Filomat 39:17 (2025), 6011–6018
https://doi.org/10.2298/FIL2517011A

Published by Faculty of Sciences and Mathematics, University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On the minimum total irregularity index of tetracyclic graphs

Hassan Ahmed[a], Akhlaq Ahmad Bhatti[a], Shamaila Yousuf[b], Akbar Ali[c,*]

[a]*Department of Sciences and Humanities, National University of Computer and Emerging Sciences, B-block, Faisal Town, Lahore, Pakistan*
[b]*Department of Mathematics, University of Gujrat, Hafiz Hayat Campus, Gujrat, Pakistan*
[c]*Department of Mathematics, College of Science, University of Ha'il, Ha'il, Saudi Arabia*

Abstract. The total irregularity of a graph G is defined as the sum of the absolute values of the differences of vertex degrees over all unordered pairs of vertices of G. In the present paper, the problem of determining graphs attaining the first two smallest values of the total irregularity index among all fixed-order tetracyclic graphs is addressed, where an n-order tetracyclic graph is a connected graph with n vertices and $n+3$ edges.

1. Introduction

Consider a graph $G = (V, E)$, where V represents the set of vertices and E represents the set of edges. The degree of a vertex v in G is denoted by $d_G(v)$. Let $V = \{v_1, v_2, \ldots, v_n\}$ and take $d_i = d_G(v_i)$ for $i = 1, 2, \ldots, n$, provided that $d_1 \geq d_2 \geq \cdots \geq d_n$. Throughout this paper, we write the degree sequence of a graph in nonincreasing order; that is, we write the degree sequence of G as $(d_1, d_2, \ldots, d_n)$. The graph-theoretical terms that we use in this paper without providing their definitions can be found in some standard books on graph theory, for example [7].

A graph in which all vertices have the same degree is known as a regular graph. A nonregular graph is a graph that is not regular. In the literature, there exist many graph invariants for measuring the nonregularity of graphs. Such graph invariants are often called irregularity measures. One of the much-studied irregularity measures is due to Albertson [4]. For a given graph G, Albertson's irregularity measure is defined [4] as

$$\mathrm{irr}(G) = \sum_{uv \in E} |d_G(u) - d_G(v)|.$$

In [4], it was shown that the star graph maximizes among all fixed-order trees. Results on irr using the computer software, namely AutoGraphiX, can be found in [12]. The problem of determining graphs maximizing irr among all fixed-order graphs was addressed in [2]. For some other existing results on irr, we refer the reader to [10, 13].

2020 *Mathematics Subject Classification*. Primary 05C07; Secondary 05C09.
Keywords. irregularity; tetracyclic graphs; topological index; total irregularity.
Received: 27 September 2024; Revised: 19 December 2024; Accepted: 06 April 2025
Communicated by Paola Bonacini
* Corresponding author: Akbar Ali
Email addresses: hassanms5664@gmail.com (Hassan Ahmed), akhlaq.ahmad@nu.edu.pk (Akhlaq Ahmad Bhatti), shumaila.yousaf@uog.edu.pk (Shamaila Yousuf), akbarali.maths@gmail.com (Akbar Ali)
ORCID iD: https://orcid.org/0000-0001-8160-4196 (Akbar Ali)

In order to overcome some of the limitations of Albertson's irregularity, Abdo et al. [1] introduced its following modified version for any nontrivial graph G:

$$\mathrm{irr}_t(G) = \sum_{\{u,v\}\subseteq V} |d_G(u) - d_G(v)|.$$

The aforementioned limitations of *irr* include the following: If H_1 and H_2 are two graphs with the same order as well as the same degree sequence, then Albertson's irregularity measure of H_1 and H_2 may have different values; however, they have the same irr_t. Also, the number of distinct elements of the degree sequence of every graph maximizing irr among all fixed-order graphs is 2 (which should be the largest possible), see [2]; however, for the case of irr_t, this number is considerably large. Finally, for a disconnected nonregular graph H, it is possible that $\mathrm{irr}(H) = 0$; however, $\mathrm{irr}_t(H) = 0$ if and only if H is regular. For details on these limitations of irr and benefits of irr_t, see [1].

Dimitrov and Škrekovski [8] established inequalities between irr and irr_t. Most of the existing extremal results and bounds related to irr_t can be found in the recent survey paper [5].

A graph of order n is called an n-order graph. A connected n-order graph of size $n + c - 1$ is known as a c-cyclic graph, where c is a nonnegative integer. If $c = 0, 1, 2, 3$ or 4, then the corresponding c-cyclic graph is called a tree, unicyclic graph, bicyclic graph, tricyclic graph or tetracyclic graph, respectively. The problems of determining graphs attaining the first three smallest values of irr_t among all fixed-order (i) trees, (ii) unicyclic graphs and (iii) bicyclic graphs, were attacked in [17]; similar problems were addressed in [3] and [11] for tricyclic graphs and c-cyclic graphs, respectively.

In this paper, we examine the characterization of graphs that attain the two smallest values of irr_t among all fixed-order tetracyclic graphs. Additionally, we address an error in [11]. Let n, k, and c be three positive integers such that $n > 2c^2 - 3c + 2$ and $c \geq 2$. In Theorem 2.15 of [11], it was established that if $1 \leq k \leq c$, then among all n-order c-cyclic graphs of maximum degree at most 4, the graphs with the following degree sequence have the kth minimum value of irr_t:

$$(\underbrace{4,\ldots,4}_{k-1}, \underbrace{3,\ldots,3}_{2(k+c-2)}, \underbrace{2,\ldots,2}_{n-(3k+2c-5)}). \tag{1}$$

Furthermore, Theorem 2.16 of [11] indicates that if $1 \leq k \leq 3$ and $c \geq 3$, then among all n-order c-cyclic graphs, the graphs with the above degree sequence (given in (1)) also achieve the kth minimum value of irr_t.

Now, consider an n-order c-cyclic graph G of maximum degree at most 4 and minimum degree 2. For $i \in \{1, \ldots, n-1\}$, let $n_i(G)$ represent the number of vertices of degree i in G. If $n_4(G) = k-1$, then the equations

$$\sum_{i=2}^{4} n_i(G) = n \quad \text{and} \quad \sum_{i=1}^{4} i \cdot n_i(G) = 2(n + c - 1),$$

yield $n_3(G) = 2(c - k)$ and $n_2(G) = n - 2c + k + 1$. This indicates some errors in the degree sequence of (1). Specifically, in Theorems 2.15 and 2.16 of [11], the degree sequence in (1) has to be replaced with the following:

$$(\underbrace{4,\ldots,4}_{k-1}, \underbrace{3,\ldots,3}_{2(c-k)}, \underbrace{2,\ldots,2}_{n-2c+k+1}).$$

This correction serves as the primary motivation for the present study. Additionally, the formulation of Theorem 2.16 in [11] for n-vertex tetracyclic graphs under the constraint $n \geq 23$ further motivates this research.

2. Results

For a given graph $G(V, E)$, we use $V(G) := V$ and $E(G) := E$. Let $N_G(v) := \{w \in V(G) : wv \in E(G)\}$ and $N_G[v] := N_G(v) \cup \{v\}$. We start this section with the following lemma, whose special case (Lemma 2.2) is used frequently in the rest of the paper.

Lemma 2.1. *Let G be a graph of minimum degree δ and maximum degree Δ such that $\Delta - \delta \geq 2$. Let $x', x, y \in V(G)$ be three different vertices such that $d_G(x) = \Delta$, $d_G(y) = \delta$ and $x' \in N_G(x) \setminus N_G(y)$. Let G' be the graph obtained from G by removing the edge $x'x$ and adding the edge $x'y$. Then,*

$$\mathrm{irr}_t(G) - \mathrm{irr}_t(G') = 2|V(G) \setminus (\{x, y\} \cup V_\Delta \cup V_\delta)| + 2,$$

where $V_\Delta = \{a \in V(G) \setminus \{x\} : d_G(a) = \Delta\}$ and $V_\delta = \{b \in V(G) \setminus \{y\} : d_G(b) = \delta\}$.

Proof. We note that $d_{G'}(x) = d_G(x) - 1$, $d_{G'}(y) = d_G(y) + 1$ and $d_{G'}(v) = d_G(v)$ for every $v \in V(G) \setminus \{x, y\}$. Since $\Delta - \delta \geq 2$, we have

$$|d_G(x) - d_G(y)| - |d_{G'}(x) - d_{G'}(y)| = 2$$

and hence

$$\begin{aligned}\mathrm{irr}_t(G) - \mathrm{irr}_t(G') &= \sum_{v \in V(G)\setminus\{x,y\}} \Big(\Delta - d_G(v) - |d_{G'}(x) - d_G(v)|\Big) + \sum_{v \in V(G)\setminus\{x,y\}} \Big(d_G(v) - \delta - |d_G(v) - d_{G'}(y)|\Big) + 2 \\ &= \sum_{v \in V(G)\setminus\{x,y\}} \Big(\Delta - \delta - |d_{G'}(x) - d_G(v)| - |d_G(v) - d_{G'}(y)|\Big) + 2 \qquad (2)\end{aligned}$$

For every $v \in V_\Delta \cup V_\delta$, it holds that

$$\Delta - \delta - |d_{G'}(x) - d_G(v)| - |d_G(v) - d_{G'}(y)| = 0,$$

and hence (2) yields

$$\begin{aligned}\mathrm{irr}_t(G) - \mathrm{irr}_t(G') &= \sum_{v \in V(G)\setminus(\{x,y\}\cup V_\Delta \cup V_\delta)} \Big(\Delta - \delta - (\Delta - d_G(v) - 1) - (d_G(v) - \delta - 1)\Big) + 2 \\ &= 2|V(G) \setminus (\{x, y\} \cup V_\Delta \cup V_\delta)| + 2\end{aligned}$$

□

The next result is a special case of Lemma 2.1, where both the considered graphs are assumed to be connected.

Lemma 2.2. *Let G be a connected graph of minimum degree δ and maximum degree Δ such that $\Delta - \delta \geq 2$. Let $x, y \in V(G)$ such that $d_G(x) = \Delta$ and $d_G(y) = \delta$. Pick $x' \in N_G(x) \setminus N_G[y]$ such that the graph G' obtained from G by removing the edge $x'x$ and adding the edge $x'y$ is connected. Then,*

$$\mathrm{irr}_t(G) - \mathrm{irr}_t(G') = 2(|V(G)| - n_\Delta(G) - n_\delta(G) + 1).$$

Remark 2.3. *In Lemma 2.2, if there are at least two paths between x and y in G then certainly G' is connected for every choice of $x' \in N_G(x) \setminus N_G[y]$. If x and y are connected via exactly one path, then by condition $\Delta - \delta \geq 2$ we can pick x' in such a way that it does not lie on the path connecting x and y. Therefore, in Lemma 2.2, we can always pick x' in such a way that G' is connected.*

Lemma 2.4. *If G is an n-order tetracyclic graph of minimum degree δ such that $n \geq 7$, then $\delta \leq 2$.*

Proof. If $\delta \geq 3$ then by the degree-sum formula, we have $3n \leq 2(n + 3)$, a contradiction. □

Theorem 2.5. *Let G_1 be the graph minimizing irr_t among all n-order tetracyclic graphs for $n \geq 7$. Then, the degree sequence of G_1 is $(3, 3, 3, 3, 3, 3, 2, 2, \ldots, 2)$ and $\mathrm{irr}_t(G_1) = 6(n - 6)$; particularly, the minimum and maximum degrees of G_1 are 2 and 3, respectively.*

Proof. If the difference between the maximum degree and minimum degree of G_1 is at least 2, then by Lemma 2.2 there exists n-order tetracyclic graph G' such that $\mathrm{irr}_t(G_1) - \mathrm{irr}_t(G') \geq 2$, which contradicts the minimality of $\mathrm{irr}_t(G_1)$. Hence, the difference between the maximum degree and minimum degree of G_1 is at most 1. Since $n \geq 7$, the minimum degree of G_1 is at most 2. Consequently, the minimum and maximum degrees of G_1 are 2 and 3, respectively. Then, $n_2(G_1) + n_3(G_1) = n$ and $2n_2(G_1) + 3n_3(G_1) = 2(n + 3)$, which yield the desired degree sequence and hence $\mathrm{irr}_t(G_1) = 6(n - 6)$. □

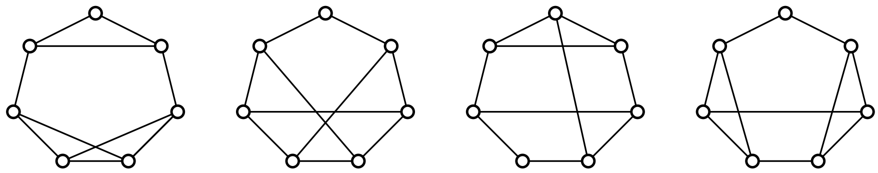

Figure 1: The graphs minimizing irr_t among all 7-order tetracyclic graphs.

Using Theorem 2.5, we obtain all graphs that minimize irr_t among all 7-order tetracyclic graphs (see Figure 1).

Next, we focus on the second-minimum value of irr_t among all n-order tetracyclic graphs.

Lemma 2.6. *Let G_2 be the graph attaining the second-minimum value of irr_t among all n-order tetracyclic graphs for $n \geq 7$. Then, the maximum degree of G_2 is at most 4.*

Proof. Since $n \geq 7$, by Lemma 2.4 the minimum degree of G_2 is at most 2. If the maximum degree of G_2 is at least 5, then the difference between the maximum degree and minimum degree of G_2 is at least 3 and hence by applying the transformation of Lemma 2.2 a finite number of times, we obtain an n-order tetracyclic graph G_2' of maximum degree 4 such that $\mathrm{irr}_t(G_2) > \mathrm{irr}_t(G_2') > \mathrm{irr}_t(G_1) = 6(n-6)$, which is a contradiction to the fact that G_2 has the second-minimum value of irr_t among all n-order tetracyclic graphs for $n \geq 7$, where G_1 is defined in Theorem 2.5. □

Lemma 2.7. *Let G be an n-order tetracyclic graph of maximum degree 4 and minimum degree 1 such that $n \geq 7$. Then, G attains neither the minimum value of irr_t nor the second-minimum value of irr_t among all n-order tetracyclic graphs for $n \geq 7$.*

Proof. If $n_1(G) > n_4(G)$ then by Lemmas 2.2 and 2.4, there exists an n-order tetracyclic graph G' of maximum degree 3 and minimum degree 1 such that $\mathrm{irr}_t(G) > \mathrm{irr}_t(G') > \mathrm{irr}_t(G_1) = 6(n-6)$, where G_1 is given in Theorem 2.5.

If $n_1(G) < n_4(G)$ then again by Lemmas 2.2 and 2.4, there exists an n-order tetracyclic graph G'' of maximum degree 4 and minimum degree 2 such that $\mathrm{irr}_t(G) > \mathrm{irr}_t(G'') > \mathrm{irr}_t(G_1) = 6(n-6)$.

In what follows, we assume that $n_1(G) = n_4(G)$. Solving the equations

$$2n_1(G) + n_2(G) + n_3(G) = n \quad \text{and} \quad 5n_1(G) + 2n_2(G) + 3n_3(G) = 2(n+3)$$

for $n_2(G)$ and $n_3(G)$ and replacing these values in

$$\mathrm{irr}_t(G) = 3n_1(G)n_2(G) + 3n_1(G)n_3(G) + 3(n_1(G))^2 + n_2(G)n_3(G),$$

we obtain

$$\mathrm{irr}_t(G) = n(n-6) + 2[n - n_1(G)]n_1(G) > \mathrm{irr}_t(G_1) = 6(n-6),$$

provided that $n \geq 7$. □

Lemma 2.8. *Let G be an n-order tetracyclic graph of maximum degree 4 and minimum degree 2 such that $n \geq 7$ and $n_4(G) \geq 2$. Then, G attains neither the minimum value of irr_t nor the second-minimum value of irr_t among all n-order tetracyclic graphs for $n \geq 7$.*

Proof. Since $n_4(G) \geq 2$, by Lemmas 2.2 and 2.4, there exists an n-order tetracyclic graph G' of maximum degree 4 and minimum degree 2 such that $\mathrm{irr}_t(G) > \mathrm{irr}_t(G') > \mathrm{irr}_t(G_1) = 6(n-6)$, where G_1 is given in Theorem 2.5. □

Lemma 2.9. *Let G be an n-order tetracyclic graph of maximum degree 3 and minimum degree 1 such that $n \geq 7$ and $n_1(G) \geq 2$. Then, G attains neither the minimum value of irr_t nor the second-minimum value of irr_t among all n-order tetracyclic graphs for $n \geq 7$.*

Proof. Since $n_1(G) \geq 2$, by Lemma 2.2, there exists an n-order tetracyclic graph G' of maximum degree 3 and minimum degree 1 such that $\mathrm{irr}_t(G) > \mathrm{irr}_t(G') > \mathrm{irr}_t(G_1) = 6(n-6)$, where G_1 is given in Theorem 2.5. □

Theorem 2.10. *Let G_2 be a graph attaining the second-minimum value of irr_t among all n-order tetracyclic graphs for $n \geq 7$. Let*

$$D_1 = (4, 3, 3, 3, 3, \underbrace{2, \ldots, 2}_{n-5}) \quad \text{and} \quad D_2 = (3, 3, 3, 3, 3, 3, 3, \underbrace{2, \ldots, 2}_{n-8}, 1).$$

If either $n = 7$ or $n \geq 13$, then the degree sequence of G_2 is D_1. If $8 \leq n \leq 11$, then the degree sequence of G_2 is D_2. For $n = 12$, the degree sequence of G_2 is either of the sequences D_1 and D_2. Also, $\mathrm{irr}_t(G_2) = 2(3n-13)$ when either $n = 7$ or $n \geq 13$, and $\mathrm{irr}_t(G_2) = 2(4n-25)$ when $8 \leq n \leq 12$.

Proof. By Lemma 2.6, the maximum degree of G_2 is at most 4.

Case 1. The maximum degree of G_2 is 4.
By Lemmas 2.4 and 2.7, the minimum degree of G_2 is 2. By Lemma 2.8, $n_4(G_2) \leq 1$. However, the choice $n_4(G_2) = 0$ yields a graph with the degree sequence

$$(3, 3, 3, 3, 3, 3, \underbrace{2, \ldots, 2}_{n-6}),$$

which corresponds to the first minimum value of irr_t. Hence, $n_4(G_2) = 1$. Consequently, from the equations

$$n_2(G_2) + n_3(G_2) + 1 = n$$

and

$$2n_2(G_2) + 3n_3(G_2) + 4 = 2(n+3),$$

we obtain $n_2(G_2) = n - 5$ and $n_3(G_2) = 4$. Therefore, $\mathrm{irr}(G_2) = 2(3n-13)$.

Case 2. The maximum degree of G_2 is 3.
The possibility $n_1(G_2) = 0$ yields a graph with the degree sequence

$$(3, 3, 3, 3, 3, 3, \underbrace{2, \ldots, 2}_{n-6}),$$

which corresponds to the first minimum value of irr_t. Hence, $n_1(G_2) \geq 1$. Now, by Lemma 2.9, we have $n_1(G_2) = 1$. Consequently, from the equations

$$1 + n_2(G_2) + n_3(G_2) = n$$

and

$$1 + 2n_2(G_2) + 3n_3(G_2) = 2(n+3),$$

we obtain $n_2(G_2) = n - 8$ and $n_3(G_2) = 7$. Hence, in the present case, we must have $n \geq 8$. Also, $\mathrm{irr}(G_2) = 2(4n-25)$, in the present case.

Now, in the following, we compare $\mathrm{irr}_t(G_2)$ obtained in both cases:

$$2(3n-13) > 2(4n-25) \quad \text{for } 8 \leq n \leq 11$$

$$2(3n-13) = 2(4n-25) \quad \text{for } n = 12$$

and

$$2(3n-13) < 2(4n-25) \quad \text{for } n \geq 13.$$

□

3. Concluding Remarks

In this section, we present two results about the graphs attaining extreme values of irr_t among all fixed-order c-cyclic graphs for $0 \le k \le 6$. Both of these results follow from the existing studies; however, to the best of authors' knowledge, neither of these results has been derived earlier in this way, but their parts have been proved in several different publications.

Keeping in mind Lemma 1 and Corollary 2 of [9], the discussion of Section 4 and the initial part of Section 5 in [6], we obtain the degree sequences of graphs attaining the extreme values of irr_t among all fixed-order k-cylic graphs for $0 \le k \le 6$. In the case of the maximum value of irr_t for $c = 4$, we have to compare irr_t of the graphs J_1 and J_2 with the following degree sequences, respectively:

$$(n-1,4,3,3,2,\underbrace{1,\dots,1}_{n-5}) \quad \text{and} \quad (n-1,5,2,2,2,2,\underbrace{1,\dots,1}_{n-6}),$$

where $n \ge 6$. However, $\mathrm{irr}_t(J_1) = n(n+5) - 40 > n(n+5) - 42 = \mathrm{irr}_t(J_2)$. Also, note that for $c = 5$, we have to compare irr_t of the graphs L_1, L_2 and L_3 with the following degree sequences, respectively:

$$(n-1,4,4,3,3,\underbrace{1,\dots,1}_{n-5}), \quad (n-1,5,3,3,2,2\underbrace{1,\dots,1}_{n-6}) \quad \text{and} \quad (n-1,6,2,2,2,2,2,2,\underbrace{1,\dots,1}_{n-7}),$$

where $n \ge 7$. However, $\mathrm{irr}_t(L_1) = \mathrm{irr}_t(L_2) = n(n+7) - 54 > n(n+7) - 58 = \mathrm{irr}_t(L_3)$. Finally, for $c = 6$, we have to compare irr_t of the graphs O_1, O_2, O_3, O_4 and O_5 with the following degree sequences, respectively:

$$(n-1,5,4,3,3,2,\underbrace{1,\dots,1}_{n-6}),\ (n-1,7,\underbrace{2,\dots,2}_{6},\underbrace{1,\dots,1}_{n-8}),\ (n-1,6,3,3,2,2,2,\underbrace{1,\dots,1}_{n-7}),$$

$$(n-1,5,3,3,3,3,\underbrace{1,\dots,1}_{n-6}) \quad \text{and} \quad (n-1,4,4,4,4,\underbrace{1,\dots,1}_{n-5}),$$

where $n \ge 8$. However,

$$\mathrm{irr}_t(O_1) = n(n+9)-68 > \mathrm{irr}_t(O_3) = \mathrm{irr}_t(O_5) = n(n+9)-70 > n(n+9)-74 = \mathrm{irr}_t(O_4) > n(n+9)-76 = \mathrm{irr}_t(O_2).$$

Therefore, we have the following result:

Theorem 3.1. *Among all n-order c-cyclic graphs, the graph maximizing irr_t has the degree sequence*

(i) $(n-1,\underbrace{1,\dots,1}_{n-1})$ *for $c = 0$ and $n \ge 4$,*

(ii) $(n-1,2,2,\underbrace{1,\dots,1}_{n-3})$ *for $c = 1$ and $n \ge 4$,*

(iii) $(n-1,3,2,2,\underbrace{1,\dots,1}_{n-4})$ *for $c = 2$ and $n \ge 5$,*

(iv) either $(n-1,4,2,2,2,\underbrace{1,\dots,1}_{n-5})$ *or* $(n-1,3,3,3,\underbrace{1,\dots,1}_{n-4})$ *for $c = 3$ and $n \ge 5$,*

(v) $(n-1,4,3,3,2,\underbrace{1,\dots,1}_{n-5})$ *for $c = 4$ and $n \ge 6$,*

(vi) either $(n-1,4,4,3,3,\underbrace{1,\dots,1}_{n-5})$ *or* $(n-1,5,3,3,2,2\underbrace{1,\dots,1}_{n-6})$ *for $c = 5$ and $n \ge 7$,*

(vii) $(n-1,5,4,3,3,2,\underbrace{1,\ldots,1}_{n-6})$ *for* $c=6$ *and* $n\geq 8$.

Theorem 3.1(i), Theorem 3.1(ii), Theorem 3.1(iii) and Theorem 3.1(iv)–(vii) were proved independently in [1], [14], [15] and [16], respectively.

Next, we have the minimal version of Theorem 3.1, which also follows from the general results of [6].

Theorem 3.2. *Among all n-order c-cyclic graphs, the graph minimizing* irr_t *has the degree sequence*

(i) $(\underbrace{2,\ldots,2}_{n-2},1,1)$ *for* $c=0$ *and* $n\geq 4$,

(ii) $(\underbrace{2,\ldots,2}_{n})$ *for* $c=1$ *and* $n\geq 4$,

(iii) $(3,3,\underbrace{2,\ldots,2}_{n-2})$ *for* $c=2$ *and* $n\geq 5$,

(iv) $(3,3,3,3,\underbrace{2,\ldots,2}_{n-4})$ *for* $c=3$ *and* $n\geq 5$,

(v) $(3,3,3,3,3,3,\underbrace{2,\ldots,2}_{n-6})$ *for* $c=4$ *and* $n\geq 6$,

(vi) $(\underbrace{3,\ldots,3}_{8},\underbrace{2,\ldots,2}_{n-8})$ *for* $c=5$ *and* $n\geq 8$,

(vi) $(\underbrace{3,\ldots,3}_{10},\underbrace{2,\ldots,2}_{n-10})$ *for* $c=6$ *and* $n\geq 10$.

Theorem 3.2(i)–(iii) and Theorem 3.2(iv) were proved independently in [17] and [3], respectively. All parts of Theorem 3.2 for sufficiently large n also follow from a general result (that is, Theorem 2.16) reported in [11].

Acknowledgment

Helpful comments and valuable suggestions from the referee are gratefully acknowledged.

References

[1] H. Abdo, S. Brandt, D. Dimitrov, *The total irregularity of a graph*, Discrete Math. Theor. Comput. Sci. **16** (2014), 201–206.
[2] H. Abdo, N. Cohen, D. Dimitrov, *Graphs with maximal irregularity*, Filomat **28** (2014), 1315–1322.
[3] H. Ahmed, A. A. Bhatti, *Minimum total irregularity index of tricyclic graphs*, Kuwait J. Sci. **50** (2023), 1–14.
[4] M. O. Albertson, *The irregularity of a graph*, Ars Combin. **46** (1997), 219–225.
[5] A. Ali, D. Dimitrov, T. Réti, A. M. Albalahi, A. E. Hamza, *Bounds and optimal results for the total irregularity measure*, MATCH Commun. Math. Comput. Chem **94** (2025), 5–29.
[6] M. Bianchi, A. Cornaro, J. L. Palacios, A. Torriero, *New bounds of degree-based topological indices for some classes of c-cyclic graphs*, Discrete Appl. Math. **184** (2015), 62–75.
[7] J. A. Bondy, U. S. Murty, *Graph Theory and Its Applications*, Macmillan, London, 1976.
[8] D. Dimitrov, R. Škrekovski, *Comparing the irregularity and the total irregularity of graphs*, Ars Math. Contemp. **9** (2015), 45–50.
[9] M. Eliasi, *The maximal total irregularity of some connected graphs*, Iranian J. Math. Chem. **6** (2015), 121–128.
[10] F. Gao, K. Xu, T. Došlić, *On the difference of Mostar index and irregularity of graphs*, Bull. Malays. Math. Sci. Soc. **44** (2021), 905–926.
[11] A. Ghalavand, A. R. Ashrafi, *Ordering of c-cyclic graphs with respect to total irregularity*, J. Appl. Math. Comput. **63** (2020), 707–715.

[12] P. Hansen, H. Mélot, Variable neighborhood search for extremal graphs. 9. Bounding the irregularity of a graph, In: S. Fajtlowicz, P. W. Fowler, P. Hansen, M. F. Janowitz, F. S. Roberts (Eds.), *Graphs and Discovery*, Am. Math. Soc., Providence, 2005, pp. 253–264.
[13] M. A. Henning, D. Rautenbach, *On the irregularity of bipartite graphs*, Discrete Math. **307** (2007), 1467–1472.
[14] L. You, J. Yang, Z. You, *The maximal total irregularity of unicyclic graphs*, Ars Combin. **114** (2014), 153–160.
[15] L. You, J. Yang, Y. Zhu, Z. You, *The maximal total irregularity of bicyclic graphs*, J. Appl. Math. **2014** (2014), #785084.
[16] S. Yousaf, A.A. Bhatti, *Maximum total irregularity index of some families of graph with maximum degree* $n-1$, Asian-European J. Math. **15** (2022), #2250069.
[17] Y. Zhu, L. You, J. Yang, *The minimal total irregularity of some classes of graphs*, Filomat **30** (2016), 1203–1211.