{B(R_2)\backslash B(R_1)}(m-4l_F)F(\frac{|du|^2}{2})F'(\frac{|du|^2}{2})g^{ij}h_{\alpha\beta}\frac{\partial u^\alpha}{\partial x_i}\frac{\partial\psi_1(\frac{r(x)}{R_1})}{\partial x_j}u^\beta dv_g\\
&=-2\frac{1}{R\varepsilon}\int_{B((1+\varepsilon)R)\backslash B(R)}(m-4l_F)F(\frac{|du|^2}{2})F'(\frac{|du|^2}{2})g^{ij}h_{\alpha\beta}\frac{\partial u^\alpha}{\partial x_i}\frac{\partial r(x)}{\partial x_j}u^\beta dv_g
\end{aligned}\tag{23}$$

By (18), (20), (21), (22), (23) and letting $\varepsilon \to 0$, we have

$$\begin{aligned}
&\int_{B(R)\setminus B(R_2)} \Big\{ \frac{4d_F^2}{\sqrt{m}} \|u^*h\|^{-1} F\Big(\frac{\sqrt{m}}{2}\|u^*h\|\Big) F'\Big(\frac{\sqrt{m}}{2}\|u^*h\|\Big) g^{ik} g^{jl} \Big[2h_{\alpha\beta} + \frac{\partial h_{\alpha\beta}}{\partial u^\zeta} u^\zeta\Big] \times \frac{\partial u^\alpha}{\partial x_i}\frac{\partial u^\beta}{\partial x_j} h_{\gamma\delta} \frac{\partial u^\gamma}{\partial x_k}\frac{\partial u^\delta}{\partial x_l}\Big\} dv_g \\
&+ \int_{B(R)\setminus B(R_2)} (m-4l_F) F\Big(\frac{|du|^2}{2}\Big) F'\Big(\frac{|du|^2}{2}\Big) g^{ij} \Big[2h_{\alpha\beta} + \frac{\partial h_{\alpha\beta}}{\partial u^\zeta} u^\zeta\Big] \frac{\partial u^\alpha}{\partial x_i}\frac{\partial u^\beta}{\partial x_j} dv_g + D(R_1) \\
&= 2\int_{\partial B(R)} \Big\{ \frac{4d_F^2}{\sqrt{m}} \|u^*h\|^{-1} F\Big(\frac{\sqrt{m}}{2}\|u^*h\|\Big) F'\Big(\frac{\sqrt{m}}{2}\|u^*h\|\Big) g^{ik} g^{jl} h_{\alpha\beta} \frac{\partial u^\alpha}{\partial x_i}\frac{\partial r(x)}{\partial x_j} \times u^\beta h_{\gamma\delta} \frac{\partial u^\gamma}{\partial x_k}\frac{\partial u^\delta}{\partial x_l}\Big\} ds_g \\
&+ 2\int_{\partial B(R)} (m-4l_F) F\Big(\frac{|du|^2}{2}\Big) F'\Big(\frac{|du|^2}{2}\Big) g^{ij} h_{\alpha\beta} \frac{\partial u^\alpha}{\partial x_i}\frac{\partial r(x)}{\partial x_j} u^\beta ds_g,
\end{aligned} \tag{24}$$

where $D(R_1)$ is given by the following

$$\begin{aligned}
D(R_1) = &\int_{B(R_2)\setminus B(R_1)} \Big\{ \frac{4d_F^2}{\sqrt{m}} \|u^*h\|^{-1} F\Big(\frac{\sqrt{m}}{2}\|u^*h\|\Big) F'\Big(\frac{\sqrt{m}}{2}\|u^*h\|\Big) g^{ik} g^{jl} \Big[2h_{\alpha\beta} + \frac{\partial h_{\alpha\beta}}{\partial u^\zeta} u^\zeta\Big] \\
&\qquad \times \Big(1-\psi_1\Big(\frac{r(x)}{R_1}\Big)\Big) \frac{\partial u^\alpha}{\partial x_i}\frac{\partial u^\beta}{\partial x_j} h_{\gamma\delta} \frac{\partial u^\gamma}{\partial x_k}\frac{\partial u^\delta}{\partial x_l}\Big\} dv_g \\
&+ \int_{B(R_2)\setminus B(R_1)} \Big\{ (m-4l_F) F\Big(\frac{|du|^2}{2}\Big) F'\Big(\frac{|du|^2}{2}\Big) g^{ij} \Big[2h_{\alpha\beta} + \frac{\partial h_{\alpha\beta}}{\partial u^\zeta} u^\zeta\Big] \times \Big(1-\psi_1\Big(\frac{r(x)}{R_1}\Big)\Big) \frac{\partial u^\alpha}{\partial x_i}\frac{\partial u^\beta}{\partial x_j}\Big\} dv_g \\
&- 2\int_{B(R_2)\setminus B(R_1)} \Big\{ \frac{4d_F^2}{\sqrt{m}} \|u^*h\|^{-1} F\Big(\frac{\sqrt{m}}{2}\|u^*h\|\Big) F'\Big(\frac{\sqrt{m}}{2}\|u^*h\|\Big) g^{ik} g^{jl} h_{\alpha\beta} \frac{\partial u^\alpha}{\partial x_i} \times \frac{\partial \psi_1(\frac{r(x)}{R_1})}{\partial x_j} u^\beta h_{\gamma\delta} \frac{\partial u^\gamma}{\partial x_k}\frac{\partial u^\delta}{\partial x_l}\Big\} dv_g \\
&- 2\int_{B(R_2)\setminus B(R_1)} (m-4l_F) F\Big(\frac{|du|^2}{2}\Big) F'\Big(\frac{|du|^2}{2}\Big) g^{ij} h_{\alpha\beta} \frac{\partial u^\alpha}{\partial x_i} \frac{\partial \psi_1(\frac{r(x)}{R_1})}{\partial x_j} u^\beta dv_g.
\end{aligned}$$

Now we estimate the term on the left hand of (24). Take any point $p \in \partial B(R)$. It is easy to known that the term

$$\begin{aligned}
&\sum_{i,j,k,l,\alpha,\beta,\gamma,\delta} \frac{4d_F^2}{\sqrt{m}} \|u^*h\|^{-1} F\Big(\frac{\sqrt{m}}{2}\|u^*h\|\Big) F'\Big(\frac{\sqrt{m}}{2}\|u^*h\|\Big) g^{ik} g^{jl} h_{\alpha\beta} \frac{\partial u^\alpha}{\partial x_i}\frac{\partial r(x)}{\partial x_j} u^\beta h_{\gamma\delta} \frac{\partial u^\gamma}{\partial x_k}\frac{\partial u^\delta}{\partial x_l} \\
&+ \sum_{i,j,\alpha,\beta} (m-4l_F) F\Big(\frac{|du|^2}{2}\Big) F'\Big(\frac{|du|^2}{2}\Big) g^{ij} h_{\alpha\beta} \frac{\partial u^\alpha}{\partial x_i}\frac{\partial r(x)}{\partial x_j} u^\beta
\end{aligned}$$

does not depend on the coordinate on M and N at point p and $u(p)$. So we choose the adapted coordinate systems on M and N such that $g_{ij}(p) = \delta_{ij}$, $g^{ij}(p) = \delta^{ij}$ and $h_{\alpha\beta}(u(p)) = \delta_{\alpha\beta}$. We compute at p.

$$\begin{aligned}
&\sum_{i,j,k,l,\alpha,\beta,\gamma,\delta} \frac{4d_F^2}{\sqrt{m}} \|u^*h\|^{-1} F\Big(\frac{\sqrt{m}}{2}\|u^*h\|\Big) F'\Big(\frac{\sqrt{m}}{2}\|u^*h\|\Big) g^{ik} g^{jl} h_{\alpha\beta} \frac{\partial u^\alpha}{\partial x_i}\frac{\partial r(x)}{\partial x_j} u^\beta h_{\gamma\delta} \frac{\partial u^\gamma}{\partial x_k}\frac{\partial u^\delta}{\partial x_l} \\
&\quad + \sum_{i,j,\alpha,\beta} (m-4l_F) F\Big(\frac{|du|^2}{2}\Big) F'\Big(\frac{|du|^2}{2}\Big) g^{ij} h_{\alpha\beta} \frac{\partial u^\alpha}{\partial x_i}\frac{\partial r(x)}{\partial x_j} u^\beta \\
&= \sum_{i,j=1}^m \frac{4d_F^2}{\sqrt{m}} \|u^*h\|^{-1} F\Big(\frac{\sqrt{m}}{2}\|u^*h\|\Big) F'\Big(\frac{\sqrt{m}}{2}\|u^*h\|\Big) \Big[\sum_{\alpha=1}^n \frac{\partial r(x)}{\partial x_j}\frac{\partial u^\alpha}{\partial x_i} u^\alpha\Big]\Big[\sum_{\gamma=1}^n \frac{\partial u^\gamma}{\partial x_i}\frac{\partial u^\gamma}{\partial x_j}\Big] \\
&\quad + \sum_{i=1}^m (m-4l_F) F\Big(\frac{|du|^2}{2}\Big) F'\Big(\frac{|du|^2}{2}\Big) \Big[\sum_{\alpha=1}^n \frac{\partial u^\alpha}{\partial x_i} u^\alpha\Big] \frac{\partial r(x)}{\partial x_i} \\
&\le \frac{4d_F^2}{\sqrt{m}} \|u^*h\|^{-1} F\Big(\frac{\sqrt{m}}{2}\|u^*h\|\Big) F'\Big(\frac{\sqrt{m}}{2}\|u^*h\|\Big) \Big[\sum_{i,j=1}^m \Big(\sum_{\alpha=1}^n \frac{\partial r(x)}{\partial x_j}\frac{\partial u^\alpha}{\partial x_i} u^\alpha\Big)^2\Big]^{\frac{1}{2}} \Big[\sum_{i,j=1}^m \Big(\sum_{\gamma=1}^n \frac{\partial u^\gamma}{\partial x_i}\frac{\partial u^\gamma}{\partial x_j}\Big)^2\Big]^{\frac{1}{2}}
\end{aligned}$$

$$
\begin{aligned}
&+(m-4l_F)F(\frac{|du|^2}{2})F'(\frac{|du|^2}{2})\Big[\sum_{i=1}^{m}(\sum_{\alpha=1}^{n}\frac{\partial u^\alpha}{\partial x_i}u^\alpha)^2\Big]^{\frac{1}{2}}\Big[\sum_{i=1}^{m}(\frac{\partial r(x)}{\partial x_i})^2\Big]^{\frac{1}{2}}\\
&=\frac{4d_F^2}{\sqrt{m}}\|u^*h\|^{-1}F(\frac{\sqrt{m}}{2}\|u^*h\|)F'(\frac{\sqrt{m}}{2}\|u^*h\|)\Big[\sum_{i=1}^{m}(\sum_{\alpha=1}^{n}\frac{\partial u^\alpha}{\partial x_i}u^\alpha)^2\Big]^{\frac{1}{2}}\Big[\sum_{i,j=1}^{m}(\sum_{\gamma=1}^{n}\frac{\partial u^\gamma}{\partial x_i}\frac{\partial u^\gamma}{\partial x_j})^2\Big]^{\frac{1}{2}}\\
&+(m-4l_F)F(\frac{|du|^2}{2})F'(\frac{|du|^2}{2})\Big[\sum_{i=1}^{m}(\sum_{\alpha=1}^{n}\frac{\partial u^\alpha}{\partial x_i}u^\alpha)^2\Big]^{\frac{1}{2}}\\
&=\frac{4d_F^2}{\sqrt{m}}F(\frac{\sqrt{m}}{2}\|u^*h\|)F'(\frac{\sqrt{m}}{2}\|u^*h\|)\Big[\sum_{i=1}^{m}(\sum_{\alpha=1}^{n}\frac{\partial u^\alpha}{\partial x_i}u^\alpha)^2\Big]^{\frac{1}{2}}\\
&+(m-4l_F)F(\frac{|du|^2}{2})F'(\frac{|du|^2}{2})\Big[\sum_{i=1}^{m}(\sum_{\alpha=1}^{n}\frac{\partial u^\alpha}{\partial x_i}u^\alpha)^2\Big]^{\frac{1}{2}}\\
&\le\frac{4d_F^2}{\sqrt{m}}F(\frac{\sqrt{m}}{2}\|u^*h\|)F'(\frac{\sqrt{m}}{2}\|u^*h\|)\Big[\sum_{i=1}^{m}\sum_{\alpha=1}^{n}\frac{\partial u^\alpha}{\partial x_i}\frac{\partial u^\alpha}{\partial x_i}\sum_{\alpha=1}^{n}(u^\alpha)^2\Big]^{\frac{1}{2}}\\
&+(m-4l_F)F(\frac{|du|^2}{2})F'(\frac{|du|^2}{2})\Big[\sum_{i=1}^{m}\sum_{\alpha=1}^{n}\frac{\partial u^\alpha}{\partial x_i}\frac{\partial u^\alpha}{\partial x_i}\sum_{\alpha=1}^{n}(u^\alpha)^2\Big]^{\frac{1}{2}}\\
&=\Big\{\frac{4d_F^2}{\sqrt{m}}F(\frac{\sqrt{m}}{2}\|u^*h\|)F'(\frac{\sqrt{m}}{2}\|u^*h\|)\Big[\sum_{i=1}^{m}(\sum_{\alpha=1}^{n}\frac{\partial u^\alpha}{\partial x_i}\frac{\partial u^\alpha}{\partial x_i})\Big]^{\frac{1}{2}}\\
&+(m-4l_F)F(\frac{|du|^2}{2})F'(\frac{|du|^2}{2})\Big[\sum_{i=1}^{m}\sum_{\alpha=1}^{n}\frac{\partial u^\alpha}{\partial x_i}\frac{\partial u^\alpha}{\partial x_i}\Big]^{\frac{1}{2}}\Big\}\Big[\sum_{\alpha=1}^{n}(u^\alpha)^2\Big]^{\frac{1}{2}}\\
&\le\Big\{\frac{4d_F^2}{\sqrt{m}}F(\frac{\sqrt{m}}{2}\|u^*h\|)F'(\frac{\sqrt{m}}{2}\|u^*h\|)\sqrt[4]{m}\Big[\sum_{i=1}^{m}(\sum_{\alpha=1}^{n}\frac{\partial u^\alpha}{\partial x_i}\frac{\partial u^\alpha}{\partial x_i})^2\Big]^{\frac{1}{4}}\\
&+(m-4l_F)F(\frac{|du|^2}{2})F'(\frac{|du|^2}{2})\Big[\sum_{i=1}^{m}\sum_{\alpha=1}^{n}\frac{\partial u^\alpha}{\partial x_i}\frac{\partial u^\alpha}{\partial x_i}\Big]^{\frac{1}{2}}\Big\}\Big[\sum_{\alpha=1}^{n}(u^\alpha)^2\Big]^{\frac{1}{2}}\\
&=\Big\{\frac{4d_F^2}{\sqrt{m}}F(\frac{\sqrt{m}}{2}\|u^*h\|)F'(\frac{\sqrt{m}}{2}\|u^*h\|)\sqrt[4]{m}\Big[\sum_{i=1}^{m}(h(du(\frac{\partial}{\partial x_i}),du(\frac{\partial}{\partial x_i}))^2\Big]^{\frac{1}{4}}\\
&+(m-4l_F)F(\frac{|du|^2}{2})F'(\frac{|du|^2}{2})\Big[\sum_{i=1}^{m}h(du(\frac{\partial}{\partial x_i}),du(\frac{\partial}{\partial x_i}))\Big]^{\frac{1}{2}}\Big\}\Big[\sum_{\alpha,\beta=1}^{n}h_{\alpha\beta}u^\alpha u^\beta\Big]^{\frac{1}{2}}\\
&\le\Big\{\frac{4d_F^2}{\sqrt{m}}F(\frac{\sqrt{m}}{2}\|u^*h\|)F'(\frac{\sqrt{m}}{2}\|u^*h\|)\sqrt[4]{m}\Big[\sum_{i,j=1}^{m}(h(du(\frac{\partial}{\partial x_i}),du(\frac{\partial}{\partial x_j}))^2\Big]^{\frac{1}{4}}\\
&+(m-4l_F)F(\frac{|du|^2}{2})F'(\frac{|du|^2}{2})\Big[\sum_{i=1}^{m}h(du(\frac{\partial}{\partial x_i}),du(\frac{\partial}{\partial x_i}))\Big]^{\frac{1}{2}}\Big\}\Big[\sum_{\alpha,\beta=1}^{n}h_{\alpha\beta}u^\alpha u^\beta\Big]^{\frac{1}{2}}\\
&=\Big\{\frac{4d_F^2}{\sqrt{m}}F(\frac{\sqrt{m}}{2}\|u^*h\|)F'(\frac{\sqrt{m}}{2}\|u^*h\|)\sqrt[4]{m}\|u^*h\|^{\frac{1}{2}}\\
&+(m-4l_F)F(\frac{|du|^2}{2})F'(\frac{|du|^2}{2})(|du|^2)^{\frac{1}{2}}\Big\}\Big[\sum_{\alpha,\beta=1}^{n}h_{\alpha\beta}u^\alpha u^\beta\Big]^{\frac{1}{2}}\\
&\le\max\{2d_F,\ \sqrt{m-4l_F}\}\Big[F(\frac{\sqrt{m}}{2}\|u^*h\|)F'(\frac{\sqrt{m}}{2}\|u^*h\|)\Big]^{\frac{1}{2}}\Big[\sum_{\alpha,\beta=1}^{n}h_{\alpha\beta}u^\alpha u^\beta\Big]^{\frac{1}{2}}
\end{aligned}
$$

$$
\begin{aligned}
&\times\Big\{\Big[\frac{4d_F^2}{\sqrt{m}}F(\frac{\sqrt{m}}{2}\|u^*h\|)F'(\frac{\sqrt{m}}{2}\|u^*h\|)\|u^*h\|\Big]^{\frac{1}{2}}+\Big[(m-4l_F)F(\frac{|du|^2}{2})F'(\frac{|du|^2}{2})|du|^2\Big]^{\frac{1}{2}}\Big\}\\
&\le \sqrt{2}\max\{2d_F,\sqrt{m-4l_F}\}\Big[F(\frac{\sqrt{m}}{2}\|u^*h\|)F'(\frac{\sqrt{m}}{2}\|u^*h\|)\sum_{\alpha,\beta=1}^{n}h_{\alpha\beta}u^\alpha u^\beta\Big]^{\frac{1}{2}}\\
&\times\Big[\frac{4d_F^2}{\sqrt{m}}F(\frac{\sqrt{m}}{2}\|u^*h\|)F'(\frac{\sqrt{m}}{2}\|u^*h\|)\|u^*h\|+(m-4l_F)F(\frac{|du|^2}{2})F'(\frac{|du|^2}{2})|du|^2\Big]^{\frac{1}{2}}.
\end{aligned}
$$

Here we have used $|\nabla r|^2=1$ and the concave property of the function $f(x)=x^{\frac{1}{2}}$ for $x\ge 0$. So the following result holds

$$
\begin{aligned}
&\sum_{i,j,k,l,\alpha,\beta,\gamma,\delta}\frac{4d_F^2}{\sqrt{m}}\|u^*h\|^{-1}F(\frac{\sqrt{m}}{2}\|u^*h\|)F'(\frac{\sqrt{m}}{2}\|u^*h\|)g^{ik}g^{jl}h_{\alpha\beta}\frac{\partial u^\alpha}{\partial x_i}\frac{\partial r(x)}{\partial x_j}u^\beta h_{\gamma\delta}\frac{\partial u^\gamma}{\partial x_k}\frac{\partial u^\delta}{\partial x_l}\\
&+\sum_{i,j,\alpha,\beta}(m-4l_F)F(\frac{|du|^2}{2})F'(\frac{|du|^2}{2})g^{ij}h_{\alpha\beta}\frac{\partial u^\alpha}{\partial x_i}\frac{\partial r(x)}{\partial x_j}u^\beta\\
&\le \sqrt{2}\max\{2d_F,\sqrt{m-4l_F}\}\Big[F(\frac{\sqrt{m}}{2}\|u^*h\|)F'(\frac{\sqrt{m}}{2}\|u^*h\|)\sum_{\alpha,\beta=1}^{n}h_{\alpha\beta}u^\alpha u^\beta\Big]^{\frac{1}{2}}\\
&\times\Big[\frac{4d_F^2}{\sqrt{m}}F(\frac{\sqrt{m}}{2}\|u^*h\|)F'(\frac{\sqrt{m}}{2}\|u^*h\|)\|u^*h\|+(m-4l_F)F(\frac{|du|^2}{2})F'(\frac{|du|^2}{2})|du|^2\Big]^{\frac{1}{2}}.
\end{aligned}
$$

Integrating this inequality over $\partial B(R)$, then applying Hölder's inequality, we have

$$
\begin{aligned}
&\int_{\partial B(R)}\frac{4d_F^2}{\sqrt{m}}\|u^*h\|^{-1}F(\frac{\sqrt{m}}{2}\|u^*h\|)F'(\frac{\sqrt{m}}{2}\|u^*h\|)g^{ik}g^{jl}h_{\alpha\beta}\frac{\partial u^\alpha}{\partial x_i}\frac{\partial r(x)}{\partial x_j}u^\beta h_{\gamma\delta}\frac{\partial u^\gamma}{\partial x_k}\frac{\partial u^\delta}{\partial x_l}ds_g\\
&+\int_{\partial B(R)}(m-4l_F)F(\frac{|du|^2}{2})F'(\frac{|du|^2}{2})g^{ij}h_{\alpha\beta}\frac{\partial u^\alpha}{\partial x_i}\frac{\partial r(x)}{\partial x_j}u^\beta ds_g\\
&\le C_1\int_{\partial B(R)}\Big[\frac{4d_F^2}{\sqrt{m}}F(\frac{\sqrt{m}}{2}\|u^*h\|)F'(\frac{\sqrt{m}}{2}\|u^*h\|)\|u^*h\|+(m-4l_F)F(\frac{|du|^2}{2})F'(\frac{|du|^2}{2})|du|^2\Big]^{\frac{1}{2}}\\
&\qquad\times\Big[F(\frac{\sqrt{m}}{2}\|u^*h\|)F'(\frac{\sqrt{m}}{2}\|u^*h\|)\sum_{\alpha,\beta=1}^{n}h_{\alpha\beta}u^\alpha u^\beta\Big]^{\frac{1}{2}}ds_g\\
&\le C_1\Big\{\int_{\partial B(R)}\Big[\frac{4d_F^2}{\sqrt{m}}F(\frac{\sqrt{m}}{2}\|u^*h\|)F'(\frac{\sqrt{m}}{2}\|u^*h\|)\|u^*h\|+(m-4l_F)F(\frac{|du|^2}{2})F'(\frac{|du|^2}{2})|du|^2\Big]ds_g\Big\}^{\frac{1}{2}}\\
&\times\Big[\int_{\partial B(R)}F(\frac{\sqrt{m}}{2}\|u^*h\|)F'(\frac{\sqrt{m}}{2}\|u^*h\|)\sum_{\alpha,\beta=1}^{n}h_{\alpha\beta}u^\alpha u^\beta ds_g\Big]^{\frac{1}{2}},
\end{aligned}
\tag{25}
$$

where $C_1=2\sqrt{2}\max\{2d_F,\sqrt{m-4l_F}\}$ is a positive constant.

By (16), we have

$$
\begin{aligned}
&\int_{B(R)\setminus B(R_2)}\frac{4d_F^2}{\sqrt{m}}\|u^*h\|^{-1}F(\frac{\sqrt{m}}{2}\|u^*h\|)F'(\frac{\sqrt{m}}{2}\|u^*h\|)g^{ik}g^{jl}[2h_{\alpha\beta}+\frac{\partial h_{\alpha\beta}}{\partial u^\zeta}u^\zeta]\frac{\partial u^\alpha}{\partial x_i}\frac{\partial u^\beta}{\partial x_j}h_{\gamma\delta}\frac{\partial u^\gamma}{\partial x_k}\frac{\partial u^\delta}{\partial x_l}dv_g\\
&+\int_{B(R)\setminus B(R_2)}(m-4l_F)F(\frac{|du|^2}{2})F'(\frac{|du|^2}{2})g^{ij}[2h_{\alpha\beta}+\frac{\partial h_{\alpha\beta}}{\partial u^\zeta}u^\zeta]\frac{\partial u^\alpha}{\partial x_i}\frac{\partial u^\beta}{\partial x_j}dv_g+D(R_1)
\end{aligned}
$$

$$\begin{aligned}
&\geq \int_{B(R)\setminus B(R_2)} \frac{4d_F^2}{\sqrt{m}} \|u^*h\|^{-1} F(\frac{\sqrt{m}}{2}\|u^*h\|) F'(\frac{\sqrt{m}}{2}\|u^*h\|) g^{ik} g^{jl} h_{\alpha\beta} \frac{\partial u^\alpha}{\partial x_i} \frac{\partial u^\beta}{\partial x_j} h_{\gamma\delta} \frac{\partial u^\gamma}{\partial x_k} \frac{\partial u^\delta}{\partial x_l} dv_g \\
&\quad + \int_{B(R)\setminus B(R_2)} (m - 4l_F) F(\frac{|du|^2}{2}) F'(\frac{|du|^2}{2}) g^{ij} h_{\alpha\beta} \frac{\partial u^\alpha}{\partial x_i} \frac{\partial u^\beta}{\partial x_j} dv_g + D(R_1) \\
&= \int_{B(R)\setminus B(R_2)} \Big[\frac{4d_F^2}{\sqrt{m}} F(\frac{\sqrt{m}}{2}\|u^*h\|) F'(\frac{\sqrt{m}}{2}\|u^*h\|)\|u^*h\| + (m - 4l_F) F(\frac{|du|^2}{2}) F'(\frac{|du|^2}{2})|du|^2\Big] dv_g \\
&\quad + D(R_1).
\end{aligned} \tag{26}$$

By (24), (25) and (26), we have

$$\begin{aligned}
&\int_{B(R)\setminus B(R_2)} \Big[\frac{4d_F^2}{\sqrt{m}} F(\frac{\sqrt{m}}{2}\|u^*h\|) F'(\frac{\sqrt{m}}{2}\|u^*h\|)\|u^*h\| + (m - 4l_F) F(\frac{|du|^2}{2}) F'(\frac{|du|^2}{2})|du|^2\Big] dv_g \\
&\quad + D(R_1) \\
&\leq C_2 \Big\{ \int_{\partial B(R)} \Big[\frac{4d_F^2}{\sqrt{m}} F(\frac{\sqrt{m}}{2}\|u^*h\|) F'(\frac{\sqrt{m}}{2}\|u^*h\|)\|u^*h\| + (m - 4l_F) F(\frac{|du|^2}{2}) F'(\frac{|du|^2}{2})|du|^2\Big] ds_g \Big\}^{\frac{1}{2}} \\
&\quad \times \Big[\int_{\partial B(R)} F(\frac{\sqrt{m}}{2}\|u^*h\|) F'(\frac{\sqrt{m}}{2}\|u^*h\|) \sum_{\alpha,\beta=1}^{n} h_{\alpha\beta} u^\alpha u^\beta ds_g \Big]^{\frac{1}{2}},
\end{aligned} \tag{27}$$

where $C_2 = 2C_1$ is a positive constant.

Set

$$\begin{aligned}
Z(R) = &\int_{B(R)\setminus B(R_2)} \Big[\frac{4d_F^2}{\sqrt{m}} F(\frac{\sqrt{m}}{2}\|u^*h\|) F'(\frac{\sqrt{m}}{2}\|u^*h\|)\|u^*h\| \\
&+ (m - 4l_F) F(\frac{|du|^2}{2}) F'(\frac{|du|^2}{2})|du|^2\Big] dv_g + D(R_1), \quad for \quad R > R_2.
\end{aligned}$$

Then

$$Z'(R) = \int_{\partial B(R)} \Big[\frac{4d_F^2}{\sqrt{m}} F(\frac{\sqrt{m}}{2}\|u^*h\|) F'(\frac{\sqrt{m}}{2}\|u^*h\|)\|u^*h\| + (m - 4l_F) F(\frac{|du|^2}{2}) F'(\frac{|du|^2}{2})|du|^2\Big] ds_g. \tag{28}$$

Using (27) and (28), we have

$$Z(R) \leq C_2 [Z'(R)]^{\frac{1}{2}} \Big[\int_{\partial B(R)} F(\frac{\sqrt{m}}{2}\|u^*h\|) F'(\frac{\sqrt{m}}{2}\|u^*h\|) \sum_{\alpha,\beta=1}^{n} h_{\alpha\beta} u^\alpha u^\beta ds_g \Big]^{\frac{1}{2}}. \tag{29}$$

On the other hand, we have the following estimate.

$$\begin{aligned}
&Z(R) - D(R_1) \\
&= \int_{B(R)\setminus B(R_2)} \Big[\frac{4d_F^2}{\sqrt{m}} F(\frac{\sqrt{m}}{2}\|u^*h\|) F'(\frac{\sqrt{m}}{2}\|u^*h\|)\|u^*h\| + (m - 4l_F) F(\frac{|du|^2}{2}) F'(\frac{|du|^2}{2})|du|^2\Big] dv_g \\
&\geq 2l_F \int_{B(R)\setminus B(R_2)} \Big[\frac{4d_F^2}{m} (F(\frac{\sqrt{m}}{2}\|u^*h\|))^2 + (m - 4l_F)(F(\frac{|du|^2}{2}))^2\Big] dv_g.
\end{aligned} \tag{30}$$

Since $l_F > 0$ and $E^R_{\Phi_{S,F}}(u) \to +\infty$ as $R \to +\infty$, there exists an $R_3 \geq R_2$ such that $Z(R) > 0$ for $R_3 > 0$. Thus from (29), we have

$$(Z(R))^2 \leq (C_2)^2 Z'(R) \int_{\partial B(R)} F(\frac{\sqrt{m}}{2}\|u^*h\|) F'(\frac{\sqrt{m}}{2}\|u^*h\|) \sum_{\alpha,\beta=1}^{n} h_{\alpha\beta} u^\alpha u^\beta ds_g \quad for \quad R > R_3.$$

If we denote

$$M(R)=\int_{\partial B(R)}F(\frac{\sqrt{m}}{2}\|u^*h\|)F'(\frac{\sqrt{m}}{2}\|u^*h\|)\sum_{\alpha,\beta=1}^{n}h_{\alpha\beta}u^{\alpha}u^{\beta}ds_g,$$

then for any $R_4>R>R_3$, it follows that

$$\int_R^{R_4}(-\frac{1}{Z(r)})'dr\geq\frac{1}{(C_2)^2}\int_R^{R_4}\frac{1}{M(r)}dr.$$

Letting $R_4\to+\infty$ and noticing that $Z(R)>0$, we have

$$\frac{1}{Z(R)}\geq\frac{1}{(C_2)^2}\int_R^{+\infty}\frac{1}{M(r)}dr.$$

Thus

$$Z(R)\leq\frac{C_2^2}{\int_R^{\infty}\frac{1}{M(r)}dr},\quad for\quad R>R_3. \tag{31}$$

Note that $F(\frac{\sqrt{m}}{2}\|u^*h\|)F'(\frac{\sqrt{m}}{2}\|u^*h\|)$ is uniformly bounded. According to the fact that $u(x)\to0$ as $r(x)\to+\infty$, we get

$$M(R)\leq\widetilde{c}\eta(R)\cdot vol(\partial B(R))$$

where $\widetilde{c}$ is a constant only depending on u and $\eta(R)$ is chosen in such a way that
(i) $\eta(R)$ is nonincreasing on $(R_3,+\infty)$ and $\eta(R)\to0$ as $R\to+\infty$;
(ii) $\eta(R)\geq\max_{r(x)=R}\sum_{\alpha,\beta=1}^{n}(h_{\alpha\beta}u^{\alpha}u^{\beta})^2$.

Then by the assumption, we derive

$$\int_R^{\infty}\frac{1}{M(r)}dr\geq\frac{1}{\widetilde{c}\eta(R)}\int_R^{\infty}\frac{1}{vol(\partial B(r))}dr\geq\frac{1}{\widetilde{c}C_0\eta(R)}R^{-\Lambda}. \tag{32}$$

Hence from (31) and (32), we have

$$Z(R)\leq C\eta(R)R^{\Lambda}\quad for\quad R\geq R_3, \tag{33}$$

where $C=C_0(C_2)^2\widetilde{c}$ is a positive constant.
From (30) and (33), we obtain

$$\begin{aligned}E^R_{\Phi_{S,F}}(u)&=\int_{B(R)}\Big[(m-4l_F)\Big(F(\frac{|du|^2}{2})\Big)^2+\frac{4d_F^2}{m}\Big(F(\frac{\sqrt{m}}{2}\|u^*h\|)\Big)^2\Big]dv_g\\&\leq\frac{1}{2l_F}(Z(R)-D(R_1))+\int_{B(R_2)}\Big[(m-4l_F)\Big(F(\frac{|du|^2}{2})\Big)^2+\frac{4d_F^2}{m}\Big(F(\frac{\sqrt{m}}{2}\|u^*h\|)\Big)^2\Big]dv_g\\&\leq\frac{C}{2l_F}\eta(R)R^{\Lambda}-\frac{D(R_1)}{2l_F}+\int_{B(R_2)}\Big[(m-4l_F)\Big(F(\frac{|du|^2}{2})\Big)^2+\frac{4d_F^2}{m}\Big(F(\frac{\sqrt{m}}{2}\|u^*h\|)\Big)^2\Big]dv_g\\&=C(\frac{\eta(R)}{2l_F}+\frac{c(u)}{R^{\Lambda}})R^{\Lambda}\end{aligned}$$

for $\quad R>R_3=R_0$. □

By Proposition 5.1 and Proposition 5.2, the following vanishing theorem for $\Phi_{S,F}$-harmonic map is established.

Theorem 5.3. *Let $u : (M^m, g) \to (N^n, h)$ be a $\Phi_{S,F}$-harmonic map from a Riemannian manifold M with a pole x_0. Suppose $l_F > 0$ and $F(\frac{\sqrt{m}}{2}\|u^*h\|)F'(\frac{\sqrt{m}}{2}\|u^*h\|) < +\infty$ .Asumme that $r(x)$ satisfies (9) and $(\int_R^{+\infty} \frac{1}{vol(\partial B_r)}dr)^{-1} \le C_0R^\Lambda$ for R large enough. If $u(x) \to p_0 \in N$ as $r(x) \to +\infty$, then u is a constant map. Furthermore, we get $S_F = 0$.*

By using Theorem 5.3 and Lemma 4.5, we can apply the above vanishing theorem to some concrete pinched manifolds.

Theorem 5.4. *Let $u : (M^m, g) \to (N^n, h)$ be a $\Phi_{S,F}$-harmonic map from a Riemannian manifold with a pole x_0. Suppose $l_F > 0$, $F(\frac{\sqrt{m}}{2}\|u^*h\|)F'(\frac{\sqrt{m}}{2}\|u^*h\|) < +\infty$ and $(\int_R^{\infty} \frac{1}{vol(\partial B_r)}dr)^{-1} \le CR^\Lambda$ for R large enough. Assume that the radial curvature K_r of M satisfies one of the following conditions:*
(i) $-\alpha^2 \le K_r \le -\beta^2$ with $\alpha > 0$, $\beta > 0$ and $(m-1)\beta - 4d_F\alpha > 0$;
(ii) $-\frac{A}{(1+r^2)^{1+\epsilon}} \le K_r \le \frac{B}{(1+r^2)^{1+\epsilon}}$ with $\epsilon > 0$, $A \ge 0$, $0 \le B < 2\epsilon$ and $1 + (m-1)(1-\frac{B}{2\epsilon}) - 4d_Fe^{\frac{A}{2\epsilon}} > 0$;
(iii) $-\frac{A(A-1)}{r^2} \le K_r \le -\frac{B(B-1)}{r^2}$ with $A \ge B \ge 1$ and $1 + (m-1)A - 4d_FB > 0$;
(iv) $-\frac{a^2}{1+r^2} \le K_r \le \frac{b^2}{1+r^2}$ with $b^2 \in [0, \frac{1}{4}]$ and $1 + (m-1)\frac{1+\sqrt{1-4b^2}}{2} - 2d_F(1+\sqrt{1+4a^2}) > 0$.
If $u(x) \to p_0 \in N$ as $r(x) \to +\infty$, where

$$\Lambda = \begin{cases} m - 4d_F\frac{\alpha}{\beta}, & if \quad K_r \quad satisfies \quad (i) \\ 1 + (m-1)(1-\frac{B}{2\epsilon}) - 4d_Fe^{\frac{A}{2\epsilon}}, & if \quad K_r \quad satisfies \quad (ii) \\ 1 + (m-1)A - 4d_FB, & if \quad K_r \quad satisfies \quad (iii) \\ 1 + (m-1)\frac{1+\sqrt{1-4b^2}}{2} - 2d_F(1+\sqrt{1+4a^2}), & if \quad K_r \quad satisfies \quad (iv) \end{cases}$$

then u is a constant map. So we have $S_F = 0$.

In [31], the authors give the volume growth estimates under Ricci curvature conditions. Hence, applying the results to the following cases, the right side of $(\int_R^{+\infty} \frac{1}{vol(\partial B_r)}dr)^{-1} \le C_0R^\Lambda$ can be expressed as a polynomial.

Corollary 5.5. *Let $u : (M^m, g) \to (N^n, h)$ be a $\Phi_{S,F}$-harmonic map from a Riemannian manifold with a pole x_0. Suppose $l_F > 0$ and $F(\frac{\sqrt{m}}{2}\|u^*h\|)F'(\frac{\sqrt{m}}{2}\|u^*h\|) < +\infty$. The radial curvature K_r of M satisfies one of the following two conditions:*
(i) $-\frac{A}{(1+r^2)^{1+\epsilon}} \le K_r \le \frac{B}{(1+r^2)^{1+\epsilon}}$ with $\epsilon > 0$, $A \ge 0$, $0 \le B < 2\epsilon$ and $1 + (m-1)(1-\frac{B}{2\epsilon}) - 2d_Fe^{\frac{A}{2\epsilon}} \ge m-2$;
(ii) $-\frac{a^2}{1+r^2} \le K_r \le \frac{b^2}{1+r^2}$ with $b^2 \in [0, \frac{1}{4}]$ and $1 + \frac{(m-1)(1+\sqrt{1-4b^2})}{2} - d_F(1+\sqrt{1+4a^2}) \ge \frac{(m-1)(1+\sqrt{1+4a^2})}{2} - 1$.
If $u(x) \to p_0 \in N$ as $r(x) \to +\infty$, then u is a constant map. So we have $S_F = 0$.

Proof. (i) From the condition of K_r, we have

$$Ric_g(x) \ge -\frac{(m-1)A}{(1+r^2)^{1+\epsilon}}, \forall x \in M.$$

By a direct calculation, we have

$$\int_0^\infty \frac{Ar}{(1+r^2)^{1+\epsilon}}dr = \frac{A}{2\epsilon}.$$

Using the volume comparison theorem (cf. Corollary 2.17 in [31]), we obtain

$$vol(\partial B(R)) \le \omega_m e^{\frac{(m-1)A}{2\epsilon}} R^{m-1}$$

where ω_m is the $(m-1)$-volume of the unit sphere in $\mathbb{R}^N$, and thus

$$(\int_R^\infty \frac{1}{vol(\partial B_r)}dr)^{-1} \le (n-2)\omega_m e^{\frac{(m-1)A}{2\epsilon}} R^{m-2} \quad for \quad r > R_0.$$

(ii) It follows that

$$Ric_g(x) \ge -\frac{(m-1)a^2}{(1+r(x)^2)}, \forall x \in M.$$

By the volume comparison theorem (cf. Corollary 2.17 in [31]), we have

$$vol(\partial B(R)) \leq CR^{(m-1)A'}$$

where $A' = \frac{1+\sqrt{1+4a^2}}{2}$. So

$$\left(\int_R^\infty \frac{1}{vol(\partial B_r)}dr\right)^{-1} \leq CR^{(m-1)A'-1} \quad for \quad r > R_0.$$

Therefore using Theorem 4.2, the conclusion is immediately proved. □

6. Constant Dirichlet boundary-value problems

In this section, we deal with constant Dirichlet boundary-value problems for $\Phi_{S,F}$-harmonic maps. As in [14], we introduce starlike domains with C^1-boundaries which generalize C^1-convex domains.

Definition 6.1. *A bounded domain $D \subset (M, g)$ with C^1-boundary ∂D is called starlike if there exists an interior point $x_0 \in D$ such that*

$$\langle \frac{\partial}{\partial r_{x_0}}, \nu \rangle \Big|_{\partial D} \geq 0,$$

where ν is the inner normal to ∂D, and for any $x \in D \setminus \{x_0\} \cup \partial D$, $\frac{\partial}{\partial r_{x_0}}(x)$ is the unit vector tangent vector tangent to the unique geodesic joining x_0 to x and pointing away from x_0.

Theorem 6.2. *Let $u : (M^m, g) \to (N^n, h)$ be a C^2 map from a Riemannian manifold M with a pole x_0 and $D \subset M$ be a starlike domain. Assume that $l_F \geq \frac{1}{4}$ and $u\big|_{\partial D}$ is constant. If u is a $\Phi_{S,F}$-harmonic map, then u is constant on D provided one of the following conditions is satisfied:*
(i) $-\alpha^2 \leq K_r \leq -\beta^2$ with $\alpha > 0$, $\beta > 0$ and $(m-1)\beta - 4d_F\alpha > 0$;
(ii) $-\frac{A}{(1+r^2)^{1+\epsilon}} \leq K_r \leq \frac{B}{(1+r^2)^{1+\epsilon}}$ with $\epsilon > 0$, $A \geq 0$, $0 \leq B < 2\epsilon$ and $1 + (m-1)(1 - \frac{B}{2\epsilon}) - 4d_F e^{\frac{A}{2\epsilon}} > 0$;
(iii) $-\frac{A(A-1)}{r^2} \leq K_r \leq -\frac{B(B-1)}{r^2}$ with $A \geq B \geq 1$ and $1 + (m-1)A - 4d_F B > 0$;
(iv) $-\frac{a^2}{1+r^2} \leq K_r \leq \frac{b^2}{1+r^2}$ with $b^2 \in [0, \frac{1}{4}]$ and $1 + (m-1)\frac{1+\sqrt{1-4b^2}}{2} - 2d_F(1 + \sqrt{1+4a^2}) > 0$.

Proof. Set $X = r\frac{\partial}{\partial r}$, where $r = r_{x_0}$. From the proof of Theorem 4.1 and Lemma 4.3, we get

$$\langle S_{\Phi_{S,F}}, \frac{1}{2}L_{r\frac{\partial}{\partial r}}g \rangle \geq \Lambda[(m - 4l_F)(F(\frac{|du|^2}{2}))^2 + \frac{4d_F^2}{m}(F(\frac{\sqrt{m}}{2}\|u^*h\|))^2], \tag{34}$$

where

$$\Lambda = \begin{cases} m - 4d_F\frac{\alpha}{\beta}, & if \quad K_r \quad satisfies \quad (i) \\ 1 + (m-1)(1 - \frac{B}{2\epsilon}) - 4d_F e^{\frac{A}{2\epsilon}}, & if \quad K_r \quad satisfies \quad (ii) \\ 1 + (m-1)A - 4d_F B, & if \quad K_r \quad satisfies \quad (iii) \\ 1 + (m-1)\frac{1+\sqrt{1-4b^2}}{2} - 2d_F(1 + \sqrt{1+4a^2}), & if \quad K_r \quad satisfies \quad (iv) \end{cases}$$

Let $x \in \partial D$ and choose a local orthonormal frame field $\{e_1, \cdots, e_{m-1}, \nu\}$ on T_xM such that $\{e_1, \cdots, e_{m-1}\}$ is a orthonormal frame field on $T_x\partial D$. Since $u\big|_{\partial D}$ is constant, we get $du(e_i) = 0, i = 1, \cdots, m-1$ and

$du(\frac{\partial}{\partial r}) = \langle \frac{\partial}{\partial r}, \nu \rangle du(\nu)$. Hence at x

$$\begin{aligned} S_{\Phi_{S,F}}(r\frac{\partial}{\partial r}, \nu) &= r\Big\{[(m-4l_F)(F(\frac{|du|^2}{2}))^2 + \frac{4d_F^2}{m}(F(\frac{\sqrt{m}}{2}\|u^*h\|))^2]\langle \frac{\partial}{\partial r}, \nu\rangle_g - h(du(\frac{\partial}{\partial r}), \sigma_{F,u}(\nu))\Big\} \\ &= r\Big\{[(m-4l_F)(F(\frac{|du|^2}{2}))^2 + \frac{4d_F^2}{m}(F(\frac{\sqrt{m}}{2}\|u^*h\|))^2]\langle \frac{\partial}{\partial r}, \nu\rangle \\ &\quad - h(du(\frac{\partial}{\partial r}), \frac{8d_F^2}{\sqrt{m}}(\|u^*h\|)^{-1}F(\frac{\sqrt{m}}{2}\|u^*h\|)F'(\frac{\sqrt{m}}{2}\|u^*h\|)h(du(\nu), du(\nu))du(\nu) \\ &\qquad + 2(m-4l_F)F(\frac{|du|^2}{2})F'(\frac{|du|^2}{2})du(\nu))\Big\} \\ &= r\Big\{[(m-4l_F)(F(\frac{|du|^2}{2}))^2 + \frac{4d_F^2}{m}(F(\frac{\sqrt{m}}{2}\|u^*h\|))^2]\langle \frac{\partial}{\partial r}, \nu\rangle \\ &\quad - \langle \frac{\partial}{\partial r}, \nu\rangle[\frac{8d_F^2}{\sqrt{m}}F(\frac{\sqrt{m}}{2}\|u^*h\|)F'(\frac{\sqrt{m}}{2}\|u^*h\|)\|u^*h\| + 2(m-4l_F)F(\frac{|du|^2}{2})F'(\frac{|du|^2}{2})|du|^2] \\ &\le r\Big\{[(m-4l_F)(F(\frac{|du|^2}{2}))^2 + \frac{4d_F^2}{m}(F(\frac{\sqrt{m}}{2}\|u^*h\|))^2]\langle \frac{\partial}{\partial r}, \nu\rangle \\ &\quad - 4l_F[\frac{4d_F^2}{m}(F(\frac{\sqrt{m}}{2}\|u^*h\|))^2 + (m-4l_F)(F(\frac{|du|^2}{2}))^2]\langle \frac{\partial}{\partial r}, \nu\rangle\Big\}. \\ &\le r\langle \frac{\partial}{\partial r}, \nu\rangle(1-4l_F)[(m-4l_F)(F(\frac{|du|^2}{2}))^2 + \frac{4d_F^2}{m}(F(\frac{\sqrt{m}}{2}\|u^*h\|))^2]. \end{aligned} \tag{35}$$

Since D is starlike, by (8) and (35)

$$\int_D \langle S_{\Phi_{S,F}}, \frac{1}{2}L_{r\frac{\partial}{\partial r}} g\rangle dv_g \le 0. \tag{36}$$

From (34) and (36), we have

$$\int_D [(m-4l_F)(F(\frac{|du|^2}{2}))^2 + \frac{4d_F^2}{m}(F(\frac{\sqrt{m}}{2}\|u^*h\|))^2]dv_g = 0.$$

Therefore u is constant on D. □

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