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ω^*-d-spaces, C-sobriety and duality of countably directed complete posets

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Abstract. C-sobriety characterizes the P-spaces which are determined by their open sets lattices. In this paper, on the one hand, we obtain the equivalent definitions of countably sober, ω^*-well-filtered and ω^*-d-spaces, and prove that a first countable P-space X is countably sober if and only if it is an ω^*-d-space. On the other hand, we establish a topological duality for countably directed complete posets via C-sobriety.

1. Introduction

Sobriety, well-filteredness and monotone convergence are three of the most important and extensively studied topological properties in non-Huasdorff topology and domain theory. Sobriety has been used in the characterization of the spaces which are determined by their open sets lattices. In the past few years, the research on sober spaces, well-filtered spaces and d-spaces has got some new breakthrough (see [9, 18]). To investigate various countability properties of these three spaces, Yang and Shi introduced a new topological property called c-sobriety, which is a weaker condition than sobriety and generalizes the concept of sober spaces [19]. Yang and Xi proved that c-sobriety can characterize the P-spaces which are determined by their open sets lattices in [20]. Xu, Shen, Xi and Zhao introduced the concepts of ω-well-filtered spaces, ω^*-well-filtered spaces and ω^*-d-spaces [18], and obtained some interesting results. For instance, a first countable T_0-space X is sober iff it is an ω-well-filtered d-space. In [21], Yang, Luo and Ye proved that a locally compact P-space X is countably sober if and only if it is ω^*-well-filtered. It is a natural question whether there are some links between c-sober spaces and ω^*-d-spaces. One of our purposes is to investigate this question and give the equivalent definitions of c-sober spaces, ω^*-well-filtered spaces and ω^*-d-spaces.

As we all know, the research on topological duality of ordered structures goes back to Stone's famous work [13, 14] on the topological duality of Boolean algebras and distributive lattices. The tool of prime ideal plays an important role in establishing topological duality for Boolean algebras. The dual space of a Boolean algebra is obtained by endowing the set of prime ideals with the Hull-Kernel topology. Stone explored the dual equivalence between the category with Boolean algebras as objects and Boolean

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lattice homomorphisms as morphisms and the category whose objects are Stone spaces (compact and totally disconnected spaces) and whose morphisms are the continuous mappings. In 1938, he developed a topological duality of distributive lattices relying on the same technology. In [12], Priestley built a duality for bounded distributive lattices taking advantage of ordered topological spaces. She showed that the category of bounded distributive lattices and lattice homomorphisms is dually equivalent to that of Priestley spaces (compact and totally order-disconnected spaces) with continuous mappings. Based on the work of Stone and Priestley, the theory of topological duality for lattices and other order structures has been widely developed. For example, in [11], Moshier and Jipsen developed topological dualities for meet-semilattices with a top element and bounded lattices. Later, González and Jansana generalized the duality given by Moshier and Jipsen for bounded lattices to posets [4]. They got the dual space of a poset by endowing the set of all filters with Scott topology, the dual spaces are abstractly characterized as the sober spaces which have the set of the compact open filters (w.r.t. the specialization order) as a base. Moreover, they built a dual equivalence between the category of posets with the maps that are order-preserving and satisfy that the inverse image of a filter is a filter and the category whose objects are the sober spaces with compact open filters as a base and whose morphisms are the continuous functions with the property that the inverse image of a compact open filter is a compact open filter.

Inspired by the work in [4], in this paper, we will use the countably sober spaces to establish a topological duality for a kind of special posets, named countably directed complete posets. A fundamental concept to build our duality is the notion of countably down-directed filters of a countably directed complete poset, where a countably down-directed filter F of a poset is an up-set with the property that every countable family of elements of F has a lower bound in F. We obtain the dual space of a countably directed complete poset by endowing the set of all countably down-directed filters with σ-Scott topology and establish a topological duality for countably directed complete posets. In addition, a duality for dcpos can be obtained by deleting the countability conditions in our duality.

The remaining parts of this paper are organized as follows. Section 2 recalls some basic concepts and results used in this paper. In Section 3, we investigate the equivalent definitions of countably sober spaces, ω^*-well-filtered spaces and ω^*-d-spaces by choosing different Θ, and prove that a first countable P-space X is countably sober if and only if it is an ω^*-d-space. In Section 4, we establish a topological representation for countably directed complete posets (c-dcpos, for short) by using countably down-directed filters. In Section 5, we develop a topological duality for c-dcpos. It is shown that the category of c-dcpos with the morphisms which preserve countably directed sups and its inverse image of a countably down-directed filter is a countably down-directed filter is dually equivalent to that of L_c-spaces with F_c-continuous mappings whose inverse maps preserves the countably directed sups of any nonempty family of compact open filters. Specially, we obtain a topological duality for dcpos by deleting the countability conditions in the duality for countably directed complete posets.

2. Preliminaries

We refer to [2] for the standard definitions and notations of order theory and domain theory, and to [1, 3] for topology.

Let $(L, \leq)$ be a poset and $A \subseteq L$. We define $A^u = \{x \in L : \forall a \in A, a \leq x\}$ and $A^l = \{x \in L : \forall a \in A, x \leq a\}$ as well as $\downarrow A = \{x \in L : (\exists a \in A) x \leq a\}$, $\downarrow x = \{y \in L : y \leq x\}$; $\uparrow A$ and $\uparrow x$ are defined dually. A nonempty subset F of L is called a *filter* if $F = \uparrow F$ and for any $x, y \in F$, there exists $z \in F$ such that $z \in \{x, y\}^l$.

Let X be a topological space. We denote the family of all open subsets of X by $\mathcal{O}(X)$. The *specialization order* $\sqsubseteq$ on X is defined as $x \sqsubseteq y$ if and only if for every open neighbourhood U of x, $y \in U$. It is easy to see that the specialization order $\sqsubseteq$ on X is reflexive and transitive. Moreover, It should be noted that $x \sqsubseteq y$ if and only if $x \in cl(\{y\})$, where cl is the closure operator.

Let A be a subset of X. The *saturation* $\mathrm{sat}(A)$ of A is the intersection of all open sets containing A. A subset A of a topological space X is said to be *saturated* if $A = \mathrm{sat}(A)$. It should be noted that $A \subseteq X$ is saturated iff it is an upper set with respect to the specialization order, that is iff $A = \uparrow A$.

Definition 2.1. *[8] Let L be a poset. A nonempty subset $D \subseteq L$ is said to be countably directed if for any countable subset $A \subseteq D$, there exists $d \in D$ such that $d \in A^u$.*

It is easy to see that each countably directed set is directed, but a directed set may not be countably directed by Example 2.2 in [8].

Definition 2.2. *[8] Let L be a poset. Then L is called a countably directed complete poset(c-dcpo, for short) if for any countably directed subset $D \subseteq L$, $\sup D$ exists.*

In [18], c-dcpo is written as ω^*-dcpo. By Definition 2.1, it is clear that each dcpo is a c-dcpo, but a c-dcpo may not be a dcpo. For example, let $L = \mathbb{N}$. Then L is a c-dcpo, but not a dcpo.

Definition 2.3. *[6] Let L, M be c-dcpos. A function $f : L \longrightarrow M$ is said to preserve countably directed sups if for any countably directed subset $D \subseteq L$, $\sup f(D)$ exists in M and $f(\sup D) = \sup f(D)$.*

Definition 2.4. *[6] Let L be a poset. A subset $U \subseteq L$ is called σ-Scott open if it satisfies the following conditions:*
(1) $U = \uparrow U$;
(2) *For any countably directed subset $D \subseteq L$, if $\sup D$ exists and $\sup D \in U$, then $D \cap U \neq \emptyset$.*

By the definition of σ-Scott open, it is obvious that each Scott open set is a σ-Scott open set, but a σ-Scott open set may not be a Scott open set. For instance, let $L = N_\omega$, where $N_\omega = \mathbb{N} \cup \{\omega\}$ with the order $1 < 2 < ... < n < ... < \omega$. Then we have $\{\omega\}$ is a σ-Scott open set, but not a Scott open set.

Let L be a poset. All σ-Scott open subsets of L form a topology called the *σ-Scott topology* and denoted as $\sigma_c(L)$.

For a c-dcpo L, we denote the set of all σ-Scott open filters of L by

$$OF_c(L) = \{F \subseteq L : F \text{ is a } \sigma\text{-Scott open filter of } L\}.$$

Definition 2.5. *A T_0-space X is called a σ-Scott space if for any subset $U \subseteq X$, U is open if and only if U is an up-set and for every countably directed subset $D \subseteq X$, if $\sup D$ exists and $\sup D \in U$, then $D \cap U \neq \emptyset$.*

In the following, the symbol $\mathbb{Z}_+$ denotes a countable set.

Definition 2.6. *[19] Let L be a complete lattice and $F \subseteq L$.*
(1) *F is called a countable filter if $F = \uparrow F$ and $\bigwedge_{i \in \mathbb{Z}_+} x_i \in F$ for any $\{x_i : i \in \mathbb{Z}_+\} \subseteq F$.*
(2) *F is called a completely prime countable filter if F is a countable filter and for any $S \subseteq L$, $\bigvee S \in F$ implies $S \cap F \neq \emptyset$.*

Definition 2.7. *Let L be a poset and $F \subseteq L$. F is called a countably down-directed filter if $F = \uparrow F$ and for any $\{x_i : i \in \mathbb{Z}_+\} \subseteq F$, there exists $x \in F$ such that $x \in \{x_i : i \in \mathbb{Z}_+\}^l$.*

Let L be a poset. The set of all countably down-directed filters of L is denoted by $Fi_c(L)$. Specially, for any $x \in L$, $\uparrow x \in Fi_c(L)$.

Definition 2.8. *[19] Let X be a topological space and $C \subseteq X$. C is said to be countably irreducible if C is nonempty and if for any closed subsets $\{B_i : i \in \mathbb{Z}_+\}$, $C \subseteq \bigcup_{i \in \mathbb{Z}_+} B_i$ implies that $C \subseteq B_i$ for some $i \in \mathbb{Z}_+$.*

We denote the set of all countably irreducible subsets of space X by $CIrr(X)$.

Definition 2.9. *[5, 10] Let X be a topological space. A point $p \in X$ is called a P-point if its filter of neighbourhoods is closed under countable intersection. A topological space X is called a P-space if every point in X is a P-point.*

3. Countably sober, ω^*-well-filtered and ω^*-d-spaces

In this section, we give the equivalent definitions of countably sober, ω^*-well-filtered and ω^*-d-spaces by using the definition of Θ-fine in [9], and establish some connections between countably sober spaces and ω^*-d-spaces.

Definition 3.1. *[9] Let Θ be a "function" which assigns a family $\Theta(X)$ of collections of subsets of X for each topological space X.*

A topological space X is called Θ-fine if for any open set U of X and $\mathcal{A} \in \Theta(X)$,

$$\bigcap\{sat(A) : A \in \mathcal{A}\} \subseteq U \text{ implies } A_0 \subseteq U \text{ for some } A_0 \in \mathcal{A}.$$

Example 3.2.

(1) *For each topological space X, let $\Theta_{cd}(X)$ comprise $\mathcal{A} = \{\{x_i\} : i \in I\}$ such that $\{x_i : i \in I\}$ is a countably directed set with respect to the specialization order.*

(2) *For each topological space X, let $\Theta_{cs}(X)$ comprise $\mathcal{A} = \{\{x_i\} : i \in I\}$ such that $\{x_i : i \in I\}$ is a countably irreducible set.*

(3) *For each topological space X, let $\Theta_{\omega^*}(X)$ comprise $\mathcal{A} = \{\{F_i\} : i \in I\}$, where every F_i is compact and $\{\{F_i\} : i \in I\}$ is countably directed (that is, for any $\{F_i : i \in \mathbb{Z}_+\} \subseteq \mathcal{A}$, there exists $F_k \in \mathcal{A}$ such that $F_k \subseteq \bigcap_{i\in\mathbb{Z}_+} \uparrow F_i$).*

Remark 3.3.

(1) *It is obvious that every countably directed subset D of a topological space X (with respect to the specialization order) is countably irreducible. Thus every Θ_{cs}-fine space is Θ_{cd}-fine.*

(2) *For every countably directed set $\{x_i : i \in I\}$, $\{\{x_i\} : i \in I\} \in \Theta_{\omega^*}$. Hence every Θ_{ω^*}-fine space is Θ_{cd}-fine.*

Definition 3.4. *[19] A T_0-space X is called countably sober, or c-sober for short, if for any countably irreducible closed set C, there exists a unique element $x \in X$ such that $C = \downarrow x$.*

By Proposition 3.3 in [21], the following result is trivial now. For the reader's convenience, we provide a brief proof.

Proposition 3.5. *A topological space X is countably sober iff it is Θ_{cs}-fine.*

Proof. Assume that the topological space X is countably sober and F a countably irreducible set of X. Then cl(F) is a countably irreducible closed set by Proposition 2.2 in [16]. Thus there exists an element $x_0 \in X$ such that cl$(F) = \downarrow x_0$. Let U be an open subset of X and

$$\bigcap\{\uparrow x : x \in F\} \subseteq U.$$

We have $x_0 \in U$ since $F \subseteq \downarrow x_0$. Thus, $F \cap U \neq \emptyset$. So there exists $x \in F$ such that $x \in U$. Therefore, X is a Θ_{cs}-fine space.

Conversely, assume that X is Θ_{cs}-fine. Let F be a countably irreducible closed set of X. Then we have $\{\{x\} : x \in F\} \in \Theta_{cs}(X)$. By Lemma 2 in [9], through the same process as the proof of Theorem 1 in [9], we can obtain that $\bigcap\{\uparrow x : x \in F\} = \uparrow a$ for some $a \in X$. Hence, $F \subseteq \downarrow a$. We claim that $\downarrow a \subseteq F$. Suppose that $a \notin F$. There is an element $x \in F$ such that $x \in X \backslash F$ by the definition of Θ_{cs}-fine spaces, which is a contradiction. Therefore, $F = \downarrow a$. □

For a topological space X, we denote the set of all nonempty compact saturated subsets of X by $\mathcal{Q}(X)$.

Definition 3.6. *[18] A T_0-space X is called ω^*-well-filtered, if for any countably filtered family $\{K_i : i \in I\} \subseteq \mathcal{Q}(X)$ and $U \in \mathcal{O}(X)$, it satisfies*

$$\bigcap_{i\in I} K_i \subseteq U \Rightarrow \exists i_0 \in I, K_{i_0} \subseteq U.$$

From definition, we deduce the following.

Proposition 3.7. *A topological space X is ω^*-well-filtered iff it is Θ_{ω^*}-fine.*

Definition 3.8. *[18] A T_0-space X is said to be an ω^*-d-space iff every subset D countably directed relative to the specialization order of X has a sup, and the relation* $\sup D \in U$ *for any open set U of X implies $D \cap U \neq \emptyset$.*

From Proposition 4.6 in [18], we have the following.

Proposition 3.9. *A topological space X is an ω^*-d-space iff it is Θ_{cd}-fine.*

The *upper Vietoris topology* on $Q(X)$ is the topology that has $\{\Box U : U \in \mathcal{O}(X)\}$ as a base, where $\Box U = \{K \in Q(X) : K \subseteq U\}$. The sets $\Diamond C = \{K \in Q(X) : K \cap C \neq \emptyset\}$ for a closed set C of X form a base for the closed sets of $Q(X)$. The set $Q(X)$ equipped with the upper Vietoris topology is called the *Smyth power space* or *upper space* of X in [7, 15].

For each topological space X, let $CS(X)$ be the collection of all countably irreducible sets of the upper space $Q(X)$. Then by Theorem 3.4 in [21] we immediately have the following.

Proposition 3.10. *A topological space X is countably sober iff X is CS-fine.*

In [22], Yang and Liu gave a variant of Rudin's Lemma.

Lemma 3.11. *[22] Let $\mathcal{F}$ be a countably directed family of nonempty finite subsets of a poset L. Then there exists a countably directed set $D \subseteq \bigcup_{F\in\mathcal{F}} F$ such that $D \cap F \neq \emptyset$ for all $F \in \mathcal{F}$.*

For each topological space X, let Θ_{cf} be the collection of all countably directed families $\mathcal{A}$ of nonempty finite subsets of X, that is, $\mathcal{A} = \{A_i : i \in I\} \in \Theta_{cf}$ if each A_i is a nonempty finite subset of X and for any $\{A_i : i \in \mathbb{Z}_+\} \subseteq \mathcal{A}$ there is a A_j such that $A_j \subseteq \bigcap_{i\in\mathbb{Z}_+} \uparrow A_i$. Therefore, it is easy to see that every Θ_{cf}-fine space is Θ_{cd}-fine.

Theorem 3.12. *A topological space X is an ω^*-d-space if and only if it is Θ_{cf}-fine.*

Proof. Assume that X is Θ_{cf}-fine, then it is clear that X is an ω^*-d-space by Proposition 3.9. Now suppose X is an ω^*-d-space and assume that $\mathcal{A} = \{A_i : i \in I\}$ is a countably directed family of nonempty finite subsets of X and $U \in \mathcal{O}(X)$ with

$$\bigcap\{\uparrow A_i : i \in I\} \subseteq U.$$

Suppose that $A_i \nsubseteq U$ for any $i \in I$. Then $\mathcal{T} = \{A_i - U : i \in I\}$ is a family of nonempty finite subsets. For any $\{A_i - U : i \in \mathbb{Z}_+\} \subseteq \mathcal{T}$, there exists $A_j \in \mathcal{A}$ such that $A_j \subseteq \bigcap_{i\in\mathbb{Z}_+} \uparrow A_i$. It follows that $A_j - U \subseteq \bigcap_{i\in\mathbb{Z}_+}(\uparrow A_i - U)$. Now we need to show that $\uparrow A_i - U \subseteq \uparrow(A_i - U)$ for all $i \in \mathbb{Z}_+$. Let $x \in \uparrow A_i - U$, then there exists $t \in A_i - U$ such that $t \sqsubseteq x$. Thus $x \in \uparrow(A_i - U)$. So, $A_j - U \subseteq \bigcap_{i\in\mathbb{Z}_+}(\uparrow A_i - U) \subseteq \bigcap_{i\in\mathbb{Z}_+} \uparrow(A_i - U)$. Hence, $\mathcal{T}$ is a countably directed family of nonempty finite subsets of a poset $(X, \sqsubseteq)$. By Lemma 3.11, there is a countably directed subset $D \subseteq \bigcup\{A_i - U : i \in I\}$ such that $D \cap (A_i - U) \neq \emptyset$ for each $i \in I$. Since X is an ω^*-d-space, $\sup D$ exists and $\sup D \notin U$. If $\sup D \in U$, then $D \cap U \neq \emptyset$. This contradicts $D \subseteq X - U$. On the other hand, $\sup D \in \bigcap\{\uparrow(A_i - U) : i \in I\} \subseteq \bigcap\{\uparrow A_i : i \in I\} \subseteq U$ implies $\sup D \in U$.

This contradiction shows that there must be a $A_i \in \mathcal{A}$ such that $A_i \subseteq U$. Therefore X is Θ_{cf}-fine. □

In what follows, we will establish some connections between countably sober spaces and ω^*-d-spaces.

Theorem 3.13. *Let X be a T_0-space. If X is a first countable P-space and $A \in CIrr(X)$, then $cl(A)$ is countably directed.*

Proof. For any $x \in X$, since X is first countable, there is an open neighborhood base $\{U_n(x) : n \in \mathbb{N}\}$. Set $U_\infty(x) = \bigcap_{n\in\mathbb{N}} U_n(x) = \uparrow x$. Since X is a P-space, $U_\infty(x)$ is an open set by Proposition 4.3 in [19] and $x \in U_\infty(x)$.

Suppose that $A \in CIrr(X)$, we prove that $cl(A)$ is countably directed. Let $\{a_i : i \in \mathbb{Z}_+\} \subseteq cl(A)$. It needs to be shown that $\bigcap_{i\in\mathbb{Z}_+} \uparrow a_i \cap cl(A) \neq \emptyset$. Since $a_i \in cl(A)$ for any $i \in \mathbb{Z}_+$, $A \cap U_\infty(a_i) \neq \emptyset$. Thus $A \cap \bigcap_{i\in\mathbb{Z}_+} U_\infty(a_i) \neq \emptyset$ by the fact that A is countably irreducible. Therefore, we conclude that

$$cl(A) \cap \bigcap_{i\in\mathbb{Z}_+} \uparrow a_i = cl(A) \cap \bigcap_{i\in\mathbb{Z}_+} U_\infty(a_i) \neq \emptyset.$$

□

Theorem 3.14. *Let X be a first countable P-space. Then X is countably sober iff it is an ω^*-d-space.*

Proof. Suppose that X is countably sober. By Proposition 3.5 and Proposition 3.9, it is straightforward to see that X is an ω^*-d-space.

Conversely, let A be a countably irreducible closed subset of X. Then by Theorem 3.13, A is countably directed. Since X is an ω^*-d-space, $\sup A \in A$, and hence $A = \downarrow \sup A$. Thus X is countably sober. □

In [18], Xu, Shen, Xi and Zhao proved that an ω^*-well-filtered space is an ω^*-d-space. And Yang, Luo and Ye showed that a countably sober space is ω^*-well-filtered in [21]. Hence, we have the following.

Corollary 3.15. *Let X be a first countable P-space. Then X is countably sober iff X is ω^*-well-filtered.*

4. Topological representation for *c*-dcpos

In this section, our main aim is to establish the topological representation of c-dcpos.

Let L be a c-dcpo. Then $(Fi_c(L), \subseteq)$ is also a c-dcpo. Hence, if $\{F_i : i \in I\}$ is a countably directed subset of $Fi_c(L)$, then

$$\sup_{i\in I} F_i = \bigcup_{i\in I} F_i.$$

Now, let us consider the c-dcpo $(Fi_c(L), \subseteq)$ with the σ-Scott topology $\tau_{Fi_c(L)}$. It is easy to show that the topological space $(Fi_c(L), \tau_{Fi_c(L)})$ is a P-space. For short, we denote $X_{L_c} = (Fi_c(L), \tau_{Fi_c(L)})$. It should be noted that the specialization order $\sqsubseteq$ of the space X_{L_c} is the order of inclusion $\subseteq$. That is, $F_1 \sqsubseteq F_2$ if and only if $F_1 \subseteq F_2$, for any $F_1, F_2 \in Fi_c(L)$. For any $x \in L$, we define the set $\varphi_x = \{F \in Fi_c(L) : x \in F\}$.

Proposition 4.1. *Let L be a c-dcpo. Then the family $\{\varphi_x : x \in L\}$ form a base for the σ-Scott topology on X_{L_c}.*

Proof. **Claim1**: φ_x is a σ-Scott open set for any $x \in L$.

It is obvious that φ_x is an upper set. Let $\{F_i\}_{i\in I}$ be a countably directed subset of $Fi_c(L)$ and $\sup_{i\in I} F_i \in \varphi_x$. Since $\sup_{i\in I} F_i = \bigcup_{i\in I} F_i$, we have that $x \in \bigcup_{i\in I} F_i$. Then there exists $i_0 \in I$ such that $x \in F_{i_0}$. Therefore, $F_{i_0} \in \varphi_x$.

Claim2: The family $\{\varphi_x : x \in L\}$ form a base for the σ-Scott topology on X_{L_c}.

Suppose $U \subseteq Fi_c(L)$ is a σ-Scott open subset of X_{L_c} and $F \in U$. Let us take the set $\mathcal{D} = \{\uparrow x : x \in F\}$. Then $\mathcal{D} \neq \emptyset$ because $F \neq \emptyset$. Now, we claim that $\mathcal{D}$ is countably directed. Assume $\{\uparrow x_i : i \in \mathbb{Z}_+\} \subseteq \mathcal{D}$, then $\{x_i : i \in \mathbb{Z}_+\} \subseteq F$. There is an element $t \in F$ such that $t \leq x_i$ for all $i \in \mathbb{Z}_+$ by the fact that F is a countably down-directed filter. Thus $\uparrow t \in \mathcal{D}$ and $\uparrow x_i \subseteq \uparrow t$ for any $i \in \mathbb{Z}_+$. We conclude that $\mathcal{D}$ is a countably directed subset. Therefore, $\sup \mathcal{D} = \bigcup \mathcal{D} = F \in U$.

Since U is a σ-Scott open set, we have that $U \cap \mathcal{D} \neq \emptyset$. Hence, there exists $x \in F$ such that $\uparrow x \in U$. So $\varphi_x \subseteq U$. Thus $F \in \varphi_x$ follows from $\uparrow x \in \varphi_x$ and $\uparrow x \subseteq F$, which implies that $F \in \varphi_x \subseteq U$.

Therefore, the family $\{\varphi_x : x \in L\}$ form a base for the σ-Scott topology on X_{L_c}. □

In the following, the topological space X_{L_c} is termed as the *dual space* of the c-dcpo L, where the topology is generated by the base $\{\varphi_x : x \in L\}$.

In [4], González and Jansana obtained the dual space of a poset P by endowing the set of all filters with Scott topology and characterized the basic open sets of the dual space of a poset P. Next, similar to Proposition 4.2 in [4], we characterize the basic open sets of the dual space of a countably directed complete poset L.

Proposition 4.2. *Let L be a c-dcpo. Then for any $x \in L$, φ_x is a compact open filter of X_{L_c}.*

Proof. Since $\uparrow x \in Fi_c(L)$ for any $x \in L$, we have $\varphi_x = \uparrow(\uparrow x)$. It is easy to see that φ_x is an open filter by the proof of Proposition 4.1. Now, let $\{U_i\}_{i\in I}$ be an open cover of φ_x. Then $\varphi_x \subseteq \bigcup_{i\in I} U_i$. Since $\uparrow x \in \varphi_x \subseteq \bigcup_{i\in I} U_i$, there exists $i_0 \in I$ such that $\uparrow x \in U_{i_0}$. Therefore, $\varphi_x \subseteq U_{i_0}$. □

Let L be a c-dcpo. We denote $KOF_c(X_{L_c})$ the set of all the compact open filters in X_{L_c}.

Proposition 4.3. *Let L be a c-dcpo. Then for any compact open filter U of the dual space X_{L_c}, there exists an element $x \in L$ such that $U = \varphi_x$.*

Proof. Suppose U is a compact open filter of X_{L_c}. Since U is a compact filter, there exists $F \in Fi_c(L)$ such that $U = \{G \in Fi_c(L) : F \subseteq G\}$. Let $\mathcal{D} = \{\uparrow x : x \in F\}$. Then $\mathcal{D}$ is a nonempty countably directed set of $Fi_c(L)$. Hence, $\sup \mathcal{D} = \bigcup \mathcal{D} = F \in U$. As U is σ-Scott open, we have that $U \cap \mathcal{D} \neq \emptyset$. So there exists $x \in F$ such that $\uparrow x \in U$ and $F \subseteq \uparrow x$. Thus we obtain that $\uparrow x = F$. Therefore,

$$U = \{G \in Fi_c(L) : \uparrow x \subseteq G\} = \{G \in Fi_c(L) : x \in G\} = \varphi_x.$$

□

By Proposition 4.2 and Proposition 4.3, we know that $KOF_c(X_{L_c}) = \{\varphi_x : x \in L\}$. In the following, we will consider the poset $(KOF_c(X_{L_c}), \subseteq)$ and give a topological representation for c-dcpos.

Theorem 4.4. *Let L be a c-dcpo. Then the mapping $\phi_L : L \longrightarrow KOF_c(X_{L_c}) : x \mapsto \varphi_x$ is an order isomorphism.*

Proof. (1) By Proposition 4.3, it is obvious that ϕ_L is surjective;

(2) The mapping ϕ_L is order-embedding. Let $x, y \in L$, we claim that

$$x \leq y \Leftrightarrow \varphi_x \subseteq \varphi_y.$$

($\Rightarrow$) Assume $F \in \varphi_x$. Then we have $x \in F$. Since $x \leq y$ and F is an upper set, we conclude that $y \in F$ and $F \in \varphi_y$. Therefore, $\varphi_x \subseteq \varphi_y$.

($\Leftarrow$) Suppose that $x \nleq y$. Then $y \notin \uparrow x$. Hence, $\uparrow x \notin \varphi_y$ but $\uparrow x \in \varphi_x$. This contradicts $\varphi_x \subseteq \varphi_y$. So $x \leq y$. □

5. Topological duality for *c*-dcpos

In Section 4, we prove that each c-dcpo L is order isomorphic to the compact open filters of its dual space X_{L_c} with respect to inclusion order. In this section, we will build the topological duality for c-dcpos.

Let X be a topological space. We define the set $\mathrm{sco}(U) = \bigcap\{B \in KOF_c(X) : U \subseteq B\}$ for any open filter U of X by using the idea of [17], where $KOF_c(X)$ is the set of all the compact open filters of X. Let $\mathrm{Fin}(X) = \{a \in X : \uparrow a \text{ is an open set}\}$. Then we have $KOF_c(X) = \{\uparrow a : a \in \mathrm{Fin}(X)\}$ by the property of compact open filters. Now, we will give the definition of L_c-spaces.

Definition 5.1. *A P-space X is said to be an L_c-space if it satisfies the following conditions:*
(L_{c1}) *X is countably sober;*
(L_{c2}) *$KOF_c(X)$ forms a base for the topology on X;*
(L_{c3}) *For any countably directed subset $\{U_i\}_{i\in I} \subseteq KOF_c(X)$, $sco(\bigcup_{i\in I} U_i) \in KOF_c(X)$.*

It is noteworthy that the sobriety is an important property for the dual space of a poset P in [4]. In our duality, c-sobriety is also a crucial property for the dual space of a countably directed complete poset L. And the condition (L_{c3}) of Definition 5.1 ensures that the set of all compact open filters of a P-space X under the order of inclusion is a c-dcpo.

Next, similar to Proposition 5.2 in [4], we provide an equivalent characterization of L_c-spaces.

Proposition 5.2. *Let X be a P-space. Then X is an L_c-space if and only if the following statements hold:*
(1) *For any countably directed set $\mathcal{D} \subseteq X$, $\sup \mathcal{D}$ exists concerning the specialization $\sqsubseteq$;*
(2) *X is a σ-Scott space;*
(3) *$KOF_c(X)$ forms a base for the topology on X;*
(4) *For any countably directed subset $\{U_i\}_{i\in I} \subseteq KOF_c(X)$, $sco(\bigcup_{i\in I} U_i) \in KOF_c(X)$.*

Proof. (Necessary) Assume that X is an L_c-space, it is clear that (3) and (4) hold. We only need to show (1) and (2).

By Proposition 3.1 and Proposition 3.2 in [16], it is easy to see that (1) holds and for any open set U, U is a σ-Scott open set.

To prove (2), it is sufficient to show that each σ-Scott open set is an open set. Let U be a σ-Scott open set and $x \in U$. Let $\mathcal{D} = \{a \in \mathrm{Fin}(X) : a \sqsubseteq x\}$. Then $\mathcal{D} \neq \emptyset$ from the fact that $KOF_c(X) = \{\uparrow a : a \in \mathrm{Fin}(X)\}$ is a base for the topology on X. Now, we prove the following claim.

Claim: $\mathcal{D}$ is countably directed.

Let $\{a_i : i \in \mathbb{Z}_+\} \subseteq \mathcal{D}$. Then $a_i \sqsubseteq x$ for every $i \in \mathbb{Z}_+$. Thus $x \in \bigcap_{i\in\mathbb{Z}_+} \uparrow a_i$. Since X is a P-space, we have $\bigcap_{i\in\mathbb{Z}_+} \uparrow a_i$ is an open set by Proposition 4.3 in [19]. Hence, there exists the family $\{c_j : j \in I\} \subseteq \mathrm{Fin}(X)$ such that $\bigcap_{i\in\mathbb{Z}_+} \uparrow a_i = \bigcup_{j\in I} \uparrow c_j$. So $x \in \bigcup_{j\in I} \uparrow c_j$. Then there is a $j_0 \in I$ such that $x \in \uparrow c_{j_0}$. Thus $c_{j_0} \in \mathcal{D}$ and $a_i \sqsubseteq c_{j_0}$ for every $i \in \mathbb{Z}_+$. We conclude that $\mathcal{D}$ is countably directed.

By (1), we know $\sup \mathcal{D}$ exists. Now we want to prove that $x = \sup \mathcal{D}$. It is easy to see that $x \in \mathcal{D}^u$. Hence, $\sup \mathcal{D} \sqsubseteq x$. Conversely, let $\uparrow s \in KOF_c(X)$ and $x \in \uparrow s$, then $s \sqsubseteq x$ and $s \in \mathcal{D}$. Thus $s \sqsubseteq \sup \mathcal{D}$, which implies that $x \sqsubseteq \sup \mathcal{D}$. Hence, $\sup \mathcal{D} = x \in U$. So $U \cap \mathcal{D} \neq \emptyset$ since U is σ-Scott open. Thus there exists $t \in U \cap \mathcal{D}$ such that $\uparrow t \subseteq U$ and $t \sqsubseteq x$. This implies that $x \in \uparrow t \subseteq U$. So U is an open set of X.

Therefore, X is a σ-Scott space.

(Sufficiency) We only need to show that X is countably sober.

Obviously, X is a T_0 space since X is a σ-Scott space. Suppose $\mathcal{F}$ is a completely prime countable filter of $\mathcal{O}(X)$. We need to prove that there exists an element $x \in X$ such that $\mathcal{F} = N^\circ(x)$.

Let $\mathcal{D} = \{x \in X : \uparrow x \in \mathcal{F}\}$. It follows from $\mathcal{F} \neq \emptyset$ that there exists $U \in \mathcal{F}$ such that $U = \bigcup_{i\in I} \uparrow x_i$ for some $\{x_i : i \in I\} \subseteq \mathrm{Fin}(X)$. Thus $\bigcup_{i\in I} \uparrow x_i \in \mathcal{F}$. Since $\mathcal{F}$ is a completely prime countable filter, there exists $i_0 \in I$ such that $\uparrow x_{i_0} \in \mathcal{F}$. So $x_{i_0} \in \mathcal{D}$. Thus $\mathcal{D} \neq \emptyset$.

We claim that $\mathcal{D}$ is countably directed. Let $\{x_i : i \in \mathbb{Z}_+\} \subseteq \mathcal{D}$. Then for every $i \in \mathbb{Z}_+$, $\uparrow x_i \in \mathcal{F}$. Since X is a P-space, we have $\bigcap_{i\in\mathbb{Z}_+} \uparrow x_i$ is an open set by Proposition 4.3 in [19]. Then $\bigcap_{i\in\mathbb{Z}_+} \uparrow x_i \in \mathcal{F}$ because $\mathcal{F}$ is a countably down-directed filter. If $\bigcap_{i\in\mathbb{Z}_+} \uparrow x_i = \emptyset$, then $\mathcal{F} = \mathcal{O}(X)$. This contradicts the fact that $\mathcal{F}$ is completely prime. Hence, $\bigcap_{i\in\mathbb{Z}_+} \uparrow x_i \neq \emptyset$. So there exists $\{c_j : j \in I\} \subseteq \mathrm{Fin}(X)$ such that $\bigcap_{i\in\mathbb{Z}_+} \uparrow x_i = \bigcup_{j\in I} \uparrow c_j$. Whence we have $\bigcup_{j\in I} \uparrow c_j \in \mathcal{F}$. Since $\mathcal{F}$ is completely prime, there exists $j_0 \in I$ such that $\uparrow c_{j_0} \in \mathcal{F}$. So $c_{j_0} \in \mathcal{D}$ and $x_i \sqsubseteq c_{j_0}$ for every $i \in \mathbb{Z}_+$. Thus $\mathcal{D}$ is countably directed.

By (1), $\sup \mathcal{D}$ exists. Let $x = \sup \mathcal{D}$. We want to prove that $\mathcal{F} = N^\circ(x)$. On the one hand, let $U \in \mathcal{F}$, there exists $\{x_i : i \in I\} \subseteq \mathrm{Fin}(X)$ such that $U = \bigcup_{i\in I} \uparrow x_i$ and $\bigcup_{i\in I} \uparrow x_i \in \mathcal{F}$. Hence, there exists $i_0 \in I$ such that $\uparrow x_{i_0} \in \mathcal{F}$ by the fact that $\mathcal{F}$ is completely prime. So $x_{i_0} \in \mathcal{D}$ and $x_{i_0} \sqsubseteq x$. Therefore, $x \in U$. So we obtain that $\mathcal{F} \subseteq N^\circ(x)$. On the other hand, let $U \in N^\circ(x)$, then $\mathcal{D} \cap U \neq \emptyset$. Thus there exists $y \in \mathcal{D} \cap U$ such that $\uparrow y \subseteq U$. Since $\uparrow y \in \mathcal{F}$, we have $U \in \mathcal{F}$. So $N^\circ(x) \subseteq \mathcal{F}$. Therefore, $\mathcal{F} = N^\circ(x)$. Thus X is countably sober by Theorem 4.5 in [19] as desired. □

Theorem 5.3. *Let L be a c-dcpo. Then the dual space X_{L_c} is an L_c-space.*

Proof. It is obvious that the dual space X_{L_c} is a P-space and satisfies (1), (2), (3) of Proposition 5.2. Hence, we just need to show that for any countably directed subset $\{\varphi_{x_i} : i \in I\} \subseteq KOF_c(X_{L_c})$, $\mathrm{sco}(\bigcup_{i\in I} \varphi_{x_i}) \in KOF_c(X_{L_c})$.

Claim: $\mathrm{sco}(\bigcup_{i\in I} \varphi_{x_i}) = \varphi_{\sup_{i\in I} x_i}$.

It follows from the countable directness of $\{\varphi_{x_i} : i \in I\}$ that $\{x_i : i \in I\}$ is a countably directed subset of L. Thus $\sup_{i\in I} x_i$ exists. Since $\bigcup_{i\in I} \varphi_{x_i}$ is an open filter of X_{L_c}, we have that

$$\mathrm{sco}(\bigcup_{i\in I} \varphi_{x_i}) = \bigcap \{\varphi_a \in KOF_c(X_{L_c}) : \bigcup_{i\in I} \varphi_{x_i} \subseteq \varphi_a\}.$$

Since $\varphi_{x_i} \subseteq \varphi_{\sup_{i\in I} x_i}$ for any $i \in I$, we have $\bigcup_{i\in I} \varphi_{x_i} \subseteq \varphi_{\sup_{i\in I} x_i}$. Hence, $\mathrm{sco}(\bigcup_{i\in I} \varphi_{x_i}) \subseteq \varphi_{\sup_{i\in I} x_i}$. Assume $\varphi_a \in KOF_c(X_{L_c})$ and $\bigcup_{i\in I} \varphi_{x_i} \subseteq \varphi_a$. Then for any $i \in I$, $\varphi_{x_i} \subseteq \varphi_a$. So $x_i \leq a$ for all $i \in I$ by Theorem 4.4. Thus $\sup_{i\in I} x_i \leq a$. By Theorem 4.4, we have $\varphi_{\sup_{i\in I} x_i} \subseteq \varphi_a$. So $\varphi_{\sup_{i\in I} x_i} \subseteq \mathrm{sco}(\bigcup_{i\in I} \varphi_{x_i})$.

Therefore, $\mathrm{sco}(\bigcup_{i\in I} \varphi_{x_i}) = \varphi_{\sup_{i\in I} x_i} \in KOF_c(X_{L_c})$. □

Obviously, by the definition of L_c-spaces, we know that if X is an L_c-space, then $(KOF_c(X), \subseteq)$ is a c-dcpo. In particular, for any countably directed set $\{U_i\}_{i\in I} \subseteq KOF_c(X)$, $\sup_{i\in I} U_i = \mathrm{sco}(\bigcup_{i\in I} U_i)$. Now, we will consider the dual space $Fi_c(KOF_c(X))$ of $KOF_c(X)$ for the L_c-space X.

Theorem 5.4. *Let X be an L_c-space. Then the mapping $\theta_X : X \longrightarrow Fi_c(KOF_c(X)) : x \mapsto \{U \in KOF_c(X) : x \in U\}$ is a homeomorphism.*

Proof. (1) θ_X is well-defined. It is obvious that $\theta_X(x)$ is an upper set. Let $\{U_i : i \in \mathbb{Z}_+\} \subseteq \theta_X(x)$. Then $x \in \bigcap_{i\in\mathbb{Z}_+} U_i$. Since X is a P-space, we know that $\bigcap_{i\in\mathbb{Z}_+} U_i$ is an open set by Proposition 4.3 in [19]. Thus there exists a compact open filter $U \in KOF_c(X)$ such that $x \in U \subseteq \bigcap_{i\in\mathbb{Z}_+} U_i$. So $U \in \theta_X(x)$. Therefore, $\theta_X(x) \in Fi_c(KOF_c(X))$.

(2) θ_X is injective. Let $x, y \in X$ and $x \neq y$, without loss of generality, we assume that $x \not\leq y$. Then there exists $U \in KOF_c(X)$ such that $x \in U$, $y \notin U$. Hence, $U \in \theta_X(x)$ and $U \notin \theta_X(y)$. So $\theta_X(x) \neq \theta_X(y)$.

(3) θ_X is surjective. Let $\mathcal{F} \in Fi_c(KOF_c(X))$ and $\mathcal{D} = \{x \in X : \uparrow x \in \mathcal{F}\}$. It follows from $\mathcal{F} \neq \emptyset$ that $\mathcal{D} \neq \emptyset$. Assume $\{x_i : i \in \mathbb{Z}_+\} \subseteq \mathcal{D}$. Then $\{\uparrow x_i : i \in \mathbb{Z}_+\} \subseteq \mathcal{F}$. Since $\mathcal{F}$ is a countably down-directed filter, there exists $\uparrow t \in \mathcal{F}$ such that $\uparrow t \subseteq \uparrow x_i$ for any $i \in \mathbb{Z}_+$. So $x_i \leq t$ for any $i \in \mathbb{Z}_+$ and $t \in \mathcal{D}$. Thus $\mathcal{D}$ is countably directed.

Obviously, $\sup \mathcal{D}$ exists since X is an L_c-space. Let $x = \sup \mathcal{D}$. We claim that $\theta_X(x) = \mathcal{F}$. On the one hand, assume $\uparrow t \in \mathcal{F}$. Then $t \in \mathcal{D}$ and $x \in \uparrow t$. Thus $\uparrow t \in \theta_X(x)$. So $\mathcal{F} \subseteq \theta_X(x)$. On the other hand, assume $\uparrow t \in \theta_X(x)$. According to $\sup \mathcal{D} = x \in \uparrow t$, we have $\mathcal{D} \cap \uparrow t \neq \emptyset$. Thus there exists $d \in \mathcal{D} \cap \uparrow t$ such that $\uparrow d \in \mathcal{F}$ and $\uparrow d \subseteq \uparrow t$. Hence, $\uparrow t \in \mathcal{F}$ from the fact that $\mathcal{F}$ is an upper set. Therefore, $\theta_X(x) \subseteq \mathcal{F}$.

(4) θ_X is continuous. Let $\varphi_U \in KOF_c(Fi_c(KOF_c(X)))$. Then we have

$$\begin{aligned}\theta_X^{-1}(\varphi_U) &= \{x \in X : \theta_X(x) \in \varphi_U\} \\ &= \{x \in X : U \in \theta_X(x)\} \\ &= \{x \in X : x \in U\} \\ &= U.\end{aligned}$$

(5) θ_X is an open map. Suppose $U \in KOF_c(X)$. We claim that $\theta_X(U) = \varphi_U$. Assume $\mathcal{F} \in \theta_X(U)$. Then there exists $x \in U$ such that $\theta_X(x) = \mathcal{F}$. So $U \in \mathcal{F}$ and $\mathcal{F} \in \varphi_U$. Therefore, $\theta_X(U) \subseteq \varphi_U$. Conversely, suppose $\mathcal{F} \in \varphi_U$. Then $U \in \mathcal{F}$. Since θ_X is surjective, there exists $x \in X$ such that $\theta_X(x) = \mathcal{F}$. Hence, $U \in \theta_X(x)$ and $\mathcal{F} \in \theta_X(U)$. So $\varphi_U \subseteq \theta_X(U)$. □

We consider the following categories:

$\mathbb{Dcpo}_c$ has c-dcpos as objects with the morphisms $f : L \to M$ preserving countably directed sups and satisfying that the inverse image of a countably down-directed filter is a countably down-directed filter.

A function $g : X \longrightarrow Y$ from the L_c-space X to the L_c-space Y is said to be *F_c-continuous* if for any $U \in KOF_c(Y)$, $g^{-1}(U) \in KOF_c(X)$.

$\mathbb{TOP}_{L_c}$ denotes the category whose objects are L_c-spaces and whose morphisms are F_c-continuous and its inverse map preserves the countably directed sups of any nonempty family of compact open filters.

Now we will build a duality between the categories $\mathbb{Dcpo}_c$ and $\mathbb{TOP}_{L_c}$ below.

Theorem 5.5. *The categories $\mathbb{Dcpo}_c$ and $\mathbb{TOP}_{L_c}$ are dually equivalent via the following functors:*

1. *$\Omega : \mathbb{Dcpo}_c \to \mathbb{TOP}_{L_c}^{op}$ defined by*
   - *$\Omega(L) := X_{L_c}$, for each c-dcpo L;*
   - *for every morphism $f : L \to M$ of $\mathbb{Dcpo}_c$, $\Omega(f) : X_{M_c} \longrightarrow X_{L_c}$ is given by $\Omega(f) := f^{-1}$.*
2. *$\Upsilon : \mathbb{TOP}_{L_c}^{op} \to \mathbb{Dcpo}_c$ defined by*

- $\Upsilon(X) := KOF_c(X)$, *for each* L_c*-space* X*;*
- *for every morphism* $g : X \to Y$ *of* $\mathbb{TOP}_{L_c}$*,* $\Upsilon(g) : KOF_c(Y) \to KOF_c(X)$ *is given by* $\Upsilon(g) := g^{-1}$*.*

Proof. (1) It is clear that Ω, Υ are well-defined.

(2) Suppose $f : L \to M$ is a morphism of $\mathbb{D}\mathbb{cpo}_c$. Then the mapping $\Omega(f) = f^{-1} : X_{M_c} \longrightarrow X_{L_c}$ is well defined by the definition of f. We only need to show that $\Omega(f)$ is F_c-continuous and its inverse maps $\Omega(f)^{-1}$ preserves the countably directed sups of any nonempty family of compact open filters. Let $\varphi_x \in KOF_c(X_{L_c})$. Then

$$\begin{aligned}\Omega(f)^{-1}(\varphi_x) &= (f^{-1})^{-1}(\varphi_x)\\ &= \{F \in Fi_c(M) : f^{-1}(F) \in \varphi_x\}\\ &= \{F \in Fi_c(M) : x \in f^{-1}(F)\}\\ &= \{F \in Fi_c(M) : f(x) \in F\}\\ &= \{F \in Fi_c(M) : F \in \varphi_{f(x)}\}\\ &= \varphi_{f(x)} \in KOF_c(X_{M_c}).\end{aligned}$$

So $\Omega(f)$ is F_c-continuous.

Let $\{\varphi_{x_i} : i \in I\} \subseteq KOF_c(X_{L_c})$ be a countably directed set. Then

$$\begin{aligned}\Omega(f)^{-1}(sco(\bigcup_{i\in I}\varphi_{x_i})) &= (f^{-1})^{-1}(sco(\bigcup_{i\in I}\varphi_{x_i}))\\ &= (f^{-1})^{-1}(\varphi_{\sup_{i\in I} x_i})\\ &= \varphi_{f(\sup_{i\in I} x_i)}\\ &= \varphi_{\sup_{i\in I} f(x_i)}\\ &= sco(\bigcup_{i\in I}\varphi_{f(x_i)})\\ &= sco(\bigcup_{i\in I}\Omega(f)^{-1}(\varphi_{x_i})).\end{aligned}$$

Thus $\Omega(f)^{-1}$ preserves the countably directed sups of any nonempty family of compact open filters.

(3) Suppose $g : X \to Y$ is a morphism of the category $\mathbb{TOP}_{L_c}$. We need to prove that for any $\mathcal{F} \in Fi_c(KOF_c(X))$, $\Upsilon(g)^{-1}(\mathcal{F}) \in Fi_c(KOF_c(Y))$.

Let $\mathcal{F} \in Fi_c(KOF_c(X))$. Then

$$\Upsilon(g)^{-1}(\mathcal{F}) = \{U \in KOF_c(Y) : g^{-1}(U) \in \mathcal{F}\}.$$

By the proof of Theorem 5.4, there exists $x \in X$ such that $\mathcal{F} = \theta_X(x)$. Thus for each $U \in KOF_c(Y)$, we have

$$\begin{aligned}U \in \Upsilon(g)^{-1}(\mathcal{F}) &\Leftrightarrow g^{-1}(U) \in \mathcal{F} \Leftrightarrow g^{-1}(U) \in \theta_X(x)\\ &\Leftrightarrow x \in g^{-1}(U) \Leftrightarrow g(x) \in U \Leftrightarrow U \in \theta_X(g(x)).\end{aligned}$$

Therefore, $\Upsilon(g)^{-1}(\mathcal{F}) = \theta_X(g(x)) \in Fi_c(KOF_c(Y))$.

(4) Obviously, Ω, Υ are functors.

(5) Consider the functors $\phi : \mathrm{id}_{\mathbb{D}\mathbb{cpo}_c} \longrightarrow \Upsilon \circ \Omega$, $\theta : \mathrm{id}_{\mathbb{TOP}_{L_c}} \longrightarrow \Omega \circ \Upsilon$. For any $L \in \mathbb{D}\mathbb{cpo}_c$, $\phi_L : L \longrightarrow KOF_c(X_{L_c})$ is an order isomorphism by Theorem 4.4. And for any $X \in \mathbb{TOP}_{L_c}$, $\theta_X : X \longrightarrow Fi_c(KOF_c(X))$ is homeomorphic and θ_X is F_c-continuous by Theorem 5.4. Now we only need to prove that ϕ_L preserves countably directed sups and θ_X^{-1} preserves the countably directed sups of any nonempty family of compact open filters.

Suppose D is a countably directed set of L. Then for any $x \in L$, $\phi_L(x) = \varphi_x$. Hence, we just need to show that

$$\varphi_{\sup D} = \sup_{d\in D}\varphi_d.$$

Since $d \le \sup D$ for any $d \in D$, we have $\varphi_d \subseteq \varphi_{\sup D}$. Thus $\varphi_{\sup D} \in \{\varphi_d : d \in \mathcal{D}\}^u$. Assume $U \in \{\varphi_d : d \in \mathcal{D}\}^u$. By Proposition 4.3, there exists $x \in L$ such that $U = \varphi_x$. Hence, $\varphi_d \subseteq \varphi_x$ for any $d \in D$. We know that $d \le x$ from Theorem 4.4. So $x \in D^u$ and $\sup D \le x$. Therefore, $\varphi_{\sup D} \subseteq \varphi_x$.

Suppose $\{\varphi_{U_i} : i \in I\}$ is a countably directed set of $KOF_c(Fi_c(KOF_c(X)))$. Then $\{U_i : i \in I\}$ is a countably directed set of $KOF_c(X)$. Hence, $\sup_{i\in I} U_i$ exists. So we have

$$\theta_X^{-1}(sco(\bigcup_{i\in I}\varphi_{U_i})) = \theta_X^{-1}(\varphi_{\sup_{i\in I} U_i}) = \sup_{i\in I} U_i = sco(\bigcup_{i\in I} U_i) = sco(\bigcup_{i\in I}\theta_X^{-1}(\varphi_{U_i})),$$

as desired.

(6) Now we want to prove that for any morphism $f : L \longrightarrow M$ of the category $\mathbb{Dcpo}_c$ as well as any morphism $g : X \longrightarrow Y$ of the category $\mathbb{TOP}_{L_c}$, the following diagrams commute:

$$\begin{array}{ccc} L & \xrightarrow{\phi_L} & KOF_c(X_{L_c}) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \Upsilon(\Omega(f))} \\ M & \xrightarrow[\phi_M]{} & KOF_c(X_{M_c}) \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{\theta_X} & Fi_c(KOF_c(X)) \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \Omega(\Upsilon(g))} \\ Y & \xrightarrow[\theta_Y]{} & Fi_c(KOF_c(Y)) \end{array}$$

For any $x \in L$,

$$\Upsilon(\Omega(f)) \circ \phi_L(x) = (f^{-1})^{-1}(\varphi_x) = \varphi_{f(x)} = \phi_M \circ f(x).$$

Then $\Upsilon(\Omega(f)) \circ \phi_L = \phi_M \circ f$.

For any $x \in X$, $\theta_X(x) = \{U \in KOF_c(X) : x \in U\}$. Thus

$$\begin{aligned} \Omega(\Upsilon(g)) \circ \theta_X(x) &= (g^{-1})^{-1}(\theta_X(x)) \\ &= \{V \in KOF_c(Y) : g^{-1}(V) \in \theta_X(x)\} \\ &= \{V \in KOF_c(Y) : x \in g^{-1}(V)\} \\ &= \{V \in KOF_c(Y) : g(x) \in V\} \\ &= \theta_Y \circ g(x). \end{aligned}$$

So $\Omega(\Upsilon(g)) \circ \theta_X = \theta_Y \circ g$.

Therefore, we conclude that the categories $\mathbb{Dcpo}_c$ and $\mathbb{TOP}_{L_c}$ are dually equivalent. □

Specially, we can obtain a topological duality for dcpos by deleting the countability conditions in the duality for countably directed complete posets.

In [4], the dual space of a poset P is obtained by considering the poset $(\mathrm{Fi}(P), \subseteq)$, where $\mathrm{Fi}(P)$ is the set of all filters of P, and endowing the set $\mathrm{Fi}(P)$ with Scott topology determined by $(\mathrm{Fi}(P), \subseteq)$. Inspired by the proofs in the duality for countably directed complete posets, it is not difficult to see that if P is a dcpo, then the dual space satisfies that for every directed family $\{U_i : i \in I\}$ of compact open filters, the intersection of the family of all compact open filters that include $\bigcup_{i\in I} U_i$ is a compact open filter. Moreover, this condition abstractly characterizes the duals of dcpos.

Definition 5.6. *A topological space (X, τ) is a P_d-space if it satisfies the following conditions:*
(P_{d1}) *X is sober;*
(P_{d2}) *$KOF_c(X)$ forms a base for the topology on X;*
(P_{d3}) *For any directed subset $\{U_i\}_{i\in I} \subseteq KOF_c(X)$, $sco(\bigcup_{i\in I} U_i) \in KOF_c(X)$.*

We denote by $\mathcal{DCPO}$ the category whose objects are dcpos and whose morphisms are Scott-continuous maps satisfying that the inverse image of a filter is a filter.

A function $g : X \longrightarrow Y$ from the P_d-space X to the P_d-space Y is said to be *F-continuous* if for any $U \in KOF_c(Y)$, $g^{-1}(U) \in KOF_c(X)$.

The symbol $\mathcal{TOP}(P_d)$ denotes the category whose objects are P_d-spaces and whose morphisms are F-continuous maps satisfying that the sup of the inverse images of the elements of any directed family $\mathcal{X}$ of compact open filters is the inverse image of the sup of $\mathcal{X}$.

The following theorem provides a topological duality for dcpos. The proof of the theorem is similar to that of Theorem 5.5.

Theorem 5.7. *There is a dual equivalence between the categories $\mathcal{DCPO}$ and $\mathcal{TOP}(P_d)$.*

In fact, another duality for dcpos of a different nature is obtained in [23]. In [23], the dual space of a dcpo is obtained by endowing the set of prime Scott open subsets with the Hull-Kernel topology, which applies only to dcpos with a top element. In contrast, in this paper, the dual space is defined by endowing the set of filters with the Scott topology, allowing for the characterization of the duals of general dcpos.

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