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On some properties for a class of deformed trace functions on finite von Neumann algebras

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Abstract. The purpose of this note is to establish some properties for a class of deformed trace functions for operators in finite von Neumann algebras. Moreover, some properties for functions related to generalized singular values are also included. As an application, we extend the results of Hansen [6] to the case of finite von Neumann algebras.

1. Introduction

The convexity or concavity of certain trace functions for the deformed logarithmic and exponential functions has been widely used in statistical mechanics, quantum information theory and operator theory ([5, 7, 13, 15]). Tsallis [16] generalized in 1988 the standard Bolzmann-Gibbs entropy to a non-extensive quantity S_q depending on a parameter q. In the quantum version, the Tsallis entropy may be written in the form

$$S_q(\rho) = -Tr\rho \log_q(\rho),$$

where the deformed logarithm $\log_q$ is given by

$$\log_q x = \begin{cases} \frac{x^{q-1}-1}{q-1}, & q > 1, \\ \log x, & q = 1, \end{cases}$$

for $x > 0$. It has the property that $S_q(\rho) \to S(\rho)$ for $q \to 1$, where $S(\rho) = -Tr\rho \log \rho$ is the von Neumann entropy. The deformed logarithm is also denoted by the p-logarithm. The range of the p-logarithm is given by the intervals

$$(-(p-1)^{-1}, \infty) \quad \text{for } p > 1;$$
$$(-\infty, -(p-1)^{-1}) \quad \text{for } p < 1;$$
$$(-\infty, \infty) \quad \text{for } p = 1.$$

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The inverse function $\exp_p$ (denoted by the p-exponential) is always positive and given by

$$\exp_p(x) = \begin{cases} (x(p-1)+1)^{\frac{1}{p-1}} & \text{for } p > 1 \text{ and } x > -(p-1)^{-1}; \\ (x(p-1)+1)^{\frac{1}{p-1}} & \text{for } p < 1 \text{ and } x < -(p-1)^{-1}; \\ \exp x & \text{for } p = 1 \text{ and } x \in \mathbb{R}. \end{cases}$$

The p-logarithm and the p-exponential functions converge, respectively, to the logarithmic and the exponential functions for $p \to 1$.

The collection of all $n \times n$ complex matrices is denoted by $\mathbb{M}_n(\mathbb{C})$. Recall that $A \in \mathbb{M}_n(\mathbb{C})$ is called positive semi-definite if all its eigenvalues are non-negative. The collection of all positive semidefinite $n \times n$ complex matrices is denoted by $\mathbb{M}_n(\mathbb{C})^+$. In [6], Hansen studied the concavity/convexity of the following trace function on $\mathbb{M}_n(\mathbb{C})^+$:

$$\phi^A_{p,q}(X) = \mathrm{Tr}[\exp_p(A + B^* \log_p(X)B)^q]. \tag{1}$$

The trace function given in (1) can be regarded as a generalization of the trace function

$$X \to \mathrm{Tr}\exp(A + \log X). \tag{2}$$

The well-known concavity theorem by Lieb [9, Theorem 6] states that the map (2) for a fixed self-adjoint matrix A, is concave in positive definite matrices. This map is the basis for the proof of strong subadditivity of the quantum mechanical entropy [10, 15]. On the other hand, Hansen-Liang-Shi [7] and Shi-Hansen [15] gave some generalizations of Peierls-Bogolyubov's and Golden-Thompson's trace inequality in terms of the so-called deformed exponential and logarithmic functions. In particular, they used these results to improve previously known lower bounds for the Tsallis relative entropy. More information about trace functions for deformed exponential and logarithmic functions can be found in [6, 7, 13, 15].

A von Neumann algebra $\mathcal{M}$ equipped with a faithful normal finite trace τ serves as the noncommutative analog of a bounded measurable space endowed with a finite measure. he matrix algebra and the L^∞-space are both two classical examples of von Neumann algebras. Note that the properties of a class of deformed trace functions are very useful in the theories of statistical mechanics, quantum information theory, and other fields. Therefore, it is highly significant to take into account the properties of a class of deformed trace functions on a finite von Neumann algebra.

To begin with, in this paper, we will investigate the geometric properties for a class of trace functions on finite von Neumann algebras. Some properties for functions related to generalized singular values are also included. The main theme of the paper is to extend earlier results of Hansen [6] and Shi-Hansen [15] to the case of finite von Neumann algebras.

This paper is organized as follows: Section 2 introduces the key notions and concepts of this work. In section 3 we provide some variational representations of trace functions. In Section 3, we adapt the techniques from [1, 6, 15] to extend earlier results of Hansen [6] and Shi-Hansen [15] to the case of finite von Neumann algebras.

2. Preliminaries

In this section, we recall some notions of the theory of noncommutative integration. In what follows, $\mathcal{H}$ is a separable Hilbert space over the field $\mathbb{C}$, and $\mathcal{B}(\mathcal{H})$ is the $*$-algebra of all bounded linear operators on $\mathcal{H}$ equipped with the uniform norm $\|\cdot\|$. Additionally, $\mathbb{I}$ represents the identity operator on $\mathcal{H}$. Let $\mathcal{M}$ be a $*$-subalgebra of $\mathcal{B}(\mathcal{H})$ containing the identity operator $\mathbb{I}$. Then $\mathcal{M}$ is called a von Neumann algebra if $\mathcal{M}$ is weak* operator closed. Let $\mathcal{M}^+$ denote the positive part of $\mathcal{M}$. We recall that a weight on $\mathcal{M}$ is a map $\tau : \mathcal{M}^+ \to [0, \infty]$ satisfying

(i) $\tau(x + y) = \tau(x) + \tau(y)$, for all $x, y \in \mathcal{M}^+$;
(ii) $\tau(\alpha x) = \alpha\tau(x)$ for all $x \in \mathcal{M}^+$ and $\alpha \in [0, \infty)$, with the convention $0 \cdot \infty = 0$.

The weight τ is called faithful if $\tau(x^*x) = 0$ implies $x = 0$, normal if $x_i \uparrow_i x$ in $\mathcal{M}^+$ implies that $0 \leq \tau(x_i) \uparrow_i \tau(x)$, and tracial if $\tau(x^*x) = \tau(xx^*)$ for all $x \in \mathcal{M}$. It is also customary to say trace instead of tracial weight. A trace τ is called finite if $\tau(\mathbb{I}) < \infty$. A von Neumann algebra $\mathcal{M}$ is called finite if the family formed of the finite normal traces separates the points of $\mathcal{M}$. Since $\mathcal{H}$ is a separable Hilbert space, $\mathcal{M}$ is finite if and only if it admits a faithful normal finite trace. For details on von Neumann algebra theory, the reader is referred to [14].

2.1. Noncommutative L^p spaces

The self-adjoint part of $\mathcal{M}$, denoted by $\mathcal{M}^{sa}$, is a partially ordered vector space under the ordering $x \geq 0$(resp. $x > 0$) defined by $\langle x\xi, \xi\rangle \geq 0, \xi \in \mathcal{H}$ (resp. $\langle x\xi, \xi\rangle > 0, \xi \in \mathcal{H}$). We denote by $\mathcal{M}^{++}$ the collection of all positive and invertible operators in $\mathcal{M}$. A closed densely defined linear operator x on $\mathcal{H}$ with domain $D(x)$ is said to be affiliated with $\mathcal{M}$ if and only if $xu = ux$ for any unitary operator u which belongs to the commutant of $\mathcal{M}$. When x is affiliated with $\mathcal{M}$, x is said to be τ-measurable if for every $\varepsilon > 0$ there exists a projection operator $e \in \mathcal{M}$ such that $e(\mathcal{H}) \subseteq D(x)$ and $\tau(e^\perp) < \varepsilon$, where $e^\perp = 1 - e$. The collection of all measurable operators with respect to $\mathcal{M}$ is denoted by $L_0(\mathcal{M})$, which is a unital $*$-algebra with respect to strong sums and products, denoted simply by $x + y$ and xy for all $x, y \in L_0(\mathcal{M})$. Since $\mathcal{M}$ is finite, the set of operators which are affiliated with $\mathcal{M}$ is equivalent to the set $L_0(\mathcal{M})$.

For $0 < p < \infty$, a noncommutative L^p space $L^p(\mathcal{M})$ is defined by

$$[L^p(\mathcal{M}) := \{x \in L_0(\mathcal{M}); \|x\|_p := [\tau(|x|^p)]^{\frac{1}{p}} < \infty\}.$$

It is well known that $L^p(\mathcal{M})$ is a Banach space under $\|\cdot\|_p$ when $1 \leq p < \infty$ and it is a quasi-Banach space when $0 < p < 1$. As usual, we set $L^\infty(\mathcal{M}) := \mathcal{M}$ equipped with the operator norm. Since $\mathcal{M}$ is finite, it follows that $\mathcal{M} \subseteq L^p(\mathcal{M})$ and $\mathcal{M}$ is dense in $L^p(\mathcal{M}), 0 < p < \infty$. We refer readers to the survey papers [3, 14] for information on non-commutative L^p spaces.

2.2. Generalized singular value function

Let $x \in L_0(\mathcal{M})$ and $t > 0$. The generalized singular value function $\mu_\cdot(x)$ of x is defined by

$$\mu_t(x) = \inf\{\|xe\| : e \text{ is a projection in } \mathcal{M} \text{ with } \tau(e^\perp) \leq t\}.$$

We denote simply by $\mu_\cdot(x)$ the function $t \to \mu_t(x)$. The generalized singular number function $t \to \mu_t(x)$ is decreasing right-continuous. See [3, 14] for basic properties and detailed information for $\mu_t(x)$.

To enhance the reader's convenience, we provide a summary of properties for $\mu(\cdot)$ without including their proofs.

Proposition 2.1. *([3]) Let $x, y \in L_0(\mathcal{M})$.*

(i) $\mu_\cdot(|x|) = \mu_\cdot(x) = \mu_\cdot(x^*)$ *and* $\mu_\cdot(\alpha x) = |\alpha|\mu_\cdot(x)$ *for* $\alpha \in \mathbb{C}$.

(ii) *Let f be a bounded continuous increasing function on* $[0, \infty)$ *with* $f(0) = 0$. *Then* $\mu_\cdot(f(x)) = f(\mu_\cdot(x))$ *and* $\tau(f(x)) = \int_0^{\tau(\mathbb{I})} f(\mu_t(x))dt$.

(iii) *If* $0 \leq x \leq y$, *then* $\mu_\cdot(x) \leq \mu_\cdot(y)$.

In what follows, we will keep all previous notations throughout the paper. Unless stated otherwise, $\mathcal{M}$ will always denote a finite von Neumann algebra acting on a separable Hilbert space $\mathcal{H}$, with a normal faithful finite tracial state τ.

3. Variational expressions

An important tool in our investigation is taken from convex analysis. These techniques are used in engineering, automatic control, signal processing, resourceal location, portfolio theory, and numerous other fields. We, in particular, use that partial minimisation of a convex function is convex. This technique was successfully applied by Carlen and Lieb in the investigation of trace functions.

In this section, we investigate the geometric properties for a class of trace functions expressed in terms of the deformed logarithmic and exponential functions. By employing a proof technique similar to [6, Lemma 1.2] and basic computations, we can establish this useful lemma. Therefore, the proof process will not be elaborated here.

Lemma 3.1. *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two finite von Neumann algebras and let $f : \mathcal{M}_1^+ \times \mathcal{M}_2^+ \to \mathbb{R}$ be a function of two variables. For $y \in \mathcal{M}^+$, we write*

$$g(y) := \inf_{x \in \mathcal{M}_1^+} f(x, y) \ \text{ and } \ h(y) := \sup_{x \in \mathcal{M}_2^+} f(x, y).$$

The following assertions are valid:

(i) *If $f(x, y)$ is jointly convex, then g is convex;*
(ii) *If $f(x, y)$ is convex in the second variable, then h is convex;*
(iii) *If $f(x, y)$ is jointly concave, then h is concave;*
(iv) *If $f(x, y)$ is concave in the second variable, then g is concave.*

To arrive at min-max theorems related to generalized singular, it is necessary to first employ the following lemma.

Proposition 3.2. *Let $\mathcal{M}$ be a finite von Neumann algebra and $x, y \in \mathcal{M}^{++}$. Then*

$$\int_0^s \mu_t(x^p y^{1-p})dt \le p \int_0^s \mu_t(x)dt + (1-p) \int_0^s \mu_t(y)dt, \quad 0 \le p \le 1,$$

and

$$\int_0^s \mu_t(x^p y^{1-p})dt \ge p \int_0^s \mu_t(x)dt + (1-p) \int_0^s \mu_t(y)dt, \quad p \le 0,\ p \ge 1.$$

Proof. Let $0 \le p \le 1$. From [11, Theorem 2.3], we obtain

$$\begin{aligned} \int_0^s \mu_t(x^p y^{1-p})dt &\le \int_0^s \mu_t(px + (1-p)y)dt \\ &\le \int_0^s \mu_t(px)dt + \int_0^s \mu_t((1-p)y)dt \\ &\le p \int_0^s \mu_t(x)dt + (1-p) \int_0^s \mu_t(y)dt. \end{aligned}$$

For $p \ge 1$, it follows from [11, Theorem 3.6] that

$$\int_0^s p\mu_t(x) + (1-p)\mu_t(y)dt \le \int_0^s \mu_t(x^p y^{1-p})dt.$$

Finally, when $p < 0$, then $1 - p > 1$. Since $x > 0,\ y > 0$, $\mu_t(x^p y^{1-p}) = \mu_t(y^{1-p} x^p)$, so we can obtain

$$\int_0^s p\mu_t(x) + (1-p)\mu_t(y)dt = \int_0^s (1-p)\mu_t(y) + [1-(1-p)]\mu_t(x)dt \le \int_0^s \mu_t(x^p y^{1-p})dt. \tag{3}$$

□

Remark 3.3. *From Proposition 3.2, we also get the following conclusion: Let $x, y \in \mathcal{M}^{++}$. Then*

$$\tau(x^p y^{1-p}) \leq p\tau(x) + (1-p)\tau(y), \quad p \in [0,1],$$

and

$$\tau(x^p y^{1-p}) \geq p\tau(x) + (1-p)\tau(y), \quad p \in (-\infty, 0] \text{ or } p \in [1, \infty).$$

Now, we are ready to showcase the validity of the min-max theorems for positive invertible operators in finite von Neumann algebras.

Lemma 3.4. *For $x, y \in \mathcal{M}^{++}$ and $s > 0$, we have*

$$\int_0^s \mu_t(y)dt = \begin{cases} \sup\limits_{x>0} \left\{ \int_0^s \mu_t(x)dt - \frac{\int_0^s \mu_t(x)dt - \int_0^s \mu_t(x^p y^{1-p})dt}{1-p} \right\}, & 0 \leq p < 1 \text{ or } 1 < p < \infty, \\ \inf\limits_{x>0} \left\{ \int_0^s \mu_t(x)dt - \frac{\int_0^s \mu_t(x)dt - \int_0^s \mu_t(x^p y^{1-p})dt}{1-p} \right\}, & p \in (-\infty, 0). \end{cases}$$

Proof. For $x, y \in \mathcal{M}^{++}$, from Proposition 3.2 we have

$$\int_0^s \mu_t(y)dt \geq \int_0^s \mu_t(x)dt - \frac{\int_0^s \mu_t(x)dt - \int_0^s \mu_t(x^p y^{1-p})dt}{1-p}, \quad 0 \leq p < 1 \text{ or } 1 < p < \infty,$$

and

$$\int_0^s \mu_t(y)dt \leq \int_0^s \mu_t(x)dt - \frac{\int_0^s \mu_t(x)dt - \int_0^s \mu_t(x^p y^{1-p})dt}{1-p}, \quad p \in (-\infty, 0).$$

For $x = y$, the above inequalities become equalities, and hence

$$\int_0^s \mu_t(y)dt = \begin{cases} \sup\limits_{x>0} \left\{ \int_0^s \mu_t(x)dt - \frac{\int_0^s \mu_t(x)dt - \int_0^s \mu_t(x^p y^{1-p})dt}{1-p} \right\}, & 0 \leq p < 1 \text{ or } 1 < p < \infty, \\ \inf\limits_{x>0} \left\{ \int_0^s \mu_t(x)dt - \frac{\int_0^s \mu_t(x)dt - \int_0^s \mu_t(x^p y^{1-p})dt}{1-p} \right\}, & p \in (-\infty, 0). \end{cases}$$

□

Remark 3.5. *From Lemma 3.4, we can obtain the following conclusion: For $x, y \in \mathcal{M}^{++}$, setting $q = 2 - p$, we have*

$$\tau(y) = \begin{cases} \sup\limits_{x>0} \left\{ \tau(x) - \tau\left(x^{2-q}(\log_q x - \log_q y)\right) \right\}, & q \leq 2, \\ \inf\limits_{x>0} \left\{ \tau(x) - \tau\left(x^{2-q}(\log_q x - \log_q y)\right) \right\}, & q > 2. \end{cases}$$

For $q = 1$, the assertion reduces to the identity

$$\tau(y) = \sup_{x>0} \left\{ \tau(x) - \tau\left(x^{2-q}(\log_q x - \log_q y)\right) \right\},$$

which entails the inequality

$$S(x|y) \geq \tau(x - y)$$

for the relative entropy $S(x|y)$.

4. Main results

Let $\mathcal{M}$ be a finite von Neumann algebra and

$$M_2(\mathcal{M}) = \left\{ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} : a_{11}, a_{12}, a_{21}, a_{22} \in \mathcal{M} \right\}.$$

Then $M_2(\mathcal{M})$ is a von Neumann algebra on $\mathcal{H} \oplus \mathcal{H}$ with trace $tr \otimes \tau$. For $a \in L_0(\mathcal{M})$, it is well known that

$$\mu_t\left(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \right) = \mu_t(a), 0 < t < \tau(\mathbb{I}). \tag{4}$$

Definition 4.1. *Let $b \in \mathcal{M}$, take $a \geq 0$, and choose exponents $p, s \in \mathbb{R}$. For $x \in \mathcal{M}^{++}$, we define functions $\Upsilon^a_{p,s}(x)$ and $\Upsilon^a_{p,s;t}(x)$ as*

$$\Upsilon^a_{p,s}(x) := \tau[(a + b^* x^p b)^s]$$

and

$$\Upsilon^a_{p,s;t}(x) := \int_0^t \mu_\lambda[(a + b^* x^p b)^s] d\lambda.$$

In the matrix case, the trace functions $\Upsilon_{p,s}(x)$ were introduced and studied by Carlen and Lieb in [2, Theorem 1.1]. Here we adopt a generalized and different definition from [6]. More information about trace functions $\Upsilon^a_{p,s}(x)$ can be found in [1, 6].

Remark 4.2. *By employing a proof technique similar to [6, Lemma 2.2] and (4), we deduce that $\Upsilon^a_{p,s}(x)$ and $\Upsilon^a_{p,s;t}(x)$ are convex (respectively, concave) for arbitrary $b \in \mathcal{M}$ and $a \geq 0$, if and only if they are convex (respectively, concave) for arbitrary $b \in \mathcal{M}$ and $a = 0$. On the other hand, if $p, s > 0$, we can consider the maps $\Upsilon^a_{p,s}(x)$ on $\mathcal{M}^+ := \{x \in \mathcal{M} : x \geq 0\}$.*

To arrive at our main conclusion, it is necessary to first employ the following lemmas. By adopting a proof method similar to that used in [6, Proposition 3.3] and the properties of operator monotone functions(see [5, 8]), we can prove the following useful lemma. For the sake of completeness, a full proof is provided here.

Lemma 4.3. *Let $0 \leq a \in \mathcal{M}$, and let $b \in \mathcal{M}$. We may define the operator map*

$$\psi^a_{p,s}(x) = (a + b^* x^p b)^s$$

in $\mathcal{M}^{++}$ for exponents p and s. If $-1 \leq p \leq 0$ and $-1 \leq s \leq 0$, then $x \to \psi^a_{p,s}(x)$ is concave on $\mathcal{M}^{++}$.

Proof. We first consider the case $-1 \leq p \leq 0$ and $s = -1$. Since

$$\psi^a_{p,-1}(x) = (a + b^* x^p b)^{-1} = a^{-1/2}(\mathbb{I} + l^* x^p l)^{-1} a^{-1/2},$$

where $l = ba^{-1/2}$, we may assume $a = \mathbb{I}$. We may also without loss generality assume that b is invertible. We then obtain

$$(\mathbb{I} + b^* x^p b)^{-1} = \frac{(b^* x^p b)^{-1}}{(b^* x^p b)^{-1} + \mathbb{I}} = \frac{b^{-1} x^{-p} (b^{-1})^*}{b^{-1} x^{-p} (b^{-1})^* + \mathbb{I}}$$

by an elementary calculation. Since the map $x \to b^{-1} x^{-p} (b^{-1})^*$ is concave and the function $t \to t(1+t)^{-1}$ is operator monotone and operator concave, we obtain that $x \to (\mathbb{I} + b^* x^p b)^{-1}$ is concave. That is, $\psi^a_{p,-1}(x)$ is concave. Since the function $t \to t^\alpha$ is both operator monotone and operator concave for $0 \leq \alpha \leq 1$, it follows that $\psi^a_{p,s}(x)$ is concave for $-1 \leq s \leq 0$. □

Let b be a contraction operator and $x \in \mathcal{M}^{++}$. Then we have

$$b^* \log_p(x)b > \frac{-1}{p-1} b^*b \geq \frac{-1}{p-1} \text{ for } p > 1$$

and

$$b^* \log_p(x)b < \frac{-1}{p-1} b^*b \leq \frac{-1}{p-1} \text{ for } p < 1.$$

Therefore, $b^* \log_p(x)b$ belongs to the domain of the p-exponential. This is true even if b is not invertible since $\exp_p(0) = 1$. Therefore,

$$\exp_p(a + b^* \log_p(x)b)$$

is well defined and positive for arbitrary contraction operator b and $p \neq 1$, provided $a \geq 0$ when $p > 1$, and $a \leq 0$ when $p < 1$. In both cases, we define the deformed trace functions and singular valued functions by

$$\phi^a_{p,q}(x) = \tau[\exp_p(a + b^* \log_p(x)b)^q] \tag{5}$$

and

$$\phi^a_{p,q;s}(x) = \int_0^s \mu_t[\exp_p(a + b^* \log_p(x)b)^q]dt \tag{6}$$

for arbitrary exponent q. It follows from the definitions of $\exp_p$ and $\log_p$ that

$$\phi^a_{p,q}(x) = \tau([1 - b^*b + (p-1)a + b^*x^{p-1}b]^{q/(p-1)}) \tag{7}$$

and

$$\phi^a_{p,q;s}(x) = \int_0^s \mu_t([1 - b^*b + (p-1)a + b^*x^{p-1}b]^{q/(p-1)})dt. \tag{8}$$

Note that $(p-1)a \geq 0$ in both cases. By using Remark 4.2, we obtain the following:

Corollary 4.4. *Suppose $q/(p-1) > 0$. Then*

(i) *$x \to \phi^a_{p,q}(x)$ is convex (respectively concave) if and only if the function $x \to \tau(b^*x^{p-1}b)^{q/(p-1)}$ is convex (respectively concave);*

(ii) *$x \to \phi^a_{p,q;s}(x)$ is convex (respectively concave) if and only if the function $x \to \int_0^s \mu_t(b^*x^{p-1}b)^{q/(p-1)}dt$ is convex (respectively concave).*

Remark 4.5. (i) *Let $1 \leq p \leq 2$. By Lemma 4.1 in [1], we have $x \to \int_0^t \mu_s((b^*x^pb)^{\frac{1}{p}})ds$ is convex in $\mathcal{M}^+$.*

(ii) *Let $1 \leq p \leq 2$ and $q \geq 1$. By Theorem 4.1 in [1], we have $x \to [\tau((bx^pb)^{\frac{q}{p}})]^{\frac{1}{q}}$ is convex in $\mathcal{M}^+$.*

Lemma 4.6. *Let $q \in \mathbb{R}$ with $q/(1-p) > 0$.*

(i) *If $\phi^a_{p,q}(x)$ is convex (respectively concave) for arbitrary contraction operator b, then so is the function $\phi^{-a}_{2-p,q}(x)$.*

(ii) *If $\phi^a_{p,q;s}(x)$ is convex (respectively concave) for arbitrary contraction operator b, then so is the function $\phi^{-a}_{2-p,q;s}(x)$.*

Proof. We only prove case (ii). The other result can be similarly obtained. Without loss of generality, we assume that b is an invertible operator. By using the calculation in (8), we obtain

$$t^{-q}\phi^a_{p,q,s}(tx) = \int_0^s \mu_\lambda[t^{1-p}(1 - b^*b + (p-1)a) + b^*x^{p-1}b]^{q/(p-1)}d\lambda \tag{9}$$

for $t > 0$. By employing a proof technique of Lemma 3.2 in [6], we can establish this useful lemma. Therefore, the proof process will not be elaborated here. □

Setting $\beta = 1 + \frac{p-1}{q}$, it follows from (7) and the same calculation in [6, Lemma 1.1] that

$$\phi^a_{p,q}(x) = \tau[\exp_\beta(qa + qb^* \log_p(x)b)].$$

By replacing q with β in Remark 3.5, we obtain that

$$\phi^a_{p,q}(x) = \begin{cases} \sup\limits_{z>0} F(z,x), & \beta \leq 2, \\ \inf\limits_{z>0} F(z,x), & \beta > 2, \end{cases} \tag{10}$$

where $F(z,x) := \tau(z) - \tau\left(z^{2-\beta}(\log_\beta(z) - \log_\beta(y))\right)$ and $y = \exp_\beta(qa + qb^* \log_p(x)b)$. Then

$$\begin{aligned} F(z,x) &= \tau(z) - \tau\left(z^{2-\beta}(\log_\beta(z) - \log_\beta(y))\right) \\ &= \tau(z) - \tau\left(z^{2-\beta}(\frac{z^{\beta-1}-1}{\beta-1} - qa - qb^*\frac{x^{p-1}-1}{p-1}b\right) \\ &= \tau(z) - \frac{1}{\beta-1}\tau\left(z - z^{2-\beta} - z^{2-\beta}(p-1)a - z^{2-\beta}b^*(x^{p-1}-1)b\right) \\ &= \left(1 - \frac{1}{\beta-1}\right)\tau(z) + G(z,x), \end{aligned} \tag{11}$$

where $q/(p-1) = 1/(\beta-1)$ and

$$G(z,x) := \frac{1}{\beta-1}\tau\left(z^{2-\beta}(1 - b^*b + (p-1)a) + z^{2-\beta}b^*x^{p-1}b\right). \tag{12}$$

Remark 4.7. *Let b be a contraction operator and $a \in \mathcal{M}^{++}$. By applying Remark 3.5, the method from [15, Theorem 2.2] shows that*

$$\tau(\exp_q(h^* \log_q(a)h)) = \begin{cases} \sup\limits_{x>0}\left\{\tau(x) - \tau(x^{2-q}(\log_q x - b^* \log_q(a)b))\right\}, & q \leq 2, \\ \inf\limits_{x>0}\left\{\tau(x) - \tau(x^{2-q}(\log_q x - b^* \log_q(a)b))\right\}, & q > 2. \end{cases}$$

To achieve our main result, we will use that the functions $t \to t^p$ are operator concave, if and only if $0 \leq p \leq 1$, and operator convex, if and only if $-1 \leq p \leq 0$ or $1 \leq p \leq 2$. More information about operator concave/convex functions can be found in [8] and references therein. We also make use of Lieb's concavity theorem for operators (see [1, Lemma 3.1(vi)]) stating that the trace functions

$$(z,x) \to \tau(z^p b^* x^q b) \tag{13}$$

are concave if $p, q \geq 0$ and $p + q \leq 1$. Ando's theorem for operators (see [1, Lemma 3.1(v)] and [12, Theorem 5]) states that the trace function in (13) for arbitrary operator b, is convex for either $-1 \leq p, q \leq 0$, or for $-1 \leq p \leq 0$ and $1 - p \leq q \leq 2$, where obviously p and q may be interchanged in the condition.

According to [1, Lemma 3.1] and Lemma 4.3, we can generalize the inequalities in [6, Theorem 4.1] for operators, as presented below.

Theorem 4.8. *The function $x \to \phi^a_{p,q}(x)$ has the following geometric properties depending on a and the parameters p and q. Then*

(i) *$\phi^a_{p,q}(x)$ is concave in $\mathcal{M}^{++}$ for*

$$0 \leq p \leq 1,\ a \leq 0,\ 0 \leq q \leq 1$$

and

$$1 \leq p \leq 2,\ a \geq 0,\ 0 \leq q \leq 1.$$

(ii) $\phi^a_{p,q}(x)$ *is convex in* $\mathcal{M}^{++}$ *for*

$$0 \le p \le 1,\ a \le 0,\ q \le 0,$$

$$1 \le p \le 2,\ a \ge 0,\ q \le 0,$$

and

$$2 \le p \le 3,\ a \ge 0,\ q \ge 1.$$

Proof. The proof is analogous to the matrix case (see [6, Theorem 4.1]). However, for the sake of completeness, we will provide a separate proof here. Here we only prove case (i); for case (ii), the proof is the same as in the matrix case, by using [1, Lemma 3.1] and [12, Theorem 5]) and properties of operator concave/convex functions.

The proof of (i) is derived from the two cases in the statement.

(1) Let $0 \le p < 1$, $a \le 0$, and $1 - p \le q \le 1$. Then

$$0 \le \beta = 1 + (p-1)/q \le p < 1.$$

It follows from Equation (10)(see also Remark 3.5) that $\phi^a_{p,q}(x) = \sup_{z>0} F(z,x)$. Therefore, by Lemma 3.1, to derive that $\phi^a_{p,q}(x)$ is concave, it suffices to show that $G(z,x)$ is jointly concave or $G(z,x)$ is concave in the second variable. Recall that $-1 \le p-1 \le 0$ and $1-(p-1) \le 2-\beta \le 2$. Then the result follows from [1, Lemma 3.1(v)].

On the other hand, let $0 \le p < 1$, $a \le 0$, and $0 \le q \le 1-p$, that is $-1 \le s \le 0$, where $s = q/(p-1)$. Then the result follows from Lemma 4.3.

(2) Let $1 < p \le 2$, $a \ge 0$, and $0 < q \le p-1$. Then

$$\beta = 1 + (p-1)/q \ge 2.$$

By Equation (10), we have $\phi^a_{p,q}(x) = \inf_{z>0} F(z,x)$. Let $1 < p \le 2$, $a \ge 0$, and $p-1 \le q \le 1$. Then

$$1 < p \le \beta = 1 + (p-1)/q \le 2.$$

Thus, Equation (10) implies that $\phi^a_{p,q}(x) = \sup_{z>0} F(z,x)$. Therefore, it suffices to show that $G(z,x)$ is jointly concave or $G(z,x)$ is concave in the second variable. Then the result follows from properties of operator concave functions and [1, Lemma 3.1]. □

Corollary 4.9. *Let b be a contraction operator and consider the map*

$$\varphi(x) = \tau(\exp_q(b^* \log_q(x)b))$$

defined in $\mathcal{M}^{++}$. *The following assertions are valid:*

(i). $x \to \varphi(x)$ *is concave for* $0 \le q \le 2$;

(ii). $x \to \varphi(x)$ *is convex for* $2 < q \le 3$.

Proof. We use the method in the proof of [15, Corollary 2.3], by using [1, Lemma 3.1] and [12, Theorem 5]), to obtain the desired result. □

Similarly, by [1, Lemma 3.1] and [12, Theorem 5]), and Remark 4.7, the same method used in the matrix case (see [15]) shows that the following two results hold.

Corollary 4.10. *Let b be a contraction operator, and let a be a self-adjoint operator. The map*

$$x \to \tau(\exp_q(a + b^* \log_q(x)b)),$$

defined in $\mathcal{M}^{++}$, *is concave for* $1 < q \le 2$ *and convex for* $2 < q \le 3$. *The map*

$$x \to \tau(\exp_q(-a + b^* \log_q(x)b)),$$

defined in $\mathcal{M}^{++}$, *is concave for* $0 \le q < 1$.

Letting $q \to 1$ in Corollary 4.10, we can get:

Corollary 4.11. *Let b be a contraction operator, and let a be a self-adjoint operator. The map*

$$x \to \tau(\exp(a + b^* \log(x)b))$$

is concave in $\mathcal{M}^{++}$.

Proposition 4.12. *Let b be a contraction operator.*

(i) *If* $1 < q \le 2$, *then for* $x \in \mathcal{M}^{++}$ *and a self-adjoint operator a such that*

$$a + b^* \log_q(x)b > -\frac{1}{q-1},$$

we have the equality

$$\begin{aligned}&\tau(\exp_q(a + b^* \log_q(x)b))\\ &= \max_{z>0}\left\{\tau(z) + \tau(z^{2-q}a) - \tau(z^{2-q}(\log_q z - b^* \log_q(x)b))\right\}.\end{aligned}$$

(ii) *If* $q > 2$, *then for* $x \in \mathcal{M}^{++}$ *and a self-adjoint operator a such that*

$$a + b^* \log_q(x)b > -\frac{1}{q-1},$$

we have the equality

$$\begin{aligned}&\tau(\exp_q(a + b^* \log_q(x)b))\\ &= \min_{z>0}\left\{\tau(z) + \tau(z^{2-q}a) - \tau(z^{2-q}(\log_q z - b^* \log_q(x)b))\right\}.\end{aligned}$$

(iii) *If* $q < 1$, *then for* $x \in \mathcal{M}^{++}$ *and a self-adjoint operator a such that*

$$a + b^* \log_q(x)b < -\frac{1}{q-1},$$

we have the equality

$$\begin{aligned}&\tau(\exp_q(a + b^* \log_q(x)b))\\ &= \max_{z>0}\left\{\tau(z) + \tau(z^{2-q}a) - \tau(z^{2-q}(\log_q z - b^* \log_q(x)b))\right\}.\end{aligned}$$

Proof. Under the assumptions of (i),(ii) and (iii), the expression $\exp_q(a + b^* \log_q(x)b)$ is well defined and positive. By setting $y = \exp_q(a + b^* \log_q(x)b)$ in Remark 3.5 (see also Remark 4.7), we obtain (i),(ii) and (iii). □

The following proposition was shown in a more general setting in [15, Proposition 2.8] for the matrix case. The whole proof of [15, Proposition 2.8] works for operator case, hence, the following proposition follows from Remark 3.5 and Lemma 3.1.

Proposition 4.13. *Let b be a contraction operator, and let a be a self-adjoint operator. The map*

$$x \to \tau(\exp_q(a + b^* \log_r(x)b))$$

is convex in $\mathcal{M}^+$ for $q, r \in [2, 3]$ with $r \geq q$.

The convexity or concavity of certain trace functions for the deformed logarithmic and exponential functions has been widely used in Tsallis relative entropy (see Section 3 and Section 4 in [15] for matrix case). We won't go into details here.

Finally, we will show the relationship between $\phi^a_{p,q}(x)$ and the generalized trace functions $\Upsilon^a_{p,s}(x)$.

Corollary 4.14. *Let $0 \leq a, x \in \mathcal{M}$ and $b \in \mathcal{M}$. Then*

(i) $\Upsilon^a_{p,s}$ *is concave in $\mathcal{M}^{++}$ for*

$$-1 \leq p \leq 0, p^{-1} \leq s \leq 0$$

and

$$0 \leq p \leq 1, 0 \leq s \leq p^{-1}.$$

(ii) $\Upsilon^a_{p,s}$ *is convex in $\mathcal{M}^{++}$ for*

$$-1 \leq p \leq 0, s \geq 0,$$

$$0 \leq p \leq 1, s \leq 0,$$

and

$$1 \leq p \leq 2, s \geq p^{-1}.$$

Proof. For given $a \geq 0$, we set $L = (p-1)^{-1}a$. Then, $L \leq 0$ for $p < 1$ and $L \geq 0$ for $p > 1$. By replacing a with $t^{p-1}L$ in Equation 9, we obtain

$$\begin{aligned} t^{-q}\phi^{t^{p-1}L}_{p,q}(tx) &= \tau((t^{1-p}(1 - b^*b) + (p-1)L + b^*x^{p-1}b)^{\frac{q}{p-1}}) \\ &= \int_0^{\tau(\mathbb{I})} [\mu_\lambda(t^{1-p}(1 - b^*b) + (p-1)L + b^*x^{p-1}b)]^{\frac{q}{p-1}} d\lambda \end{aligned}$$

and $s = \frac{q}{p-1}$. Let $T := (p-1)L + b^*x^{p-1}b$ and

$$T_t := t^{1-p}(1 - b^*b) + (p-1)L + b^*x^{p-1}b.$$

If $s > 0$, then

$$\begin{aligned} \|T_t - T\|_{L^s(\mathcal{M})} &= (\int_0^{\tau(\mathbb{I})} [\mu_\lambda(T_t - T)]^s d\lambda)^{\frac{1}{s}} \\ &= (\int_0^{\tau(\mathbb{I})} [\mu_\lambda(t^{1-p}(1 - b^*b))]^s d\lambda)^{\frac{1}{s}} \\ &\leq t^{1-p}\|1 - b^*b\|_s. \end{aligned}$$

It follows from [3, Theorem 3.7] that $\|T_t\|_{L^s(\mathcal{M})} \to \|T\|_{L^s(\mathcal{M})}$, that is

$$\tau((t^{1-p}(1-b^*b)+(p-1)L+b^*x^{p-1}b)^{\frac{q}{p-1}}) \to \tau((a+b^*x^{p-1}b)^{\frac{q}{p-1}}),$$

by letting $t \to 0$ in the case $p < 1$, and letting $t \to \infty$ in the case $p > 1$.

If $s < 0$, then $0 \leq T_t^{-1} \leq T^{-1}$. From [4, Lemma 3.6] and [3, Lemma 3.4], we have $\mu_\lambda(T_t^{-1}) \uparrow \mu_\lambda(T^{-1})$ as $t \to 0$ in the case $p < 1$, and $t \to \infty$ in the case $p > 1$. Thus, monotone convergence theorem shows that

$$\int_0^{\tau(\mathbb{I})} \mu_\lambda(T_t^{-1})^{-s} d\lambda \uparrow \int_0^{\tau(\mathbb{I})} \mu_\lambda(T^{-1})^{-s} d\lambda$$

as $t \to 0$ in the case $p < 1$, and $t \to \infty$ in the case $p > 1$. That is

$$\|T_t^{-1}\|_{L^{-s}(\mathcal{M})} \uparrow \|T^{-1}\|_{L^{-s}(\mathcal{M})}.$$

Therefore,

$$\tau((t^{1-p}(1-b^*b)+(p-1)L+b^*x^{p-1}b)^{\frac{q}{p-1}}) \to \tau((a+b^*x^{p-1}b)^{\frac{q}{p-1}}).$$

With these choices, we realize that $\Upsilon^a_{p-1,s}(x)$ has the same geometric properties as $\phi^a_{p,q}(x)$. We may now replace p with $p+1$ and obtain that $\Upsilon^a_{p,s}(x)$ has the same geometric properties as $\phi^a_{p+1,q}(x)$, where $s = \frac{q}{p}$. Then the result follows from Theorem 4.8. □

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