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On the modified Szász-Chlodovsky operator on weighted spaces

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Abstract. In this paper, we suggest the modified Szász-Chlodovsky operators. Also, we establish some results in the weighted space of continuous functions defined on $\mathbb{R}^+$. Moreover, the Voronovskaja type theorem and the rate of convergence are provided in detailed proofs. Furthermore, we substantiate some shape preserving properties of the Szász-Chlodovsky operators for instance the monotonicity and the convexity. Lastly, we associate this modified operator with its classical counter part to display that the modified one has better properties.

1. Introduction

Approximation theory remains among of the important branches of mathematical analysis which scrutinizes the approximation to a given function which is obscure with more useful and simpler computable functions. Generally, the three significant situations should be well known in an approximation problem. These includes the function f , the space to which the function to be approximated belongs and to determine how near this approximation is to the function f. Ever since at the end of the 19th century, several mathematicians have introduced distinct kinds of operators aiming to evaluate simpler functions to estimate this logic. Weierstrass in 1885 [24] developed a theorem substantiating the occurence of polynomials that converge to a continuous function in a closed and bounded domain. Gupta *et al.* [15] studied semi-exponential type Gauss-Weierstrass operators and proved some results. The central moments of these operators are constant functions. Furthermore, Bernstein in 1912 [10] extended this theorem via polynomials called the Bernstein polynomials, which were introduced by himself on $[0, 1]$. Because of its significant features and plain construction remains the reason why Bernstein polynomials are extensively utilized. Many fields for example computer aided geometric design, probability theory, approximation theory, and number theory use the knowledge of Bernstein polynomials. One can find some others recent work in [14].

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During 1937, Bernstein's student Chlodovsky [13] had shifted polynomials from $[0,1]$ to $[0,b_m](b_m \to \infty)$ by attaining a new change of the Bernstein polynomials.

For further information, for $x \geq 0$ and $m \geq 1$, Chlodovsky developed the following Bernstein type operators termed Bernstein-Chlodovsky operators:

$$B_m(f(x)) = \sum_{r=0}^{m} P_{r,m}\left(\frac{x}{b_m}\right) f\left(\frac{r}{m}\right) \tag{1}$$

where f indicates a function explained on $[0,\infty)$ and bounded on each finite interval
$[0,b_m] \subset [0,\infty)$ at a specific rate, and

$$P_{r,m}\left(\frac{x}{b_m}\right) = \binom{m}{r}\left(\frac{x}{b_m}\right)^r \left(1-\frac{x}{b_m}\right)^{m-r} \tag{2}$$

where $(b_m)_{m=1}^{\infty}$ denotes a positive increasing sequence of reals with the following properties:

$$\lim_{m\to\infty} b_m = \infty, \qquad \lim_{m\to\infty} \frac{b_m}{m} = 0 \tag{3}$$

I. Chlodovsky in 1937 [13], developed these polynomials in generalizations of the Bernstein polynomials $(B_m f)(x)$, for the case $(b_m = 1)$, $m \geq 1$, which estimate the functions f on $[0,1]$.

By means of a function f which satisfy certain general assumptions, we deliver a similar modification of the Szász-Mirakyan operator, as in (1) for Bernstein-Chlodowsky operator. Keep in mind that the Bernstein operators operate on functions stated on $[0,1]$, compared to the classical Szász-Mirakyan operator which are stated for functions on the unbounded interval $[0,\infty)$.
The Szász-Mirakyan operator was developed in [23] as a generality of the Bernstein operators, with appropriate functions f constructed on the infinite interval
$\mathbb{R}^+ = [0,\infty)$:

$$S_m(f(x)) = e^{-mx} \sum_{r=0}^{\infty} f\left(\frac{r}{m}\right) \frac{(mx)^r}{r!}. \tag{4}$$

Notice that, the Szász-Mirakyan operators are modified in order to define the operators known as Szász-Chlodovsky operators, which are given below,

$$S_m^*(f(x)) = e^{\frac{-mx}{b_m}} \sum_{r=0}^{\infty} f\left(\frac{rb_m}{m}\right)\left(\frac{mx}{b_m}\right)^r \frac{1}{r!}. \tag{5}$$

An explicit expression for $S_m^*(t^m;x)$ for $m = 0,1,2$ is given by direct calculations as follows:

$$S_m^*(1;x) = 1 \tag{6}$$

$$S_m^*(t;x) = x \tag{7}$$

$$S_m^*(t^2;x) = x^2 + \frac{b_m}{m}x. \tag{8}$$

Recently some important modifications of Szász and other operators have been studied in [3–9, 18, 19, 21, 22]. The existing paper aims to investigate the convergence features of the modified Szász-Chlodowsky operators on weighted spaces when the convergence interval extends as $m \to \infty$, to acquire asymptotic formula for the $S_m^*(f)$ utilizing Taylor's theorem, to provide quantitative type theorems in order to acquire the degree of the weighted convergence by employing weighted modulus of continuity, to derive a certain shape preserving properties of modified operators for instance the monotonicity and the convexity. Finally, we compare this modified operator with classical Szász-Mirakyan operators in order to show that the modified one has superior properties.

2. Basic notations and preliminary results

Throughout this paper, $\mathbb{R}$ indicates the set of all real numbers, $\mathbb{R}^+$ indicates the set of all positive real numbers, $\mathbb{N}$ denotes the set of all natural numbers, $\mathbb{Z}$ will stand for the set of all integers, $\forall$ will mean for all, $\exists$ will stand for there exist(s), $\in$ will stand for the member of,

Definition 2.1. ***Weighted function space***
Suppose that $\Psi(x) = 1 + x^2$ represents a weight function and also let $B_\Psi(\mathbb{R}^+)$ represents the space [4], determined by

$$B_\Psi(\mathbb{R}^+) = \{f : \mathbb{R}^+ \to \mathbb{R} \Big| \ |f(x)| \leq N_f \Psi(x), \ x \geq 0\}$$

with N_f denoting any constant depending on f. $B_\Psi(\mathbb{R}^+)$ stands for a normed space having the norm

$$\|f\|_\Psi = \sup_{x\in\mathbb{R}^+} \frac{|f(x)|}{\Psi(x)}.$$

$C_\Psi(\mathbb{R}^+)$ stands for the subspaces of all continuous functions in $B_\Psi(\mathbb{R}^+)$,
$C^r_\Psi(\mathbb{R}^+)$ indicates the subspaces of all functions $f \in C_\Psi(\mathbb{R}^+)$ with $\lim_{m\to\infty} \frac{f(x)}{\Psi(x)} = \tau_f$, where τ_f is any constant depending on f.
Moreover, suppose $U_\Psi(\mathbb{R}^+)$ is the space of functions $f \in C_\Psi(\mathbb{R}^+)$ so that $\frac{f(x)}{\Psi(x)}$ is uniformly continuous. It can be clearly seen that

$$C^r_\Psi(\mathbb{R}^+) \subset U_\Psi(\mathbb{R}^+) \subset C_\Psi(\mathbb{R}^+) \subset B_\Psi(\mathbb{R}^+).$$

Lemma 2.2. *[4] The linear positive operator H_m where $m \geq 1$, shift from $C_\Psi(\mathbb{R}^+)$ to $B_\Psi(\mathbb{R}^+)$ if and only in the event that the inequality*

$$|H_m(\Psi; x)| \leq J_m \Psi(x), \quad x \geq 0, \ m \geq 1,$$

maintains, where J_m denotes any positive constant.

Theorem 2.3. *[4] Assume that the sequence of linear positive operator $(H_m)_{m\geq1}$ taking $C_\Psi(\mathbb{R}^+)$ to $B_\Psi(\mathbb{R}^+)$ and fulfilling the prerequisites*

$$\lim_{m\to\infty} \|H_m(t^w; x) - x^w\|_\Psi = 0, \quad w = 0, 1, 2$$

then for every function $f \in C^r_\Psi(\mathbb{R}^+)$

$$\lim_{m\to\infty} \|H_m f - f\|_\Psi = 0.$$

Remark 2.4. *As observed from the equations (6), (8) and in Lemma 2.2, S^*_m are the positive linear operators operating from $C_\Psi(\mathbb{R}^+)$ into $B_\Psi(\mathbb{R}^+)$.*

We have the following for the generalized Szász–Chlodovsky operators:

Theorem 2.5. *[16] Assume that $\{b_m\}$ represents a sequence with $\lim_{m\to\infty} b_m = \infty$ and $\{S^*_m\}$ represents the sequence of linear positive operator acting from $C_{\Psi,[0,b_m]}(\mathbb{R}^+)$ into $B_{\Psi,[0,b_m]}(\mathbb{R}^+)$. If for $w = 0, 1, 2$*

$$\lim_{m\to\infty} \|S^*_m(t^w; x) - x^w\|_{\Psi,[0,b_m]} = 0$$

then for every function $f \in C^r_{\Psi,[0,b_m]}(\mathbb{R}^+)$

$$\lim_{m\to\infty} \|S^*_m f - f\|_{\Psi,[0,b_m]} = 0$$

where $B_{\Psi,[0,b_m]}(\mathbb{R}^+)$, $C_{\Psi,[0,b_m]}(\mathbb{R}^+)$ and $C^r_{\Psi,[0,b_m]}(\mathbb{R}^+)$ indicates the similar way as $B_\Psi(\mathbb{R}^+)$, $C_\Psi(\mathbb{R}^+)$ and $C^r_\Psi(\mathbb{R}^+)$ respectively, however, the functions have been taken on $[0, b_m]$ rather than the positive real axis $\mathbb{R}^+$ and the norm is regarded as:

$$\|f\|_{\Psi,[0,b_m]} = \sup_{0\leq x\leq b_m} \frac{|f(x)|}{\Psi(x)}.$$

Definition 2.6. ***Weighted modulus of continuity***
Since it is more comfortable for us to take the supremum on $|\Psi(x) - \Psi(t)| \leq \sigma$ *instead of* $|x - t| \leq \sigma$*, we employ the weighted modulus of continuity provided in [1]. It is described as follow:*

$$\omega_{\Psi}(f;\sigma) = \omega_{\Psi}(f;\sigma)_{\mathbb{R}^+} = \sup_{x,t\in\mathbb{R}^+, |\Psi(x)-\Psi(t)|\leq\sigma} \frac{|f(x) - f(t)|}{\Psi(t) + \Psi(x)} \tag{9}$$

for every $f \in C_{\Psi}(\mathbb{R}^+)$ *and for each* $\sigma > 0$.

The function $\omega_{\Psi}(f;\sigma)$ *is called weighted modulus of continuity. It is clearly observed that* $\omega_{\Psi}(f;0) = 0$ *for each* $f \in C_{\Psi}(\mathbb{R}^+)$*. Also, the function* $\omega_{\Psi}(f;\sigma)$ *denotes a non-negative non-decreasing with respect to* σ *for* $f \in C_{\Psi}(\mathbb{R}^+)$*. In this case, we focus on the spaces* $C^r_{\Psi}(\mathbb{R}^+)$*,* $U_{\Psi}(\mathbb{R}^+)$ *,* $C_{\Psi}(\mathbb{R}^+)$ *and* $B_{\Psi}(\mathbb{R}^+)$ *with assuming that* $\Psi(0) = 1$ *and* $\inf_{x\geq 0} \Psi'(x) \geq 1$*. In these circumstances, we observe that* $|t - x| \leq |\Psi(t) - \Psi(x)|$*, for each* $t, x \in \mathbb{R}^+$.

There are certain characteristics of the weighted modulus of continuity, ω_{Ψ}*, that are comparable to those of the classically defined modulus of continuity. The following summarizes few of the basic characteristics of* $\omega_{\Psi}(f;\sigma)$*:*

Lemma 2.7. *[17] Assume that* $f \in C^r_{\Psi}(\mathbb{R}^+)$*,*

(i) $\omega_{\Psi}(f;\sigma)$ *is monotonically increasing function of* σ*,* $\sigma \geq 0$.

(ii) For every $f \in C^r_{\Psi}(\mathbb{R}^+)$*,* $\lim_{\sigma\to 0} \omega_{\Psi}(f;\sigma) = 0$.

(iii) For every positive value of α

$$\omega_m(f;\alpha\sigma) \leq 2(1 + \alpha)(1 + \sigma^2)\omega_{\Psi}(f;\sigma). \tag{10}$$

Using the inequality (2.7) and definition of $\omega_{\Psi}(f;\sigma)$ *we obtain*

$$|f(x) - f(t)| \leq 2\left(1 + \frac{|x - t|}{\sigma}\right)(1 + \sigma^2)\omega_{\Psi}(f;\sigma)(1 + x^2)(1 + (t - x)^2) \tag{11}$$

for each $f \in C^r_{\Psi}(\mathbb{R}^+)$ *and* $t, x \in [0, \infty)$.

Lemma 2.8. *[4]* $\lim_{\sigma\to 0} \omega_{\Psi}(f;\sigma) = 0$ *for each* $f \in U_{\Psi}(\mathbb{R}^+)$.

Theorem 2.9. *[4] Assume that* $H_m : C_{\Psi}(\mathbb{R}^+) \to B_{\Psi}(\mathbb{R}^+)$ *denotes a sequence of linear operators with*

$$||H_m(t^0;x) - x^0||_{\Psi^0} = g_m \tag{12}$$

$$||H_m(t;x) - x||_{\Psi^{\frac{1}{2}}} = h_m \tag{13}$$

$$||H_m(t^2;x) - x^2||_{\Psi} = i_m \tag{14}$$

$$||H_m(t^3;x) - x^3||_{\Psi^{\frac{3}{2}}} = j_m \tag{15}$$

where g_m, h_m, i_m, j_m *tends to zero only if* $m \to \infty$.
Now,

$$||H_m(f) - f||_{\Psi^{\frac{3}{2}}} \leq (7 + 4g_m + 2i_m)\omega_{\Psi}(f;\sigma_m) + ||f||_{\Psi} g_m$$

for all $f \in C_{\Psi}(\mathbb{R}^+)$*, where*

$$\sigma_m = 2\sqrt{(g_m + 2h_m + i_m)(1 + g_m)} + g_m + 3h_m + 3i_m + j_m.$$

3. Main Results

In the existing section, we introduce the main results of the study based on the weighted space of continuous functions defined on the interval $[0,\infty)$, for example convergence properties, Voronovskaja type theorem, shape preserving properties and the rate of convergence for the modified Szász-Chlodovsky operators.

3.1. Convergence properties on weighted spaces

In the present subsection we develop the convergence properties of modified Szász-Chlodovsky operators. This is accomplished via the subsequent theorems.

Theorem 3.1. *For every function $f \in C^r_\Psi(\mathbb{R}^+)$,*

$$\lim_{m\to\infty} \|S^*_m(f) - f\|_\Psi = 0.$$

Proof. Utilizing Theorem 2.2 we observe that it is enough to prove the following three conditions as follows:

$$\lim_{m\to\infty} \|S^*_m(t^w;x) - x^w\|_\Psi = 0, \quad w = 0,1,2. \tag{16}$$

From equations (6) and (7) it is clearly seen that, $\|S^*_m(1;x) - 1\|_\Psi = 0$ and $\|S^*_m(t;x) - x\|_\Psi = 0$.

Therefore the requirements (16) are satisfied for $w = 0,1$. Moreover, utilizing the property (8) we obtain

$$\|S^*_m(t^2;x) - x^2\|_\Psi = \sup_{x\in\mathbb{R}^+} \frac{x}{(1+x^2)m} \le \frac{1}{m}. \tag{17}$$

This implies that the requirement (16) also holds for $w = 2$ and by using Theorem 2.2 the proof ends here. □

Theorem 3.2. *If for all $f \in C_\Psi(\mathbb{R}^+)$, then*

$$\|S^*_m(f) - f\|_{\Psi^{\frac{3}{2}}} \le \left(7 + \frac{2}{m}\right)\omega_\Psi\left(f; \frac{2}{\sqrt{m}} + \frac{7}{m}\right).$$

Proof. Utilizing Theorem 2.9, we will evaluate the sequences g_m, h_m, i_m, and j_m. From (6) and (7) it is clear that

$$\|S^*_m(t^0;x) - x^0\|_{\Psi^0} = g_m = 0$$

and

$$\|S^*_m(t;x) - x\|_{\Psi^{\frac{1}{2}}} = h_m = 0$$

Also from (17), we know that

$$i_m = \|S^*_m(t^2;x) - x^2\|_\Psi \le \frac{1}{m}.$$

Since

$$S^*_m(t^3;x) = x^3 + 3\frac{b_m}{m}x^2 + \frac{b_m^2}{m^2}x$$

, we can write

$$\begin{aligned} j_m &= \|S^*_m(t^3;x) - x^3\|_{\Psi^{\frac{3}{2}}} \\ &= \sup_{x\in\mathbb{R}^+}\left\{\frac{3x^2}{(1+x^2)^{\frac{3}{2}}}\frac{1}{m} + \frac{x}{(1+x^2)^{\frac{3}{2}}}\frac{1}{m^2}\right\} \\ &\le \frac{4}{m}. \end{aligned}$$

Hence, the conditions (12) to (15) are fulfilled. Furthermore, from the Theorem 2.9 we obtain

$$\|S^*_m(f) - f\|_{\Psi^{\frac{3}{2}}} \le \left(7 + \frac{2}{m}\right)\omega_\Psi\left(f; \frac{2}{\sqrt{m}} + \frac{7}{m}\right).$$

□

Remark 3.3. *Using Lemma 2.8 and the Theorem 3.2, it yields*

$$\lim_{m\to\infty} \|S_m^*(f) - f\|_{\Psi^{\frac{3}{2}}} = 0$$

for every $f \in U_\Psi(\mathbb{R}^+)$.

Theorem 3.4. *For every* $f \in C_\Psi^r(\mathbb{R}^+)$*, the following inequality*

$$\sup_{x\geq 0} \frac{|S_m^*(f;x) - f(x)|}{(1+x^2)^3} \leq J_1 \omega_\Psi \left(f; \sqrt{\frac{b_m}{m}}\right)$$

is fulfilled for a sufficiently large m*, with* J_1 *being any constant independent of* b_m.

Proof. Using (11), we have

$$\begin{aligned}
|S_m^*(f;x) - f(x)| &\leq 2(1+\sigma_m^2)\omega_\Psi(f;\sigma_m)(1+x^2)e^{\frac{-mx}{b_m}} \sum_{r=0}^{\infty} \left(\frac{mx}{b_m}\right)^r \frac{1}{r!} \\
&\times \left(1 + \frac{|\frac{rb_m}{m} - x|}{\sigma_m}\right)\left(1 + \left(\frac{rb_m}{m} - x\right)^2\right) \\
&\leq 4\omega_\Psi(f;\sigma_m)(1+x^2)\Bigg\{1 + \frac{1}{\sigma_m} e^{\frac{-mx}{b_m}} \sum_{r=0}^{\infty} \left(\frac{mx}{b_m}\right)^r \frac{1}{r!} \left|\frac{rb_m}{m} - x\right| \\
&+ e^{\frac{-mx}{b_m}} \sum_{r=0}^{\infty} \left(\frac{mx}{b_m}\right)^r \frac{1}{r!} \left(\frac{rb_m}{m} - x\right)^2 \\
&+ \frac{1}{\sigma_m} e^{\frac{-mx}{b_m}} \sum_{r=0}^{\infty} \left(\frac{mx}{b_m}\right)^r \frac{1}{r!} \left|\frac{rb_m}{m} - x\right| \left(\frac{rb_m}{m} - x\right)^2 \Bigg\}
\end{aligned}$$

for every $\sigma_m > 0$.

Utilizing the Cauchy-Schwartz inequality, it yields

$$|S_m^*(f;x) - f(x)| \leq 4\omega_\Psi(f;\sigma_m)(1+x^2)\left(1 + \frac{2}{\sigma_m}\sqrt{R_1} + R_1 + \frac{1}{\sigma_m}R_2\right) \tag{18}$$

where,

$$R_1 = e^{\frac{-mx}{b_m}} \sum_{r=0}^{\infty} \left(\frac{mx}{b_m}\right)^r \frac{1}{r!} \left(\frac{rb_m}{m} - x\right)^2$$

and

$$R_2 = e^{\frac{-mx}{b_m}} \sum_{r=0}^{\infty} \left(\frac{mx}{b_m}\right)^r \frac{1}{r!} \left(\frac{rb_m}{m} - x\right)^4.$$

Employing an easily understood calculation, we acquire that

$$S_m^*(t^3;x) = x^3 + 3\frac{b_m}{m}x^2 + \frac{b_m^2}{m^2}x \tag{19}$$

and

$$S_m^*(t^4;x) = x^4 + 6\frac{b_m}{m}x^3 + 7\frac{b_m^2}{m^2}x^2 + \frac{b_m^3}{m^3}x. \tag{20}$$

From (6), (7), (8), (19) and (20), we obtain

$$R_1 = S_m^*((t-x)^2; x) = \frac{b_m}{m} x \tag{21}$$

and

$$R_2 = S_m^*((t-x)^4; x) = -5\frac{b_m}{m} x^3 + 3\frac{b_m^2}{m^2} x^2 + \frac{b_m^3}{m^3} x. \tag{22}$$

Using condition (3) in (21) and (22), we can write

$$R_1 = O\left(\frac{b_m}{m}\right)(x)$$

and

$$R_2 = O\left(\frac{b_m}{m}\right)(x^3 + x^2 + x)$$

Substituting these inequalities into (18), it gives

$$|S_m^*(f;x) - f(x)| \leq 4\omega_\Psi(f;\sigma_m)(1+x^2)\Big\{1 + \frac{2}{\sigma_m}\sqrt{O\left(\frac{b_m}{m}\right)(x)} + O\left(\frac{b_m}{m}\right)(x) + \frac{1}{\sigma_m} O\left(\frac{b_m}{m}\right)(x^3+x^2+x)\Big\}.$$

Choosing $\sigma_m = \sqrt{\frac{b_m}{m}}$, whenever m's are large enough, we get

$$\sup_{x \geq 0} \frac{|S_m^*(f;x) - f(x)|}{(1+x^2)^3} \leq J_1 \omega_\Psi\left(f; \sqrt{\frac{b_m}{m}}\right),$$

where J_1 is a constant that exists independently of b_m. □

3.2. A Voronovskaya type theorem

In this subsection, the pointwise convergence of the Szász-Chlodovsky operator is provided. To present the convergence, we develop a Voronovskaya type theorem by utilizing the method introduced of [11].

Theorem 3.5. *Let $f \in C_\Psi(\mathbb{R}^+)$, $x \in \mathbb{R}^+$. Assume that f has the first and second derivative at x. Provided that the second derivative of f is bounded on $\mathbb{R}^+$, then we acquire the following*

$$\lim_{m\to\infty} m(S_m^*(f;x) - f(x)) = \frac{1}{2} b_m x f''(x).$$

Proof. Utilizing Taylor's expansion for the function f at point $x \in \mathbb{R}^+$, $\exists\, \eta$ located between t and x so that

$$f(t) = f(x) + f'(x)(t-x) + \frac{1}{2} f''(x)(t-x)^2 + \lambda_x(t)(t-x)^2, \tag{23}$$

$$\lambda_x(t) = \frac{f''(\eta) - f''(x)}{2}. \tag{24}$$

Notice that, the assumption on f along with the definition (24) guaranteed that,

$$|\lambda_x(t)| \leq N, \quad \forall t,$$

and it converge to zero whenever $t \to x$, notice that $N > 0$ is any constant.

Apply the operator (5) to the above inequality, we obtain

$$S_m^*(f;x) - f(x) = f'(x)S_m^*((t-x);x) + \frac{1}{2}f''(x)S_m^*((t-x)^2;x) + S_m^*(\lambda_x(t)(t-x)^2;x).$$

Utilizing (6), (7) and (8), it leads to

$$\lim_{m\to\infty} mS_m^*((t-x);x) = 0.$$

$$\lim_{m\to\infty} mS_m^*((t-x)^2;x) = b_m x.$$

and thus

$$\lim_{m\to\infty} m(S_m^*(f;x) - f(x)) = \frac{1}{2}b_m x f''(x) + \lim_{m\to\infty} mS_m^*(\lambda_x(t)(t-x)^2;x).$$

We now calculate the final term on the RHS of the equality above as follows:
Let $\varepsilon > 0$ and choose $\alpha > 0$ so that, $|\lambda_x(t)| < \varepsilon$ for $|x-t| < \alpha$. Again it is clearly observed by the condition $\Psi(0) = 1$, $\inf_{x\in\mathbb{R}^+} \Psi'(x) \geq 1$ that $|\Psi(x) - \Psi(t)| = \Psi'(\xi)|x-t| \geq |x-t|$.

Therefore, if $|x-t| < \alpha$, then, $|\lambda_x(t)(t-x)^2| < \varepsilon(t-x)^2$, while if $|x-t| \geq \alpha$, then because $|\lambda_x(t)| \leq N$, we obtain,

$$|\lambda_x(t)(t-x)^2| \leq \frac{N}{\alpha^2}(t-x)^4.$$

Consequently, we are able to write

$$S_m^*(\lambda_x(t)(t-x)^2;x) < \varepsilon S_m^*((t-x)^2;x) + \frac{N}{\alpha^2}S_m^*((t-x)^4;x).$$

Direct computations demonstrate that

$$S_m^*((t-x)^4;x) = O\left(\frac{1}{m^2}\right),$$

and we terminate

$$\lim_{m\to\infty} mS_m^*(\lambda_x(t)(t-x)^2;x) = 0.$$

Therefore, the proof of the theorem completes here. □

3.3. *The shape preserving properties of $S_m^*(f)$*

This subsection investigates the distinctive features of the Szász-Chlodovsky operator in the circumstance that the real valued function $f(x)$, stated on the unbounded interval $[0,\infty)$, is increasing (decreasing) or convex. Stated differently, the shape preserving properties of the Szász-Chlodovsky operators are demonstrated by verifying that $S_m^*(f)$ maintains the convexity [2, 12, 20] .
Before we provide other characteristics of $S_m^*(f)$, we first infer a few features of the functions given by

$$P_{m,r}(x) = e^{\frac{-mx}{b_m}}\left(\frac{mx}{b_m}\right)^r \frac{1}{r!}, \quad x \in [0,\infty),\ m \in \mathbb{N},\ r \geq 0.$$

Lemma 3.6. *[4] If $x \in [0,\infty)$, $m \in \mathbb{N}$, $r \geq 0$, then the following identities holds*

(i) $P'_{m,r}(x) = m\left(\frac{x}{b_m}\right)'(P_{m,r-1}(x) - P_{m,r}(x))$,

(ii) $\left(\frac{x}{b_m}\right)P'_{m,r}(x) = \left(\frac{x}{b_m}\right)' P_{m,r}(x)\left(r - \frac{mx}{b_m}\right)$, *where* $P_{m,-1}(x) = 0$.

To compare $S_m^*(f)$ by $f(x)$, we want the symmetric forms of divided difference for the function f. Suppose that $x_0, x_1, ..., x_m$ denotes the distinct points in the domain of f.
Denote

$$f(x_0, x_1, ..., x_m) = \sum_{j=0}^{m} \frac{f(x_j)}{\prod_{i\neq j}^{m}(x_j - x_i)}$$

where, j remains fixed while i taking all the values starting from 0 up to m, with j excluded.

Now we suggest an illustration of the modified Szász-Chlodovsky operator $S_m^*(f)$ by means of the divided differences for the function f.

Theorem 3.7. *For every $f \in C_\Psi(\mathbb{R}^+)$, $m \in \mathbb{N}$ and $x \in [0, \infty)$, so that $\frac{x}{b_m} \neq \frac{r}{m}$, $r = 0, 1, ...$, then the following identity maintains:*

$$S_m^*(f; x) - f(x) = \frac{x}{mb_m} \sum_{r=0}^{\infty} f\left[\frac{x}{b_m}, \frac{rb_m}{m}, \frac{(r+1)b_m}{m}\right] P_{m,r}(x)$$

Proof. Since $S_m^*(1; x) = 1$, we obtain,

$$\begin{aligned} S_m^*(f; x) - f(x) &= \sum_{r=0}^{\infty} \left(f\left(\frac{rb_m}{m}\right) - f(x)\right) P_{m,r}(x) \\ &= \sum_{r=0}^{\infty} \left(\frac{rb_m}{m} - \frac{x}{b_m}\right) f\left[\frac{x}{b_m}, \frac{rb_m}{m}\right] P_{m,r}(x). \end{aligned}$$

Utilizing Lemma 3.6 (*i*) and (*ii*), it yields

$$\begin{aligned} S_m^*(f; x) - f(x) &= \frac{x}{mb_m} \left(\frac{b_m}{x}\right)' \sum_{r=0}^{\infty} P'_{m,r}(x) f\left[\frac{x}{b_m}, \frac{rb_m}{m}\right] \\ &= \frac{x}{b_m} \sum_{r=0}^{\infty} f\left[\frac{x}{b_m}, \frac{rb_m}{m}\right] (P_{m,r-1}(x) - P_{m,r}(x)) \\ &= \frac{x}{b_m} \sum_{r=0}^{\infty} \left(f\left[\frac{x}{b_m}, \frac{(r+1)b_m}{m}\right] - f\left[\frac{x}{b_m}, \frac{rb_m}{m}\right]\right) P_{m,r}(x). \end{aligned}$$

By recalling the divided difference concept, we have

$$f\left[\frac{x}{b_m}, \frac{(r+1)b_m}{m}\right] - f\left[\frac{x}{b_m}, \frac{rb_m}{m}\right] = \frac{1}{m} f\left[\frac{x}{b_m}, \frac{rb_m}{m}, \frac{(r+1)b_m}{m}\right]$$

So that

$$S_m^*(f; x) - f(x) = \frac{x}{mb_m} \sum_{r=0}^{\infty} f\left[\frac{x}{b_m}, \frac{rb_m}{m}, \frac{(r+1)b_m}{m}\right] P_{m,r}(x).$$

□

Consequently, from Theorem 3.7, we obtain the following Corollary.

Corollary 3.8. *If the function $f(x)$ is convex on the interval $[0, \infty)$, then we have*

$$S_m^*(f; x) \geq f(x)$$

for $x \in [0, \infty)$ and $\forall m \geq 0$ so that $\frac{x}{b_m} \neq \frac{r}{m}$, $(r = 0, 1, 2, ...)$. This leads to the next theorem.

Theorem 3.9. *Suppose that $f(x)$ is a convex function on the unbounded interval $[0,\infty)$, then we have*

$$S^*_m(f;x) \geq S^*_{m+1}(f;x)$$

$\forall m \geq 0$ *and* $0 \leq x < \infty$. *Moreover, if the same function f is linear then* $S^*_m(f;x) = S^*_{m+1}(f;x)$.

Proof. Firstly, notice that

$$S^*_m(f;x) - S^*_{m+1}(f;x) = e^{-(m+1)\frac{x}{b_m}} \times \left[\frac{e^{-\frac{mx}{b_m}}}{e^{-(m+1)\frac{x}{b_m}}}\sum_{r=0}^{\infty} f\left(\frac{rb_m}{m}\right)\left(\frac{mx}{b_m}\right)^r \frac{1}{r!} - \sum_{r=0}^{\infty} f\left(\frac{rb_m}{m+1}\right)\left(\frac{(m+1)x}{b_m}\right)^r \frac{1}{r!}\right].$$

Consequently, we obtain

$$\begin{aligned} S^*_m(f;x) - S^*_{m+1}(f;x) = e^{-(m+1)\frac{x}{b_m}}\Big[e^{\frac{x}{b_m}}\sum_{r=0}^{\infty} f\left(\frac{rb_m}{m}\right)\left(\frac{mx}{b_m}\right)^r \frac{1}{r!} \\ - \sum_{r=0}^{\infty} f\left(\frac{rb_m}{m+1}\right)\left((m+1)\frac{x}{b_m}\right)^r \frac{1}{r!}\Big]. \end{aligned} \tag{25}$$

Furthermore, if we employ Cauchy rule for multiplication for the two series, it gives

$$\begin{aligned} e^{\frac{x}{b_m}}\sum_{r=0}^{\infty} f\left(\frac{rb_m}{n}\right)\left(\frac{mx}{b_m}\right)^r \frac{1}{r!} &= \sum_{r=0}^{\infty}\left(\frac{x}{b_m}\right)^r \frac{1}{r!}\sum_{r=0}^{\infty} f\left(\frac{rb_m}{m}\right)\left(\frac{mx}{b_m}\right)^r \frac{1}{r!} \\ &= \sum_{r=0}^{\infty}\sum_{w=0}^{r} f\left(\frac{w}{m}\right)\frac{\left(\frac{x}{b_m}\right)^r m^w}{w!(r-w)!}, \end{aligned}$$

so that when joined to (25), it yields,

$$\begin{aligned} S^*_m(f;x) - S^*_{m+1}(f;x) = e^{-(m+1)\frac{x}{b_m}}\sum_{r=0}^{\infty}\Big(\sum_{w=0}^{r} f\left(\frac{w}{m}\right)\frac{m^w}{w!(r-w)!} \\ - f\left(\frac{rb_m}{m+1}\right)\left(\frac{(m+1)^r}{r!}\right)\Big)\left(\frac{x}{b_m}\right)^r. \end{aligned} \tag{26}$$

It will be adequate to demonstrate that the inequality

$$f\left(\frac{rb_m}{m+1}\right) \leq \frac{r!}{(m+1)^r}\sum_{w=0}^{r} f\left(\frac{w}{m}\right)\frac{m^w}{w!(r-w)!}. \tag{27}$$

holds. Let $\lambda_w = \frac{r!}{(m+1)^r}\frac{m^w}{w!(r-w)!}$ and $x_w = \frac{w}{m}$ for $w \in [0,r]$. We can write

$$\begin{aligned} \sum_{w=0}^{r}\lambda_w x_w &= \frac{r!}{(m+1)^r}\sum_{w=0}^{r}\frac{m^w}{w!(r-w)!}\frac{w}{m} \\ &= \frac{r!}{(m+1)^r}\sum_{w=1}^{r}\frac{m^{w-1}}{(w-1)!(r-w)!} \\ &= \frac{r}{(m+1)^r}\sum_{w=0}^{r-1}\binom{r-1}{w}m^w \\ &= \frac{r}{m+1} \end{aligned}$$

and,

$$\sum_{w=0}^{r} \lambda_w = 1.$$

Notice that $f(x)$ is convex as given, So,the inequality (27) could be verified via the above equalities. Then, utilizing (26), we acquire

$$S_m^*(f;x) \geq S_{m+1}^*(f;x)$$

$\forall m \geq 0$ and for $x \in [0, \infty)$

Moreover, if f is linear then we can write:

$$f\left(\frac{rb_m}{m+1}\right) = \frac{r!}{(m+1)^r} \sum_{w=0}^{r} f\left(\frac{w}{m}\right) \frac{m^w}{w!(r-w)!}.$$

Hence, this finishes the proof of the theorem. □

3.4. The rate of convergence

This subsection compares the rates of convergence of the Szász-Chlodovsky operator and the classical Szász-Mirakyan operators in terms of the degree of estimation it is observed that the modified operators (Szász-Chlodovsky operator) has atleast a good estimatimation in comparison with the classically defined Szász-Mirakyan operators $S_m(f(x))$ for a particular interval.

For this reason, let us start by summarizing the rates of convergence of both the modified operators and its corresponding classical. Let f be a continuous function defined on the interval $[0, \infty)$, again let $\omega_\Psi(f;\sigma)$ denote a standard modulus of continuity. So, the rates of convergence of these two operators are defined as:

$\|S_m(f(x)) - f(x)\| \leq 2\omega_\Psi(f;\sigma^*)$ and $\|S_m^*(f(x)) - f(x)\| \leq 2\omega_\Psi(f;\sigma)$,

with $\sigma^* = \sqrt{\frac{x}{m}}$ and $\sigma = \sqrt{\frac{x}{mb_m}}$ respectively.

Let us now provide the following theorem in order to verify that the modified operator (Szász-Chlodovsky operator) have better rate of convergence in comparison with the classical Szász-Mirakyan operator.

Theorem 3.10. *Let $C[0, \infty)$ be the class of continuous functions, the rate of convergence of the Szász-Chlodovsky operator is better compared with the classical Szász-Mirakyan operator because the inequality*

$$\sigma^* \geq \sigma,$$

holds true $\forall x \in [0, \infty)$.

Proof. Let $f \in C[0, \infty$. So, to display that the Szász-Chlodovsky operator has better rate of convergence than its classical correspondence, we want to be capable to display that the subsequent inequality also holds true $\forall x \in [0, \infty)$:

$$\sqrt{\frac{x}{m}} \geq \sqrt{\frac{x}{mb_m}}.$$

For this reason, we introduce a function which is

$$\Omega(x) = \sqrt{\frac{x}{m}} - \sqrt{\frac{x}{mb_m}}.$$

If we can be able to illustrate that $\Omega(x)$ is positive, then our claim will be verified as correct.

$\forall x \in [0, \infty)$, $m \in \mathbb{N}$, it is clearly that $\sqrt{\frac{x}{m}}$ is greater than $\sqrt{\frac{x}{mb_m}}$ for the case $(b_m)_{m=1}^{\infty} = \{1, 2, 3, ...\}$. It is clearly seen that $\Omega(x)$ does not change the sign on $[0, \infty)$. Therefore, under this circumstance it can be generalized that the function $\Omega(x)$ is a positive. Hence, the proof is completed. □

4. Conclusion

In the existing paper, the modified Szász-Chlodovsky operators are developed and their estimation properties are presented as well. Moreover, some shape-preserving properties for instance the convexity and the monotonicity of the Szász-Chlodovsky operators denoted by $S_m^*(f(x))$ are scrutinized. Furthermore, the asymptotic formula of operators are established by considering the Voronovskaya type theorem. Finally, the modified Szász-Chlodovsky operators and the classical Szász-Mirakyan operators are compared to validate the theoretical results.

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