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On $\mathcal{I}$-convergence of sequences of functions

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Abstract. Let X be a topological space, Y a uniform space and $\mathcal{I}$ an admissible ideal on the set $\mathbb{N}$ of natural numbers. In this paper, we mainly study the conditions that be added to pointwise $\mathcal{I}$-convergence of a sequence of (continuous) functions in Y^X to preserve the continuity of the $\mathcal{I}$-limit function. Ideal versions of weak exhaustiveness, semi-exhaustiveness, semi-uniform convergence, α-convergence and semi-α convergence of sequences of functions are introduced. Their relationships are clarified. Assume that a sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ pointwise $\mathcal{I}$-converges to f, we prove that:

(a) f is continuous if and only if the sequence $\{f_n\}_{n\in\mathbb{N}}$ is weakly $\mathcal{I}$-exhaustive.

(b) If the sequence $\{f_n\}_{n\in\mathbb{N}}$ is semi-$\mathcal{I}$-exhaustive, then f is continuous.

(c) If the sequence $\{f_n\}_{n\in\mathbb{N}}$ semi-uniformly $\mathcal{I}$-converges to f and f_n is continuous for every $n \in \mathbb{N}$, then f is continuous.

(d) If $\mathcal{I}$ is "good" and X is first countable, then $\{f_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-α convergent to f if and only if $\{f_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-exhaustive.

(e) If the sequence $\{f_n\}_{n\in\mathbb{N}}$ semi-$\mathcal{I}$-α converges to f, then f is continuous.

1. Introduction

It is well known that the pointwise limit of a sequence of continuous functions is not necessarily a continuous function. So one of the central questions in analysis is what precisely must be added to pointwise convergence of a sequence of continuous functions to preserve the continuity of the limit function. In 1841, Weierstrass gave a sufficient condition called uniform convergence which yields the continuity of the limit function. After that, Dini [16], Arzelà [2], Bartel [5] and Alexandroff [1] et al. further studied this problem.

The notion of α-convergence (also known as continuous convergence) had been known at the beginning of the 20th century. One of the interesting facts about α-convergence (proved by Stoilov [35]) is that it preserves the continuity of the limit function of sequences of functions in metric spaces. In 1993, Ewert [19] introduced almost uniform convergence of sequences of functions, which preserves the continuity of the pointwise limit of a sequence of continuous functions and is weaker than uniform convergence.

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In 2008, Gregoriades and Papanastassiou [23] introduced the concepts of exhaustiveness and weak exhaustiveness of a sequence of functions in metric spaces. These notions view the convergence of a sequence of functions in terms of properties of the sequence and not of properties of functions as single members. Particularly, a necessary and sufficient condition for the continuity of the pointwise limit of a sequence of functions (not necessarily continuous) was given by the notion of weak exhaustiveness.

Recently, Papanastassiou [32] introduced semi-exhaustiveness, semi-uniform convergence and semi-α convergence of sequences of functions in metric spaces, which are strictly weaker than exhaustiveness, almost uniform convergence and α-convergence, respectively. He proved that: (1) If a sequence of functions is semi-exhaustive and pointwise convergent, then the limit function is continuous; (2) If a sequence of functions is semi-α convergent, then the limit function is continuous.

Statistical convergence is a generalization of usual convergence, which was first proposed by Zygmund in the first edition of his monograph [44] in 1935, and formally introduced by Fast [20] and Steinhaus [34] independently in 1951. Statistical convergence has wide applications in different fields of mathematics, see [4, 6, 7, 9, 15, 18, 21, 22, 25, 28, 33, 36] etc. Based on the relevant concepts of statistical convergence, Caserta and Kocinač [8] defined statistical versions of exhaustiveness, weak exhaustiveness and α-convergence of sequences of functions in metric spaces. Particularly, they obtained some results about the continuity of the statistical pointwise limit of a sequence of functions.

The idea of statistical convergence had been extended to $\mathcal{I}$-convergence by Kostyrko et al. in [26] with the help of ideals. $\mathcal{I}$-convergence includes ordinary convergence and statistical convergence when $\mathcal{I}$ is the ideal of all finite subsets of the set of natural numbers and all subsets of the set of natural numbers of natural density zero, respectively. P. Das [13, p.78] said that "This approach is much more general as most of the known convergence methods become special cases, but many questions regarding this convergence still remain open as most results involving statistical convergence where the density function has been used explicitly cannot be obtained for general ideals. So one of the most interesting areas of investigation is the determination of those ideals for which these properties can be established." Over the last 20 years, a lot of work has been done on $\mathcal{I}$-convergence and associated topics, it has turned out to be one of the most active research areas in Topology and Analysis, for more details see [3, 10–12, 14, 24, 27, 29, 31, 37–43] etc.

Ideal versions of exhaustiveness, weak exhaustiveness and α-convergence (of a sequence of functions) were given by Papachristodoulos et al. in [31]. They found that for an $\mathcal{I}$-convergence of sequence of functions in metric spaces, the $\mathcal{I}$-limit function is continuous if and only if the sequence is weakly $\mathcal{I}$-exhaustive. Megaritis in [30] introduced ideal versions of uniform convergence and almost uniform convergence of nets of functions from a topological space to a uniform space. Megaritis proved that uniform $\mathcal{I}$-convergence and almost uniform $\mathcal{I}$-convergence preserve the continuity of the $\mathcal{I}$-limit function of a sequence of continuous functions. He also defined the notion of $\mathcal{I}$-equicontinuous family of functions on which pointwise and uniform $\mathcal{I}$-convergence coincide on compact sets, and gave a necessary and sufficient condition for a net of continuous functions from a compact space to a uniform space pointwise $\mathcal{I}$-converges to a continuous function.

Let X be a topological space and Y a uniform space. In this paper, we mainly study the conditions that must be added to pointwise $\mathcal{I}$-convergence of a sequence of (continuous) functions in Y^X to preserve the continuity of the $\mathcal{I}$-limit function. Ideal versions of weak exhaustiveness, semi-exhaustiveness, semi-uniform convergence, α-convergence and semi-α convergence of sequences of functions are introduced. Their relations are studied in detail. The results we obtained generalize the related results in the literature. The paper is organized as follows.

In Section 3, we introduce the notion of weak $\mathcal{I}$-exhaustiveness and semi-$\mathcal{I}$-exhaustiveness of sequences of functions. Some examples are constructed to clarify their relations. We prove that: (a) The $\mathcal{I}$-limit function of a pointwise $\mathcal{I}$-convergence of sequence of functions is continuous if and only if the sequence $\{f_n\}_{n\in\mathbb{N}}$ is weakly $\mathcal{I}$-exhaustive. (b) If a sequence $\{f_n\}_{n\in\mathbb{N}}$ pointwise $\mathcal{I}$-converges to f and $\{f_n\}_{n\in\mathbb{N}}$ is semi-$\mathcal{I}$-exhaustive, then f is continuous.

In Section 4, we define the semi-uniform $\mathcal{I}$-convergence of sequences of functions. The relations of uniform convergence, almost uniform convergence, semi-uniform convergence and their ideal versions are clarified. We prove that: if a sequence of continuous functions $\{f_n\}_{n\in\mathbb{N}}$ semi-uniformly $\mathcal{I}$-converges to f, then f is continuous. Also, the connections among uniform $\mathcal{I}$-convergence, almost uniform $\mathcal{I}$-convergence and $\mathcal{I}$-exhaustiveness are studied.

In Section 5, $\mathcal{I}$-α convergence and semi-$\mathcal{I}$-α convergence of sequences of functions are considered. We show that: (a) If $\mathcal{I}$ is "good" and X is first countable, then a sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-α convergent to f if and only if $\{f_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-exhaustive and pointwise $\mathcal{I}$-convergent to f. (b) If a sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ semi-$\mathcal{I}$-α converges to f, then f is continuous. (c) The following (1)-(3) are equivalent: (1) The sequence $\{f_n\}_{n\in\mathbb{N}}$ is semi-$\mathcal{I}$-exhaustive and $\{f_n\}_{n\in\mathbb{N}}$ pointwise $\mathcal{I}$-converges to f. (2) The sequence $\{f_n\}_{n\in\mathbb{N}}$ semi-$\mathcal{I}$-α converges to f. (3) The sequence $\{f_n\}_{n\in\mathbb{N}}$ semi-uniformly $\mathcal{I}$-converges to f and the function f is continuous.

2. Preliminaries

Throughout the paper, $\mathbb{N}$ denotes the set of all positive integers, X denotes a topological space and Y a uniform space with the uniformly $\mathscr{U}$, unless stated otherwise. Y^X (resp. $C(X,Y)$) denotes the set of all mapping (resp. all continuous functions) from X to Y. The sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ always defined in Y^X. Readers may consult [17] for notation and terminology not given here.

Definition 2.1. ([26]) Let $\mathcal{I}$ be a family of non-empty subsets on $\mathbb{N}$, $\mathcal{I}$ is said to be an *ideal* if:

(1) $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$;

(2) $A \in \mathcal{I}$, $B \subseteq A$ implies $B \in \mathcal{I}$.

An ideal is said to be *non-trivial* if $\mathcal{I} \neq \{\emptyset\}$ and $\mathbb{N} \notin \mathcal{I}$. A non-trivial ideal $\mathcal{I}$ is called *admissible* if $\mathcal{I} \supseteq \{\{n\} : n \in \mathbb{N}\}$. Clearly, every non-trivial ideal $\mathcal{I}$ defines a *dual filter* $\mathcal{F}(\mathcal{I}) = \{A \subseteq \mathbb{N} : \mathbb{N} \setminus A \in \mathcal{I}\}$ on $\mathbb{N}$.

In this paper, $\mathcal{I}$ denotes an admissible ideal on $\mathbb{N}$ unless stated otherwise.

Definition 2.2. ([26]) Let X be a topological space. A sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is said to be *$\mathcal{I}$-convergent* to a point $x \in X$ if for each neighborhood U of x, we have the set $\{n \in \mathbb{N} : x_n \notin U\} \in \mathcal{I}$. In this case we write $\mathcal{I}\text{-}\lim x_n = x$ or $x_n \xrightarrow{\mathcal{I}} x$.

If $\mathcal{I}$ is the class $\mathcal{I}_f$ of all finite subsets of $\mathbb{N}$, then $\mathcal{I}_f$ is an admissible ideal and $\mathcal{I}_f$-convergence coincides with the usual convergence of a sequence; if $\mathcal{I}_d$ is the class of all $A \subseteq \mathbb{N}$ with $d(A) = 0$, where $d(A)$ denotes the asymptotic density of the set A, then $\mathcal{I}_d$ is an admissible ideal and $\mathcal{I}_d$-convergence coincides with the statistical convergence.

Definition 2.3. ([17]) A *uniformity* on a set X is a collection $\mathscr{U}$ of subsets of $X \times X$ satisfying the following properties:

(1) If $V \in \mathscr{U}$, $V \subseteq W$, then $W \in \mathscr{U}$.

(2) If $V \in \mathscr{U}$, then $V^{-1} \in \mathscr{U}$, where $V^{-1} = \{(x,y) : (y,x) \in V\}$.

(3) If $V_1, V_2 \in \mathscr{U}$, then $V_1 \cap V_2 \in \mathscr{U}$.

(4) For every $V \in \mathscr{U}$ there exists $W \in \mathscr{U}$ such that $2W = W + W \subseteq \mathscr{U}$, where $W + W = \{(x,z)$: there exists $y \in X$, such that $(x,y) \in W,\ (y,z) \in W\}$.

(5) $\bigcap_{U\in\mathscr{U}} U = \triangle$, where $\triangle = \{(x,x) : x \in X\}$.

A *uniform space* is a pair $(X, \mathscr{U})$ consisting of a set X and a uniformity $\mathscr{U}$ on the set X. The elements of $\mathscr{U}$ are called *entourages*. An entourage V is called symmetric if $V^{-1} = V$. For every $V \in \mathscr{U}$ and $(x,y) \in X \times X$, if $(x,y) \in V$, we say that the distance between x and y is less than V and we write $|x - y| < V$; otherwise we write $|x - y| \geq V$.

For every $U \in \mathscr{U}$ and $x_0 \in X$, the set $U[x_0] = \{x \in X : |x_0 - x| < U\}$ is called *the ball with centre x_0 and radius U*, or briefly *the U-ball about x_0*.

A mapping f from a topological space X to a uniform space $(Y, \mathscr{U})$ is called *continuous at x_0* if for every $U \in \mathscr{U}$ there exists an open neighborhood O_{x_0} of x_0 such that $f(O_{x_0}) \subseteq U[f(x_0)]$ or equivalently $|f(x_0) - f(x)| < U$, for each $x \in O_{x_0}$. The mapping f is called *continuous* if it is continuous at each point of X.

Definition 2.4. ([30]) Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of functions from a non-empty set X to a uniform space $(Y, \mathscr{U})$. The sequence $\{f_n\}_{n\in\mathbb{N}}$ is said to *pointwise $\mathcal{I}$-converge to f on X* if for each $x \in X$ and $U \in \mathscr{U}$, the set $\{n \in \mathbb{N} : |f(x) - f_n(x)| \geq U\} \in \mathcal{I}$ or equivalently for each $x \in X$ and $U \in \mathscr{U}$, there exists a set $A \in \mathcal{I}$, such that for every $n \notin A$ we have $|f(x) - f_n(x)| < U$. In this case we write $f_n \xrightarrow{\mathcal{I}} f$ and f is called the $\mathcal{I}$-limit function of $\{f_n\}_{n\in\mathbb{N}}$.

Definition 2.5. Let (X, d), (Y, ρ) be metric spaces, $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of functions in Y^X and $f \in Y^X$.

(1) The sequence $\{f_n\}_{n\in\mathbb{N}}$ is said to *α-converge* ([23]) to f if for every $x \in X$ and for every sequence $\{x_n\}_{n\in\mathbb{N}}$ of points of X converging to x, the sequence $\{f_n(x_n)\}_{n\in\mathbb{N}}$ converges to $f(x)$.

(2) The sequence $\{f_n\}_{n\in\mathbb{N}}$ is said to be *exhaustive* ([23]) if for each $x \in X$ and $\varepsilon > 0$, there exists $\delta > 0$ and $n_0 \in \mathbb{N}$ such that for each $y \in S(x, \delta)$ and $n \geq n_0$ we have $\rho(f_n(y), f_n(x)) < \varepsilon$.

(3) The sequence $\{f_n\}_{n\in\mathbb{N}}$ is said to be *weakly exhaustive* ([23]) if for each $x \in X$ and $\varepsilon > 0$ there exists $\delta > 0$ such that for each $y \in S(x, \delta)$ there exists $n_y \in \mathbb{N}$ such that for every $n \geq n_y$ we have $\rho(f_n(y), f_n(x)) < \varepsilon$.

(4) The sequence $\{f_n\}_{n\in\mathbb{N}}$ is said to *converge almost uniformly to f* ([19]) on X if for each $x \in X$ and $\varepsilon > 0$, there exists $\delta > 0$ and $n_0 \in \mathbb{N}$ such that $\rho(f_n(y), f(y)) < \varepsilon$ for each $y \in S(x, \delta)$ and $n \geq n_0$.

(5) The sequence $\{f_n\}_{n\in\mathbb{N}}$ is said to *semi-α converge to f at x* if

(5a) $f_n(x) \to f(x)$;

(5b) for each $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, x) > 0$ such that for every $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$, $m \geq n$ such that $\rho(f_m(y), f(x)) < \varepsilon$ for each $y \in S(x, \delta)$.

The sequence $\{f_n\}_{n\in\mathbb{N}}$ is said to be *semi-α convergent* ([32]) if it is semi-α convergent at each $x \in X$. A sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ is said to have the *semi-α property* with respect to f if $\{f_n\}_{n\in\mathbb{N}}$ satisfies the condition (5b).

Definition 2.6. ([32]) Let X be a topological space, (Y, ρ) be a metric space, $x \in X$, $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of functions in Y^X and $f \in Y^X$. The sequence $\{f_n\}_{n\in\mathbb{N}}$ is said to be *semi-uniformly convergent to f at x* if

(1) $f_n(x) \to f(x)$;

(2) for each $\varepsilon > 0$ there exists a neighborhood O of x such that for every $n \in \mathbb{N}$ there exists $m \in \mathbb{N}, m \geq n$ such that $\rho(f_m(y), f(y)) < \varepsilon$ for each $y \in O$.

The sequence $\{f_n\}_{n\in\mathbb{N}}$ is said to be *semi-uniformly convergent* if it is semi-uniformly convergent at each $x \in X$.

The notions in Definitions 2.5 and 2.6 can be extended to sequences of functions from topological spaces to uniform spaces.

Definition 2.7. ([30]) Let X be a topological space, $(Y, \mathscr{U})$ be a uniform space, $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of functions in Y^X and $f \in Y^X$.

(1) The sequence $\{f_n\}_{n\in\mathbb{N}}$ is said to *$\mathcal{I}$-converge uniformly to f* on X if for every $U \in \mathscr{U}$ there exists $A \in \mathcal{F}(\mathcal{I})$ such that for every $n \in A$ and $x \in X$ we have $|f_n(x) - f(x)| < U$.

(2) The sequence $\{f_n\}_{n\in\mathbb{N}}$ is said to *$\mathcal{I}$-converge almost uniformly to f* on X if for each $x \in X$ and $U \in \mathscr{U}$ there exists $A \in \mathcal{F}(\mathcal{I})$ and an open neighborhood O_x of x such that for every $n \in A$ and $y \in O_x$ we have $|f_n(y) - f(y)| < U$.

(3) The sequence $\{f_n\}_{n\in\mathbb{N}}$ is said to be *$\mathcal{I}$-exhaustive at $x \in X$* if for every $U \in \mathscr{U}$, there exists a neighborhood O of x and a set $M \in \mathcal{F}(\mathcal{I})$ such that $|f_n(y) - f_n(x)| < U$ for every $n \in M$ and $y \in O$. The sequence $\{f_n\}_{n\in\mathbb{N}}$ is said to be *$\mathcal{I}$-exhaustive* if it is $\mathcal{I}$-exhaustive at each $x \in X$.

Remark 2.8. (1) The notions in Definition 2.7 can be defined in general forms [30], we state here only for sequences of functions.

(2) In [30], the author called $\mathcal{I}$-exhaustiveness (of a sequence of functions) as $\mathcal{I}$-equicontinuity, it is the same as the notion of $\mathcal{I}$-exhaustiveness defined in [31]. Thus, we also use the notion of $\mathcal{I}$-exhaustiveness.

3. $\mathcal{I}$-exhaustive, weakly $\mathcal{I}$-exhaustive and semi-$\mathcal{I}$-exhaustive sequences of functions

In the first part of this section, we define ideal version of weak exhaustiveness of sequences of functions with values in uniform spaces. Many examples are constructed to clarify the relations among exhaustiveness, weak exhaustiveness and their ideal versions. We mainly prove that the pointwise $\mathcal{I}$-limit of a sequence of functions is continuous if and only if the sequence is weakly $\mathcal{I}$-exhaustive.

Definition 3.1. A sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ is said to be *weakly $\mathcal{I}$-exhaustive at $x \in X$* if for every $U \in \mathscr{U}$, there exists a neighborhood O of x such that for each $y \in O$ there is a set $M_y \in \mathcal{F}(\mathcal{I})$ such that $|f_n(y)-f_n(x)| < U$ for every $n \in M_y$. The sequence $\{f_n\}_{n\in\mathbb{N}}$ is said to be *weakly $\mathcal{I}$-exhaustive* if it is weakly $\mathcal{I}$-exhaustive at each $x \in X$.

Obviously, each $\mathcal{I}$-exhaustive sequence of functions is weakly $\mathcal{I}$-exhaustive. However, there exists a weakly $\mathcal{I}$-exhaustive sequence of functions which is not $\mathcal{I}$-exhaustive for any ideal $\mathcal{I}$.

Example 3.2. There is a weakly $\mathcal{I}$-exhaustive sequence of functions which is not $\mathcal{I}$-exhaustive.

Proof. Consider the sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ in $\mathbb{R}^{\mathbb{R}}$ defined as follows:

$$f_n(x) = \begin{cases} 1/4n, & \text{if } x \le 0; \\ n, & \text{if } x = 1/n; \\ 1/n, & \text{if } x > 0, x \neq 1/n. \end{cases}$$

Obviously, $\{f_n\}_{n\in\mathbb{N}}$ pointwise converges to $f \equiv 0$. By [23, Theorem 4.2.3], $\{f_n\}_{n\in\mathbb{N}}$ is weakly exhaustive, and then it is weakly $\mathcal{I}$-exhaustive. However, taking $x_0 = 0$ and $\varepsilon = 1/4$. For each $\delta > 0$ and $A \in \mathcal{F}(\mathcal{I})$, since the set A is infinite, there exists $n_0 \in A$ such that $1/n_0 < \delta$. Taking $y_0 = 1/n_0$, then $|f_{n_0}(y_0) - f_{n_0}(0)| = n_0 - 1/4n_0 \ge 1/4$. It follows that $\{f_n\}_{n\in\mathbb{N}}$ is not $\mathcal{I}$-exhaustive at $x_0 = 0$. □

An exhaustive sequence of functions is $\mathcal{I}$-exhaustive, and a weakly exhaustive sequence of functions is weakly $\mathcal{I}$-exhaustive, but the converse is not true for any $\mathcal{I} \neq \mathcal{I}_f$. Actually, we have the following example.

Example 3.3. If $\mathcal{I} \neq \mathcal{I}_f$, there is an $\mathcal{I}$-exhaustive sequence of functions which is not weakly exhaustive.

Proof. Since $\mathcal{I} \neq \mathcal{I}_f$, there exists an infinite set $A \in \mathcal{I}$. Consider the sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ in $\mathbb{R}^{\mathbb{R}}$ defined as follows:

$$f_n(x) = \begin{cases} 1, & \text{if } n \notin A; \\ 1, & \text{if } n \in A,\ x < 0; \\ 1/n, & \text{if } n \in A,\ x \ge 0. \end{cases}$$

For each $x \in \mathbb{R}$ and $\varepsilon > 0$, let $M = \mathbb{N} \setminus A \in \mathcal{F}(\mathcal{I})$. Choose a neighborhood O of x, then we have $|f_n(y) - f_n(x)| = 0 < \varepsilon$ for every $n \in M$ and $y \in O$. It follows that the sequence $\{f_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-exhaustive. However, the sequence $\{f_n\}_{n\in\mathbb{N}}$ is not weakly exhaustive at $x_0 = 0$. Indeed, taking $\varepsilon = 1/2$, for each $\delta > 0$, pick $y_0 \in (-\delta, 0)$. Since the set A is infinite, for every $n \in \mathbb{N}$, there exists $n_0 \in A$ with $n_0 > n$ such that $|f_{n_0}(y_0) - f_{n_0}(0)| = |1 - 1/n_0| \ge 1/2$. It shows that $\{f_n\}_{n\in\mathbb{N}}$ is not weakly exhaustive at $x_0 = 0$. Therefore, the sequence $\{f_n\}_{n\in\mathbb{N}}$ is not weakly exhaustive. □

Since every exhaustive sequence of functions is weakly exhaustive and every $\mathcal{I}$-exhaustive sequence of functions is weakly $\mathcal{I}$-exhaustive, Example 3.3 shows that if $\mathcal{I} \neq \mathcal{I}_f$, there is an $\mathcal{I}$-exhaustive sequence of functions which is not exhaustive, and there is a weakly $\mathcal{I}$-exhaustive sequence of functions which is not weakly exhaustive.

The following theorem is one of the main results in this section.

Theorem 3.4. *If $f_n \xrightarrow{\mathcal{I}} f \in Y^X$, then f is continuous if and only if the sequence $\{f_n\}_{n\in\mathbb{N}}$ is weakly $\mathcal{I}$-exhaustive.*

Proof. (⇒) For every $U \in \mathscr{U}$, there exists a symmetric entourage $V \in \mathscr{U}$ such that $3V \subseteq U$. For each $x \in X$, from the continuity of f at x there exists a neighborhood O of x such that

$$|f(y) - f(x)| < V \tag{i}$$

for each $y \in O$. Since $\{f_n\}_{n\in\mathbb{N}}$ pointwise $\mathcal{I}$-converges to f, it follows that there exists an $A_y \in \mathcal{F}(\mathcal{I})$ such that for every $n \in A_y$ we have

$$|f_n(y) - f(y)| < V,\ |f_n(x) - f(x)| < V. \tag{ii}$$

Thus, for every $n \in A_y$, it follows from (i) and (ii) that

$$|f_n(y) - f_n(x)| < 3V \subseteq U,$$

which implies that $\{f_n\}_{n\in\mathbb{N}}$ is weakly $\mathcal{I}$-exhaustive at x. Therefore, $\{f_n\}_{n\in\mathbb{N}}$ is weakly $\mathcal{I}$-exhaustive.

($\Leftarrow$) For every $U \in \mathscr{U}$, there exists a symmetric entourage $V \in \mathscr{U}$ such that $3V \subseteq U$. For each $x \in X$, since $\{f_n\}_{n\in\mathbb{N}}$ is weakly $\mathcal{I}$-exhaustive at x, there is a neighborhood O of x such that for each $y \in O$ there is a set $M_y \in \mathcal{F}(\mathcal{I})$ such that

$$|f_n(y) - f_n(x)| < V \tag{iii}$$

for every $n \in M_y$. Since $\{f_n\}_{n\in\mathbb{N}}$ pointwise $\mathcal{I}$-converges to f, it follows that there is a set $A_y \in \mathcal{F}(\mathcal{I})$ such that

$$|f_n(y) - f(y)| < V,\ |f_n(x) - f(x)| < V \tag{iv}$$

for every $n \in A_y$. Taking an arbitrary $n \in A_y \cap M_y$, from (iii) and (iv), it follows that

$$|f(y) - f(x)| < 3V \subseteq U,$$

which shows that f is continuous at x, and then f is continuous. □

Since an $\mathcal{I}$-exhaustive sequence of functions is weakly $\mathcal{I}$-exhaustive, the following corollary can be obtained directly from Theorem 3.4.

Corollary 3.5. *If $f_n \xrightarrow{\mathcal{I}} f \in Y^X$ and $\{f_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-exhaustive, then f is continuous.*

Example 3.2 shows that there is a sequence of functions $\{f_n\}_{n\in\mathbb{N}}$, which is not $\mathcal{I}$-exhaustive, pointwise $\mathcal{I}$-convergent to a continuous function.

In the rest of this section, we will consider ideal version of semi-exhaustiveness of sequences of functions. In [32], the author considered the semi-exhaustiveness of sequences of functions in metric spaces. Similarly, we can define the semi-exhaustiveness of sequences of functions from a topological space X to a uniform space $(Y, \mathscr{U})$.

Definition 3.6. A sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ is said to be *semi-exhaustive at $x \in X$* if for every $U \in \mathscr{U}$, there exists a neighborhood O of x such that for every $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$, $m > n$ such that for each $y \in O$ we have $|f_m(y) - f_m(x)| < U$. The sequence $\{f_n\}_{n\in\mathbb{N}}$ is said to be *semi-exhaustive* if it is semi-exhaustive at each $x \in X$.

According to definitions, an $\mathcal{I}$-exhaustive sequence of functions is semi-exhaustive, but the converse is not true. In [32], the author gave an example showing that there exists a semi-exhaustive sequence of functions which is not exhaustive, but we can check that Example 3.3 in [32] is exhaustive at each point. Thus, we construct the following example.

Example 3.7. There is a semi-exhaustive sequence of functions which is not weakly exhaustive.

Proof. Consider the sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ in $\mathbb{R}^{\mathbb{R}}$ defined as follows:

$$f_n(x) = \begin{cases} 1, & \text{if n is odd;} \\ 1, & \text{if n is even and } x < 0; \\ 1/n, & \text{if n is even and } x \geq 0. \end{cases}$$

For each $x \in \mathbb{R}$ and $\varepsilon > 0$, taking an arbitrary neighborhood O of x. For every $n \in \mathbb{N}$ there exists $m > n$ with m is odd such that $|f_m(y) - f_m(x)| = 0 < \varepsilon$ for all $y \in O$. It shows that $\{f_n\}_{n\in\mathbb{N}}$ is semi-exhaustive. However, the sequence $\{f_n\}_{n\in\mathbb{N}}$ is not weakly exhaustive at $x_0 = 0$. Indeed, taking $\varepsilon = 1/2$, for each $\delta > 0$, pick $y_0 \in (-\delta, 0)$. For every $n \in \mathbb{N}$, there is $n_0 > n$ with n_0 is even such that $|f_{n_0}(y_0) - f_{n_0}(0)| = 1 - 1/n_0 \geq 1/2$. □

Example 3.8. If $\mathcal{I} \neq \mathcal{I}_f$, there is a semi-exhaustive sequence of functions which is not weakly $\mathcal{I}$-exhaustive.

Proof. Since $\mathcal{I} \neq \mathcal{I}_f$, there exists an infinite set $A \in \mathcal{I}$. Consider the sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ in $\mathbb{R}^{\mathbb{R}}$ defined as follows:

$$f_n(x) = \begin{cases} 1, & \text{if } n \in A; \\ 1, & \text{if } n \notin A,\ x < 0; \\ 1/n, & \text{if } n \notin A,\ x \geq 0. \end{cases}$$

For each $x \in \mathbb{R}$ and $\varepsilon > 0$, taking an arbitrary neighborhood O of x. For every $n \in \mathbb{N}$, since the set A is infinite, there is $m \in A$ with $m > n$ such that $|f_m(y) - f_m(x)| < \varepsilon$ for each $y \in O$. It follows that $\{f_n\}_{n\in\mathbb{N}}$ is semi-exhaustive. However, taking $\varepsilon = 1/2$, for each $\delta > 0$, pick $y_0 \in (-\delta, 0)$. For every $M \in \mathcal{F}(\mathcal{I})$, since the set $B = M \cap (\mathbb{N} \setminus A) \in \mathcal{F}(\mathcal{I})$ is infinite, there is $n_0 \in B$ such that $|f_{n_0}(y_0) - f_{n_0}(0)| = 1 - 1/n_0 \geq 1/2$. It follows that the sequence $\{f_n\}_{n\in\mathbb{N}}$ is not weakly $\mathcal{I}$-exhaustive at $x_0 = 0$. □

Let (X, d) and (Y, ρ) be metric spaces, if the sequence $\{f_n\}_{n\in\mathbb{N}}$ is semi-exhaustive at $x \in X$ and pointwise convergent to f, then f is continuous at x [32, Proposition 4.3]. When $\mathcal{I} \neq \mathcal{I}_f$, let $\{f_n\}_{n\in\mathbb{N}}$ be the sequence of functions in Example 3.8. Then $\{f_n\}_{n\in\mathbb{N}}$ is semi-exhaustive and pointwise $\mathcal{I}$-convergent to $f(x) = \begin{cases} 1, & x < 0; \\ 0, & x \geq 0. \end{cases}$ This shows that semi-exhaustiveness is too weak to preserve the continuity of the $\mathcal{I}$-limit of $\mathcal{I}$-convergence. Thus we define ideal version of semi-exhaustiveness (of a sequence of functions).

Definition 3.9. A sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ is said to be *semi-$\mathcal{I}$-exhaustive at $x \in X$* if for every $U \in \mathscr{U}$, there exists a neighborhood O of x such that for every $A \in \mathcal{F}(\mathcal{I})$ there exists $m \in A$ such that for each $y \in O$ we have $|f_m(y) - f_m(x)| < U$. The sequence $\{f_n\}_{n\in\mathbb{N}}$ is said to be *semi-$\mathcal{I}$-exhaustive* if it is semi-$\mathcal{I}$-exhaustive at each $x \in X$.

We can check that semi-$\mathcal{I}$-exhaustiveness implies semi-exhaustiveness, and semi-$\mathcal{I}$-exhaustiveness coincides with semi-exhaustiveness whenever $\mathcal{I} = \mathcal{I}_f$. However, there exists a semi-exhaustive sequence of functions which is not semi-$\mathcal{I}$-exhaustive for any $\mathcal{I} \neq \mathcal{I}_f$.

Example 3.10. If $\mathcal{I} \neq \mathcal{I}_f$, there is a semi-exhaustive sequence of functions which is not semi-$\mathcal{I}$-exhaustive.

Proof. Consider the sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ defined in Example 3.8, then the sequence $\{f_n\}_{n\in\mathbb{N}}$ is semi-exhaustive. However, it is not semi-$\mathcal{I}$-exhaustive at $x_0 = 0$. Indeed, taking $\varepsilon = 1/2$, then for each $\delta > 0$, put $M = (\mathbb{N} \setminus A) \cap \{n \in \mathbb{N} : n > 1\} \in \mathcal{F}(\mathcal{I})$. For every $n \in M$, there exists $y_0 \in (-\delta, 0)$ such that $|f_n(y_0) - f_n(0)| = 1 - 1/n \geq 1/2$. It follows that the sequence $\{f_n\}_{n\in\mathbb{N}}$ is not semi-$\mathcal{I}$-exhaustive at $x_0 = 0$. □

According to the definition of $\mathcal{I}$-exhaustiveness, an $\mathcal{I}$-exhaustive sequence of functions is semi-$\mathcal{I}$-exhaustive. A natural question is whether a semi-$\mathcal{I}$-exhaustive sequence of functions is $\mathcal{I}$-exhaustive? We will give a complete answer to this question.

Using Zorn's lemma, we can show that in the family of all ideal of $\mathbb{N}$, there exists a maximal ideal (with respect to inclusion).

Lemma 3.11. *[13] Let $\mathcal{I}_0$ be an ideal on $\mathbb{N}$. Then $\mathcal{I}_0$ is a maximal ideal if and only if $(A \in \mathcal{I}_0) \vee (\mathbb{N} \setminus A \in \mathcal{I}_0)$ holds for each $A \subseteq \mathbb{N}$.*

Example 3.12. If $\mathcal{I}$ is not a maximal ideal, there exists a semi-$\mathcal{I}$-exhaustive sequence of functions which is not weakly $\mathcal{I}$-exhaustive.

Proof. Since $\mathcal{I}$ is not maximal, by Lemma 3.11, there exists an infinite set $M \subseteq \mathbb{N}$ such that $M \notin \mathcal{I}$ and $\mathbb{N} \setminus M \notin \mathcal{I}$. Consider the sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ in $\mathbb{R}^{\mathbb{R}}$ defined as follows:

$$f_n(x) = \begin{cases} 1, & \text{if } n \in M; \\ 0, & \text{if } n \notin M,\ x = 0; \\ 1, & \text{if } n \notin M,\ x \neq 0. \end{cases}$$

For each $x \in \mathbb{R}$ and $\varepsilon > 0$, taking an arbitrary neighborhood O of x. For every $A \in \mathcal{F}(\mathcal{I})$, since the set $M \notin \mathcal{I}$, there is $m \in A \cap M$ such that $|f_m(y) - f_m(x)| = 0 < \varepsilon$ for each $y \in O$. It follows that $\{f_n\}_{n\in\mathbb{N}}$ is semi-$\mathcal{I}$-exhaustive. However, taking $\varepsilon = 1/2$, then for each $\delta > 0$, pick $y_0 \in S(0, \delta) \setminus \{0\}$. For every $A \in \mathcal{F}(\mathcal{I})$, since the set $\mathbb{N} \setminus M \notin \mathcal{I}$, there exists $n_0 \in A \cap (\mathbb{N} \setminus M)$ such that$|f_{n_0}(y_0) - f_{n_0}(0)| = 1 \geq 1/2$. It follows that the sequence $\{f_n\}_{n\in\mathbb{N}}$ is not weakly $\mathcal{I}$-exhaustive at $x_0 = 0$. □

When $\mathcal{I}$ is a maximal ideal, we have the following result.

Theorem 3.13. *Let $\mathcal{I}$ be a maximal ideal on $\mathbb{N}$. If a sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ is semi-$\mathcal{I}$-exhaustive, then it is also $\mathcal{I}$-exhaustive.*

Proof. Since $\{f_n\}_{n\in\mathbb{N}}$ is semi-$\mathcal{I}$-exhaustive, for each $x \in X$ and $U \in \mathscr{U}$, there exists a neighborhood O of x such that for every $A \in \mathcal{F}(\mathcal{I})$ there exists $m \in A$ such that for each $y \in O$ we have $|f_m(y) - f_m(x)| < U$. Let

$$M = \{n \in \mathbb{N} : |f_n(y) - f_n(x)| < U \text{ for each } y \in O\},$$

then $|f_n(y) - f_n(x)| < U$ for each $n \in M$ and $y \in O$. We claim that $M \in \mathcal{F}(\mathcal{I})$. Suppose that $M \notin \mathcal{F}(\mathcal{I})$, since $\mathcal{I}$ is maximal, this means that $\mathbb{N} \setminus M \in \mathcal{F}(\mathcal{I})$. Therefore, there exists $m \in \mathbb{N} \setminus M$ such that $|f_m(y) - f_m(x)| < U$ for all $y \in O$. By the construction of M, we can conclude that $m \in M$, which is a contradiction. Hence, $\{f_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-exhaustive at x, and then $\{f_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-exhaustive. □

The following theorem is the other main result in this section.

Theorem 3.14. *If $f_n \xrightarrow{\mathcal{I}} f \in Y^X$ and $\{f_n\}_{n\in\mathbb{N}}$ is semi-$\mathcal{I}$-exhaustive, then f is continuous.*

Proof. For every $U \in \mathscr{U}$ there exists a symmetric entourage $V \in \mathscr{U}$ such that $3V \subseteq U$. For each $x \in X$, since $\{f_n\}_{n\in\mathbb{N}}$ is semi-$\mathcal{I}$-exhaustive at x, there exists a neighborhood O of x such that for every $A \in \mathcal{F}(\mathcal{I})$ there is $m \in A$ such that for each $y \in O$ we have

$$|f_m(y) - f_m(x)| < V.$$

By assumption, $\{f_n\}_{n\in\mathbb{N}}$ pointwise $\mathcal{I}$-converges to f, then for each $y \in O$ there exists $A_y \in \mathcal{F}(\mathcal{I})$ such that

$$|f_n(y) - f(y)| < V, \ |f_n(x) - f(x)| < V$$

for every $n \in A_y$. Thus, for each $y \in O$, there exists $m \in A \cap A_y$ such that

$$|f_m(y) - f_m(x)| < V, \ |f_m(y) - f(y)| < V, \ |f_m(x) - f(x)| < V.$$

It follows that $|f(y) - f(x)| < 3V \subseteq U$, which implies that f is continuous at x. Thus f is continuous. □

The following example shows that if a sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ pointwise $\mathcal{I}$-converges to a continuous function, then the sequence is not necessarily semi-$\mathcal{I}$-exhaustive.

Example 3.15. There is a sequence of functions which is pointwise $\mathcal{I}$-convergent to a continuous function but the sequence is not semi-exhaustive.

Proof. Consider the sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ defined in Example 3.2. Let $f(x) = 0$ for each $x \in \mathbb{R}$. It is easy to check that the sequence $\{f_n\}_{n\in\mathbb{N}}$ pointwise $\mathcal{I}$-converges to f. Taking $\varepsilon = 1/4$. For each $\delta > 0$ there exists $n_0 \in \mathbb{N}$ such that $1/n_0 < \delta$. Then for every $n > n_0$ there exists $y_0 = 1/n < 1/n_0 < \delta$ such that $|f_n(y_0) - f_n(0)| = |n - 1/4n| \geq 1/4$. It follows that the sequence $\{f_n\}_{n\in\mathbb{N}}$ is not semi-exhaustive at $x_0 = 0$. □

Since a semi-$\mathcal{I}$-exhaustive sequence of functions is semi-exhaustive, it follows that the sequence $\{f_n\}_{n\in\mathbb{N}}$ defined in Example 3.15 is not semi-$\mathcal{I}$-exhaustive.

The following figure shows the main relations between all types of exhaustiveness (of a sequence of functions) discussed in this section. The symbol $\mathcal{M}$ denotes the family of all maximal ideal on $\mathbb{N}$. The arrows connecting to "f is continuous " in this figure need the sequences of functions pointwise $\mathcal{I}$-converges to f.

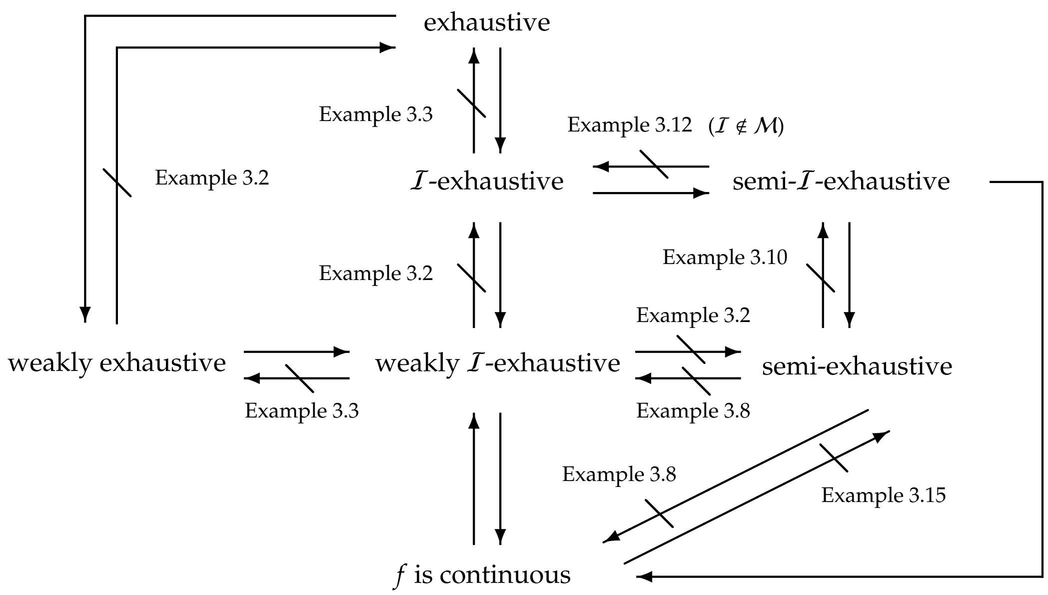

Figure. 1

4. Uniform $\mathcal{I}$-convergence, Almost uniform $\mathcal{I}$-convergence and semi-uniform $\mathcal{I}$-convergence

In the first part of this section, we further study the notions of uniform convergence, almost uniform convergence and their ideal versions, which were introduced in [30]. Some examples are constructed to clarify their relations. The connections among uniform $\mathcal{I}$-convergence, almost uniform $\mathcal{I}$-convergence and $\mathcal{I}$-exhaustiveness are also studied.

Obviously, a uniform convergence of sequence of functions is uniform $\mathcal{I}$-convergence, and an almost uniform convergence of sequence of functions is almost uniform $\mathcal{I}$-convergence, but the converse is not true for any $\mathcal{I} \neq \mathcal{I}_f$. Actually, we have the following example.

Example 4.1. If $\mathcal{I} \neq \mathcal{I}_f$, there is a uniformly $\mathcal{I}$-convergent sequence of functions which is not almost uniformly convergent.

Proof. Since $\mathcal{I} \neq \mathcal{I}_f$, there exists an infinite set $A \in \mathcal{I}$. Consider the sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ in $\mathbb{R}^{\mathbb{R}}$ defined as follows:

$$f_n(x) = \begin{cases} 1, & \text{if } n \in A; \\ \dfrac{x}{1+n^2x^2}, & \text{if } n \notin A. \end{cases}$$

Let $f(x) = 0$ for every $x \in \mathbb{R}$, then

$$|f_n(x) - f(x)| \leq \frac{|x|}{1+n^2x^2} \leq \frac{1}{2n}$$

for each $x \in \mathbb{R}$ and $n \notin A$. For each $\varepsilon > 0$, taking $n_0 = [\frac{1}{2\varepsilon}] + 1$. Then

$$|f_n(x) - f(x)| \leq \frac{1}{2n} < \varepsilon$$

for each $x \in \mathbb{R}$ and $n \in M = (\mathbb{N} \setminus A) \cap \{n \in \mathbb{N} : n > n_0\}$. Since $M \in \mathcal{F}(\mathcal{I})$, it follows that $\{f_n\}_{n\in\mathbb{N}}$ $\mathcal{I}$-converges uniformly to f. However, $\{f_n\}_{n\in\mathbb{N}}$ does not converge to f, thus $\{f_n\}_{n\in\mathbb{N}}$ does not almost uniform converge to f. □

Uniform $\mathcal{I}$-convergence of a sequence of functions implies almost uniform $\mathcal{I}$-convergence, but the converse is not true for any ideal $\mathcal{I}$.

Example 4.2. There is an almost uniformly $\mathcal{I}$-convergent sequence of functions which is not uniformly $\mathcal{I}$-convergent.

Proof. Let $X = (0, 1)$. Consider the sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ in $\mathbb{R}^X$ defined as follows:

$$f_n(x) = \frac{1}{nx}, \text{ for every } n \in \mathbb{N}.$$

Let $f(x) = 0$ for each $x \in X$ and take $\varepsilon = 1$. For every $A \in \mathcal{F}(\mathcal{I})$, pick $n_0 \in A$ and $x_0 = 1/(n_0 + 1) \in (0, 1)$. Then

$$|f_{n_0}(x_0) - f(x_0)| = |\frac{n_0 + 1}{n_0}| = 1 + \frac{1}{n_0} > 1.$$

It follows that $\{f_n\}_{n\in\mathbb{N}}$ does not $\mathcal{I}$-converge uniformly to f.

On the other hand, for each $x \in (0, 1)$ and $\varepsilon > 0$, there exists a neighborhood O_x of x and $a, b \in (0, 1)$ with $a < b$ such that $O_x \subseteq (a, b) \subseteq (0, 1)$. Thus

$$|f_n(y) - f(y)| = \frac{1}{ny} < \frac{1}{na}$$

for each $y \in O_x$. Pick $n_0 = [\frac{1}{a\varepsilon}] + 1$ and $A = \{n \in \mathbb{N} : n > n_0\}$, then $|f_n(y) - f(y)| = \frac{1}{ny} < \varepsilon$ for each $y \in O_x$ and $n \in A$. Therefore, $\{f_n\}_{n\in\mathbb{N}}$ $\mathcal{I}$-converges almost uniformly to f. □

The following theorem shows the connection among $\mathcal{I}$-exhaustiveness, uniform $\mathcal{I}$-convergence and almost uniform $\mathcal{I}$-convergence.

Theorem 4.3. *Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of functions in Y^X and $f \in Y^X$, then the following are equivalent:*

(1) *$f_n \xrightarrow{\mathcal{I}} f$ and the sequence $\{f_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-exhaustive.*

(2) *The sequence $\{f_n\}_{n\in\mathbb{N}}$ $\mathcal{I}$-converges almost uniformly to f and f is continuous.*

If X is compact, then (1) *and* (2) *are equivalent also to:*

(3) *The sequence $\{f_n\}_{n\in\mathbb{N}}$ $\mathcal{I}$-converges uniformly to f.*

Proof. (1) ⇒ (2) and (2) ⇔ (3) see [30, Theorems 5.3 and 6.2], we prove (2) ⇒ (1).

(2) ⇒ (1) It is obvious that (2) implies $f_n \xrightarrow{\mathcal{I}} f$, hence we only need to prove that $\{f_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-exhaustive. For every $U \in \mathscr{U}$ there exists a symmetric entourage $V \in \mathscr{U}$ such that $3V \subseteq U$. Since $\{f_n\}_{n\in\mathbb{N}}$ $\mathcal{I}$-converges almost uniformly to f, for each $x \in X$ there is a neighborhood P_x of x and $A_1 \in \mathcal{F}(\mathcal{I})$ such that

$$|f_n(y) - f(y)| < V \tag{v}$$

for every $n \in A_1$ and $y \in P_x$. By the continuity of f, there exists a neighborhood Q_x of x such that

$$|f(y) - f(x)| < V \tag{vi}$$

for each $y \in Q_x$. Since $f_n \xrightarrow{\mathcal{I}} f$, there exists $A_2 \in \mathcal{F}(\mathcal{I})$ such that

$$|f_n(x) - f(x)| < V \tag{vii}$$

for every $n \in A_2$. Let $O_x = P_x \cap Q_x$ and $A = A_1 \cap A_2$, it follows from (v)-(vii) that

$$|f_n(y) - f_n(x)| < 3V \subseteq U$$

for each $y \in O_x$ and $n \in A$. Thus $\{f_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-exhaustive at x. Therefore, the sequence $\{f_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-exhaustive. □

The author in [32] introduced the concept of semi-uniform convergence. He pointed out that the notion of semi-uniform convergence lies between almost uniform and simple uniform convergence. Thus it is natural to consider the properties of ideal version of semi-uniform convergence, which is the main topic in the rest of this section.

Definition 4.4. A sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ is said to *semi-uniformly $\mathcal{I}$-converge to f at x* if

(1) $f_n(x) \xrightarrow{\mathcal{I}} f(x)$;

(2) for every $U \in \mathscr{U}$ there exists a neighborhood O of x such that for every $A \in \mathcal{F}(\mathcal{I})$ there exists $m \in A$ such that $|f_m(y) - f(y)| < U$ for all $y \in O$.

The sequence $\{f_n\}_{n\in\mathbb{N}}$ is said to *semi-uniformly $\mathcal{I}$-converge* if it semi-uniformly $\mathcal{I}$-converges at each $x \in X$.

From the definitions of semi-uniform $\mathcal{I}$-convergence and semi-uniform convergence, it is easy to check that when $\mathcal{I} = \mathcal{I}_f$, semi-uniform $\mathcal{I}$-convergence coincides with semi-uniform convergence. However, they are very different whenever $\mathcal{I} \neq \mathcal{I}_f$.

Example 4.5. If $\mathcal{I} \neq \mathcal{I}_f$, there is a semi-uniformly convergent sequence of functions which is not semi-uniformly $\mathcal{I}$-convergent.

Proof. Since $\mathcal{I} \neq \mathcal{I}_f$, there exists an infinite set $A \in \mathcal{I}$. Consider the sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ in $\mathbb{R}^{\mathbb{R}}$ defined as follows:

$$f_n(x) = \begin{cases} 1/n, & \text{if } n \in A; \\ 0, & \text{if } n \notin A,\ x \in (-\infty, -1/n] \cup [0, +\infty); \\ 2nx + 2, & \text{if } n \notin A,\ x \in (-1/n, -1/2n]; \\ -2nx, & \text{if } n \notin A,\ x \in (-1/2n, 0). \end{cases}$$

Let $f(x) = 0$ for each $x \in \mathbb{R}$. Then the sequence $\{f_n\}_{n\in\mathbb{N}}$ pointwise converges to f, which means that it is pointwise $\mathcal{I}$-convergent to f. For each $x \in \mathbb{R}$ and $\varepsilon > 0$, taking an arbitrary neighborhood O of x. For every $n \in \mathbb{N}$, since the set A is infinite, there exists $m \in A$ with $m \geq n$ such that $|f_m(y) - f(y)| = |1/m - 0| < \varepsilon$ for all $y \in O$. Thus, $\{f_n\}_{n\in\mathbb{N}}$ is semi-uniformly convergent to f.

However, $\{f_n\}_{n\in\mathbb{N}}$ does not semi-uniformly $\mathcal{I}$-converge to f at $x_0 = 0$. Indeed, taking $\varepsilon = 1/2$, then for each $\delta > 0$ there exists $n_0 \in \mathbb{N}$ such that $1/2n < \delta$ for every $n \geq n_0$. For every $n \in M = (\mathbb{N} \setminus A) \cap \{n \in \mathbb{N} : n \geq n_0\} \in \mathcal{F}(\mathcal{I})$, pick $y = -1/2n \in (-\delta, \delta)$, then $|f_n(y) - f(y)| = 1 > 1/2$. It follows that $\{f_n\}_{n\in\mathbb{N}}$ does not semi-uniformly $\mathcal{I}$-converge to f at $x_0 = 0$. □

Example 4.6. If $\mathcal{I} \neq \mathcal{I}_f$, there is a semi-uniformly $\mathcal{I}$-convergent sequence of functions which is not semi-uniformly convergent.

Proof. Since $\mathcal{I} \neq \mathcal{I}_f$, there exists an infinite set $A \in \mathcal{I}$. Consider the sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ in $\mathbb{R}^{\mathbb{R}}$ defined as follows:

$$f_n(x) = \begin{cases} 1, & \text{if } n \in A; \\ 1/n, & \text{if } n \notin A. \end{cases}$$

Let $f(x) = 0$ for each $x \in \mathbb{R}$, then the sequence $\{f_n\}_{n\in\mathbb{N}}$ pointwise $\mathcal{I}$-converges to f. For each $x \in \mathbb{R}$ and $\varepsilon > 0$, taking an arbitrary neighborhood O of x. Then there exists $n_0 \in \mathbb{N}$ such that $1/n < \varepsilon$ for every $n \geq n_0$. For every $M \in \mathcal{F}(\mathcal{I})$, since $M_1 = M \cap (\mathbb{N} \setminus A) \cap \{n \in \mathbb{N} : n \geq n_0\} \neq \emptyset$, pick $m \in M_1$, thus $|f_m(y) - f(y)| = 1/n < \varepsilon$ for each $y \in O$. It follows that the sequence $\{f_n\}_{n\in\mathbb{N}}$ semi-uniformly $\mathcal{I}$-converges to f. However, since the set A is infinite, the sequence $\{f_n\}_{n\in\mathbb{N}}$ does not pointwise converge to f. Thus it is not semi-uniformly convergent. □

Next we will show that semi-uniform $\mathcal{I}$-convergence preserves the continuity of the pointwise $\mathcal{I}$-limit of a sequence of continuous functions.

Theorem 4.7. *Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of functions in $C(X, Y)$ and $f \in Y^X$. If $\{f_n\}_{n\in\mathbb{N}}$ semi-uniformly $\mathcal{I}$-converges to f, then f is continuous.*

Proof. For every $U \in \mathscr{U}$ there is a symmetric entourage $V \in \mathscr{U}$ such that $3V \subseteq U$. For each $x \in X$, since $\{f_n\}_{n\in\mathbb{N}}$ semi-uniformly $\mathcal{I}$-converges to f at x, there exists a neighborhood O_1 of x such that for every $A \in \mathcal{F}(\mathcal{I})$ there exists $m \in A$ such that for each $y \in O_1$ we have

$$|f_m(y) - f(y)| < V.$$

As the sequence $\{f_n\}_{n\in\mathbb{N}}$ pointwise $\mathcal{I}$-converges to f at x, there is $A' \in \mathcal{F}(\mathcal{I})$ such that $|f_n(x) - f(x)| < V$ for every $n \in A'$. Thus there is $m \in A \cap A'$ such that

$$|f_m(y) - f(y)| < V, \ |f_m(x) - f(x)| < V. \tag{viii}$$

By the continuity of f_m, there exists a neighborhood O_2 of x, such that

$$|f_m(y) - f_m(x)| < V \tag{ix}$$

for each $y \in O_2$. Let $O = O_1 \cap O_2$, then for each $y \in O$, from (viii) and (ix) we have

$$|f(y) - f(x)| < 3V \subseteq U.$$

Hence f is continuous at x, and then f is continuous. □

The condition $\{f_n\}_{n\in\mathbb{N}}$ in $C(X, Y)$ is necessary in Theorem 4.7.

Example 4.8. There is a sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ semi-uniformly $\mathcal{I}$-converging to a function f, which is not continuous.

Proof. Consider the sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ in $\mathbb{R}^{\mathbb{R}}$ defined as follows:

$$f_n(x) = \begin{cases} 1, & \text{if } x = 0; \\ 1/n, & \text{if } x \neq 0. \end{cases}$$

Let $f(x) = \begin{cases} 1, & x = 0; \\ 0, & x \neq 0. \end{cases}$ It is obvious that $\{f_n\}_{n\in\mathbb{N}}$ pointwise converges to f, which also means that $\{f_n\}_{n\in\mathbb{N}}$ pointwise $\mathcal{I}$-converges to f. For each $x \in X$, $\varepsilon > 0$, $\delta > 0$ and $A \in \mathcal{F}(\mathcal{I})$, since A is infinite, there exists $m \in A$ such that $1/m < \varepsilon$. Then for each $y \in S(x, \delta)$, if $y = 0$, we have $|f_m(y) - f(y)| = |1 - 1| = 0 < \varepsilon$; if $y \neq 0$, we have $|f_m(y) - f(y)| = 1/m < \varepsilon$. It follows that $\{f_n\}_{n\in\mathbb{N}}$ semi-uniformly $\mathcal{I}$-converges to f. However, f is not continuous. □

If a sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ is pointwise $\mathcal{I}$-convergent to a continuous function, the sequence may not be semi-uniformly $\mathcal{I}$-convergent.

Example 4.9. There is a sequence of continuous functions, which is pointwise $\mathcal{I}$-convergent to a continuous function, but the sequence is not semi-uniformly $\mathcal{I}$-convergent.

Proof. Let $X = [0, 1]$. Consider the sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ in $\mathbb{R}^X$ defined as follows:

$$f_n(x) = \frac{nx}{1 + n^2x^2}.$$

Obviously the sequence $\{f_n\}_{n\in\mathbb{N}}$ pointwise $\mathcal{I}$-converges to $f \equiv 0$. Take $\varepsilon = 1/2$. For each neighborhood O of $x_0 = 0$, there exists $n_0 \in \mathbb{N}$ such that $1/n \in O$ for every $n > n_0$. Put $A = \{n \in \mathbb{N} : n > n_0\} \in \mathcal{F}(\mathcal{I})$. Then for every $n \in A$ there is $y_0 = 1/n \in O$ such that $|f_n(y_0) - f(y_0)| = 1/2 \geq \varepsilon$. It follows that $\{f_n\}_{n\in\mathbb{N}}$ does not semi-uniformly $\mathcal{I}$-converge to f at $x_0 = 0$. □

An almost uniform $\mathcal{I}$-convergence of sequence of functions is semi-uniform $\mathcal{I}$-convergence, but the converse is not true when $\mathcal{I}$ is not a maximal ideal.

Example 4.10. If $\mathcal{I}$ is not maximal, there exists a semi-uniformly $\mathcal{I}$-convergent sequence of functions which is not almost uniformly $\mathcal{I}$-convergent.

Proof. Since $\mathcal{I}$ is not a maximal ideal, by Lemma 3.11, there exists an infinite set $M \subseteq \mathbb{N}$ such that $M \notin \mathcal{I}$ and $\mathbb{N} \setminus M \notin \mathcal{I}$. Consider the sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ in $\mathbb{R}^{\mathbb{R}}$ defined as follows:

$$f_n(x) = \begin{cases} 1/n, & \text{if } n \in M; \\ 0, & \text{if } n \notin M,\ x \in (-\infty, -1/n] \cup [0, +\infty); \\ 2nx + 2, & \text{if } n \notin M,\ x \in (-1/n, -1/2n]; \\ -2nx, & \text{if } n \notin M,\ x \in (-1/2n, 0). \end{cases}$$

Let $f(x) = 0$ for each $x \in \mathbb{R}$. It is obvious that $\{f_n\}_{n\in\mathbb{N}}$ pointwise converges to f, then it pointwise $\mathcal{I}$-converges to f. For each $x \in \mathbb{R}$ and $\varepsilon > 0$, taking an arbitrary neighborhood O of x. For every $A \in \mathcal{F}(\mathcal{I})$, since $M \notin \mathcal{I}$, $A \cap M$ is infinite. Therefore, there exists $m \in A \cap M$ such that $|f_m(y) - f(y)| = |1/m - 0| < \varepsilon$ for each $y \in O$. Thus, $\{f_n\}_{n\in\mathbb{N}}$ semi-uniformly $\mathcal{I}$-converges to f.

However, $\{f_n\}_{n\in\mathbb{N}}$ does not almost uniformly $\mathcal{I}$-converge to f at $x_0 = 0$. Indeed, taking $\varepsilon = 1/2$, then for each $\delta > 0$, there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ we have $1/2n < \delta$. For every $A \in \mathcal{F}(\mathcal{I})$, since $\mathbb{N} \setminus M \notin \mathcal{I}$, there exists $m \in A \cap (\mathbb{N} \setminus M) \cap \{n \in \mathbb{N} : n \geq n_0\}$. Pick $y_0 = -1/2m \in (-\delta, \delta)$, then we have $|f_m(y_0) - f(y_0)| = 1 > 1/2$. It follows that $\{f_n\}_{n\in\mathbb{N}}$ does not almost uniformly $\mathcal{I}$-converge to f at $x_0 = 0$. □

However, when $\mathcal{I}$ is a maximal ideal, a semi-uniform $\mathcal{I}$-convergent sequence of functions is almost uniform $\mathcal{I}$-convergent. The proof is similar to Theorem 3.13, so we omit it.

Theorem 4.11. *For each maximal ideal $\mathcal{I}$, semi-uniform $\mathcal{I}$-convergence implies almost uniform $\mathcal{I}$-convergence.*

The following figure shows the relations among all types of uniform convergence discussed in this section. The symbol $\mathcal{M}$ denotes the family of all maximal ideal on $\mathbb{N}$. The arrows connecting to "f is continuous " in this figure means that the sequence of continuous functions pointwise $\mathcal{I}$-converges to f. The symbols *u-c*, *u-$\mathcal{I}$-c*, *a-u-c*, *a-u-$\mathcal{I}$-c*, *s-u-c* and *s-u-$\mathcal{I}$-c* denote uniform convergence, uniform $\mathcal{I}$-convergence, almost uniform convergence, almost uniform $\mathcal{I}$-convergence, semi-uniform convergence and semi-uniform $\mathcal{I}$-convergence, respectively.

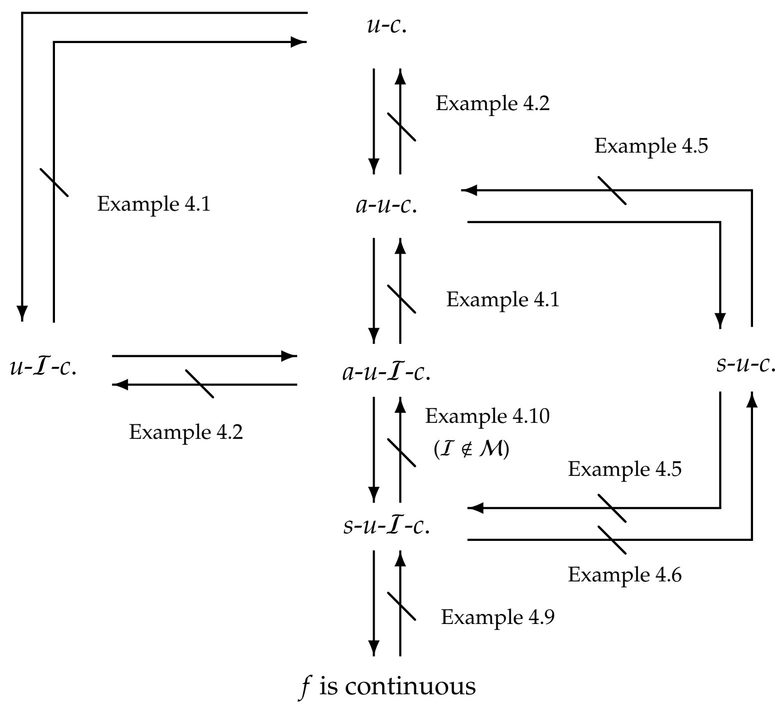

Figure. 2

5. $\mathcal{I}$-α convergence and semi-$\mathcal{I}$-α convergence of sequences of functions

In this section, we define ideal versions of α-convergence and semi-α convergence of sequences of functions. We will show that: (a) If $\mathcal{I}$ is "good" and X is first countable, then a sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-α convergent to $f \in Y^X$ if and only if $\{f_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-exhaustive and pointwise $\mathcal{I}$-convergent to f. (b) semi-$\mathcal{I}$-α convergence preserves the continuity of the pointwise $\mathcal{I}$-limit. In addition, the connections among semi-$\mathcal{I}$-exhaustiveness, semi-uniform $\mathcal{I}$-convergence and semi-$\mathcal{I}$-α convergence are given.

We begin with the notion of $\mathcal{I}$-α convergence.

Definition 5.1. A sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ is said to *$\mathcal{I}$-α converge to* f, if for every $x \in X$ and for every sequence $\{x_n\}_{n\in\mathbb{N}}$ in X $\mathcal{I}$-converging to x, the sequence $\{f_n(x_n)\}_{n\in\mathbb{N}}$ $\mathcal{I}$-converges to $f(x)$.

In [23], the authors show that in metric spaces, a sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ is α-convergent to f if and only if $\{f_n\}_{n\in\mathbb{N}}$ is exhaustive and pointwise convergent to f. Ideal versions of their relation in metric spaces was given in [31]. We extend these results to sequences of functions from a topological space to a uniform space.

Proposition 5.2. *If an $\mathcal{I}$-exhaustive sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ is pointwise $\mathcal{I}$-convergent to f, then $\{f_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-α convergent to f.*

Proof. For each $U \in \mathscr{U}$, there exists a symmetric entourage $V \in \mathscr{U}$ such that $2V \subseteq U$. Suppose that $x_n \xrightarrow{\mathcal{I}} x_0$. Since the sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ pointwise $\mathcal{I}$-converges to f at x_0, there is $A_1 \in \mathcal{F}(\mathcal{I})$ such that $|f_n(x_0) - f(x_0)| < V$ for each $n \in A_1$. On the other hand, since the sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-exhaustive at x_0, there exists a neighborhood O of x_0 and $A_2 \in \mathcal{F}(\mathcal{I})$ such that $|f_n(x) - f_n(x_0)| < V$ for each $x \in O$ and $n \in A_2$. By hypothesis, $x_n \xrightarrow{\mathcal{I}} x_0$, there exists $A_3 \in \mathcal{F}(\mathcal{I})$ such that $x_n \in O$ for each $n \in A_3$. Hence, for every $n \in A_1 \cap A_2 \cap A_3$ we have $|f_n(x_n) - f_n(x_0)| < V$ and $|f_n(x_0) - f(x_0)| < V$. It follows that $|f_n(x_n) - f(x_0)| < 2V \subseteq U$. Thus, $\{f_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-α convergent to f at x_0, and then it is $\mathcal{I}$-α convergent to f. □

Definition 5.3. ([31]) An ideal $\mathcal{I}$ is said to be *"good"*, if for every sequence $\{A_n\}_{n\in\mathbb{N}}$ of sets such that $A_n \notin \mathcal{I}$, there exists a sequence $\{B_n\}_{n\in\mathbb{N}}$ of pairwise disjoint sets such that $B_n \subseteq A_n$, $B_n \in \mathcal{I}$ and $\bigcup_{n=1}^{\infty} B_n \notin \mathcal{I}$.

It was shown in [31] that $\mathcal{I}_f$ is a "good" ideal and the ideal defined in [26, Example 3.1 (g)] is also a "good" ideal. However, $\mathcal{I}_d$ is not a "good" ideal.

Theorem 5.4. *Let X be a first countable space and $\mathcal{I}$ be a "good" ideal. Then a sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-α convergent to $f \in Y^X$ if and only if $\{f_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-exhaustive and pointwise $\mathcal{I}$-convergent to f.*

Proof. ($\Leftarrow$) Follows from Proposition 5.2.

($\Rightarrow$) Since the sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ $\mathcal{I}$-α converges to f, it is pointwise $\mathcal{I}$-convergent to f. Next we show that $\{f_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-exhaustive.

If $\{f_n\}_{n\in\mathbb{N}}$ is not $\mathcal{I}$-exhaustive, then there exists $x_0 \in X$ and $U \in \mathscr{U}$ such that for every neighborhood O of x_0 and every $A \in \mathcal{I}$, there exists $x \in O$ and $n \in \mathbb{N} \setminus A$ such that $|f_n(x) - f_n(x_0)| \geq U$. Since X is first countable, there is a neighborhood base $\{V_k\}_{k\in\mathbb{N}}$ of x_0 satisfies $V_{k+1} \subseteq V_k$ for every $k \in \mathbb{N}$. Let $A_k = \{n \in \mathbb{N} : |f_n(x) - f_n(x_0)| \geq U$ for some $x \in V_k\}$ for each $k \in \mathbb{N}$. If $A_k \in \mathcal{I}$, then there exists $n \in \mathbb{N} \setminus A_k$ and $x \in V_k$ such that $|f_n(x) - f_n(x_0)| \geq U$, which contradicts to the definition of A_k. Thus $A_k \notin \mathcal{I}$ for each $k \in \mathbb{N}$. Since $\mathcal{I}$ is a "good" ideal, there exists a countable sequence $\{B_k\}_{k\in\mathbb{N}}$ of pairwise disjoint sets such that $B_k \subseteq A_k$, $B_k \in \mathcal{I}$ for each $k \in \mathbb{N}$ and $\bigcup_{k=1}^{\infty} B_k \notin \mathcal{I}$.

Consider the sequence $\{y_n\}_{n\in\mathbb{N}}$ as follows: $y_n = x_0$ if $n \notin \bigcup_{k=1}^{\infty} B_k$; otherwise, if $n \in B_k$, pick an $y_n \in V_k$ such that $|f_n(y_n) - f_n(x_0)| \geq U$. Let V_k be given, if $k = 1$, then $\{n \in \mathbb{N} : y_n \notin V_1\} = \varnothing \in \mathcal{I}$; if $k \geq 2$, then $\{n \in \mathbb{N} : y_n \notin V_k\} \subseteq \bigcup_{k=1}^{k-1} B_k \in \mathcal{I}$. It follows that $y_n \xrightarrow{\mathcal{I}} x_0$.

Choosing a symmetric entourage $V \in \mathscr{U}$ such that $2V \subseteq U$. Since $y_n \xrightarrow{\mathcal{I}} x_0$ and $\{f_n\}_{n\in\mathbb{N}}$ $\mathcal{I}$-α converges to f at x_0, we have $D_1 = \{n \in \mathbb{N} : |f_n(y_n) - f(x_0)| \geq V\} \in \mathcal{I}$ and $D_2 = \{n \in \mathbb{N} : |f_n(x_0) - f(x_0)| \geq V\} \in \mathcal{I}$. Clearly, $\{n \in \mathbb{N} : |f_n(y_n) - f_n(x_0)| \geq U\} \subseteq D_1 \cup D_2$, which implies that $\{n \in \mathbb{N} : |f_n(y_n) - f_n(x_0)| \geq U\} \in \mathcal{I}$. However, since $\bigcup_{k=1}^{\infty} B_k \subseteq \{n \in \mathbb{N} : |f_n(y_n) - f_n(x_0)| \geq U\}$ and $\bigcup_{k=1}^{\infty} B_k \notin \mathcal{I}$, it follows that $\{n \in \mathbb{N} : |f_n(y_n) - f_n(x_0)| \geq U\} \notin \mathcal{I}$, which is a contradiction. Hence the sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-exhaustive. □

It is worth noticing that from [31, Theorem 2.10 and Example 2.12], Theorem 5.4 is not true for $\mathcal{I}_d$.

In the rest of the paper, we consider ideal version of semi-α convergence of sequences of functions.

Let (X, d) and (Y, ρ) be metric spaces. Papanastassiou proved that if a sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ in Y^X is pointwise convergent to f and satisfies semi-α property with respect to f at $x \in X$, then f is continuous at x [32, Proposition 4.3]. One may consider that we could define semi-$\mathcal{I}$-α convergence by changing $f_n \to f$ to $f_n \xrightarrow{\mathcal{I}} f$. However, the following example shows that it fails to preserve the continuity of the pointwise $\mathcal{I}$-limit function in this case, that is the semi-α property is too weak to preserve the continuity of the pointwise $\mathcal{I}$-limit of $\mathcal{I}$-convergence.

Example 5.5. If $\mathcal{I} \neq \mathcal{I}_f$, there is a sequence of functions which is pointwise $\mathcal{I}$-convergent to a function f and satisfies the semi-α property with respect to f, but f is not continuous.

Proof. Since $\mathcal{I} \neq \mathcal{I}_f$, there exists an infinite set $A \in \mathcal{I}$. Consider the sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ in $\mathbb{R}^\mathbb{R}$ defined as follows:

$$f_n(x) = \begin{cases} 1, & \text{if } n \in A; \\ 1, & \text{if } n \notin A,\ x = 0; \\ 1/n, & \text{if } n \notin A,\ x \neq 0. \end{cases}$$

Let $f(x) = \begin{cases} 1, & x = 0; \\ 0, & x \neq 0. \end{cases}$ It is clear that $\{f_n\}_{n\in\mathbb{N}}$ pointwise $\mathcal{I}$-converges to f. Next we show that $\{f_n\}_{n\in\mathbb{N}}$ satisfies the semi-α property with respect to f. For each $x \in X$, if $x = 0$, fix arbitrarily $\varepsilon > 0$ and take $\delta = 1$. For every $n \in \mathbb{N}$, since the set A is infinite, there exists $m \in A$ with $m \geq n$ such that $|f_m(y) - f(0)| = |1-1| = 0 < \varepsilon$ whenever $y \in (-1, 1)$. If $x \neq 0$, for each $\varepsilon > 0$, there is a neighborhood O of x such that $0 \notin O$. For every $n \in \mathbb{N}$, since the set $\mathbb{N} \setminus A$ is infinite, there is $m \in \mathbb{N} \setminus A$ such that $|f_m(y) - f(x)| = 1/m < \varepsilon$ for each $y \in O$. It follows that $\{f_n\}_{n\in\mathbb{N}}$ satisfies the semi-α property with respect to f. Obviously, f is not continuous at $x = 0$. □

For this reason, we define ideal version of semi-α convergence of sequences of functions as follow.

Definition 5.6. A sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ is said to *semi-$\mathcal{I}$-α converge to f at x* if

(1) $f_n(x) \xrightarrow{\mathcal{I}} f(x)$;

(2) for every $U \in \mathscr{U}$ there exists a neighborhood O of x such that for every $A \in \mathcal{F}(\mathcal{I})$ there exists $m \in A$ such that $|f_m(y) - f(x)| < U$ for all $y \in O$.

The sequence $\{f_n\}_{n\in\mathbb{N}}$ is said to be *semi-$\mathcal{I}$-α convergent* if it is semi-$\mathcal{I}$-α convergent at each $x \in X$. A sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ is said to have the *semi-$\mathcal{I}$-α property* with respect to f if it satisfies the condition (2).

It is clear that semi-$\mathcal{I}$-α convergence coincides with semi-α convergence whenever $\mathcal{I} = \mathcal{I}_f$.

Theorem 5.7. *If a sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ semi-$\mathcal{I}$-α converges to $f \in Y^X$, then f is continuous.*

Proof. For every $U \in \mathscr{U}$ there exists a symmetric entourage $V \in \mathscr{U}$ such that $2V \subseteq U$. For each $x \in X$, since the sequence $\{f_n\}_{n\in\mathbb{N}}$ satisfies the semi-$\mathcal{I}$-α property at x with respect to f, there exists a neighborhood O of x such that for every $A \in \mathcal{F}(\mathcal{I})$ there is $m \in A$ such that

$$|f_m(y) - f(x)| < V$$

for each $y \in O$. Since $\{f_n\}_{n\in\mathbb{N}}$ pointwise $\mathcal{I}$-converges to f, there exists $A_y \in \mathcal{F}(\mathcal{I})$ such that

$$|f_n(y) - f(y)| < V$$

for every $n \in A_y$. Then, for each $y \in O$, there is $m \in A \cap A_y$ such that

$$|f_m(y) - f(x)| < V,\ |f_m(y) - f(y)| < V.$$

Thus, $|f(y) - f(x)| < 2V \subseteq U$. It follows that f is continuous at x, then f is continuous. □

Theorem 5.8. *Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of functions in Y^X and $f \in Y^X$, then the following are equivalent:*

(1) *The sequence $\{f_n\}_{n\in\mathbb{N}}$ is semi-$\mathcal{I}$-exhaustive and $f_n \xrightarrow{\mathcal{I}} f$.*
(2) *The sequence $\{f_n\}_{n\in\mathbb{N}}$ semi-$\mathcal{I}$-α converges to f.*
(3) *The sequence $\{f_n\}_{n\in\mathbb{N}}$ semi-uniformly $\mathcal{I}$-converges to f and the function f is continuous.*

Proof. (1) ⇒ (2) For every $U \in \mathscr{U}$ there exists a symmetric entourage $V \in \mathscr{U}$ such that $2V \subseteq U$. For each $x \in X$, since $\{f_n\}_{n\in\mathbb{N}}$ is semi-$\mathcal{I}$-exhaustive at x, there exists a neighborhood O of x such that for every $A \in \mathcal{F}(\mathcal{I})$ there is $m \in A$ such that

$$|f_m(y) - f_m(x)| < V.$$

for each $y \in O$. By assumption, $\{f_n\}_{n\in\mathbb{N}}$ pointwise $\mathcal{I}$-converges to f, then there exists $A' \in \mathcal{F}(\mathcal{I})$ such that

$$|f_n(x) - f(x)| < V$$

for every $n \in A'$. Hence, for every $A \in \mathcal{F}(\mathcal{I})$, since $A \cap A' \in \mathcal{F}(\mathcal{I})$, there is $m \in A \cap A'$ such that

$$|f_m(y) - f_m(x)| < V,\ |f_m(x) - f(x)| < V$$

for all $y \in O$. It follows that $|f_m(y) - f(x)| < 2V \subseteq U$. Therefore, the sequence $\{f_n\}_{n\in\mathbb{N}}$ is semi-$\mathcal{I}$-α convergent to f at x, and then it is semi-$\mathcal{I}$-α convergent to f.

(2) ⇒ (3) From Theorem 5.7 and the definition of semi-$\mathcal{I}$-α convergence, it is obvious that (1) implies that $f_n \xrightarrow{\mathcal{I}} f$ and f is continuous. For every $U \in \mathscr{U}$ there exists a symmetric entourage $V \in \mathscr{U}$ such that $2V \subseteq U$. For each $x \in X$, since $\{f_n\}_{n\in\mathbb{N}}$ satisfies semi-$\mathcal{I}$-α property at x with respect to f, there exists a neighborhood O_1 of x such that for every $A \in \mathcal{F}(\mathcal{I})$ there exists $m \in A$ such that

$$|f_m(y) - f(x)| < V \tag{x}$$

for each $y \in O_1$. By the continuity of f, there exists a neighborhood O_2 of x such that

$$|f(y) - f(x)| < V \tag{xi}$$

for each $y \in O_2$. Let $O = O_1 \cap O_2$, then for all $y \in O$, from (x) and (xi) we have

$$|f_m(y) - f(y)| < 2V \subseteq U.$$

It follows that the sequence is semi-uniformly $\mathcal{I}$-convergent to f at x, then it is semi-uniformly $\mathcal{I}$-convergent to f.

(3) ⇒ (1) It is obvious that (3) implies that $f_n \xrightarrow{\mathcal{I}} f$. For every $U \in \mathscr{U}$ there exists a symmetric entourage $V \in \mathscr{U}$ such that $3V \subseteq U$. For each $x \in X$, since $\{f_n\}_{n\in\mathbb{N}}$ semi-uniformly $\mathcal{I}$-converges to f, there exists a neighborhood O_1 of x such that for every $A \in \mathcal{F}(\mathcal{I})$ there exists $m \in A$ such that

$$|f_m(y) - f(y)| < V \tag{xii}$$

for each $y \in O_1$. By the continuity of f, there exists a neighborhood O_2 of x such that

$$|f(y) - f(x)| < V \tag{xiii}$$

for each $y \in O_2$. It follows from (xii) and (xiii) that

$$|f_m(x) - f(x)| < V,\ |f_m(y) - f(y)| < V,\ |f(y) - f(x)| < V$$

for all $y \in O = O_1 \cap O_2$. Therefore, $|f_m(y) - f_m(x)| < 3V \subseteq U$ for each $y \in O$. Thus the sequence is semi-$\mathcal{I}$-exhaustive at x, so it is semi-$\mathcal{I}$-exhaustive. □

From Theorem 5.4 and 5.8, we have the following corollary.

Corollary 5.9. *Let X be first countable and $\mathcal{I}$ be a "good" ideal, the following statements hold: (1) $\mathcal{I}$-α convergence implies semi-$\mathcal{I}$-α convergence.*

(2) If a sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-α convergent to f, then f is continuous.

From Theorem 3.13, 5.8 and Proposition 5.2, we have the following corollary.

Corollary 5.10. *If $\mathcal{I}$ is a maximal ideal, then semi-$\mathcal{I}$-α convergence implies $\mathcal{I}$-α convergence.*

Remark 5.11. (a) Example 4.5, Example 4.6 and Theorem 5.8 also shows that: (1) if a sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ pointwise $\mathcal{I}$-converges to a continuous function, then the sequence is not necessarily semi-$\mathcal{I}$-α convergent; (2) semi-$\mathcal{I}$-α convergence and semi-α convergence are different whenever $\mathcal{I} \neq \mathcal{I}_f$.

(b) Example 4.10 and Theorem 5.8 also shows that if $\mathcal{I}$ is not a maximal ideal, then there exists a semi-$\mathcal{I}$-α convergent sequence of functions which is not $\mathcal{I}$-α convergent (Considering $\{x_n\}_{n\in\mathbb{N}} = \{-\frac{1}{2n}\}_{n\in\mathbb{N}}$).

The following figure shows the relations among $\mathcal{I}$-α convergence, semi-$\mathcal{I}$-α convergence, semi-$\mathcal{I}$-exhaustiveness, semi-uniform $\mathcal{I}$-convergence and the continuity of the $\mathcal{I}$-limit function of pointwise $\mathcal{I}$-convergence. The symbol $\mathcal{M}$ denotes the family of all maximal ideal on $\mathbb{N}$.

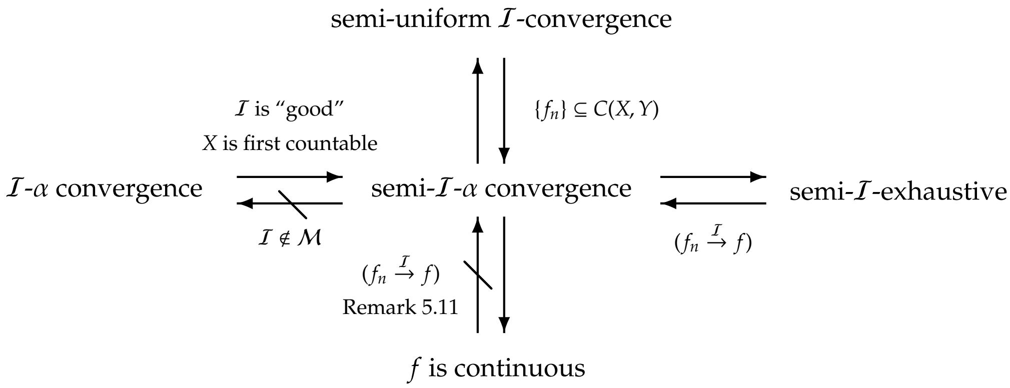

Figure. 3

6. Conclusions

Let X be a topological space, Y a uniform space and $\mathcal{I}$ an admissible ideal on the set $\mathbb{N}$ of natural numbers. In this paper, we mainly study the conditions that be added to pointwise $\mathcal{I}$-convergence of a sequence of (continuous) functions in Y^X to preserve the continuity of the $\mathcal{I}$-limit function.

In Section 3, ideal versions of exhaustiveness, weak exhaustiveness and semi-exhaustiveness (of a sequence of functions) are studied. Some examples are constructed to clarify the relations of the related concepts. We mainly prove that:

(a) If $f_n \xrightarrow{\mathcal{I}} f \in Y^X$, then f is continuous if and only if the sequence $\{f_n\}_{n\in\mathbb{N}}$ is weakly $\mathcal{I}$-exhaustive.

(b) If $f_n \xrightarrow{\mathcal{I}} f \in Y^X$ and $\{f_n\}_{n\in\mathbb{N}}$ is semi-$\mathcal{I}$-exhaustive, then f is continuous.

The relations between all types of exhaustiveness (of a sequence of functions) and the continuity of the $\mathcal{I}$-limit functions can be shown as Figure 1.

In Section 4, ideal versions of uniform convergence, almost uniform convergence and semi-uniform convergence (of a sequence of functions) are considered. Based on the existing literature, we further study the properties of uniform $\mathcal{I}$-convergence and almost uniform $\mathcal{I}$-convergence. Also, ideal version of semi-uniform convergence is introduced and studied. We mainly show that: If $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of functions in $C(X, Y)$, $f \in Y^X$ and $\{f_n\}_{n\in\mathbb{N}}$ semi-uniformly $\mathcal{I}$-converges to f, then f is continuous.

The relations among all types of uniform convergence (of a sequence of functions) discussed in this section can be shown as Figure 2.

In Section 5, ideal versions of α-convergence and semi-α-convergence are introduced and studied. We mainly obtain the following sufficient conditions that preserve the continuity of the pointwise $\mathcal{I}$-limit of $\{f_n\}_{n\in\mathbb{N}}$:

(a) If $\{f_n\}_{n\in\mathbb{N}}$ semi-$\mathcal{I}$-α converges to $f \in Y^X$, then f is continuous.

(b) If X is first countable, $\mathcal{I}$ is a "good" ideal and $\{f_n\}_{n\in\mathbb{N}}$ is $\mathcal{I}$-α convergent to $f \in Y^X$, then f is continuous.

The relations among $\mathcal{I}$-α convergence, semi-$\mathcal{I}$-α convergence, semi-uniform $\mathcal{I}$-convergence, semi-$\mathcal{I}$-exhaustiveness and the continuity of the $\mathcal{I}$-limit function of pointwise $\mathcal{I}$-convergence can be shown as Figure 3.

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