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Multidimensional characterization of asymptotically $\mathcal{I}_2^s-$ equivalent sequences

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Abstract. In 1951 Fast [12] introduced the concept statistical convergence of sequences which is a generalization of convergence. Afterward, Kostyrko et al. [14] extended the notion of statistical convergence to ideal convergence and established some basic theorems. Taking inspiration from this new approach, in this paper we introduce the matrix characterization of asymptotically $\mathcal{I}_2-$equivalent and asymptotically $\mathcal{I}_2-$statistical equivalent double sequences with an up-to-date perspective on multidimensional matrix transformation. Consequently, we will obtain conditions on $(a_{m,n,k,l})$ which assure us that the transformation is asymptotically $\mathcal{I}_2-$regular. This will be accomplished through a series of regularity-type theorems.

1. Introduction

The concept of statistical convergence was used to prove theorems on the statistical convergence of Fourier series by Zygmund [39] in the first edition of his celebrated monograph published in Warsaw. He used the term "almost convergence" place of statistical convergence and at that time this idea was not recognized much. Since the term "almost con vergence" was already in use Lorentz [17], Fast [7] had to choose a different name for his concept and "statistical convergence" was mostly the suitable one. Active research on this topic started after the paper of Fridy [9] and since then a large collection of literature has appeared. At the last quarter of the 20th century, statistical convergence has been discussed and captured important aspects in creating the basis of several investigations conducted in main branches of mathematics such as the theory of number [5], measure theory [19], trigonometric series [39], probability theory [4], and approximation theory [10]. In addition, it was further investigated from the sequence space point of view and linked with summability theory by Connor [2], Et at. al. [6], Kolk [13], Šalát [31], and many others made substantial contributions to the theory.

In 1980, Pobyvanets [29] introduced the concept of asymptotically regular matrices. This concept assess the preservation of asymptotic equivalence of two non-negative number sequences x and y. Thus in general, if $\frac{x}{y}$ has the limit 1, then $\frac{Ax}{Ay}$ of the transformed sequences also has a limit. In short, $x \sim y$ implies $Ax \sim Ay$. Fridy [8] introduced the following methods of comparing rates of convergence. Let x in ℓ_∞ and define the m-th term of the remainder sum as $R_m x = \sum_{n=m}^{\infty} |x_n|$, and examined the ratio $\frac{R_m x}{R_m y}$

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as $m \to \infty$. For the case where x is not in ℓ_∞, Fridy used the following partial sums $S_m x := \sum_{n \le m} |x_n|$ to examine the ratio. If x is a bounded sequence then Fridy used the supremum of the remaining term which is presented by $\mu_m t = \sup_{n \ge m} |t_n|$. Additionally, Marouf [18] and Li [16] presented significant results which are similar to Pobyvanets' by considering summability matrices that map sequences x and y with $\frac{x}{y}$ tends to 1 to sequences with property that the ratio of the sums of their tails or the ratio of the supremum of their tails also tends to 1. In 2003, Patterson [24] presented Pobyvalents' type result by replacing ordinary convergence with statistical convergence. Additionally, Patterson and Savas [27] presented necessary and sufficient conditions for sequences to be asymptotically equivalent with respect to statistical convergence, lacunary statistical convergence, and lacunary strong summability.

Let us consider the definitions and notations. Let c_0, c denote the collections of real-valued sequences which, respectively, converge to 0, converge to a limit and

$\ell_\infty = \left\{ x_k : \sum_{k=1}^{\infty} |x_k| < \infty \right\}$, $d_A = \left\{ x_k : \lim_n \sum_{k=1}^{\infty} a_{n,k} x_k \text{ exists.} \right\}$,

P_δ = {The set of all real number sequences such that $x_k \ge \delta > 0$ for all k} , and

P_0 ={The set of all non-negative sequences which have at most a finite number of zero terms}.

The definition of statistical convergence of a real sequence $x = (x_k)$ first studied by Fast [7] and Fridy in [9] as follows:

Definition 1.1. *Let A be a subset of $\mathbb{N}$, the set of natural numbers. Then the natural density of A denoted by $\delta(A)$. This is defined as*

$$\delta(A) = \lim_{p \to \infty} \frac{1}{p} \left| \{ k \le p : k \in A \} \right|$$

where $|A|$ denotes the cardinality of the enclosed set. The sequence $x = (x_k)$ has statistic limit L, denoted by $st - \lim x = L$ provided that for every $\varepsilon > 0$,

$$\lim_n \frac{1}{n} \left| \{ \text{the number of } k \le n : |x_k - L| \ge \varepsilon \} \right| = 0.$$

In 2003, R. F. Patterson introduced the concept of asymptotically equivalent as follows:

Definition 1.2. *[24] Two non-negative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically equivalent if*

$$\lim_k \frac{x_k}{y_k} = 1.$$

In this case, we denote this by $x \sim y$.

Patterson also extended this concept by presented the following definition which is natural combination of the concept of statistical convergence and asymptotically equivalent.

Definition 1.3. *[24] Two non-negative sequence (x_k) and (y_k) are said to be asymptotically statistical equivalent if, for every $\varepsilon > 0$,*

$$\lim_m \frac{1}{m} \left| \left\{ \text{the number of } k \le n : \left| \frac{x_k}{y_k} - 1 \right| \ge \varepsilon \right\} \right| = 0.$$

In this case we write $x \overset{S}{\sim} y$.

We shall now consider the widely used form of $\mathcal{I}$−convergence, which was presented in [14]. Additional results of this concept was presented in the following papers [36], [37], [22], [35], [38] and, [1], [3], [12], [15], [20], [21], [32], [33], [34]. Be begin the process by inspecting the following definitions of ideals:

Definition 1.4. *[14] If X is a non-empty set then a family of sets $\mathcal{I} \subset 2^X$ is called an "proper ideal" if and only if for each $A, B \in \mathcal{I}$ we have $A \cup B \in \mathcal{I}$, and for each $A \in \mathcal{I}$, $B \subset A$ implies $B \in \mathcal{I}$.*

Definition 1.5. *[14] A non-empty family $\mathcal{F} \subset 2^X$ is said to be a "filter" of X if and only if $\emptyset \notin \mathcal{F}$, for each $A, B \in \mathcal{F}$ we have $A \cap B \in \mathcal{F}$, and $A \in \mathcal{F}$, and $B \supset A$ we have $B \in \mathcal{F}$.*

Definition 1.6. *[14] An proper ideal $\mathcal{I}$ is called "non-trivial" if $\mathcal{I} \neq \emptyset$ and $X \notin \mathcal{I}$.*

Definition 1.7. *[14] $\mathcal{I} \subset 2^X$ is non-trivial ideal if and only if*

$$\mathcal{F} = \mathcal{F}(\mathcal{I}) = \{X \backslash A : A \in \mathcal{I}\}$$

is a filter of X. It is called the filter associated with the ideal.

Definition 1.8. *[14] A non-trivial ideal $\mathcal{I} \subset 2^X$ is called admissible if and only if $\mathcal{I} \supset \{\{x\} : x \in X\}$.*

Throughout $\mathcal{I}$ will represent a non-trivial admissible ideal of $\mathbb{N}$.

Definition 1.9. *[14] Let $\mathcal{I}$ be an ideal of subsets of $\mathbb{N}$, $x = (x_n)$ be a real-valued sequence and $L \in \mathbb{R}$. The sequence x is said to be $\mathcal{I}-$convergent to L, and we write $\mathcal{I} - \lim x = L$, provided that $A(\varepsilon) = \{n \in \mathbb{N} : |x_n - L| \geq \varepsilon\} \in \mathcal{I}$ for every $\varepsilon > 0$.*

Definition 1.10. *[15] Let (x_k) and (y_k) be non-negative real sequences and let $\mathcal{I}$ be an ideal in $\mathbb{N}$. If, for every $\varepsilon > 0$,*

$$\left\{k \in \mathbb{N} : \left|\frac{x_k}{y_k} - 1\right| \geq \varepsilon\right\} \in \mathcal{I}$$

then x and y are called asymptotically $\mathcal{I}-$equivalent sequences, this is denoted by $x \overset{\mathcal{I}}{\sim} y$.

Theorem 1.11. *(Pobyvancts [29]). A non-negative matrix is asymptotically regular if and only if for each fixed m,*

$$\lim_{n\to\infty} \frac{a_{n,m}}{\sum_{k=1}^{\infty} a_{n,k}} = 0.$$

The following concept of convergence for double sequences was presented by Pringsheim.

Definition 1.12. *[28] A double sequence $x = (x_{k,l})$ of giving complex (or real) numbers is called convergent to a scalar L in Pringsheim's sense(denoted by $P - \lim x = L$) provided that for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $\left|x_{k,l} - L\right| < \varepsilon$ whenever $k, l > N$. Such an x is described more briefly as "P − convergence".*

Definition 1.13. *Let $\mathcal{I}_2$ be an ideal in $\mathbb{N} \times \mathbb{N}$. A double sequence $x = (x_{k,l})$ of real numbers is said to be convergent to a number L with respect to the ideal $\mathcal{I}_2$, if for every $\varepsilon > 0$,*

$$A(\varepsilon) = \left\{(k, l) \in \mathbb{N} \times \mathbb{N} : \left|x_{k,l} - L\right| \geq \varepsilon\right\} \in \mathcal{I}_2.$$

In this case, we write $\mathcal{I}_2 - \lim_{k,l} x_{k,l} = L$.

Note that if $\mathcal{I}_2$ is the ideal

$$\mathcal{I}_0 = \{A \subset \mathbb{N} \times \mathbb{N} : (\exists m(A) \in \mathbb{N}) \ \ (i, j \geq m(A) \Rightarrow m(A) \Rightarrow (i, j) \notin A)\},$$

then $\mathcal{I}_2-$convergence coincides with the convergence in Pringsheim's sense and if we take $\mathcal{I}_d = \{A \subset \mathbb{N} \times \mathbb{N} : \delta_2(A) = 0\}$, then $\mathcal{I}_d-$convergence becomes statistical convergence for double sequences. In 2003, Patterson extended asymptotically statistical equivalent of multiple L to double sequences.

Definition 1.14. *[25] Two non-negative double sequence* $(x_{k,l})$ *and* $(y_{k,l})$ *are said to be asymptotically equivalent if*

$$P-\lim_{k,l}\frac{x_{k,l}}{y_{k,l}}=1.$$

denoted by $x\overset{P}{\sim}y$.

Definition 1.15. *[23] Let* $A=(a_{m,n,k,l})$ *denote a four dimensional summability method that maps the complex double sequence* $x=(x_{k,l})$ *into the double sequence* Ax *where the* $mn-th$ *term to* Ax *is as follows:*

$$(Ax)_{m,n}=\sum_{k,l=0,0}^{\infty,\infty}a_{m,n,k,l}x_{k,l}.$$

Let us consider the following notations:
$\ell^2=\left\{x_{k,l}:\sum_{k,l=1,1}^{\infty,\infty}\left|x_{k,l}\right|<\infty\right\}$,
$d_A=\left\{x_{k,l}:P-\lim_{m,n}\sum_{k,l=1,1}^{\infty,\infty}a_{m,n,k,l}x_{k,l}=\text{exists}\right\}$,
$P''_\delta=$ {The set of all real double number sequences such that $x_{k,l}\geq\delta>0$ for all k and l}, and
$P''_0=${The set of all nonnegative sequences which have at most a finite number of columns and/or rows with zero entries},

Let us define c''_0 as follows: a double sequence $x=(x_{k,l})$ belong to the set c''_0 provided that $P-\lim_{k,l}x_{k,l}=0$. The four dimensional matrix transformation A is called an $c''_0-c''_0$ if Az is in the set c''_0 whenever z is in c''_0 and z is bounded. Furthermore, the following is a characterization of $c''_0-c''_0$ matrices.

Theorem 1.16. (*Hamilton, [11]*) *A four dimensional matrix A is an* $c''_0-c''_0$ *if and only if*

1. $\sum_{p,q=1,1}^{\infty,\infty}\left|a_{k,l,p,q}\right|<\infty$ for all k,l;
2. Let $q=q_0$ then there exists $C_q(k,l)$ such that $a_{k,l,p,q}=0$ whenever $q>C_q(k,l)$ for all k,l,p;
3. Let $p=p_0$ then then there exists $C_p(k,l)$ such that $a_{k,l,p,q}=0$ whenever $p>C_p(k,l)$ for all k,l,q;
4. $P-\lim_{k,l}a_{k,l,p,q}=0$ for all p, and q.

The following Hamilton's ideas in [11], Patterson presented the following definitions in [26].

Definition 1.17. *[26] For each* $x=(x_{k,l})\in\ell^2$ *the "Pringsheim remainder sequence"* $R(x)$ *is the double whose* $m,n-th$ *term is*

$$R_{m,n}(x):=\sum_{k,l\geq m,n}\left|x_{k,l}\right|.$$

Definition 1.18. *[26] Let* $x=(x_{k,l})$ *be a convergent double sequence with limit L. Then the "maximum remaining Pringsheim difference" is given by*

$$\rho_{m,n}x=\max_{k,l>m,n}\left|x_{k,l}-L\right|.$$

Also if x is in ℓ^2*, let* $\mu(x)$ *denote the double sequence given by*

$$\mu_{m,n}(x)=\sup_{k,l>m,n}\left|x_{k,l}\right|.$$

Definition 1.19. *[26] Let* $x=(x_{k,l})$ *be a double sequence of real numbers and for each n, let* $\alpha_n=\sup_n\{x_{k,l}:k,l\geq n\}$. *The Pringsheim limit superior of* $x=(x_{k,l})$ *is defined as follows:*

1. If $\alpha=+\infty$ for each n, then $P-\limsup(x_{k,l}):=+\infty$;
2. If $\alpha<\infty$ for some n, then $P-\limsup(x_{k,l}):=\inf_n\{\alpha_n\}$.

Similarly, let $\beta_n=\inf_n\{(x_{k,l}):k,l\geq n\}$ then the Pringheim limit inferior of $(x_{k,l})$ is defined as follows:

1. If $\beta_n=-\infty$ for each n, then $P-\liminf(x_{k,l}):=-\infty$;
2. If $\beta_n>-\infty$ for some n, then $P-\liminf(x_{k,l}):=\sup_n\{\beta_n\}$.

2. Main Results

In this section, we will present the matrix characterization of asymptotically $\mathcal{I}_2-$equivalent and asymptotically $\mathcal{I}_2-$ statistical equivalent double sequences with an up-to-date perspective on multidimensional matrix transformation. At the end, we will obtain conditions on $(a_{m,n,k,l})$ which assures us that the transformation is asymptotically $\mathcal{I}_2-$regular.

Definition 2.1. *Let $(x_{k,l})$ and $(y_{k,l})$ be two real double sequences, then we say that $x_{k,l} = y_{k,l}$ for almost all k and l related to $\mathcal{I}_2$ (a.a.k.l. $\mathcal{I}_2$) if the set*

$$\{(k,l) \in \mathbb{N} \times \mathbb{N} : x_{k,l} \neq y_{k,l}\} \in \mathcal{I}_2.$$

Definition 2.2. *If $A = (a_{m,n,k,l})$ is a four dimensional matrix, then a double sequence $x = (x_{k,l}) \in \ell^2$ is said to be $A-$summable to L if*

$$P - \lim_{m,n} (Ax)_{m,n} = P - \lim_{m,n} \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l} x_{k,l} = L.$$

Definition 2.3. *Two nonnegative double sequences $x = (x_{k,l})$ and $y = (y_{k,l})$ are said to be asymptotically $\mathcal{I}_2-$equivalent of multiple $L \in \mathbb{R}$, if every $\varepsilon > 0$,and $y_{k,l} \neq 0$, the set*

$$\left\{(k,l) \in \mathbb{N} \times \mathbb{N} : \left|\frac{x_{k,l}}{y_{k,l}} - L\right| \geq \varepsilon\right\} \in \mathcal{I}_2.$$

Whenever this occurs, it will be denoted by $x \overset{\mathcal{I}_2}{\sim} y$, and we write $\mathcal{I}_2 - \lim_{k,l} \frac{x_{k,l}}{y_{k,l}} = L$, simply asymptotically $\mathcal{I}_2-$equivalent if $L = 1$.

Let us consider the following example.

Example 2.4. *Let us consider the following two nonnegative sequences:*

$$x_{k,l} = \begin{cases} \theta\,(kl)^{-2}\text{, if } k \text{ and } l \text{ are even} \\ \left(\sqrt{\theta}\,\sqrt{kl} + \sqrt{\theta}\right)^2\text{, if } k \text{ and } l \text{ are odd} \\ \left(-\sqrt{\theta}kl + \sqrt{\theta}\right)^2\text{, if } k \text{ is even, } l \text{ is odd} \\ \theta\left(k^2 l^3\right)^{-4}\text{, if } k \text{ is odd, } l \text{ is even} \end{cases}$$

and

$$y_{k,l} = \begin{cases} (kl)^{-2}\text{, if } k \text{ and } l \text{ are even} \\ \left(\sqrt{kl} + 1\right)^2\text{, if } k \text{ and } l \text{ are odd} \\ (kl - 1)^2\text{, if } k \text{ is even, } l \text{ is odd} \\ \left(k^2 l^3\right)^{-4}\text{, if } k \text{ is odd, } l \text{ is even} \end{cases}$$

where $\theta \in \mathbb{R} \setminus \{0\}$. Therefore, we obtain

$$\frac{x_{k,l}}{y_{k,l}} = \begin{cases} \theta,\ \text{if } k \text{ and } l \text{ are even} \\ \theta,\ \text{if } k \text{ and } l \text{ are odd} \\ \theta,\ \text{if } k \text{ is even, } l \text{ is odd} \\ \theta,\ \text{if } k \text{ is odd, } l \text{ is even.} \end{cases}$$

Thus,

$$\left\{(k,l) \in \mathbb{N} \times \mathbb{N} : \left|\frac{x_{k,l}}{y_{k,l}} - \theta\right| \geq \varepsilon\right\} \in \mathcal{I}_2.$$

Hence, $x = (x_{k,l})$ and $y = (y_{k,l})$ are asymptotically $\mathcal{I}_2-$equivalent of multiple θ.

Definition 2.5. *A nonnegative $c_0'' - c_0''$ summability matrix A is said to be asymptotically $\mathcal{I}_2-$regular of multiple $\phi \in \mathbb{R}$, if $Ax \overset{\mathcal{I}_2}{\sim} Ay$ whenever $x \overset{\mathcal{I}_2}{\sim} y$, $x \in P_0''$, and $y \in P_\delta''$ for some $\delta > 0$.*

Example 2.6. *Let us consider two double sequences $x = (x_{k,l})$ and $y = (y_{k,l})$ as follows:*

$$x_{k,l} = 12 \text{ and } y_{k,l} = 4 \text{ for all } k, l \in \mathbb{N} \times \mathbb{N},$$

thus $\left\{(k,l) \in \mathbb{N} \times \mathbb{N} : \left|\frac{x_{k,l}}{y_{k,l}} - 3\right| \geq \varepsilon\right\} \in \mathcal{I}_2$. We shall define four dimensional matrix A as the following:

$$A = (a_{m,n,k,l}) = \begin{cases} \frac{1}{m}, \text{ for } n = 3m-2, k = l = 1 \\ \frac{1}{m}, \text{ for } n = 3m-1, k = 1, l = 2 \\ \frac{1}{m}, \text{ for } n = 3m, k = 1, l = 3 \\ 0, \text{ otherwise.} \end{cases}$$

Since A is $c_0'' \to c_0''$ matrix and $x \overset{\mathcal{I}_2}{\sim} y$, we can say that $Ax \overset{\mathcal{I}_2}{\sim} Ay$. Therefore, we obtain the following

$$\mathcal{I}_2 - \lim_{k,l} \frac{x_{k,l}}{y_{k,l}} = 3 \text{ and } \mathcal{I}_2 - \lim_{k,l} \frac{(Ax)_{k,l}}{(Ay)_{k,l}} = 3.$$

As a result, A is asymptotically $\mathcal{I}_2-$regular of multiple 3.

Definition 2.7. *Two double sequences $x = (x_{k,l})$ and $y = (y_{k,l})$ are said to be asymptotically $\mathcal{I}_2^s-$ statistical equivalent of multiple L, if for every $\varepsilon > 0$,and $\delta > 0$,*

$$\left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn}\left|\left\{k \leq m \text{ and } l \leq n : \left|\frac{x_{k,l}}{y_{k,l}} - L\right| \geq \varepsilon\right\}\right| \geq \delta\right\} \in \mathcal{I}_2^s$$

(denoted $x \overset{\mathcal{I}_2^s}{\sim} y$).

Theorem 2.8. *Let A be a nonnegative $c_0'' - c_0''$ summability matrix, also let x and y be member of ℓ^2, and $\mathcal{I}_2^s$ and $\mathcal{J}_2^s$ be ideals in $\mathbb{N} \times \mathbb{N}$. Suppose that $x \overset{\mathcal{I}_2^s}{\sim} y$ with $x, y \in P_\delta''$ for some $\delta > 0$, then $\mu_{m,n}(Ax) \overset{\mathcal{J}_2^s}{\sim} \mu_{m,n}(Ay)$ if and only if for each $i, j = 1, 2, 3, \ldots$ and for some $\varepsilon > 0$ and $\gamma > 0$ such that*

$$\mathcal{J}_2^s - \lim_{m,n} \frac{1}{mn}\left\{\left|\left\{k \leq m, l \leq n : \left|\frac{a_{m,n,i,j}}{\sum\limits_{r,s=1,1}^{\infty,\infty} a_{m,n,r,s}}\right| \geq \varepsilon\right\}\right| \geq \gamma\right\} = 0,$$

for $\left\{(k,l) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn}\left|\left\{k \leq m, l \leq n : \left|\frac{x_{k,l}}{y_{k,l}} - L\right| \geq \varepsilon\right\}\right| \geq \gamma\right\} \in \mathcal{I}_2^s$.

Proof. Suppose for $\varepsilon > 0$ and $\gamma > 0$, and for each $i, j = 1, 2, 3, \ldots$ such that

$$\left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn}\left|\left\{k \leq m, l \leq n : \left|\frac{a_{m,n,i,j}}{\sum\limits_{r,s=1,1}^{\infty,\infty} a_{m,n,r,s}}\right| \geq \varepsilon\right\}\right| \geq \gamma\right\} \in \mathcal{J}_2^s,$$

then we will show that $\mu_{m,n}(Ax) \overset{\mathcal{J}_2^s}{\sim} \mu_{m,n}(Ay)$.

Since $x \overset{\mathcal{I}_2^s}{\sim} y$, let $\varepsilon > 0$ and $\gamma > 0$ then there exists a Pringsheim bounded null sequence $\eta = (\eta_{k,l})$. Let $\varepsilon > 0$ and

$$S = \left\{ (k,l) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn} \left| \left\{ k \leq m, l \leq n : \left| \frac{x_{k,l}}{y_{k,l}} - L \right| \geq \varepsilon \right\} \right| \geq \gamma \right\}.$$

Observe that $S \in \mathcal{I}_2^s$ when $x_{k,l} = y_{k,l}(1 + \eta_{k,l})$ for $k, l = 1, 2, 3, ...$ Additionally,

$$\left(L - \frac{\varepsilon}{5} \right) y_{k,l} < x_{k,l} < \left(L + \frac{\varepsilon}{5} \right) y_{k,l}$$

for each $(k,l) \notin S$. Also,

$$\left\{ (m,n) \in \mathbb{N} \times \mathbb{N} : \left| \frac{a_{m,n,i,j}}{\sum_{i,j=1,1}^{\infty,\infty} a_{m,n,r,s}} \right| \geq \frac{\varepsilon}{5} \right\} \in F(\mathcal{U}).$$

Then for each ordered pair (m,n), we obtain the following

$$\begin{aligned}
\frac{\mu_{m,n}(Ax)}{\mu_{m,n}(Ay)} &= \frac{\sup_{k,l \geq m,n} (Ax)_{k,l}}{\sup_{k,l \geq m,n} (Ay)_{k,l}} \\
&= \frac{\sup_{k,l \geq m,n} \sum_{i,j=1,1}^{\infty,\infty} a_{k,l,i,j} x_{i,j}}{\sup_{k,l \geq m,n} \sum_{i,j=1,1}^{\infty,\infty} a_{k,l,i,j} y_{i,j}} \\
&= \frac{\sup_{k,l \geq m,n} \left(\sum_{i,j \in S} a_{k,l,i,j} x_{i,j} + \sum_{i,j \notin S} a_{k,l,i,j} x_{i,j} \right)}{\sup_{k,l \geq m,n} \sum_{i,j=1,1}^{\infty,\infty} a_{k,l,i,j} y_{i,j}} \\
&= \frac{\sup_{k,l \geq m,n} \left(\sum_{i,j \in S} a_{k,l,i,j} \left(y_{i,j} + y_{i,j} \eta_{i,j} \right) \right)}{\sup_{k,l \geq m,n} \sum_{i,j=1,1}^{\infty,\infty} a_{k,l,i,j} y_{i,j}} + \frac{\sup_{k,l \geq m,n} \left(\sum_{i,j \notin S} a_{k,l,i,j} x_{i,j} \right)}{\sup_{k,l \geq m,n} \sum_{i,j=1,1}^{\infty,\infty} a_{k,l,i,j} y_{i,j}} \\
&\leq \frac{\sup_{k,l \geq m,n} \left| \sum_{i,j \in S} a_{k,l,i,j} \left(y_{i,j} + y_{i,j} \eta_{i,j} \right) \right|}{\sup_{k,l \geq m,n} \sum_{i,j=1,1}^{\infty,\infty} a_{k,l,i,j} y_{i,j}} + \frac{(L + (\varepsilon/5)) \sup_{k,l \geq m,n} \left(\sum_{i,j \notin S} a_{k,l,i,j} y_{i,j} \right)}{\sup_{k,l \geq m,n} \sum_{i,j=1,1}^{\infty,\infty} a_{k,l,i,j} y_{i,j}} \\
&\leq L + \frac{\sup_{k,l \geq m,n} \sum_{i,j \in S} a_{k,l,i,j} y_{i,j} \left| \eta_{i,j} \right|}{\sup_{k,l \geq m,n} \sum_{i,j=1,1}^{\infty,\infty} a_{k,l,i,j} y_{i,j}} + \left(L + \frac{\varepsilon}{5} \right) \\
&\leq L + \frac{\sup_{k,l \geq m,n} \sum_{i,j=1,1}^{h_0,t_0} a_{k,l,i,j} y_{i,j} \left| \eta_{i,j} \right|}{\sup_{k,l \geq m,n} \sum_{i,j=1,1}^{\infty,\infty} a_{k,l,i,j} y_{i,j}} + \frac{\sup_{k,l \geq m,n} \sum_{i,j=h_0+1,1}^{\infty,t_0} a_{k,l,i,j} y_{i,j} \left| \eta_{i,j} \right|}{\sup_{k,l \geq m,n} \sum_{i,j=1,1}^{\infty,\infty} a_{k,l,i,j} y_{i,j}}
\end{aligned}$$

$$+\frac{\sup_{k,l\geq m,n}\sum_{i,j=1,t_0+1}^{h_0,\infty} a_{k,l,i,j}y_{i,j}\left|\eta_{i,j}\right|}{\sup_{k,l\geq m,n}\sum_{i,j=1,1}^{\infty,\infty} a_{k,l,i,j}y_{i,j}}+\frac{\sup_{k,l\geq m,n}\sum_{i,j=h_0+1,t_0+1}^{\infty,\infty} a_{k,l,i,j}y_{i,j}\left|\eta_{i,j}\right|}{\sup_{k,l\geq m,n}\sum_{i,j=1,1}^{\infty,\infty} a_{k,l,i,j}y_{i,j}}$$
$$+\left(L+\frac{\varepsilon}{5}\right)$$

where h_0 and t_0 are positive integers. Since η is bounded double null sequence, $\sup_{i,j}\left|\eta_{i,j}\right|<\infty$. Therefore, there exist positive integers t_0 and h_0 for $i\geq t_0$, $j\geq h_0$, then $\left|\eta_{i,j}\right|<\frac{\varepsilon}{5}$. Thus,

$$\begin{aligned}
\frac{\mu_{m,n}(Ax)}{\mu_{m,n}(Ay)} &\leq L+\sup_{i,j}\left|\eta_{i,j}\right|\sum_{i,j=1,1}^{h_0,t_0}\frac{\sup_{k,l\geq m,n}a_{k,l,i,j}y_{i,j}}{\sup_{k,l\geq m,n}\sum_{i,j=1,1}^{\infty,\infty}a_{k,l,i,j}y_{i,j}}\\
&\quad+\frac{\frac{\varepsilon}{5}\sup_{k,l\geq m,n}\sum_{i,j=h_0+1,1}^{\infty,t_0}a_{k,l,i,j}y_{i,j}}{\sup_{k,l\geq m,n}\sum_{i,j=1,1}^{\infty,\infty}a_{k,l,i,j}y_{i,j}}\\
&\quad+\frac{\frac{\varepsilon}{5}\sup_{k,l\geq m,n}\sum_{i,j=1,t_0+1}^{h_0,\infty}a_{k,l,i,j}y_{i,j}}{\sup_{k,l\geq m,n}\sum_{i,j=1,1}^{\infty,\infty}a_{k,l,i,j}y_{i,j}}\\
&\quad+\frac{\frac{\varepsilon}{5}\sup_{k,l\geq m,n}\sum_{i,j=h_0+1,t_0+1}^{\infty,\infty}a_{k,l,i,j}y_{i,j}}{\sup_{k,l\geq m,n}\sum_{i,j=1,1}^{\infty,\infty}a_{k,l,i,j}y_{i,j}}+L+\frac{\varepsilon}{5}\\
&\leq L+\sup_{r,s}\left|\eta_{r,s}\right|\sum_{i,j=1,1}^{h_0,t_0}\frac{y_{i,j}\sup_{k,l\geq m,n}a_{k,l,i,j}}{\delta\sup_{k,l\geq m,n}\sum_{i,j=1,1}^{\infty,\infty}a_{k,l,i,j}}\\
&\quad+\frac{\varepsilon}{5}+\frac{\varepsilon}{5}+\frac{\varepsilon}{5}+L+\frac{\varepsilon}{5}\\
&\leq L+\sup_{i,j}\left|\eta_{r,s}\right|\sup_{\{(r,s):1\leq r\leq h_0\ \&\ 1\leq s\leq t_0\}}y_{i,j}\sum_{i,j=1,1}^{h_0,t_0}\sup\frac{a_{k,l,i,j}}{\sum_{i,j=0,0}^{\infty,\infty}a_{k,l,i,j}}\\
&\quad+\frac{\varepsilon}{5}+\frac{\varepsilon}{5}+\frac{\varepsilon}{5}+L+\frac{\varepsilon}{5}.
\end{aligned}$$

According to the hypothesis, there exists z_0 such that $k,l\geq z_0$, then we obtain the following:

$$\frac{a_{m,n,i,j}}{\sum_{r,s=1,1}^{\infty,\infty}a_{m,n,r,s}}<\frac{\frac{\varepsilon}{5}}{t_0h_0\sup_{r,s}\left|\eta_{r,s}\right|\sup_{\{(r,s):1\leq r\leq h_0\ \&\ 1\leq s\leq t_0\}}y_{i,j}}\text{ for } a.a.k.l.\ \mathcal{J}_2^s$$

for $m\geq h_0$ and $n\geq t_0$ we have

$$\frac{\mu_{m,n}(Ax)}{\mu_{m,n}(Ay)}\leq 2L+\frac{\varepsilon}{5}+\frac{\varepsilon}{5}+\frac{\varepsilon}{5}+\frac{\varepsilon}{5}+\frac{\varepsilon}{5}\text{ for } a.a.k.l.\ \mathcal{J}_2^s.$$

This implies that,

$$\frac{\mu_{m,n}(Ax)}{\mu_{m,n}(Ay)} \leq 2L + \varepsilon \text{ for } a.a.k.l. \ \mathcal{J}_2^s.$$

Hence,

$$\lim_{m,n} \frac{\mu_{m,n}(Ax)}{\mu_{m,n}(Ay)} \leq L_1 \text{ for } a.a.k.l. \ \mathcal{J}_2^s,$$

where $L_1 = 2L$. In a similar manner, let us establish the following inequality

$$\lim_{m,n} \frac{\mu_{m,n}(Ax)}{\mu_{m,n}(Ay)} \geq L, \text{ for } a.a.k.l. \ \mathcal{J}_2^s,$$

Thus, we have

$$\lim_{m,n} \frac{\mu_{m,n}(Ax)}{\mu_{m,n}(Ay)} = L, \text{ for } a.a.k.l. \ \mathcal{J}_2^s,$$

i.e.

$$\left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn} \left| \left\{ k \leq m, l \leq n : \left| \frac{\mu_{m,n}(Ax)}{\mu_{m,n}(Ay)} - L \right| \geq \frac{\varepsilon}{5} \right\} \right| \geq \gamma \right\} \in \mathcal{J}_2^s.$$

Hence $\mu_{m,n}(Ax) \overset{\mathcal{J}_2^s}{\sim} \mu_{m,n}(Ay)$.

Suppose that $\mu_{m,n}(Ax) \overset{\mathcal{J}_2^s}{\sim} \mu_{m,n}(Ay)$, for $x \overset{\mathcal{I}_2^s}{\sim} y$ and $x, y \in P''_\delta$ for some $\delta > 0$. If we consider the sequences x and y defined by

$$x_{k,l} = y_{k,l} = L \text{ for all } k, l \in \mathbb{N},$$

then $\mu_{m,n}(Ax) \overset{\mathcal{J}_2^s}{\sim} \mu_{m,n}(Ay)$ .i.e

$$\mathcal{J}_2^s - \lim_{m,n} \frac{\sup_{k,l \geq m,n} \sum\limits_{i,j=1,1}^{\infty,\infty} a_{k,l,i,j} x_{i,j}}{\sup_{k,l \geq m,n} \sum\limits_{i,j=1,1}^{\infty,\infty} a_{k,l,i,j} y_{i,j}} = L,$$

Therefore, there exists $\mathcal{M} > 0$ such that $\left\{\sum_{i,j=1,1}^{\infty,\infty} a_{k,l,i,j}\right\}_{k,l=1,1}^{\infty,\infty}$ is bounded by $\mathcal{M}$. Suppose that

$$\left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \left| \frac{a_{m,n,i,j}}{\sum\limits_{i,j=1,1}^{\infty,\infty} a_{m,n,r,s}} \right| < \varepsilon \right\} \in F(\mathcal{U})$$

for some i, j and $\varepsilon > 0$. Then, the exists $\psi > 0$ and a index sequence $m_1 < m_2 < m_3 < \ldots$ and $n_1 < n_2 < n_3 < \ldots$ such that

$$\frac{a_{u,v,i,j}}{\sum\limits_{r,s=1,1}^{\infty,\infty} a_{u,v,r,s}} \geq \psi \text{ for } u, v = 1, 2, 3, \ldots .$$

For $r, s > 0$ and define the sequences x and y by

$$x_{k,l} = \begin{cases} 1 + rs, \text{ if } k = i \text{ and } l = j \\ 1, \text{ otherwise} \end{cases}$$

and $y_{k,l} = 1$ for all $(k,l) \in \mathbb{N} \times \mathbb{N}$. Since $x \overset{\mathcal{I}_2^s}{\sim} y$, $x, y \in P_1''$, let us consider the following limit:

$$\begin{aligned} & \mathcal{J}_2^s - \lim_{u,v\to\infty} \frac{\sup_{k,l\geq u,v} \sum_{r,s=1,1}^{\infty,\infty} a_{m_k,n_l,r,s} x_{r,s}}{\sup_{k,l\geq u,v} \sum_{i,j=1,1}^{\infty,\infty} a_{m_k,n_l,r,s} y_{r,s}} \\ = \quad & \mathcal{J}_2^s - \lim_{u,v\to\infty} \frac{\sup_{k,l\geq u,v} \sum_{r,s=1,1}^{\infty,\infty} a_{m_k,n_l,r,s} + rs \cdot a_{m_k,n_l,r,s}}{\sup_{k,l\geq u,v} \sum_{i,j=1,1}^{\infty,\infty} a_{m_k,n_l,r,s}} \\ \geq \quad & \mathcal{J}_2^s - \lim_{u,v\to\infty} \frac{\sup_{k,l\geq u,v} \left(\sum_{r,s=1,1}^{\infty,\infty} a_{m_k,n_l,r,s} + rs\psi \sum_{r,s=1,1}^{\infty,\infty} a_{m_k,n_l,r,s} \right)}{\sup_{k,l\geq u,v} \sum_{i,j=1,1}^{\infty,\infty} a_{m_k,n_l,r,s}} = 1 + rs\psi. \end{aligned}$$

Choose $rs = \frac{1}{\psi}$, then we have

$$\mathcal{J}_2^s - \lim_{u,v\to\infty} \frac{\mu_{m_u,n_v}(Ax)}{\mu_{m_u,n_v}(Ay)} \geq 2.$$

Therefore, $\mu_{m,n}(Ax) \overset{\mathcal{J}_2^s}{\sim} \mu_{m,n}(Ay)$. $\square$

Now, let us consider the following example.

Example 2.9. *Let us consider two double sequences $x = (x_{k,l})$ and $y = (y_{k,l})$ which are defined by*

$$x_{k,l} = 10 \text{ and } y_{k,l} = 2 \text{ for all } k, l \in \mathbb{N}$$

thus, for every $\varepsilon > 0$ and $\delta > 0$, $\left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \left|\left\{k \leq m, l \leq n : \left|\frac{x_{k,l}}{y_{k,l}} - 5\right| \geq \varepsilon\right\}\right| \geq \gamma\right\} \in \mathcal{I}_2^s$. Additionally, we shall define B summable matrix as follows:

$$B = (b_{m,n,k,l}) = \begin{cases} 3, \text{ for } n = m, k = l = 1 \text{ or } n = m+2, k = 1, l = 3; \text{ where } m, n \text{ are odd} \\ 1, \text{ for } n = m, k = l = 1 \text{ or } n = m+2, k = 1, l = 3; \text{ where } m, n \text{ are even} \\ 2, \text{ for } n = m+1, k = 1, l = 2 \text{ or } n = m+3, k = 1, l = 4 \\ 0, \text{ otherwise.} \end{cases}$$

Then, we obtain

$$Bx = \begin{bmatrix} 30mn & 0 & 30mn & 0 & 0 & 0 & 0 & \cdots \\ 0 & 10mn & 20mn & 10mn & 20mn & 0 & 0 & \cdots \\ 0 & 0 & 30mn & 0 & 30mn & 0 & 0 & \cdots \\ 0 & 0 & 10mn & 20mn & 10mn & 20mn & 0 & \cdots \\ 0 & 0 & 0 & 0 & 30mn & 0 & 30mn & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

and

$$By=\begin{bmatrix} 6mn & 0 & 6mn & 0 & 6mn & 0 & 0 & \cdots \\ 0 & 2mn & 4mn & 2mn & 4mn & 0 & 0 & \cdots \\ 0 & 0 & 6mn & 0 & 6mn & 0 & 0 & \cdots \\ 0 & 0 & 0 & 2mn & 4mn & 2mn & 4mn & \cdots \\ 0 & 0 & 0 & 0 & 6mn & 0 & 6mn & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

where (m,n) *is corresponded the index of* $a_{m,n,k,l}$*. Since* $\mu_{m,n}x=\sup_{k,l\geq m,n}\left|x_{k,l}\right|$ *for* $m,n\geq 0$*, we obtain the following*

$$\mu_{m,n}(Bx)=\begin{bmatrix} 30mn \\ 20mn \\ 30mn \\ 20mn \\ 30mn \\ 20mn \\ \vdots \end{bmatrix} \text{ and } \mu_{m,n}(By)=\begin{bmatrix} 6mn \\ 4mn \\ 6mn \\ 4mn \\ 6mn \\ 4mn \\ \vdots \end{bmatrix}.$$

Thus,

$$\mathcal{I}_2^s-\lim_{k,l}\frac{x_{k,l}}{y_{k,l}}=5 \text{ and } \mathcal{J}_2^s-\lim_{k,l}\frac{\mu_{m,n}(Bx)}{\mu_{m,n}(By)}=5.$$

That means $x\overset{\mathcal{I}_2^s}{\sim}y$ *implies* $\mu_{m,n}(Bx)\overset{\mathcal{J}_2^s}{\sim}\mu_{m,n}(By)$*. Moreover,*

$$\left\{(m,n)\in\mathbb{N}\times\mathbb{N}:\frac{1}{mn}\left|\left\{k\leq m \text{ and } l\leq n:\left|\frac{a_{m,n,i,j}}{\sum_{r,s=1,1}^{\infty,\infty}a_{m,n,r,s}}\right|\geq\varepsilon\right\}\right|\geq\gamma\right\}\in\mathcal{J}_2^s.$$

Let us also consider the following example.

Example 2.10. *In this example, we examine the following cases; if* $\mu x\overset{\mathcal{I}_2^s}{\sim}\mu y$*, then* $x\overset{\mathcal{I}_2^s}{\nsim}y$ *and if* $\mu x\overset{\mathcal{I}_2^s}{\sim}\mu$*, then* $\mu_{m,n}(Ax)\overset{\mathcal{J}_2^s}{\nsim}\mu_{m,n}(Ay)$*. Now, let us consider two double sequences* $x=(x_{k,l})$ *and* $y=(y_{k,l})$ *defined by*

$$x_{k,l}=6\ \forall k,l \text{ and } y_{k,l}=\begin{cases}6, & \text{if } k \text{ is odd } \forall l\\ 3, & \text{if } k \text{ is even } \forall l\end{cases}$$

and let us define a four dimensional matrix which is as follows:

$$A=(a_{m,n,k,l})=\begin{cases}1, & m=n, k=1, l=1;\\ \frac{1}{3}, & m=n, k=1, l=2;\\ \frac{1}{6}, & m=n, k=1, l=3;\\ \frac{1}{12}, & m=n, k=1, l=4;\\ \frac{1}{24}, & m=n, k=1, l=5;\\ 0, & \text{otherwise.}\end{cases}$$

Therefore,

$$Ax = \begin{bmatrix} 6mn & 0 & 0 & 0 & 0 & \cdots \\ 0 & 2mn & 0 & 0 & 0 & \cdots \\ 0 & 0 & mn & 0 & 0 & \cdots \\ 0 & 0 & 0 & \frac{mn}{2} & 0 & \cdots \\ 0 & 0 & 0 & 0 & \frac{mn}{4} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

and

$$Ay = \begin{bmatrix} 6mn & 0 & 0 & 0 & 0 & \cdots \\ 0 & mn & 0 & 0 & 0 & \cdots \\ 0 & 0 & mn & 0 & 0 & \cdots \\ 0 & 0 & 0 & \frac{mn}{4} & 0 & \cdots \\ 0 & 0 & 0 & 0 & \frac{mn}{4} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

where (m,n) *is corresponded the index of* $a_{m,n,k,l}$. *Hence, we obtain the following:*

$$\mu_{m,n}(Ax) = \begin{bmatrix} 6mn \\ 2mn \\ mn \\ \frac{mn}{2} \\ \frac{mn}{4} \\ \vdots \end{bmatrix} \text{ and } \mu_{m,n}(Ay) = \begin{bmatrix} 6mn \\ mn \\ mn \\ \frac{mn}{4} \\ \frac{mn}{4} \\ \vdots \end{bmatrix}.$$

We can say that

$$\frac{\mu x}{\mu y} = 1, m = n = 1, 2, 3, ...$$

i.e. $\mu x \overset{\mathcal{I}_2^s}{\sim} \mu y$. *Additionally,*

$$\frac{\mu_{m,n}(Ax)}{\mu_{m,n}(Ay)} = \begin{cases} 1, \text{ if } m, n \text{ are odd} \\ 2, \text{ if } m, n \text{ are even} \end{cases}$$

However, $\frac{\mu_{m,n}(Ax)}{\mu_{m,n}(Ay)}$ *has no Pringsheim limit as* $m, n \to \infty$, *and also* $\frac{x}{y}$ *has no Prinsheim limit as* $k, l \to \infty$. *This status demostrates that* $x \overset{\mathcal{I}_2^s}{\nsim} y$ *and* $\mu_{m,n}(Ax) \overset{\mathcal{J}_2^s}{\nsim} \mu_{m,n}(Ay)$.

Theorem 2.11. *Let* $x = (x_{k,l})$ *and* $y = (y_{k,l})$ *belong to* $P''_\delta \cap \ell^2$ *and* $\mathcal{I}_2$ *be an ideal in* $\mathbb{N} \times \mathbb{N}$. *Then necesarry and sufficient condition for* $x \overset{\mathcal{I}_2}{\sim} y$ *is* $\mathcal{I}_2 - \lim(x - y) = 0$.

Proof. Let us assume that $x \overset{\mathcal{I}_2}{\sim} y$, then

$$\mathcal{R} = \left\{(k,l) \in \mathbb{N} \times \mathbb{N} : \left|\frac{x_{k,l}}{y_{k,l}} - L\right| < \varepsilon\right\} \in \mathcal{F}(\mathcal{I}_2)$$

and for $(k,l) \in \mathcal{R}$ and $L \in \mathbb{R}$ we have the following

$$(L - \varepsilon) y_{k,l} < x_{k,l} < (L + \varepsilon) y_{k,l}.$$

Therefore,

$$\begin{aligned} -\varepsilon y_{k,l} &< x_{k,l} - L y_{k,l} < \varepsilon y_{k,l} \\ &= 0 < x_{k,l} - L y_{k,l} - \varepsilon y_{k,l} < 2\varepsilon y_{k,l} \\ &= x_{k,l} - L y_{k,l} < 3\varepsilon y_{k,l} \end{aligned}$$

when $L = 1$,

$$x_{k,l} - y_{k,l} < 3\varepsilon \left(\sup_{k,l} \left| y_{k,l} \right| \right) = \widetilde{\varepsilon} \text{ for each } (k,l) \in \mathcal{R}$$

Hence, $\mathcal{I}_2 - \lim (x - y) = 0$.

Now, let us consider $\mathcal{I}_2 - \lim (x - y) = 0$. Then

$$\mathcal{S}_\varepsilon = \left\{ (k,l) : \left| x_{k,l} - y_{k,l} \right| < \varepsilon \right\} \in \mathcal{F}(\mathcal{I}_2).$$

For each $(k,l) \in \mathcal{S}_\varepsilon$, we have

$$\begin{aligned} -\varepsilon &< x_{k,l} - y_{k,l} < \varepsilon \\ &= \frac{-\varepsilon}{y_{k,l}} < \frac{x_k}{y_k} - 1 < \frac{\varepsilon}{y_{k,l}} \\ &= \frac{-\varepsilon}{\psi} < \frac{x_k}{y_k} - 1 < \frac{\varepsilon}{\psi} \\ &= 1 - \frac{\varepsilon}{\psi} < \frac{x_k}{y_k} < 1 + \frac{\varepsilon}{\psi} \end{aligned}$$

Hence, $\left\{ (k,l) : \left| \frac{x_{k,l}}{y_{k,l}} - 1 \right| < \frac{\varepsilon}{\psi} \right\} \in \mathcal{F}(\mathcal{I}_2)$. Therefore, $x \overset{\mathcal{I}_2}{\sim} y$. □

Let us now present an $\mathcal{I}_2-$ regularity theorem.

Theorem 2.12. *Let $\mathcal{I}_2$ be an ideal in $\mathbb{N} \times \mathbb{N}$ and consider the following double sequences $x = (x_{k,l})$ and $y = (y_{k,l})$. Then a nonnegative four dimensional matrix $A = \left(a_{i,j,k,l} \right)$ is asymptotically $\mathcal{I}_2-$regular if and only if*

$$\mathcal{I}_2 - \lim \frac{\sum_{k,l \in \mathcal{R}} a_{m,n,k,l}}{\sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}} = 0$$

for every $\mathcal{R} \in \mathcal{I}_2$.

Proof. Let us assume that A is asymptotically $\mathcal{I}_2-$regular. Let $\mathcal{R} \in \mathcal{I}_2$ and define the sequences x and y as follows:

$$x_{k,l} = \left\{ \begin{array}{l} 1, \text{if } (k,l) \notin \mathcal{R} \\ 0, \text{if } (k,l) \in \mathcal{R} \end{array} \right\}$$

and $y_{k,l} = 1$, for every $(k,l) \in \mathbb{N} \times \mathbb{N}$.

Since x and y are bounded sequences, $x \in P_0''$, and $y \in P_\delta''$. Therefore,

$$\begin{aligned} \frac{(Ax)_{m,n}}{(Ay)_{m,n}} &= \frac{\sum_{(k,l) \in \mathcal{R}} a_{m,n,k,l} x_{k,l} + \sum_{(k,l) \notin \mathcal{R}} a_{m,n,k,l} x_{k,l}}{\sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l} y_{k,l}} \\ &= \frac{\sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l} y_{k,l} - \sum_{(k,l) \in \mathcal{R}} a_{m,n,k,l}}{\sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}} \\ &= 1 - \frac{\sum_{(k,l) \in \mathcal{R}} a_{m,n,k,l}}{\sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}}. \end{aligned}$$

Since $Ax \overset{\mathcal{I}_2}{\sim} Ay$, we obtain the following

$$\mathcal{I}_2 - \lim \frac{\sum_{(k,l)\in\mathcal{R}} a_{m,n,k,l}}{\sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}} = 0.$$

Now, we will show that $\mathcal{I}_2 - \lim \frac{\sum_{k,l\in\mathcal{R}} a_{m,n,k,l}}{\sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}} = 0$ is sufficient for A to be asymptotically $\mathcal{I}_2-$regular. Let us define the matrix $C = (c_{m,n,k,l}) = \frac{a_{m,n,k,l}}{\sum\limits_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}}$ because of that the row sums equal 1 and $\mathcal{I}_2 - \lim \frac{\sum_{k,l\in\mathcal{R}} a_{m,n,k,l}}{\sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}} = 0$ is satisfied with $\left\{(m,n) : \left|\sum_{(k,l)\in\mathcal{R}} a_{m,n,k,l}\right| \geq \varepsilon\right\} \in \mathcal{I}_2$ for all $\mathcal{R} \in \mathcal{I}_2$, C is an $\mathcal{I}_2-$regular matrix. In addition, the matrix C maps members of P''_δ to P''_δ, and since $x \overset{\mathcal{I}_2}{\sim} y$, we have $\mathcal{I}_2 - \lim (x-y) = 0$. Thus, $\mathcal{I}_2 - \lim C(x-y) = 0 \Longrightarrow \mathcal{I}_2 - \lim (Cx - Cy) = 0$. Consequently, we obtain $Cx \overset{\mathcal{I}_2}{\sim} Cy$. Furthermore,

$$\begin{aligned}
\frac{(Cx)_{m,n}}{(Cy)_{m,n}} &= \frac{\sum_{k,l=1,1}^{\infty,\infty} c_{m,n,k,l} x_{k,l}}{\sum_{k,l=1,1}^{\infty,\infty} c_{m,n,k,l} y_{k,l}} \\
&= \frac{\sum\limits_{k,l=1,1}^{\infty,\infty} \left(\frac{a_{m,n,k,l}}{\sum\limits_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}} \right) x_{k,l}}{\sum\limits_{k,l=1,1}^{\infty,\infty} \left(\frac{a_{m,n,k,l}}{\sum\limits_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}} \right) y_{k,l}} \\
&= \frac{\sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l} x_{k,l}}{\sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l} y_{k,l}}.
\end{aligned}$$

Since $Cx \overset{\mathcal{I}_2}{\sim} Cy$, we have $Ax \overset{\mathcal{I}_2}{\sim} Ay$. □

3. Conclusion

In this article, we introduce the matrix characterization of asymptotically $\mathcal{I}_2-$equivalent and asymptotically $\mathcal{I}_2-$ statistical equivalent double sequences with an up-to-date perspective on multidimensional matrix transformation. Additionally, we obtain conditions on $(a_{m,n,k,l})$ which assure us that the transformation is asymptotically $\mathcal{I}_2-$regular. This result is an open problem to extend the concept of deferred statistical convergence, and also some application areas.

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