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Berger measures for the Schur product of weighted shifts

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Abstract. In this paper we present a concrete Berger measure for the Schur product of weighted shifts. This measure will be seen as a convolution of their Berger measures. This is related to the p-th power problem for measures. For $n \in \mathbb{N}$, the n-th power of a Berger measure is a convolution of all same measures and this case was solved partially by the author of this paper. We will extend this problem to a convolution of any Berger measures. We investigate a convolution of mutually distinct measures and then we discuss any combination of Berger measures, and any two more general weighted shifts. Since a Berger measure is closely related to subnormal weighted shifts, our result is helpful for the study of subnormality and add to the very small list of subnormal weighted shifts for which Berger measure is known concretely.

1. Introduction and preliminaries

Let $\mathcal{H}$ be an infinite dimensional complex Hilbert space and let $\mathcal{B}(\mathcal{H})$ be an algebra of all bounded linear operators on $\mathcal{H}$. A bounded operator $T \in \mathcal{B}(\mathcal{H})$ is said to be *normal* if $TT^* = T^*T$, *hyponormal* if $TT^* - T^*T \geq 0$ and *subnormal* if it has a normal extension of T, i.e., $T = N|_{\mathcal{H}}$ for some normal operator N on a Hilbert space $\mathcal{K}$ including $\mathcal{H}$. Now we give an equivalent condition for subnormality, which is called the Bram-Halmos' criterion ([7],[16],[8, III.1.9]):

$$T \in \mathcal{B}(\mathcal{H}) \text{ is subnormal if and only if } M_n(T) \geq 0 \text{ for all } n \in \mathbb{N},$$

where an $(n+1) \times (n+1)$ operator matrix $M_n(T)$ is denoted by

$$M_n(T) := \left[T^{*j}T^i\right]_{i,j=0}^n = \begin{bmatrix} I & T^* & \cdots & T^{*n} \\ T & T^*T & \cdots & T^{*n}T \\ \vdots & \vdots & \ddots & \vdots \\ T^n & T^*T^n & \cdots & T^{*n}T^n \end{bmatrix}, \ n \in \mathbb{N}.$$

For each $n \in \mathbb{N}$, an operator $T \in \mathcal{B}(\mathcal{H})$ is said to be *n-hyponormal* if $M_n(T) \geq 0$, i.e.,

$$T \text{ is subnormal } \iff T \text{ is } n\text{-hyponormal for all } n \in \mathbb{N}.$$

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It is easy to know that 1-hyponormality is equivalent to hyponormality, so we have following implications;

$$\text{subnormal} \Longrightarrow \cdots \Longrightarrow n\text{-hyponormal} \Longrightarrow \cdots \Longrightarrow 2\text{-hyponormal} \Longrightarrow \text{hyponormal}.$$

Recall that a *weighted shift* W_α with a weight sequence $\alpha = \{\alpha_k\}_{k=0}^\infty$ is defined by

$$W_\alpha(e_k) = \alpha_k e_{k+1} \quad \text{for all} \quad k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\},$$

where $\{e_k\}_{k=0}^\infty$ is the canonical orthonormal basis for ℓ^2. It is easy to see that W_α can never be normal, and that W_α is hyponormal if and only if $\alpha_n \leq \alpha_{n+1}$ for all $n \in \mathbb{N}_0$. For a weight sequence $\alpha = \{\alpha_k\}_{k=0}^\infty$, formally we define the *moment sequence* $\gamma \equiv \gamma(\alpha) = \{\gamma_k\}_{k=0}^\infty \equiv \{\gamma_k(\alpha)\}_{k=0}^\infty$ of α (or W_α) by

$$\gamma_k \equiv \gamma_k(\alpha) = \begin{cases} 1, & k = 0, \\ \alpha_0^2 \cdots \alpha_{k-1}^2, & k \in \mathbb{N}. \end{cases}$$

For the sequence γ, we denote a Hankel matrix of γ by

$$H_{n,k}(\gamma) := \left[\gamma_{k+i+j}\right]_{i,j=0}^n = \begin{bmatrix} \gamma_k & \gamma_{k+1} & \cdots & \gamma_{k+n} \\ \gamma_{k+1} & \gamma_{k+2} & \cdots & \gamma_{k+n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{k+n} & \gamma_{k+n+1} & \cdots & \gamma_{k+2n} \end{bmatrix}, \quad \text{for} \quad n, k \in \mathbb{N}_0.$$

From [10, Theorem 4], we can obtain that an equivalent condition for the subnormality of weighted shifts as follows:

$$W_\alpha \text{ is subnormal if and only if } H_{n,k}(\gamma) \geq 0 \text{ for all } k \in \mathbb{N}_0 \text{ and } n \in \mathbb{N}.$$

Recall that another equivalent condition for the subnormality of weighted shifts as follows:

Theorem 1.1 ([8], [15], Berger's Theorem). *Let W_α be a weighted shift with a weight sequence α and $\gamma = \{\gamma_k\}_{k=0}^\infty$ be a moment sequence of α. Then W_α is subnormal if and only if there exists a probability Borel measure μ (called the Berger measure) on $\left[0, \|W_\alpha\|^2\right]$ with $\|W_\alpha\|^2 \in \operatorname{supp} \mu$ such that*

$$\gamma_k = \int_{[0, \|W_\alpha\|^2]} t^k d\mu(t), \quad k \in \mathbb{N}_0.$$

From the Berger's theorem, we can see that there is a relationship between the subnormality of weighted shifts and the moment problem. By using the moment problems, we can understand the subnormality.

For two sequences $\alpha = \{\alpha_n\}_{n=0}^\infty$ and $\beta = \{\beta_n\}_{n=0}^\infty$, $\alpha \circ \beta := \{\alpha_n \beta_n\}_{n=0}^\infty$ is called the *Schur product* of α and β. Similarly, for two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ of the same size, the matrix $A \circ B := [a_{ij} b_{ij}]$ is called the *Schur product* of A and B. Then we can see

$$W_\alpha \circ W_\beta = W_{\alpha \circ \beta}.$$

It follows from [12, Theorem 2.3, Corollary 2.4] that if W_α and W_β are n-hyponormal (subnormal, respectively), then $W_{\alpha\circ\beta}$ is n-hyponormal (subnormal, respectively). For moment sequences $\gamma(\alpha)$ and $\gamma(\beta)$ of α and β, respectively, we can see

$$\gamma(\alpha \circ \beta) = \gamma(\alpha) \circ \gamma(\beta) = \{\gamma_n(\alpha)\gamma_n(\beta)\}_{n=0}^\infty. \tag{1.1}$$

In this paper we consider the moment problem for the Schur product of weighted shifts and we are interested in what the Berger measure for the Schur product is, and how such measure is obtained.

If μ and ν are probability measures on $\mathbb{R}_+ := \{x \in \mathbb{R} : x \geq 0\}$, then $\mu * \nu$ is called the *convolution* of μ and ν and is defined by, for all Borel set $E \subset \mathbb{R}_+$,

$$(\mu * \nu)(E) := (\mu \times \nu)\left(p^{-1}(E)\right),$$

where $p(s,t) = st$ for $s, t \geq 0$ (see [17]). Then $\mu * \nu$ is well-defined probability measure on $\mathbb{R}_+$. By the Fubini's theorem, we have

$$\left(\int t^n d\mu\right)\left(\int t^n d\nu\right) = \iint s^n t^n d\mu(s) d\nu(t)$$
$$= \iint (st)^n d\left((\mu \times \nu) \circ p^{-1}\right)(st)$$
$$= \int u^n d(\mu * \nu)(u).$$

Then we can see that the Berger measure of $W_{\alpha \circ \beta}$ is the convolution of the Berger measures of W_α and W_β. This is our goal for this paper:

Given 2 *or more Berger measures, what is their convolution?*

We now introduce problems about convolutions of measures. In [11], Curto and Exner introduce the *square and square root problems for measure* as follows:

Problem 1.2 (**Square and square root problems for measure**)**.** Let μ and ν be positive probability Borel measures on $\mathbb{R}_+$. Suppose

$$\mu = \nu * \nu.$$

Then μ is called the *square* of ν and ν is called the *square root* of μ.

(i) Given ν, find μ.
(ii) Given μ, find ν if it exists.

As a consequence, the square and square root problems can be combined to extend generally as follows:

Problem 1.3 (**The p-th power problem for measure**)**.** Suppose $p > 0$. Let ν be a positive probability Borel measure with $\operatorname{supp} \nu \subseteq \mathbb{R}_+$. Is there a positive Borel measure μ satisfying

$$\int t^n d\mu(t) = \left(\int t^n d\nu(t)\right)^p \tag{1.2}$$

for all $n \in \mathbb{N}_0$? If so, find such measure μ.

If two probability Borel measures μ and ν are satisfying (1.2), μ is called the *p-th power* of ν and ν is called the *p-th power root* of μ. The equation (1.2) can be rewritten in terms of measures as $\mu = \nu^{*p}$ (see [18][1)]). For $p > 0$ and a sequence $\alpha = \{\alpha_n\}_{n=0}^\infty$, we denote by $\alpha^p := \left\{\alpha_n^p\right\}_{n=0}^\infty$. Then the p-th power of the Berger measure of W_α is the Berger measure of W_{α^p} if it exists. We now introduce a class of weighted shifts whose Berger measure is infinitely divisible in the classical sense.

Definition 1.4 ([5])**.** Let α be a weight sequence. The associated weighted shift W_α is called *moment infinitely divisible* (MID) if W_{α^p} is subnormal for all $p > 0$.

It is easy to see from the Berger's theorem that every MID-shift has the Berger measure and its p-th powers for all $p > 0$.

To study the operator theory, many operator theorists use weighted shifts, especially, they use the Bergman weight sequence $\left\{\sqrt{\frac{n+1}{n+2}}\right\}_{n=0}^\infty$. This is because the moment sequence of the Bergman sequence is

[1)] The authors in [18] consider p as an integer greater than 1, but we extend to positive real numbers.

$\left\{\frac{1}{n+1}\right\}_{n=0}^{\infty}$ and its Hankel determinant can be obtained easily. In this paper we will use more generalized Bergman weight sequences, which is defined by

$$\alpha = \left\{ \sqrt{\frac{an+b}{cn+d}} \right\}_{n=0}^{\infty},$$

where $a, b, c, d > 0$ and $ad > bc$. In [13] Curto-Poon-Yoon defined a new class of generalized Bergman shifts, whose weight sequence is such α. They denote $S(a, b, c, d)$ for the weighted shift W_α. In [6] the authors named it the *homographic shift*. This is our main model in this paper. We can see some properties from [13], [9], [14] and [4]:

1° ([13, Theorem 2.7]) $S(a, b, c, d)$ is subnormal.

2° ([9, Theorem 2.4]) The Berger measure of $S(a, b, c, d)$ is

$$d\mu(t) = \frac{\Gamma\left(\frac{d}{c}\right)}{\Gamma\left(\frac{b}{a}\right)\Gamma\left(\frac{d}{c}-\frac{b}{a}\right)} \left(\frac{c}{a}\right)^{\frac{b}{a}} t^{\frac{b}{a}-1} \left(1-\frac{c}{a}t\right)^{\frac{d}{c}-\frac{b}{a}-1} dt \tag{1.3}$$

with $\operatorname{supp} \mu = \left[0, \frac{a}{c}\right]$, where Γ is the classical gamma function.

3° ([14, Theorem 3.4]) $S(a, b, c, d)$ is MID.

4° ([4, Theorem 2.1]) If μ is the Berger measure of the shift $S(1, q, 1, q+1)$ and $p > 0$, then its p-th power is

$$(d\mu(t))^p = d(\mu^{*p})(t) = \frac{q^p}{\Gamma(p)} t^{q-1} \left(-\ln t\right)^{p-1} dt \tag{1.4}$$

with $\operatorname{supp} \mu = [0, 1]$.

5° ([4, Corollary 3.3]) If μ_j is the Berger measure of $S(1, q, 1, q + j - 1)$ and $j = 2, 3, \ldots$, then its measure is

$$d\mu_j(t) = \frac{\Gamma(q+j-1)}{\Gamma(q)\Gamma(j-1)} t^{q-1}(1-t)^{j-2} dt = \frac{q(q+1)\cdots(q+j-2)}{(j-2)!} \sum_{i=0}^{j-2} (-1)^i \binom{j-2}{i} t^{i+q-1} dt \tag{1.5}$$

with $\operatorname{supp} \mu = [0, 1]$, where $\binom{n}{k}$ denotes the usual binomial coefficient and its square measure, for $j = 2, 3, 4$, can be obtain as follows:

(i) $(d\mu_2(t))^2 = (qt^{q-1}dt)^2 = -q^2 t^{q-1} \ln t dt$,

(ii) $(d\mu_3(t))^2 = q^2(q+1)^2 t^{q-1} \left[2(t-1) - (t+1)\ln t\right] dt$,

(iii) $(d\mu_4(t))^2 = \frac{q^2(q+1)^2(q+2)^2}{4} t^{q-1} \left[3(t^2-1) - (t^2+4t+1)\ln t\right] dt$.

6° ([4, Theorem 4.2]) If μ is the Berger measure of $S(1, q, 1, q + 2)$, then its square root measure is

$$\sqrt{d\mu(t)} = \sqrt{q(q+1)} t^{q-\frac{1}{2}} \sum_{m=0}^{\infty} \frac{(\ln t)^{2m}}{16^m (m!)^2} dt$$

with $\operatorname{supp} \mu = [0, 1]$.

The above properties are related to Berger measures of homographic shifts. We also studied properties for Berger measures of the Schur product of specific homographic shifts. The equation (1.4) with a positive integer p shows the Berger measure for the p-th Schur power of a homographic shift. On the other hand, we consider the Schur product of mutually distinct homographic shifts in this paper.

This paper is organized as follows. In Section 2, we present a concrete Berger measure of the Schur product of finitely many second Agler-type shifts (the definition is shown in Section 3), which are mutually distinct. Furthermore, we give the result for the mixed case which is the Schur product of weighted shifts, where some shifts have the same weight sequence and the others are mutually distinct. In Section 3, we give the concrete Berger measure of any two generalized Agler-type shifts.

2. Berger measures of Schur products

In this section we will find the Berger measures for the Schur product of finitely many generalized Bergman shifts. Firstly, we consider Schur product of two shifts. Let α and β be weight sequences associated to weighted shifts $W_\alpha \equiv S(1, q_1, 1, q_1 + 1)$ and $W_\beta \equiv S(1, q_2, 1, q_2 + 1)$, respectively. Then we can see easily that the Schur product of α and β is

$$\alpha \circ \beta = \left\{ \sqrt{\frac{(n+q_1)(n+q_2)}{(n+q_1+1)(n+q_2+1)}} \right\}_{n=0}^{\infty},$$

and its associated weighted shift $W_{\alpha\circ\beta}$ is subnormal. Now we find the Berger measure for $W_{\alpha\circ\beta}$. Actually the case of $q_1 = q_2$ is the square problem which is same as (1.4) with $p = 2$, so we assume that $q_1 \neq q_2$.

Lemma 2.1. *Suppose q_1 and q_2 are positive real numbers. Let α and β be weight sequences associated to weighted shifts $W_\alpha \equiv S(1, q_1, 1, q_1 + 1)$ and $W_\beta \equiv S(1, q_2, 1, q_2 + 1)$, respectively. If $q_1 \neq q_2$, then the Berger measure μ of $W_{\alpha\circ\beta}$ is given by*

$$d\mu(t) = \frac{q_1 q_2}{q_2 - q_1}\left(t^{q_1-1} - t^{q_2-1}\right)dt$$

with $\operatorname{supp}\mu = [0, 1]$.

Proof. For the reader's convenience, we give an elementary proof. Let $\gamma_n(\alpha)$, $\gamma_n(\beta)$ and $\gamma_n(\alpha\circ\beta)$ be moments for W_α, W_β and $W_{\alpha\circ\beta}$, respectively. It follows from (1.1) and (1.3) that

$$\begin{aligned}
\gamma_n(\alpha\circ\beta) = \gamma_n(\alpha)\gamma_n(\beta) &= \int_0^1 t^n q_1 t^{q_1-1}dt \int_0^1 s^n q_2 s^{q_2-1}ds \\
&= q_1 q_2 \int_0^1 \int_0^1 (ts)^{n+q_1-1} s^{q_2-q_1} dtds \\
&= q_1 q_2 \int_0^1 \int_0^s u^{n+q_1-1} s^{q_2-q_1-1} duds \quad \text{(substituting } u = ts) \\
&= q_1 q_2 \int_0^1 \int_u^1 u^{n+q_1-1} s^{q_2-q_1-1} dsdu \\
&= \int_0^1 u^n \left(\frac{q_1 q_2}{q_2 - q_1} u^{q_1-1}\left(1 - u^{q_2-q_1}\right)\right) du.
\end{aligned}$$

Hence, $d\mu(t) = \frac{q_1 q_2}{q_2 - q_1} t^{q_1-1}\left(1 - t^{q_2-q_1}\right)dt$. □

This lemma is related to the Aluthge transform. A operator $T \in \mathcal{B}(\mathcal{H})$ can be represented as the polar decomposition $T = U|T|$, where $|T| = (T^*T)^{\frac{1}{2}}$ and U is a partial isometry with $\ker U = \ker T$ and $\ker U^* = \ker T^*$. Then the *Aluthge transform* of T is defined by $\widetilde{T} = |T|^{\frac{1}{2}} U |T|^{\frac{1}{2}}$ [2]. It is well-known that the Aluthge transforms of weighted shifts are also weighted shifts. For a weight sequence $\alpha = \{\alpha_n\}_{n=0}^{\infty}$, the weight sequence of $\widetilde{W}_\alpha$ is $\left\{\sqrt{\alpha_n \alpha_{n+1}}\right\}_{n=0}^{\infty}$, $n \in \mathbb{N}$. See the following example.

Example 2.2. In Lemma 2.1, if $q_1 = q$ and $q_2 = q + 1$, then $W_{\alpha\circ\beta} = \widetilde{W}_{\alpha^2}$ and its Berger measure μ is

$$d\mu = q(q+1)t^{q-1}(1-t)dt.$$

Since $S(1, q, 1, q + 2) = W_{\alpha\circ\beta}$, we can also obtain same result from (1.5) with $j = 3$.

To obtain the Berger measure for the Schur product of finitely many generalized Bergman shifts, we need the following notation and lemma.

Notation 2.3. For a sequence $x = \{x_n\}_{n=1}^{\infty}$ and $i, \ell \in \mathbb{N}$, we set

$$P_x(i,\ell) := \prod_{\substack{1\le j\le \ell \\ j\neq i}} (x_j - x_i).$$

Lemma 2.4. *For any sequence $x = \{x_n\}_{n=1}^{\infty}$ of mutually distinct real numbers, it holds that*

$$\sum_{i=1}^{k} \frac{1}{P_x(i,k)} = 0, \quad \textit{for all} \quad k \geq 2.$$

Proof. For $k = 2$, it holds that

$$\frac{1}{P_x(1,2)} + \frac{1}{P_x(2,2)} = \frac{1}{x_2 - x_1} + \frac{1}{x_1 - x_2} = 0,$$

for any sequence $x = \{x_n\}_{n=1}^{\infty}$ of mutually distinct real numbers.

To use the induction, we suppose that the sum of the first m terms is 0 for any sequence of mutually distinct real numbers (i.e., the statement holds for $k = m$.) and we now will show that

$$\sum_{i=1}^{m+1} \frac{1}{P_x(i,m+1)} = 0,$$

for any sequence $x = \{x_n\}_{n=1}^{\infty}$ of mutually distinct real numbers. By assumption, we have

$$\sum_{i=1}^{m} \frac{1}{P_x(i,m)} = 0.$$

Observe that

$$\sum_{i=1}^{m} \frac{1}{P_x(i,m)} = \sum_{i=1}^{m+1} \frac{x_{m+1} - x_i}{P_x(i,m+1)} = x_{m+1} \sum_{i=1}^{m+1} \frac{1}{P_x(i,m+1)} - \sum_{i=1}^{m+1} \frac{x_i}{P_x(i,m+1)},$$

it follows that

$$x_{m+1} \sum_{i=1}^{m+1} \frac{1}{P_x(i,m+1)} = \sum_{i=1}^{m+1} \frac{x_i}{P_x(i,m+1)}. \tag{2.1}$$

Let $y = \{x_{n+1}\}_{n=1}^{\infty}$. Since y is also a sequence of mutually distinct real numbers,

$$\sum_{i=1}^{m} \frac{1}{P_y(i,m)} = 0,$$

by assumption. Observe that

$$\begin{aligned}
\sum_{i=1}^{m} \frac{1}{P_y(i,m)} &= \sum_{i=1}^{m} \frac{x_1 - x_{i+1}}{(x_1 - x_{i+1})P_y(i,m)} \\
&= \sum_{i=0}^{m} \frac{x_1 - x_{i+1}}{P_x(i+1,m+1)} \\
&= \sum_{i=1}^{m+1} \frac{x_1 - x_i}{P_x(i,m+1)}
\end{aligned}$$

$$= x_1 \sum_{i=1}^{m+1} \frac{1}{P_x(i, m+1)} - \sum_{i=1}^{m+1} \frac{x_i}{P_x(i, m+1)},$$

it follows that

$$x_1 \sum_{i=1}^{m+1} \frac{1}{P_x(i, m+1)} = \sum_{i=1}^{m+1} \frac{x_i}{P_x(i, m+1)}. \tag{2.2}$$

From (2.1) and (2.2), we have

$$(x_{m+1} - x_1) \sum_{i=1}^{m+1} \frac{1}{P_x(i, m+1)} = 0.$$

Hence, $\sum_{i=1}^{m+1} \frac{1}{P_x(i,m+1)} = 0$, which completes the proof by the induction. □

We are ready to see our main result; finding the Berger measure for the Schur product of finitely many generalized Bergman shifts.

Theorem 2.5. *Suppose $\ell \geq 2$. Let $\sigma = \{q_n\}_{n=1}^{\infty}$ be a sequence of mutually distinct positive real numbers and let $\alpha^{(i)}$ be a weight sequence associated to weighted shift $W_{\alpha^{(i)}} \equiv S(1, q_i, 1, q_i + 1)$ for $i = 1, 2, \ldots, \ell$. Let $\alpha = \alpha^{(1)} \circ \cdots \circ \alpha^{(\ell)}$ be a Schur product of $\alpha^{(i)}$'s. Then a Berger measure μ of W_α is given by*

$$d\mu(t) = q_1 \cdots q_\ell \sum_{i=1}^{\ell} \frac{t^{q_i - 1}}{P_\sigma(i, \ell)} dt$$

and supp $\mu = [0, 1]$.

Proof. Without loss of generality, we assume that $0 < q_1 < q_2 < q_3 < \cdots$. Let $\gamma_n(\alpha^{(i)})$ be the moment of $\alpha^{(i)}$ and μ_i be the associated Berger measure for $W_{\alpha^{(i)}}$. Then we can see that

$$\gamma_n(\alpha^{(i)}) = \frac{q_i}{n + q_i} = \int_{[0,1]} t^n d\mu_i(t) = \int_0^1 t^n \left(q_i t^{q_i - 1}\right) dt$$

and the moment $\gamma_n(\alpha^{(1)} \circ \cdots \circ \alpha^{(\ell)})$ of α is obtained as

$$\gamma_n(\alpha^{(1)} \circ \cdots \circ \alpha^{(\ell)}) = \prod_{i=1}^{\ell} \gamma_n(\alpha^{(i)}) = \prod_{i=1}^{\ell} \int_{[0,1]} t^n d\mu_i(t) = \prod_{i=1}^{\ell} \int_0^1 t^n \left(q_i t^{q_i - 1}\right) dt.$$

Considering $d\mu = d(\mu_1 * \cdots * \mu_\ell)$, we will use the induction in $\ell \geq 2$. Firstly,

$$d(\mu_1 * \mu_2)(t) = \frac{q_1 q_2}{q_2 - q_1} \left(t^{q_2 - 1} - t^{q_1 - 1}\right) dt$$

and $\operatorname{supp}(\mu_1 * \mu_2) = [0, 1]$ by Lemma 2.1. Secondly, we suppose that

$$d(\mu_1 * \cdots * \mu_k)(t) = q_1 \cdots q_k \sum_{i=1}^{k} \frac{t^{q_i - 1}}{P_\sigma(i, k)} dt,$$

and $\operatorname{supp}(\mu_1 * \cdots * \mu_k) = [0, 1]$. Then we obtain the moment $\gamma_n(\alpha^{(1)} \circ \cdots \circ \alpha^{(k+1)})$ of the Schur product $\alpha^{(1)} \circ \cdots \circ \alpha^{(k+1)}$ by

$$\gamma_n(\alpha^{(1)} \circ \cdots \circ \alpha^{(k+1)}) = \gamma_n(\alpha^{(1)} \circ \cdots \circ \alpha^{(k)}) \gamma_n(\alpha^{(k+1)})$$

$$
\begin{aligned}
&= \int_{[0,1]} t^n d(\mu_1 * \cdots * \mu_k)(t) \int_{[0,1]} s^n d\mu_{k+1}(s) \\
&= \int_0^1 t^n q_1 \cdots q_k \sum_{i=1}^{k} \frac{t^{q_i-1}}{P_\sigma(i,k)} dt \int_0^1 q_{k+1} s^{n+q_{k+1}-1} ds \\
&= \int_0^1 \int_0^1 (ts)^n q_1 \cdots q_{k+1} \sum_{i=1}^{k} \frac{(ts)^{q_i-1} s^{q_{k+1}-q_i}}{P_\sigma(i,k)} dt ds \\
&= \int_0^1 \int_0^s u^n q_1 \cdots q_{k+1} \sum_{i=1}^{k} \frac{u^{q_i-1} s^{q_{k+1}-q_i-1}}{P_\sigma(i,k)} du ds \quad \text{(substituting } u = ts\text{)} \\
&= \int_0^1 \int_u^1 u^n q_1 \cdots q_{k+1} \sum_{i=1}^{k} \frac{u^{q_i-1} s^{q_{k+1}-q_i-1}}{P_\sigma(i,k)} ds du \\
&= \int_0^1 u^n q_1 \cdots q_{k+1} \sum_{i=1}^{k} \frac{u^{q_i-1} - u^{q_{k+1}-1}}{(q_{k+1}-q_i) P_\sigma(i,k)} du \\
&= \int_0^1 u^n q_1 \cdots q_{k+1} \left(\sum_{i=1}^{k} \frac{u^{q_i-1}}{P_\sigma(i,k+1)} - \sum_{i=1}^{k} \frac{u^{q_{k+1}-1}}{P_\sigma(i,k+1)} \right) du \\
&= \int_0^1 u^n q_1 \cdots q_{k+1} \left(\sum_{i=1}^{k+1} \frac{u^{q_i-1}}{P_\sigma(i,k+1)} \right) du;
\end{aligned}
$$

note that the last equality holds by lemma 2.4. Hence, the proof is complete by the induction. □

This theorem shows a Berger measure for *mutually distinct* weighted shifts, and a case where all shifts have *the same weight* is n-th power problem, which was solved in (1.4) ([4]). Now some questions have arisen; in Lemma 2.1, if q_2 approaches q_1, dose the convolution of their measures converges to square measure? If so, is it true for the convolution of finitely many measures, generally? The answer is affirmative as follows.

Proposition 2.6. *Under the hypothesis in Theorem 2.5, it holds that for* $\mathbf{x}_\ell = (q_1, \ldots, q_\ell)$ *and* $\mathbf{q}_\ell = (q, \ldots, q)$ *in* $\mathbb{R}^\ell$,

$$
\lim_{\mathbf{x}_\ell \to \mathbf{q}_\ell} q_1 \cdots q_\ell \sum_{i=1}^{\ell} \frac{t^{q_i-1}}{P_\sigma(i,\ell)} dt = \frac{q^\ell}{(\ell-1)!} t^{q-1} (-\ln t)^{\ell-1} dt
$$

with $\operatorname{supp}\mu = [0,1]$, *which is the* ℓ*-th power of Berger measure of* $S(1, q, 1, q+1)$.

Proof. For each $k \in \mathbb{N}$, let $F_k(t; \mathbf{x}_k) = \sum_{i=1}^{k} \frac{t^{q_i-1}}{P_\sigma(i,k)}$ and $G_k(t; q) = \frac{t^{q-1}}{(k-1)!} (-\ln t)^{k-1}$. Then

$$
\begin{aligned}
\lim_{(q_1,q_2)\to(q,q)} F_2(t; q_1, q_2) &= \lim_{(q_1,q_2)\to(q,q)} \frac{t^{q_1-1} - t^{q_2-1}}{q_2 - q_1} \\
&= \lim_{(q_1,q_2)\to(q,q)} \frac{t^{q_1-1}(1 - t^{q_2-q_1})}{q_2 - q_1} \\
&= G_2(t; q),
\end{aligned}
$$

which shows the case of $\ell = 2$. Suppose that $\lim_{\mathbf{x}_k \to \mathbf{q}_k} F_k(t; \mathbf{x}_k) dt = G_k(t; q) dt$. By the proof of Theorem 2.5, we have

$$
\lim_{\mathbf{x}_{k+1} \to \mathbf{q}_{k+1}} \int_0^1 t^n F_{k+1} dt = \lim_{\mathbf{x}_{k+1} \to \mathbf{q}_{k+1}} \left(\int_0^1 t^n F_k dt \int_0^1 s^{n+q_{k+1}-1} ds \right)
$$

$$
\begin{aligned}
&= \int_0^1 t^n G_k dt \int_0^1 s^{n+q-1} ds \\
&= \frac{1}{(k-1)!} \int_0^1 \int_0^1 (ts)^{n+q-1} (-\ln t)^{k-1} ds dt \\
&= \frac{1}{(k-1)!} \int_0^1 \int_0^t u^{n+q-2} (-\ln t)^{k-1} \frac{u}{t} du dt \quad \text{(substituting } u = ts\text{)} \\
&= \frac{1}{(k-1)!} \int_0^1 \int_u^1 u^{n+q-2} (-\ln t)^{k-1} \frac{u}{t} dt du \\
&= \frac{1}{(k-1)!} \int_0^1 \int_0^{-\ln u} u^{n+q-1} v^{k-1} dv du \quad \text{(substituting } v = -\ln t\text{)} \\
&= \int_0^1 u^n G_{k+1}(u; q) du,
\end{aligned}
$$

which completes the proof by the induction. □

By using this proposition, we can obtain mixed cases, which mean that q_i's are neither *mutually distinct* nor *all the same*. See the following example.

Example 2.7. Consider three weighted shifts $S(1, q_i, 1, q_i + 1)$, $i = 1, 2, 3$. If q_i's are mutually distinct, the Berger measure of their Schur product is

$$
d\mu(t) = q_1 q_2 q_3 \left(\frac{t^{q_1-1}}{(q_2-q_1)(q_3-q_1)} + \frac{t^{q_2-1}}{(q_1-q_2)(q_3-q_2)} + \frac{t^{q_3-1}}{(q_1-q_3)(q_2-q_3)} \right) dt
$$

with supp $\mu = [0, 1]$. If q_3 approaches q_2, we have

$$
\lim_{q_3 \to q_2} \frac{t^{q_1-1}}{(q_2-q_1)(q_3-q_1)} = \frac{t^{q_1-1}}{(q_2-q_1)^2},
$$

and

$$
\begin{aligned}
&\lim_{q_3 \to q_2} \left(\frac{t^{q_2-1}}{(q_1-q_2)(q_3-q_2)} + \frac{t^{q_3-1}}{(q_1-q_3)(q_2-q_3)} \right) \\
&\quad = \lim_{q_3 \to q_2} \frac{(q_1-q_3)t^{q_2-1} - (q_1-q_2)t^{q_3-1}}{(q_1-q_2)(q_1-q_3)(q_3-q_2)} \\
&\quad = \lim_{q_3 \to q_2} \frac{q_2 t^{q_3-1} - q_3 t^{q_2-1} + q_1(t^{q_2-1} - t^{q_3-1})}{(q_1-q_2)(q_1-q_3)(q_3-q_2)} \\
&\quad = \frac{q_2^2}{(q_1-q_2)^2} \lim_{q_3 \to q_2} \frac{\frac{t^{q_3-1}}{q_3} - \frac{t^{q_2-1}}{q_2}}{q_3 - q_2} - \frac{q_1}{(q_1-q_2)^2} \lim_{q_3 \to q_2} \frac{t^{q_3-1} - t^{q_2-1}}{q_3 - q_2} \\
&\quad = \frac{1}{(q_1-q_2)^2} t^{q_2-1}(q_2 \ln t - 1) - \frac{q_1}{(q_1-q_2)^2} t^{q_2-1} \ln t.
\end{aligned}
$$

It follows that

$$
\lim_{q_3 \to q_2} d\mu(t) = \frac{q_1 q_2^2}{(q_2-q_1)^2} \left(t^{q_1-1} - t^{q_2-1} + t^{q_2-1}(q_2 - q_1) \ln t \right) dt
$$

on $[0, 1]$, which is the Berger measure for a weight sequence $\sqrt{\frac{(n+q_1)(n+q_2)^2}{(n+q_1+1)(n+q_2+1)^2}}$ by Proposition 2.6. In particular, if $q_1 = 1$ and $q_2 = 2$, a weight sequence $\sqrt{\frac{(n+1)(n+2)}{(n+3)^2}}$ corresponds to the Berger measure $d\mu(t) = 4\,(1 - t + t \ln t)\, dt$ with supp $\mu = [0, 1]$.

Consider $S(a,b,c,d)$ with $\frac{d}{c} - \frac{b}{a} = 1$, whose weights are given by

$$\alpha_n = \sqrt{\frac{a(n+q)}{c(n+q+1)}}, \ \ n = 0,1,2,\ldots.$$

Now we discuss the Berger measure of the Schur product of weighted shifts, whose type is given above.

Lemma 2.8 ([3], [9]). *Let $\alpha = \{\alpha_n\}_{n=0}^{\infty}$ be a weight sequence and W_α be a associated subnormal weighted shift with $\|W_\alpha\| = 1$. Suppose μ is the Berger measure of W_α. Then for $k > 0$, $W_{k\alpha}$ is subnormal and its Berger measure ν is given by $d\nu(t) = d\mu\left(\frac{t}{k^2}\right)$ with $\operatorname{supp}\nu \subset \left[0, k^2\right]$, where $k\alpha := \{k\alpha_n\}_{n=0}^{\infty}$.*

Lemma 2.8 is an elementary computational property which seems to be well-known. By using this lemma, we can obtain a more general result as follows.

Theorem 2.9. *Suppose $\ell \geq 2$. Let $\sigma = \{q_n\}_{n=1}^{\infty}$ be a sequence of mutually distinct positive real numbers and let $\alpha^{(i)}$ be a weight sequence associated to weighted shift $W_{\alpha^{(i)}} \equiv S(a, aq_i, c, cq_i + c)$ for $i = 1, 2, \ldots, \ell$. Let $\alpha = \alpha^{(1)} \circ \cdots \circ \alpha^{(\ell)}$ be a Schur product of $\alpha^{(i)}$'s. Then the Berger measure μ of W_α is given by*

$$d\mu(t) = q_1 \cdots q_\ell \sum_{i=1}^{\ell} \left(\frac{c}{a}\right)^{\ell q_i} \frac{t^{q_i - 1}}{P_\sigma(i,\ell)} dt$$

and $\operatorname{supp}\mu = \left[0, \left(\frac{a}{c}\right)^{\ell}\right]$. *Moreover, if $q_1, \ldots, q_\ell$ approach q, then*

$$\lim_{q_1,\ldots,q_\ell \to q} d\mu(t) = \left(\frac{c}{a}\right)^{\ell q} \frac{q^\ell}{(\ell-1)!} t^{q-1} \left(-\ln\left(\frac{c}{a}\right)^{\ell} t\right)^{\ell-1} dt,$$

which is the ℓ-th power of the Berger measure of $S(a, aq, c, cq + c)$.

3. The *j*-th Agler-type shifts

For $j = 2, 3, \ldots$, the *j-th Agler shift* is the weighted shift with weight $\sqrt{\frac{n+1}{n+j}}$ [1]. In this section we discuss generalized Agler shifts. For $j = 2, 3, 4, \ldots$, consider $S(a,b,c,d)$ with $\frac{d}{c} - \frac{b}{a} = j - 1$. Then its weights form of

$$\alpha_n = \sqrt{\frac{a(n+q)}{c(n+q+j-1)}}, \ \ n = 0,1,2,\ldots.$$

These weighted shifts are called *j-th Agler-type (weighted) shifts* [4]. Theorem 2.9 shows a result for the second Agler-type shifts. We now discuss the Berger measure of Schur product of the third Agler-type shifts. By (1.5), the property 5°(ii) in Section 1 and Lemma 2.8, we can obtain the following corollary.

Corollary 3.1. *Let $W_\alpha \equiv S(a, aq, c, c(q+2))$. Then the Berger measure μ of W_α is given by*

$$d\mu(t) = q(q+1)\left(\frac{c}{a}\right)^{q} t^{q-1}\left(1 - \frac{c}{a}t\right) dt$$

with $\operatorname{supp}\mu = \left[0, \frac{a}{c}\right]$ *and the Berger measure $\mu * \mu$ of W_{α^2} is given by*

$$d(\mu*\mu)(t) = (d\mu(t))^2 = q^2(q+1)^2\left(\frac{c}{a}\right)^{2q} t^{q-1}\left[2\left(\left(\frac{c}{a}\right)^2 t - 1\right) - \left(\left(\frac{c}{a}\right)^2 t + 1\right)\ln\left(\frac{c}{a}\right)^2 t\right] dt$$

with $\operatorname{supp}\mu = \left[0, \left(\frac{a}{c}\right)^2\right]$.

The above corollary presents the square of the Berger measure for the second Agler-type shifts. The following theorem is related to convolutions of the Berger measures for two different second Agler-type shifts.

Theorem 3.2. *Let $q_1 \leq q_2$ be positive real numbers and let $W_\alpha \equiv S(1, q_1, 1, q_1 + 2)$ and $W_\beta \equiv S(1, q_2, 1, q_2 + 2)$. Then the Berger measure of $W_{\alpha\circ\beta}$ is*

$$d\mu(t) = \begin{cases} q_1^2(q_1+1)^2 t^{q-1}\left[2(t-1)-(t+1)\ln t\right]dt, & q_2 = q_1, \\ \frac{q_1(q_1+1)^2(q_1+2)}{2} t^{q_1-1}\left(1-t^2+2t\ln t\right)dt, & q_2 = q_1+1, \\ q_1q_2(q_1+1)(q_2+1)\frac{(q_2-q_1+1)\left(t^{q_2-1}-t^{q_1}\right)-(q_2-q_1-1)\left(t^{q_2}-t^{q_1-1}\right)}{(q_2-q_1-1)(q_2-q_1)(q_2-q_1+1)}dt, & \text{otherwise}, \end{cases}$$

with $\operatorname{supp}\mu = [0,1]$. *Moreover, for $q_2 \neq q_1, q_1+1$, if q_2 approaches q_1 (resp. q_1+1), then $d\mu(t)$ converges to the Berger measure for the case $q_1 = q_2$ (resp. $q_2 = q_1 + 1$).*

Proof. From Corollary 3.1 with $a = c = 1$, the case of $q_1 = q_2$ was proved. Now we assume $q_1 < q_2$. By (1.3), the n-th moment of $\alpha \circ \beta$ is

$$\begin{aligned}\gamma_n(\alpha\circ\beta) &= q_1q_2(q_1+1)(q_2+1)\int_0^1 t^{n+q_1-1}(1-t)dt\int_0^1 s^{n+q_2-1}(1-s)ds \\ &= q_1q_2(q_1+1)(q_2+1)\int_0^1\int_0^1 (ts)^{n+q_1-1}s^{q_2-q_1}(1-t-s+ts)dtds.\end{aligned}$$

By using the idea of the proof of Lemma 2.1, we have

$$\begin{aligned}\int_0^1\int_0^1 (ts)^{n+q_1-1}s^{q_2-q_1}dtds &= \int_0^1 u^n\frac{u^{q_1-1}-u^{q_2-1}}{q_2-q_1}du, \\ \int_0^1\int_0^1 (ts)^{n+q_1-1}s^{q_2-q_1}(-t)dtds &= \begin{cases}\int_0^1 u^n\frac{u^{q_2-1}-u^{q_1}}{q_2-q_1-1}du, & q_2\neq q_1+1, \\ \int_0^1 u^n\left(u^{q_1}\ln u\right)du, & q_2 = q_1+1,\end{cases} \\ \int_0^1\int_0^1 (ts)^{n+q_1-1}s^{q_2-q_1}(-s)dtds &= \int_0^1 u^n\frac{u^{q_2}-u^{q_1-1}}{q_2-q_1+1}du, \\ \int_0^1\int_0^1 (ts)^{n+q_1-1}s^{q_2-q_1}(ts)dtds &= \int_0^1 u^n\frac{u^{q_1}-u^{q_2}}{q_2-q_1}du.\end{aligned}$$

By direct computation, we have, for $q_2 \neq q_1 + 1$,

$$\gamma_n(\alpha\circ\beta) = q_1q_2(q_1+1)(q_2+1)\int_0^1 u^n\frac{(q_2-q_1+1)\left(u^{q_2-1}-u^{q_1}\right)-(q_2-q_1-1)\left(u^{q_2}-u^{q_1-1}\right)}{(q_2-q_1-1)(q_2-q_1)(q_2-q_1+1)}du,$$

and for $q_2 = q_1 + 1$,

$$\gamma_n(\alpha\circ\beta) = q_1(q_1+1)^2(q_1+2)\int_0^1 u^{n+q_1-1}\frac{1-u^2+2u\ln u}{2}du.$$

This is as desired. The convergence of measure, as $q_2 \to q_1, q_1+1$, holds by simple computation. □

By Lemma 2.8, we can obtain a general version for the third Agler-type shifts $S(a, aq, c, c(q+2))$.

Consider the Schur product of two j-th Agler-type shifts ($j \geq 4$); $S(1, q_1, 1, q_1+j-1)$ and $S(1, q_2, 1, q_2+j-1)$. By (1.5), their Berger measures are $d\mu_1(t) = \frac{(q_1+j-2)!}{(q_1-1)!(j-2)!}t^{q_1-1}(1-t)^{j-2}dt$ and $d\mu_2(t) = \frac{(q_2+j-2)!}{(q_2-1)!(j-2)!}t^{q_2-1}(1-t)^{j-2}dt$, respectively. If $q_1 = q_2 =: q$, we can obtain the Berger measure of Schur square of j-th Agler-type shift by [4, Corollary 3.3]; for $j = 4$, the Berger measures are shown in the property 5°(iii) in Section 1. Without loss of generality, we assume that $q_1 < q_2$. Then the n-th moment of Schur product is

$$\begin{aligned}\gamma_n &= \int_0^1 \int_0^1 K t^{n+q_1-1} s^{n+q_2-1} (1-t)^{j-2}(1-s)^{j-2} dtds \\ &= K \int_0^1 \int_0^1 t^{n+q_1-1} s^{n+q_2-1} (1-t-s+ts)^{j-2} dtds \\ &= K \int_0^1 \int_0^1 \sum_{\substack{k_1+\cdots+k_4=j-2 \\ k_1,\dots,k_4 \geq 0}} \frac{(j-2)!}{k_1!k_2!k_3!k_4!} (-1)^{k_2+k_3} t^{n+p_1-1} s^{n+p_2-1} dtds,\end{aligned}$$

where $K = \frac{(q_1+j-2)!}{(q_1-1)!(j-2)!} \frac{(q_2+j-2)!}{(q_2-1)!(j-2)!}$, $p_1 = q_1 + k_2 + k_4$ and $p_2 = q_2 + k_3 + k_4$, and then $p_1, p_2 > 0$. As you can see, since the moment is a sum of finite terms, it suffices to find the Berger measure if we can express $\int_0^1 \int_0^1 t^{n+p_1-1} s^{n+p_2-1} dtds$ as a single integral. By the idea of the proof in Theorem 3.2, we have

$$\int_0^1 \int_0^1 t^{n+p_1-1} s^{n+p_2-1} dtds = \begin{cases} \int_0^1 u^{n+p_1-1}(-\ln u)du, & p_1 = p_2, \\ \int_0^1 u^n \frac{u^{p_1-1}-u^{p_2-1}}{p_2-p_1} du, & p_1 \neq p_2. \end{cases} \tag{3.1}$$

By using (3.1), we can obtain the n-th moment of the Schur product two distinct j-th Agler-type shifts and its Berger measure.

In [4], for $p > 0$, the p-th power measure for the second Agler-type shifts $S(1, q, 1, q+1)$ is given but it is difficult to find the p-th power measures for the j-th Agler-type shifts $S(1, q, 1, q+j-1)$, $j \geq 3$. We conclude this section with an interesting open problem:

Problem 3.3. Let $j \geq 3$ be a positive integer.

(i) Find the p-th power of Berger measure of the specific j-th Agler shift for $p > 0$.
(ii) Find the Berger measure for the Schur product of mutually n distinct j-th Agler shifts for $n \geq 3$.

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