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Almost periodic solutions in distribution to a complete second-order neutral stochastic differential equation with damped term

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Abstract. In this paper, we introduce an innovative class of second-order stochastic differential equations. We examine fundamental properties such as the existence, uniqueness of almost periodic solutions in distribution (trajectory distribution). Moreover, to verify these properties, we propose a new modification of Grönwall's Lemma tailored for this type of solution. Finally, a meticulously chosen example is presented to illustrate the effectiveness of our results.

Introduction

We consider the following second-order stochastic differential equation ($SDE_{(2)}$) on a separable Hilbert space:

$$d(Y'(t) - F(t, Y(t))) = (AY(t) + BY'(t) + G(t, Y(t)))\,dt + H(t, Y(t))dW(t), \quad t \in \mathbb{R}, \tag{1}$$

where A and B are non-zero closed defined linear operators, $(W(t))_{t\in\mathbb{R}}$ is a Wiener process with covariance operator Q (a Q-Wiener process with Trace$(Q) < \infty$), and $F(\cdot,\cdot)$, $G(\cdot,\cdot)$, and $H(\cdot,\cdot)$ are continuous functions.

Equation (1) offers a versatile framework that encompasses a versatile array of semilinear $SDE_{(2)}$. Its significance shines particularly bright in capturing phenomena like stochastic wave equations, showcasing its practical value. These equations are vital in modeling a plethora of phenomena across disciplines such as physics, chemistry, and biology.

An entirely different approach to tackling the deterministic second-order differential equation ($DDE_{(2)}$), involving the formulation of propagators or solution operators, is extensively explored by several researchers as in works [9, 11, 22, 29] in the case where $B = 0$, and in [10, 12, 13, 31–36] in the general case. This method establishes necessary and sufficient conditions for the well-posedness of the $DDE_{(2)}$, particularly in scenarios involving cases of incomplete data, by exploiting the existence of a cosine and exponential-cosine

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functions. This method also allows for the establishment of an analog to the Miyadera-Feller-Phillips-Hille-Yosida (MFPHY) Theorem with $B = I$, focusing on the resolvent of A.

The lack of a definitive characterization for the well-posedness of the $DDE_{(2)}$ remains contentious. The difficulty in determining whether the $DDE_{(2)}$ is well-posed stems from two main factors. Firstly, conventional methods of reducing the problem to a system have not proved fruitful in yielding comprehensive outcomes. Secondly, the concept of well-posedness for the $DDE_{(2)}$ has remained ambiguous. Fattorini [10] demonstrated that the unique solution of the $DDE_{(2)}$ may not be exponentially bounded, unlike the case of the $DDE_{(1)}$ or the $DDE_{(2)}$ with $B = 0$. Therefore, the conventional approach of using the Laplace transform technique faced limitations in its universal application.

In the deterministic case, several authors have explored the concept of well-posedness for the $DDE_{(2)}$ by using the theory of cosine and exponential-cosine functions (for more details, see e.g., [3, 28]), which was further developed by incorporating the theory of $\mathcal{M}$, $\mathcal{N}$-functions (see [13, 17–19, 31, 36]). In the stochastic case, many authors have delved into this topic, as seen in [15, 16, 23, 25–27]. To our knowledge, until now, no work has addressed the existence and uniqueness of almost periodic solution in distribution of $SDE_{(2)}$ from model (1), with $B \neq 0$.

This study directly addresses second-order formulations, allowing for unbounded operators and preserving the system's natural structure, thereby simplifying analysis and interpretation. It also bridges gaps left by earlier methods, particularly by establishing the existence and uniqueness of almost periodic solutions, which were often overlooked. Building on [15, 21, 25–27], this work studies the almost periodicity of solutions by using the $\mathcal{M}$-$\mathcal{N}$-family framework, which generalizes the mild solution formula in unbounded time domains and facilitates the exploration of various properties of solutions, such as the existence of time-optimal controls in $\mathbb{R}$ (controllability results). This approach offers a more robust and flexible framework for studying second-order SDEs.

In this work, we investigate the existence and uniqueness of almost periodic solution in distribution of the $SDE_{(2)}$ (1) with $A \neq 0$, and $B \neq 0$. The main results of this paper are summarized as follows:

1. A novel set of sufficient conditions has been established for the existence, uniqueness, and almost periodicity of the second-order stochastic system.
2. Banach's fixed-point theorem has been effectively applied to derive results in unbounded time domains.
3. The almost periodicity in the distribution of the solution to the $SDE_{(2)}$ evolution system has been demonstrated for the first time.

More precisely:

- By assuming the commutativity of operators A and B, along with the exponential stability of $\mathcal{M}$, $\mathcal{N}$-functions (which is considered more natural than that of cosine, as motivated by the special case and the example provided at the end of this paper), we prove that the mild solution of type (1) can be expressed in a simplified form, and by using a novel variant of Grönwall's Lemma, we establish that equation (1) has a unique bounded solution. It is important to highlight that some researchers misapply the cosine function family when B is non-zero.

- We also show that if $F(\cdot,\cdot)$, $G(\cdot,\cdot)$, and $H(\cdot,\cdot)$ are almost periodic, then the unique bounded solution to equation (1) exhibits almost periodicity in distribution (i.e., trajectory distribution). Our results can be interpreted by drawing an analogy with the first-order findings presented in [1, 5, 14, 20, 30], using an integrated semigroup approach alongside a newly introduced modification of Grönwall's Lemma.

We now provide some clarifications and notes on the difficulties we encountered during our research.

When $B = 0$, the $\mathcal{M}$ and $\mathcal{N}$ families reduce to cosine and sine functions. However, the exponential stability condition for these functions is often not satisfied in many cases. For instance, in $L^2[0,\pi]$, the cosine function is bounded by 1, which conflicts with the requirement for exponential stability. Despite this, many studies assume exponential stability to validate the exponential stability of solutions to certain $SDE_{(2)}$.

However, our study leverages a non-zero value of B to more effectively formulate the solution, particularly as practical examples support this approach. By using the non-zero B, we address theoretical challenges associated with the cosine family's assumed exponential stability, which can lead to misleading results. Therefore, we focus on the $\mathcal{M}$ and $\mathcal{N}$ families, which are more suitable for studying solutions in $\mathbb{R}$, with a particular emphasis on almost-periodic solutions, a critical aspect of our research.

Our study, particularly in the special case, builds upon the comprehensive framework provided by [24], which aligns well with our findings. To tackle the limitations of previous approaches, we utilize the non-zero value of B to adapt the solution model to the infinite domain, specifically focusing on the almost periodicity of the solution. By combining the semi-group generated by $B/2$ with the cosine family defined by $A + (B/2)^2$, we manage the problem at infinity by selecting an appropriate constant that harmonizes the semi-group with the cosine family. This method ensures the exponential stability of the solution family, as demonstrated in both our specific case and the practical example. Nevertheless, proving the boundedness of the solution remains a significant challenge.

Although direct calculations can establish the boundedness of the solution, we generalize the method described in [14] to enhance rigor. Typically, the cosine (or sine) operator is bounded by $\exp(\delta t)$ (or $|t|\exp(\delta t)$, with $\delta \in \mathbb{R}$). In a bounded domain, it is feasible to use the same constant to bound both the cosine and sine families. However, in $\mathbb{R}$, applying identical constants for bounding is not practical. This distinction underscores the necessity of employing a novel variant of Grönwall's Lemma.

Following a discussion of notations and key concepts such as the Q-Wiener process, almost periodic functions, and $\mathcal{M}$ and $\mathcal{N}$-functions, the subsequent section introduces a crucial Theorem establishing the existence and uniqueness of solutions. It also highlights another significant result related to almost periodicity. To further illustrate the practical application of the theory, the next section provides a detailed example demonstrating its relevance to real-world scenarios.

1. Notations and Introductory Concepts

Let $(\mathbb{S}_1, d_{\mathbb{S}_1})$ and $(\mathbb{S}_2, d_{\mathbb{S}_2})$ be a separable and complete metric spaces. When $\mathbb{S}_1$ and $\mathbb{S}_2$ are Hilbert spaces, we denote them by $(\mathbb{H}_1, \|\cdot\|_{\mathbb{H}_1})$ and $(\mathbb{H}_2, \|\cdot\|_{\mathbb{H}_2})$ respectively, where $\|\cdot\|_{\mathbb{H}_1}$ and $\|\cdot\|_{\mathbb{H}_2}$ are the associated norms. The space of linear (resp. linear bounded) operators from $\mathbb{H}_i$ to $\mathbb{H}_j$ is denoted by $\mathcal{L}(\mathbb{H}_i, \mathbb{H}_j)$ (resp. $\mathcal{L}_b(\mathbb{H}_i, \mathbb{H}_j)$) and by $\mathcal{L}(\mathbb{H}_i)$ (resp. $\mathcal{L}_b(\mathbb{H}_i)$) if $\mathbb{H}_i := \mathbb{H}_j$. If $A \in \mathcal{L}(\mathbb{H}_i, \mathbb{H}_j)$, then A^* denotes its adjoint operator.

Q-Wiener Process

Here, we delve into key concepts from [6]. Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. An $\mathbb{H}_2$-valued stochastic process $(X(t))_{t\geqslant 0}$ is deemed Gaussian if, for any $n \in \mathbb{N}$ and arbitrary positive numbers $t_1, t_2, ..., t_n$, the $\mathbb{H}_2^n$-valued random variable $(X(t_1), ..., X(t_n))$ follows a Gaussian distribution. Let Q be a nonnegative trace class operator on a Hilbert space $\mathbb{H}_1$. Now, let's define a stochastic process $(W(t))_{t\geqslant 0}$ taking values in $\mathbb{H}_1$. We call W a Q-Wiener process if it satisfies the following conditions:

1. $W(0) = 0$,
2. W exhibits continuous trajectories,
3. W has independent increments, and
4. The law of $(W(t) - W(r))$ follows the Gaussian measure $\mathcal{N}(0, (t - r)Q)$.

It's noteworthy that there exists a complete orthonormal system $\{e_j\}$ in $\mathbb{H}_1$ and a bounded sequence of nonnegative real numbers $\{\lambda_j\}$ such that $Qe_j := \lambda_j e_j$. Assuming W is a Q-Wiener process, several statements follow: W is a Gaussian process on $\mathbb{H}_1$, $\mathbb{E}(W(t)) = 0$, $\mathrm{Cov}(W(t)) = tQ$, and,

$$\hat{W}(t) := \sum_{j=1}^{\infty} \sqrt{\lambda_j}\beta_j(t)e_j,$$

where

$$\beta_j(t) := \frac{1}{\sqrt{\lambda_j}} \left\langle \hat{W}(t), e_j \right\rangle_{\mathbb{H}_1}, \quad t \geqslant 0, \quad j \in \mathbb{N},$$

representing independent $\mathbb{R}$-valued standard Brownian motions that are mutually independent on $(\Omega, \mathcal{F}, \mathbb{P})$. At this juncture, let's introduce the subspace $\mathbb{U}_0 := Q^{1/2}(\mathbb{H}_1)$ of $\mathbb{H}_1$, which forms a Hilbert space. Additionally, we consider the space of all Hilbert–Schmidt operators $\mathcal{L}_2^0 := \mathcal{L}_2(\mathbb{U}_0, \mathbb{H}_2)$ from $\mathbb{U}_0$ into $\mathbb{H}_2$, which is a separable Hilbert space equipped with the norm,

$$\| \cdot \|^2_{\mathcal{L}_2(\mathbb{U}_0,\mathbb{H}_2)} := \|(\cdot)Q^{1/2}\|^2_{\mathcal{L}_2(\mathbb{H}_1,\mathbb{H}_2)} := \text{Trace}\,[(\cdot)Q(\cdot)^*]\,.$$

Consider two independent Q-Wiener processes denoted by $\hat{W}_1$ and $\hat{W}_2$. We now define a new process W as follows:

$$W(t) := \hat{W}_1(t)1_{\{t\geqslant 0\}} + \hat{W}_2(-t)1_{\{t\leqslant 0\}}.$$

Clearly, the process $(W(t))_{t\in\mathbb{R}}$ constitutes a Q-Wiener process with time parameters.

Probability Space

Consider a probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}}, \mathbb{P})$, on which a Q-Wiener process $(W(t))_{t\in\mathbb{R}}$ is defined on $\mathbb{H}_1$. Also, consider the right continuous filtration $\{\hat{\mathcal{F}}_t\}_{t\in\mathbb{R}}$. We further define $\{\mathcal{F}_t\}_{t\in\mathbb{R}}$ as the completed filtration of $\{\hat{\mathcal{F}}_t\}_{t\in\mathbb{R}}$, including the $\mathbb{P}$-null sets within the σ-algebra $\mathcal{F}$. For $p \geqslant 2$, consider the following spaces:

1. $\mathrm{C}(\mathbb{S}_1, \mathbb{S}_2)$ is the set of continuous functions $h : \mathbb{S}_1 \longrightarrow \mathbb{S}_2$.
2. $\mathrm{C_b}(\mathbb{H}_1, \mathbb{H}_2)$ consists of continuous functions $h : \mathbb{H}_1 \longrightarrow \mathbb{H}_2$ such that $\|h\|_\infty := \sup_{y\in\mathbb{H}_1} \|h(y)\|_{\mathbb{H}_2} < \infty$.
3. $\mathcal{M}^p(\mathbb{R}, \mathbb{H}_2)$ is the Banach space of continuous and progressively measurable stochastic processes $X : \Omega \times \mathbb{R} \to \mathbb{H}_2$, such that $\sup_{t\in\mathbb{R}} \mathbb{E}\|X(t)\|^p_{\mathbb{H}_2} < \infty$.

Almost Periodic Function (Single Variable or Two Variables)

Let $\mathcal{K}_1$ be a set of compact subsets of $\mathbb{S}_1$ and $\mathbb{K}_1 \in \mathcal{K}_1$. Suppose $h \in \mathrm{C}(\mathbb{R}, \mathbb{S}_2)$ (resp. $\hat{h} \in \mathrm{C}(\mathbb{R} \times \mathbb{S}_1, \mathbb{S}_2)$). The function h (resp. $\hat{h}$) is considered $d_{\mathbb{S}_2}$−almost periodic (with respect to $t \in \mathbb{R}$ (resp. with respect to $t \in \mathbb{R}$, uniformly with respect to $y \in \mathbb{K}_1$, for any $\mathbb{K}_1 \in \mathcal{K}_1$)) if any of the following equivalent definitions holds [2, 5],

(1) **Bohr definition:** For every $\varepsilon > 0$ (resp. and for every $\mathbb{K}_1 \in \mathcal{K}_1$), there exists a constant $l := l(\varepsilon) > 0$ (resp. $l := l(\varepsilon,\ \mathbb{K}_1) > 0$) such that,

$$\mathcal{V}(h, \varepsilon) := \left\{ \tau \in \mathbb{R} \,\middle|\, \sup_{t\in\mathbb{R}} \mathrm{d}_{\mathbb{S}_2}(h(t + \tau), h(t)) < \varepsilon \right\}, \tag{2}$$

is relatively dense, respectively,

$$\mathcal{V}(\hat{h}, \varepsilon, \mathbb{K}_1) := \left\{ \tau \in \mathbb{R} \,\middle|\, \sup_{t\in\mathbb{R}} \left[\sup_{y\in\mathbb{K}_1} \mathrm{d}_{\mathbb{S}_2}(\hat{h}(t + \tau, y), \hat{h}(t, y)) \right] < \varepsilon \right\},$$

is relatively dense.

(2) **Bochner characterization** (**single sequences**) **:** For any sequence $\{u_n'\} \subset \mathbb{R}$, (resp. and for every $\mathbb{K}_1 \in \mathcal{K}_1$), there exist subsequences $\{u_n\} \subset \{u_n'\}$ and a function $h_\infty \in \mathrm{C}(\mathbb{R}, \mathbb{S}_2)$ (resp. $\hat{h}_\infty \in \mathrm{C}(\mathbb{R} \times \mathbb{S}_1, \mathbb{S}_2)$), such that,

$$\lim_{n\to\infty} h\,(t + u_n) := h_\infty(t), \quad \text{resp.} \quad \lim_{n\to\infty} \hat{h}\,(t + u_n, y) := \hat{h}_\infty(t, y), \tag{3}$$

exists, with respect to $\mathrm{d}_{\mathbb{S}_2}$, and uniformly on $\mathbb{R}$ (resp. uniformly on $\mathbb{R} \times \mathbb{K}_1$) .

(3) **Bochner characterization** (**double sequences**) **:** For any sequences $\{u'_n\} \subset \mathbb{R}$ and $\{v'_n\} \subset \mathbb{R}$, (resp. and for every $\mathbb{K}_1 \in \mathcal{K}_1$), there exist sub-sequences $\{u_n\} \subset \{u'_n\}$ and $\{v_n\} \subset \{v'_n\}$ respectively with the same indexes, such that:

$$\lim_{n\to\infty} \lim_{m\to\infty} h\left(t + u_n + v_m\right), \quad \text{and} \quad \lim_{n\to\infty} h\left(t + u_n + v_n\right), \tag{4}$$

respectively,

$$\lim_{n\to\infty} \lim_{m\to\infty} \hat{h}\left(t + u_n + v_m, y\right), \quad \text{and} \quad \lim_{n\to\infty} \hat{h}\left(t + u_n + v_n, y\right),$$

exist and are equal, with respect to $\mathrm{d}_{\mathbb{S}_2}$, pointwise on $\mathbb{R}$ (resp. uniformly on $\mathbb{R} \times \mathbb{K}_1$).

Almost Periodicity In Distributions

Let's consider $\mathcal{B}_{\mathbb{S}_2}$ as the Borel σ-algebra and $\mathcal{P}(\mathbb{S}_2)$ as the set of probability measures $\mu : \mathcal{B}_{\mathbb{S}_2} \longrightarrow [0, 1]$ on $\mathbb{S}_2$. Consider a $\mathcal{B}_{\mathbb{S}_2}$−measurable mapping $h : (\mathbb{S}_2, \mathcal{B}_{\mathbb{S}_2}, \mu) \to \mathbb{R}$. The expectation of h under μ is denoted as

$$\mu(h) := \mathbb{E}_\mu(h) := \int_{\mathbb{S}_2} h \mathrm{d}\mu.$$

We say that a sequence of probability measures μ_m converges to μ weakly if $\mu_m(h)$ converges $\mu(h)$ for all $h \in \mathrm{C_b}(\mathbb{S}_2, \mathbb{R})$.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, $X : \Omega \to \mathbb{S}_2$ be a random variable, and $\mathrm{Law}(X) := \mathbb{P}_X : \mathcal{B}_{\mathbb{S}_2} \longrightarrow [0, 1]$ be the law of X defined by,

$$\mathbb{P}_X(A) := \mathbb{P}(X \in A) := \mathbb{P} \circ X^{-1}(A).$$

For all $\mu \in \mathcal{P}(\mathbb{S}_2)$, there exists $(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathrm{X} : \Omega \to \mathbb{S}_2$ such that $\mathrm{Law}(X) := \mu$ and $\mu(h) := \mathbb{E}(h(X))$. Now, let's proceed with the definition,

Definition 1.1 ([7, 8]). *For $h \in \mathrm{C_b}(\mathbb{S}_2, \mathbb{R})$, the Lipschitz seminorm is:*

$$|h|_{\mathrm{L}} := \sup_{y_1 \neq y_2} \left[\frac{|h(y_1) - h(y_2)|}{\mathrm{d}_{\mathbb{S}_2}(y_1, y_2)} \right].$$

We introduce this notation:

- $\mathrm{BL}(\mathbb{S}_2, \mathbb{R}) := \{h : \mathbb{S}_2 \longrightarrow \mathbb{R} \,|\, h$ is a continuous function such that $|h|_{\mathrm{BL}} := \max\{|h|_{\mathrm{L}}, |h|_\infty\} < \infty\}$.

For any $\mu_1, \mu_2 \in \mathcal{P}(\mathbb{S}_2)$ and $h \in \mathrm{C_b}(\mathbb{S}_2, \mathbb{R})$, define,

$$\int_{\mathbb{S}_2} h \mathrm{d}(\mu_1 - \mu_2) := \int_{\mathbb{S}_2} h \mu_1 - \int_{\mathbb{S}_2} h \mathrm{d}\mu_2.$$

Moreover, for any metrizable and separable topological spaces $(\mathbb{S}_2, d_{\mathbb{S}_2})$, the metric on $\mathcal{P}(\mathbb{S}_2)$ is defined as follows,

$$\mathrm{d}_{\mathrm{BL}}(\mu_1, \mu_2) := \sup_{|h|_{\mathrm{BL}} \leqslant 1} \left| \int_{\mathbb{S}_2} h \mathrm{d}(\mu_1 - \mu_2) \right|.$$

It's noteworthy that [20], d_{BL} serves as a complete metric on $\mathcal{P}(\mathbb{S}_2)$ and induces the weak topology. In simpler terms, it is the coarsest topology on $\mathcal{P}(\mathbb{S}_2)$ ensuring the mappings $\mu \longrightarrow \mu(h)$ are continuous for all $h \in \mathrm{C_b}(\mathbb{S}_2, \mathbb{R})$.

Remark 1.2 ([5]). *The definitions* (2)−(4) *hold for the metric spaces* $(\mathcal{P}(\mathbb{S}_2), \mathrm{d}_{\mathrm{BL}})$ *and* $(\mathcal{P}(\mathrm{C}(\mathbb{R}, \mathbb{S}_2)), \mathrm{d}_{\mathrm{BL}})$, *where* $\mathrm{C}(\mathbb{R}, \mathbb{S}_2)$ *is endowed with the topology of uniform convergence on compact subsets of* $\mathbb{R}$.

Definition 1.3 ([4]). *Consider the mapping:* $X : \Omega \times \mathbb{R} \to \mathbb{S}_2$ *. We say that* $X(t)$ (resp. $\hat{X}(t) := \hat{X}(t+\cdot)$ *in the case of continuity of* X)*, is almost periodic in one*−(resp. *multi*)−*dimensional distributions if the mapping* $\left[t \longrightarrow \mathbb{P}_{X(t)}\right]$ (resp. $[t \longrightarrow \mathbb{P}_{X(t+\cdot)}]$) *is* $\mathrm{d}_{\mathrm{BL}}-$ *almost periodic. Here, the mapping takes t from* $\mathbb{R}$ *to the space of probability measures* $\mathcal{P}(\mathbb{S}_2)$ (resp. $\mathcal{P}\left(\mathrm{C}(\mathbb{R},\mathbb{S}_2)\right)$).

Lemma 1.4 ([14]). *For* $p \geqslant 2$ *and* $X \in \mathcal{L}_2(\mathbb{U}_0, \mathbb{H}_2)$*, the following inequality holds for every* $t \geqslant 0$*:*

$$\mathbb{E}\left\|\int_0^t X(s)\mathrm{d}W(s)\right\|_{\mathbb{H}_2}^p \leqslant K_{p/2}\mathbb{E}\left[\left(\int_0^t \mathrm{Trace}\left(X(s)QX^*(s)\right)\mathrm{d}s\right)^{\frac{p}{2}}\right],$$

where $K_{p/2} > 0$*. Specifically,* $K_1 := 1$.

2. Second-Order Differential Equation

Consider the following second-order differential equation:

$$d(y'(t) - f(t)) = Ay(t) + By'(t) + g(t), \quad t \in \mathbb{R}, \tag{5}$$

where $f, g : \mathbb{R} \longrightarrow \mathbb{H}_2$ are continuously differentiable functions, and A and B are closed linear operators on Hilbert space $\mathbb{H}_2$. According to [13, p.p. 28–50], (or [18, Translated version]), we define:

Definition 2.1. *A one-parameter families of bounded commuting operators* $\mathcal{M}(t), \mathcal{N}(t)$ *is termed a strongly continuous* $\mathcal{M}, \mathcal{N}$*-families (generated by operators A and B) if it satisfies the following conditions:*

1. *The composition law holds,*

$$\mathcal{M}(t+s) = \mathcal{M}(s)\mathcal{M}(t) + A\mathcal{N}(s)\mathcal{N}(t),$$
$$\mathcal{N}(t+s) = \mathcal{M}(s)\mathcal{N}(t) + \mathcal{M}(t)\mathcal{N}(s) + B\mathcal{N}(s)\mathcal{N}(t), \quad s, t \geqslant 0.$$

2. *Initial conditions:* $\mathcal{N}(0) = 0, \mathcal{M}(0) = I$*, and the derivatives* $\mathcal{N}'(0)$ *and* $\mathcal{M}'(0)$ *exist and equal* I *and* 0*, respectively.*
3. $\mathcal{M}(t), \mathcal{N}(t)$ *are strongly continuous with respect to* $t \geqslant 0$.
4. *There exist* $M_\delta > 0$ *for* $\delta \in \mathbb{R}$ *such that* $\|\mathcal{M}(t)\|_{\mathcal{L}(\mathbb{H}_2)}, \|\mathcal{N}(t)\|_{\mathcal{L}(\mathbb{H}_2)} \leqslant M_\delta \exp(\delta t)$ *for all* $t \geqslant 0$.

The operators A and B, defined by:

$$Au := \mathcal{M}''(0)u := \lim_{h\to 0} h^{-2}[\mathcal{M}(2h) - 2\mathcal{M}(h) + I]u, \quad D(A) := \left\{u \in \mathbb{H}_2 : \lim_{h\to 0} h^{-2}[\mathcal{M}(2h) - 2\mathcal{M}(h) + I]u \text{ exists}\right\},$$
$$Bu := \mathcal{N}''(0)u := \lim_{h\to 0} h^{-2}[\mathcal{N}(2h) - 2\mathcal{N}(h)]u, \quad D(B) := \left\{u \in \mathbb{H}_2 : \lim_{h\to 0} h^{-2}[\mathcal{N}(2h) - 2\mathcal{N}(h)]u \text{ exists}\right\},$$

are known as the generators of the $\mathcal{M}, \mathcal{N}$-families.

2.1. Properties of Strongly Continuous $\mathcal{M}, \mathcal{N}-$Families

Let A and B be closed commuting linear operators on a Hilbert space $\mathbb{H}_2$, and let $\mathcal{M}(t), \mathcal{N}(t)$, $t \geqslant 0$, be a strongly continuous $\mathcal{M}, \mathcal{N}$-families with generators A and $B : \mathcal{M}''(0) = A, \mathcal{N}''(0) = B$. Then,

$$\begin{aligned}
&\forall u \in D(A), && \mathcal{M}(t)Au = A\mathcal{M}(t)u, \quad \mathcal{N}(t)Au = A\mathcal{N}(t)u.\\
&\forall u \in D(B), && \mathcal{M}(t)Bu = B\mathcal{M}(t)u, \quad \mathcal{N}(t)Bu = B\mathcal{N}(t)u.\\
&\forall u \in D(A), && \mathcal{M}'(t)u = \mathcal{N}(t)Au = A\mathcal{N}(t)u.\\
&\forall u \in D(B), && \mathcal{N}'(t)u = \mathcal{M}(t)u + \mathcal{N}(t)Bu = (\mathcal{M}(t) + B\mathcal{N}(t))\,u.\\
&\forall u \in D(AB), && \mathcal{M}''(t)u = (A\mathcal{M}(t) + AB\mathcal{N}(t))\,u = (B\mathcal{M}'(t) + A\mathcal{M}(t))\,u.\\
&\forall u \in D\left(B^2\right) \cap D(A), && \mathcal{N}''(t)u = \left(A\mathcal{N}(t) + B\mathcal{M}(t) + B^2\mathcal{N}(t)\right)u = (B\mathcal{N}'(t) + A\mathcal{N}(t))\,u.
\end{aligned} \tag{6}$$

For further details, see e.g., [10, 13].

2.2. Interrelations with Other Families and Specialized Cases

For additional specialized cases, please refer to [36].

- When $A = 0$, A and B generate $\mathcal{M},\mathcal{N}$-families if and only if B generates a strongly continuous semigroup $(\mathcal{T}(t))_{t\geqslant 0}$. Additionally, $\mathcal{M}(t) = \mathrm{I}$ and $\mathcal{N}(t) = \int_0^t \mathcal{T}(s)ds$ for $t \geqslant 0$.
- Now, considering the case where $B = 0$. A and B generate $\mathcal{M},\mathcal{N}$-families if and only if A generates a strongly continuous cosine function $(C(t))_{t\geqslant 0}$. Furthermore, $\mathcal{M}(t) = C(t)$ and $\mathcal{N}(t) = \int_0^t C(s)ds := \mathcal{S}(t)$ for $t \geqslant 0$.

Now, let's delve into the correlation between the $\mathcal{M},\mathcal{N}$-families. In [13, 17], the $\mathcal{EC}(t),\mathcal{ES}(t)$-families was introduced alongside $\mathcal{M}(t),\mathcal{N}(t)$, defined as:

$$\mathcal{M}(t) := \mathcal{EC}(t) - (B/2)\mathcal{ES}(t), \quad \mathcal{N}(t) := \mathcal{ES}(t).$$

Definition 2.2. *A family of bounded commuting operators, denoted as $\mathcal{EC}(t)$ and $\mathcal{ES}(t)$, parameterized by $t \geqslant 0$ is termed a set of strongly continuous $\mathcal{EC},\mathcal{ES}$-functions if it satisfies the following criteria:*

1. $\mathcal{EC}(t+s) = \mathcal{EC}(t)\mathcal{EC}(s) + \left((B/2)^2 + A\right)\mathcal{ES}(t)\mathcal{ES}(s)$.
2. $\mathcal{ES}(t+s) = \mathcal{ES}(t)\mathcal{EC}(s) + \mathcal{EC}(t)\mathcal{ES}(s)$.
3. *Initial conditions:* $\mathcal{EC}(0) = I$, $\mathcal{ES}(0) = 0$, $\mathcal{EC}'(0) = (B/2)$, $\mathcal{ES}'(0) = I$.
4. *Both* $\mathcal{EC}(t) - (B/2)\mathcal{ES}(t)$ *and* $\mathcal{ES}(t)$ *exhibit strong continuity for* $t \geqslant 0$.
5. *There exist* $M_\delta > 0$ *for* $\delta \in \mathbb{R}$ *such that* $\|\mathcal{EC}(t) - (B/2)\mathcal{ES}(t)\|_{\mathcal{L}(\mathbb{H}_2)}, \|\mathcal{ES}(t)\|_{\mathcal{L}(\mathbb{H}_2)} \leqslant M_\delta \exp(\delta t)$ *holds for all* $t \geqslant 0$.

In the papers [3, 28], the concept of the "exponential-cosine families" was explored, introduced through the equation,

$$\mathcal{E}(t+s) = 2\mathcal{E}(s)\mathcal{E}(t) + \left[\mathcal{E}(2s) - 2\mathcal{E}^2(s)\right]\mathcal{E}(t-s), \qquad 0 \leqslant s \leqslant t, \quad \mathcal{E}(0) = I. \tag{7}$$

If the linear commuting operators A and B generate strongly continuous families $\mathcal{EC}$ and $\mathcal{ES}$ of functions, then the function $\mathcal{ES}$ satisfies equation 7 (see [13, p.p. 44.]). Inspired by [13] and [10, Lemma 3.3.], we introduce the concept of a mild solution for equation (5).

Definition 2.3. *A function y is termed the mild solution of the equation* (5) *if, for all $t \geqslant u$ with $t \in \mathbb{R}$, the following holds,*

$$y(t) = \mathcal{M}(t-u)y(u) + \mathcal{N}(t-u)(y'(u) - f(u)) + \int_u^t \mathcal{N}_t(t-s)f(s)\,ds + \int_u^t \mathcal{N}(t-s)g(s)\,ds,$$

where $\mathcal{N}_t(t-s) := \dfrac{d}{dt}\mathcal{N}(t-s)$.

3. Principal results

We investigate a semilinear stochastic differential equation given by,

$$\mathrm{d}(Y'(t) - F(t,Y(t))) = AY(t)\mathrm{d}t + BY'(t)\,\mathrm{d}t + G(t,Y(t))\mathrm{d}t + H(t,Y(t))\mathrm{d}W(t), \quad t \in \mathbb{R}, \tag{8}$$

where A and B are non-zero closed defined linear operators on a Hilbert space $\mathbb{H}_2$. F and G are mappings from $\mathbb{R}\times\mathbb{H}_2$ to $\mathbb{H}_2$, and H is a mapping from $\mathbb{R}\times\mathbb{H}_2$ to $\mathcal{L}_2(\mathbb{U}_0,\mathbb{H}_2)$, $(W(t))_{t\in\mathbb{R}}$ is an $\mathbb{H}_1-$valued Q-Wiener process with Trace$(Q) < \infty$. Let $\mathcal{B}_2$ denote a collection of bounded subsets of $\mathbb{H}_2$, and $\mathbb{B}_2 \in \mathcal{B}_2$. We consider the following conditions for $p \geqslant 2$.

(H1) **Assumptions**

(1) Let $A, B \in \mathcal{L}(\mathbb{H}_2)$ be such that it generates a strongly continuous $\mathcal{M}, \mathcal{N}$ -families on $\mathbb{H}_2$. Additionally, there exist constants $M_\delta, \delta > 0$ such that,

$$\|\mathcal{M}(t-s)\|_{\mathcal{L}(\mathbb{H}_2)} + \|\mathcal{N}(t-s)\|_{\mathcal{L}(\mathbb{H}_2)} + \|\mathcal{N}_t(t-s)\|_{\mathcal{L}(\mathbb{H}_2)} \leqslant M_\delta e^{-\delta(t-s)}, \quad t \geqslant s, \quad t \in \mathbb{R}.$$

(2) F, G and H, are continuous (in y), and exhibit continuous differentiability with respect to t.

(H2) **Lipschitz conditions**

There exists some constant $L_p > 0$, for all $t \in \mathbb{R}$, $y_1, y_2 \in \mathbb{H}_2$,

$$\|F(t,y_1) - F(t,y_2)\|^p_{\mathbb{H}_2} + \|G(t,y_1) - G(t,y_2)\|^p_{\mathbb{H}_2} + \|H(t,y_1) - H(t,y_2)\|^p_{\mathcal{L}_2(\mathbb{U}_0,\mathbb{H}_2)} \leqslant L_p\|y_1 - y_2\|^p_{\mathbb{H}_2}.$$

(H3) **Almost Periodicity conditions**

For every $\varepsilon > 0$, and for every $\mathbb{B}_2 \in \mathcal{B}_2$, there exists a constant $l := l(\varepsilon, \mathbb{B}_2) > 0$ such that for any interval of length l, there exists a number τ satisfying,

$$\sup_{t\in\mathbb{R}}\left[\sup_{y\in\mathbb{B}_2}(\|F(t+\tau,y) - F(t,y)\|^p_{\mathbb{H}_2} + \|G(t+\tau,y) - G(t,y)\|^p_{\mathbb{H}_2} + \|H(t+\tau,y) - H(t,y)\|^p_{\mathcal{L}_2(\mathbb{U}_0,\mathbb{H}_2)})\right] < \varepsilon^p.$$

Remark 3.1. *Under condition* (H2) – (H3) *it is not difficult to verify that,*

$$\|F(t,y)\|^p_{\mathbb{H}_2} + \|G(t,y)\|^p_{\mathbb{H}_2} + \|H(t,y)\|^p_{\mathcal{L}_2(\mathbb{U}_0,\mathbb{H}_2)} \leqslant 3^p(L_p + \hat{C}^p_p)(1 + \|y\|^p_{\mathbb{H}_2}).$$

In the following Lemma we present a generalization of the well-known Grönwall's Lemma, as discussed in [14] (for the case where $m := 0$).

Lemma 3.2. *Consider a continuous function $h : \mathbb{R} \to \mathbb{R}$. Assume that for all $t \in \mathbb{R}$, the following inequality holds:*

$$0 \leqslant h(t) \leqslant \beta(t) + \sum_{i=1}^{n} \beta_i \sum_{j=0}^{m} \int_{-\infty}^{t} (t-s)^j e^{-\delta_i(t-s)} h(s)\,\mathrm{d}s. \tag{9}$$

Here, $\beta : \mathbb{R} \to \mathbb{R}$ is a locally integrable function, and $\beta_1, \ldots, \beta_n \geqslant 0$, $\delta_1, \ldots, \delta_n > m + \beta$, where $\beta := \sum_{i=1}^{n} \beta_i$. We assume the convergence of the integrals in the above inequality.

Let $\delta := \min_{1\leqslant i\leqslant n} \delta_i$. Then, for any γ in $(0, \delta - m - \beta]$ such that $\int_{-\infty}^{0} t^j e^{\gamma t}\beta(t)\,\mathrm{d}t$, $j = 0, 1, 2, \ldots, m$ converges, and for all $t \in \mathbb{R}$, we have:

$$h(t) \leqslant \beta(t) + \beta \int_{-\infty}^{t} e^{-\gamma(t-s)}\beta(s)\,\mathrm{d}s. \tag{10}$$

In particular, if $\beta(t) := \beta_0$ is constant, then,

$$h(t) \leqslant \beta_0 \frac{\delta - m}{\delta - m - \beta}. \tag{11}$$

Proof. Let $\beta_i' := \beta_i/\beta, i = 1, \dots, n$. Define

$$A_i(t) := \sum_{j=0}^{m} \int_{-\infty}^{t} (t-s)^j e^{-\delta_i(t-s)} h(s) \mathrm{d}s.$$

Then, differentiating, we find,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{\gamma t} A_i(t)\right) = e^{\gamma t} \Bigg\{ h(t) + \int_{-\infty}^{t} (\gamma - \delta_i) h(s) e^{-\delta_i(t-s)} \mathrm{d}s + \sum_{j=1}^{m} \int_{-\infty}^{t} (t-s)^{j-1}\, j\, h(s) e^{-\delta_i(t-s)} \mathrm{d}s + \sum_{j=1}^{m} (\gamma - \delta_i) \int_{-\infty}^{t} (t-s)^j\, h(s) e^{-\delta_i(t-s)} \mathrm{d}s \Bigg\}.$$

Utilizing a change of indexes, we obtain:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{\gamma t} A_i(t)\right) = e^{\gamma t} \Bigg\{ h(t) + \sum_{j=0}^{m} \int_{-\infty}^{t} (t-s)^j e^{-\delta_i(t-s)} h(s) \mathrm{d}s + (\gamma - \delta_i) \int_{-\infty}^{t} (t-s)^m\, h(s) e^{-\delta_i(t-s)} \mathrm{d}s + \sum_{j=0}^{m-1} \int_{-\infty}^{t} (\gamma - \delta_i + j + 1)\, (t-s)^j\, h(s) e^{-\delta_i(t-s)} \mathrm{d}s \Bigg\}.$$

Given that $\gamma - \delta_i \leqslant -m - \beta$, and $h \geqslant 0$, we have,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} \sum_{i=1}^{n} \beta_i' \left(e^{\gamma t} A_i(t)\right) &= e^{\gamma t} \Bigg\{ h(t) + \sum_{i=1}^{n} \beta_i' (\gamma - \delta_i) \int_{-\infty}^{t} (t-s)^m\, h(s) e^{-\delta_i(t-s)} \mathrm{d}s \\ &\quad + \sum_{i=1}^{n} \beta_i' \sum_{j=0}^{m-1} \int_{-\infty}^{t} (\gamma - \delta_i + j + 1)\, (t-s)^j\, h(s) e^{-\delta_i(t-s)} \mathrm{d}s \Bigg\} \\ &\leqslant e^{\gamma t} \beta(t).\end{aligned}$$

Integrating over $]-\infty, t]$, we obtain,

$$\sum_{i=1}^{n} \beta_i' e^{\gamma t} \sum_{j=0}^{m} \int_{-\infty}^{t} (t-s)^j e^{-\delta_i(s-s)} h(s) \mathrm{d}s \leqslant \int_{-\infty}^{t} e^{\gamma s} \beta(s) \mathrm{d}s,$$

because both terms approach 0 as $t \to -\infty$, i.e.,

$$\sum_{i=1}^{n} \beta_i' \sum_{j=0}^{m} \int_{-\infty}^{t} (t-s)^j e^{-\delta_i(t-s)} h(s) \mathrm{d}s \leqslant e^{-\gamma t} \int_{-\infty}^{t} e^{\gamma s} \beta(s) \mathrm{d}s. \tag{12}$$

Using (12) in (9) yields,

$$h(t) \leqslant \beta(t) + \beta \sum_{i=1}^{n} \beta_i' \sum_{j=0}^{m} \int_{-\infty}^{t} (t-s)^j e^{-\delta_i(t-s)} h(s) \mathrm{d}s \leqslant \beta(t) + \beta \int_{-\infty}^{t} e^{-\gamma(t-s)} \beta(s) \mathrm{d}s.$$

Inequality (11) follows directly from (10), with $\gamma := \delta - m - \beta$. □

Remark 3.3. *It can be demonstrated analogously that Lemma 3.2 holds within the interval $[u, t]$ instead of $]-\infty, t]$.*

Definition 3.4. *Consider a stochastic process $Y \in \mathcal{M}^p(\mathbb{R}, \mathbb{H}_2)$. We say that Y is a mild solution of the differential equation* (8), *if, $\mathbb{P}$-a.s., for all $t \geqslant u$ with $t \in \mathbb{R}$, the following holds,*

$$Y(t) = \mathcal{M}(t-u)Y(u) + \mathcal{N}(t-u)(Y'(u) - F(u, Y(u))) + \int_u^t \mathcal{N}_t(t-s)F(s, Y(s))\mathrm{d}s + \int_u^t \mathcal{N}(t-s)G(s, Y(s))\mathrm{d}s$$
$$+ \int_u^t \mathcal{N}(t-s)H(s, Y(s))\mathrm{d}W(s),$$

where $\mathcal{N}_t(t-s) := \dfrac{d}{dt}\mathcal{N}(t-s)$.

3.1. Existence and Uniqueness

Now we give the first main result. Define the parameters $\beta_{1.p}(\delta)$ and $\beta_{2.p}(\delta)$ as,

$$\begin{cases} \beta_{1.p}(\delta) := \left\{3^{p-1}C_p M_\delta^p \left(2\left(\frac{1}{\delta}\right)^p + K_{p/2}\left(\frac{1}{2\delta}\right)^{\frac{p}{2}}\right)\right\}, \\ \beta_{2.p}(\delta) := \left\{3^{p-1}C_p M_\delta^p \left(2\left(\frac{1}{\delta}\right)^{p-1} + K_{p/2}\left(\frac{1}{2\delta}\right)^{\frac{p}{2}-1}\right)\right\} \\ C_p := 3^p(L_p + \hat{C}_p^p). \end{cases}$$

The parameter $K_{p/2}$ is as shown in Lemma 1.4.

Theorem 3.5. *Assuming that conditions* (H1) − (H3) *hold and $\beta_{1.p}(\delta) < 1$ for $p \geqslant 2$, then there exists a unique mild solution $Y \in \mathcal{M}^p(\mathbb{R}, \mathbb{H}_2)$ to equation* (8) *over $\mathbb{R}$. Additionally, if $\delta > \beta_{2.p}(\delta)$, $p \geqslant 2$, the following inequalities hold: For $p > 2$,*

$$\mathbb{E}\,\|Y(t)\|^p \leqslant \frac{\delta}{\delta - \left\{3^{p-1}C_p M_\delta^p \left(2\left(\frac{1}{\delta}\right)^{p-1} + K_{p/2}\left(\frac{1}{2\delta}\right)^{\frac{p}{2}-1}\right)\right\}}.$$

For $p = 2$,

$$\mathbb{E}\,\|Y(t)\|^2 \leqslant \frac{\delta}{\delta - \left\{3C_2 M_\delta^2 \left(2\left(\frac{1}{\delta}\right) + 1\right)\right\}}.$$

Moreover, this solution can be expressed as:

$$Y(t) = \int_{-\infty}^t \mathcal{N}_t(t-s)F(s, Y(s))\mathrm{d}s + \int_{-\infty}^t \mathcal{N}(t-s)G(s, Y(s))\mathrm{d}s + \int_{-\infty}^t \mathcal{N}(t-s)H(s, Y(s))\mathrm{d}W(s).$$

To prove this Theorem, we apply Banach's classical fixed-point principle.

Proof. **Step 2:** Explicit formula.

Let's begin by proving that the function Y, defined by the expression,

$$Y(t) := \int_{-\infty}^t \mathcal{N}_t(t-s)F(s, Y(s))\mathrm{d}s + \int_{-\infty}^t \mathcal{N}(t-s)G(s, Y(s))\mathrm{d}s + \int_{-\infty}^t \mathcal{N}(t-s)H(s, Y(s))\mathrm{d}W(s). \tag{13}$$

is well-defined at $-\infty$ and satisfies the equation, $\mathbb{P}$ − a.s., for all $t \geqslant u$, $t \in \mathbb{R}$,

$$Y(t) = \mathcal{M}(t-u)Y(u) + \mathcal{N}(t-u)(Y'(u) - F(u, Y(u))) + \int_u^t \mathcal{N}_t(t-s)F(s, Y(s))\mathrm{d}s + \int_u^t \mathcal{N}(t-s)G(s, Y(s))\mathrm{d}s$$

$$+\int_u^t \mathcal{N}(t-s)H(s,Y(s))\mathrm{d}W(s).$$

Taking $u_1 < u_2$ and $Y \in \mathcal{M}^p(\mathbb{R},\mathbb{H}_2)$, let's define,

$$\Theta(u) := \int_u^t \mathcal{N}_t(t-s)F(s,Y(s))\mathrm{d}s + \int_u^t \mathcal{N}(t-s)G(s,Y(s))\mathrm{d}s + \int_u^t \mathcal{N}(t-s)H(s,Y(s))\mathrm{d}W(s).$$

We aim to demonstrate that $(\Theta(u))_{u\in\mathbb{R}}$ is defined at $-\infty$. Starting with,

$$\begin{aligned}
\mathbb{E}\|\Theta(u_2)-\Theta(u_1)\|^p_{\mathbb{H}_2} &\leqslant 3^{p-1}\mathbb{E}\left\|\int_{u_1}^{u_2} \mathcal{N}_t(t-s)F(s,Y(s))\mathrm{d}s\right\|^p_{\mathbb{H}_2}\\
&+ 3^{p-1}\mathbb{E}\left\|\int_{u_1}^{u_2} \mathcal{N}(t-s)G(s,Y(s))\mathrm{d}s\right\|^p_{\mathbb{H}_2}\\
&+ 3^{p-1}\mathbb{E}\left\|\int_{u_1}^{u_2} \mathcal{N}(t-s)H(s,Y(s))\mathrm{d}W(s)\right\|^p_{\mathbb{H}_2}\\
&:= 3^{p-1}(\mathrm{I}^1_p + \mathrm{I}^2_p + \mathrm{I}^3_p). \qquad (14)
\end{aligned}$$

For $p \geqslant 2$, applying Hölder's inequality with exponents $(p,(p-1)^{-1}p)$ under (H1) – (H3), we obtain,

$$\begin{aligned}
\mathrm{I}^1_p &= \mathbb{E}\left\|\int_{u_1}^{u_2} \mathcal{N}_t(t-s)F(s,Y(s))\mathrm{d}s\right\|^p_{\mathbb{H}_2}\\
&\leqslant \mathbb{E}\left(\int_{u_1}^{u_2}\left(\|\mathcal{N}_t(t-s)\|^{1-p^{-1}}_{\mathcal{L}(\mathbb{H}_2)}\right)\left(\|\mathcal{N}_t(t-s)\|^{p^{-1}}_{\mathcal{L}(\mathbb{H}_2)}\|F(s,Y(s))\|_{\mathbb{H}_2}\right)\mathrm{d}s\right)^p\\
&\leqslant \left(\int_{u_1}^{u_2}\|\mathcal{N}_t(t-s)\|_{\mathcal{L}(\mathbb{H}_2)}\mathrm{d}s\right)^{p-1}\int_{u_1}^{u_2}\|\mathcal{N}_t(t-s)\|_{\mathcal{L}(\mathbb{H}_2)}\mathbb{E}\|F(s,Y(s))\|^p_{\mathbb{H}_2}\mathrm{d}s\\
&\leqslant C_p\left(\int_{u_1}^{u_2}\|\mathcal{N}_t(t-s)\|_{\mathcal{L}(\mathbb{H}_2)}\mathrm{d}s\right)^{p-1}\int_{u_1}^{u_2}\|\mathcal{N}_t(t-s)\|_{\mathcal{L}(\mathbb{H}_2)}\mathbb{E}\left(1+\|Y(s)\|^p_{\mathbb{H}_2}\right)\mathrm{d}s\\
&\leqslant C_p\left(\int_{u_1}^{u_2}\|\mathcal{N}_t(t-s)\|_{\mathcal{L}(\mathbb{H}_2)}\mathrm{d}s\right)^{p} + C_p\left(\int_{u_1}^{u_2}\|\mathcal{N}_t(t-s)\|_{\mathcal{L}(\mathbb{H}_2)}\mathrm{d}s\right)^{p-1}\int_{u_1}^{u_2}\|\mathcal{N}_t(t-s)\|_{\mathcal{L}(\mathbb{H}_2)}\mathbb{E}\|Y(s)\|^p_{\mathbb{H}_2}\mathrm{d}s. \qquad (15)
\end{aligned}$$

Furthermore, we also have for I^2_p,

$$\begin{aligned}
\mathrm{I}^2_p &= \mathbb{E}\left\|\int_{u_1}^{u_2} \mathcal{N}(t-s)G(s,Y(s))\mathrm{d}s\right\|^p_{\mathbb{H}_2}\\
&\leqslant \mathbb{E}\left(\int_{u_1}^{u_2}\left(\|\mathcal{N}(t-s)\|^{1-p^{-1}}_{\mathcal{L}(\mathbb{H}_2)}\right)\left(\|\mathcal{N}(t-s)\|^{p^{-1}}_{\mathcal{L}(\mathbb{H}_2)}\|G(s,Y(s))\|_{\mathbb{H}_2}\right)\mathrm{d}s\right)^p\\
&\leqslant \left(\int_{u_1}^{u_2}\|\mathcal{N}(t-s)\|_{\mathcal{L}(\mathbb{H}_2)}\mathrm{d}s\right)^{p-1}\int_{u_1}^{u_2}\|\mathcal{N}(t-s)\|_{\mathcal{L}(\mathbb{H}_2)}\mathbb{E}\|G(s,Y(s))\|^p_{\mathbb{H}_2}\mathrm{d}s\\
&\leqslant C_p\left(\int_{u_1}^{u_2}\|\mathcal{N}(t-s)\|_{\mathcal{L}(\mathbb{H}_2)}\mathrm{d}s\right)^{p-1}\int_{u_1}^{u_2}\|\mathcal{N}(t-s)\|_{\mathcal{L}(\mathbb{H}_2)}\mathbb{E}\left(1+\|Y(s)\|^p_{\mathbb{H}_2}\right)\mathrm{d}s\\
&\leqslant C_p\left(\int_{u_1}^{u_2}\|\mathcal{N}(t-s)\|_{\mathcal{L}(\mathbb{H}_2)}\mathrm{d}s\right)^{p} + C_p\left(\int_{u_1}^{u_2}\|\mathcal{N}(t-s)\|_{\mathcal{L}(\mathbb{H}_2)}\mathrm{d}s\right)^{p-1}\int_{u_1}^{u_2}\|\mathcal{N}(t-s)\|_{\mathcal{L}(\mathbb{H}_2)}\mathbb{E}\|Y(s)\|^p_{\mathbb{H}_2}\mathrm{d}s. \qquad (16)
\end{aligned}$$

For $p > 2$, employing the $(\frac{p}{2}, (\frac{p}{2}-1)^{-1}\frac{p}{2})$−Hölder's inequality and applying Lemma 1.4, we obtain,

$$\begin{aligned}
I_p^3 &= \mathbb{E}\left\|\int_{u_1}^{u_2} \mathcal{N}(t-s)H(s,Y(s))\mathrm{d}W(s)\right\|_{\mathbb{H}_2}^p \\
&\leqslant K_{p/2}\mathbb{E}\left(\int_{u_1}^{u_2}\left(\|\mathcal{N}(t-s)\|_{\mathcal{L}(\mathbb{H}_2)}^{2(1-2p^{-1})}\|\right)\left(\mathcal{N}(t-s)\|_{\mathcal{L}(\mathbb{H}_2)}^{2(2p^{-1})}\|H(s,Y(s))\|_{\mathcal{L}_2(\mathbb{U}_0,\mathbb{H}_2)}^2\right)\mathrm{d}s\right)^{\frac{p}{2}} \\
&\leqslant K_{p/2}\left(\int_{u_1}^{u_2}\|\mathcal{N}(t-s)\|_{\mathcal{L}(\mathbb{H}_2)}^2\mathrm{d}s\right)^{\frac{p}{2}-1}\int_{u_1}^{u_2}\|\mathcal{N}(t-s)\|_{\mathcal{L}(\mathbb{H}_2)}^2\mathbb{E}\|H(s,Y(s))\|_{\mathcal{L}_2(\mathbb{U}_0,\mathbb{H}_2)}^p\mathrm{d}s \\
&\leqslant C_pK_{p/2}\left(\int_{u_1}^{u_2}\|\mathcal{N}(t-s)\|_{\mathcal{L}(\mathbb{H}_2)}^2\mathrm{d}s\right)^{\frac{p}{2}-1}\int_{u_1}^{u_2}\|\mathcal{N}(t-s)\|_{\mathcal{L}(\mathbb{H}_2)}^2\mathbb{E}\left(1+\|Y(s)\|_{\mathbb{H}_2}^p\right)\mathrm{d}s \qquad (17)\\
&\leqslant C_pK_{p/2}\left(\int_{u_1}^{u_2}\|\mathcal{N}(t-s)\|_{\mathcal{L}(\mathbb{H}_2)}^2\mathrm{d}s\right)^{\frac{p}{2}} + C_pK_{p/2}\left(\int_{u_1}^{u_2}\|\mathcal{N}(t-s)\|_{\mathcal{L}(\mathbb{H}_2)}^2\mathrm{d}s\right)^{\frac{p}{2}-1}\int_{u_1}^{u_2}\|\mathcal{N}(t-s)\|_{\mathcal{L}(\mathbb{H}_2)}^2\mathbb{E}\|Y(s)\|_{\mathbb{H}_2}^p\mathrm{d}s. \qquad (18)
\end{aligned}$$

If $p = 2$, we have,

$$\begin{aligned}
I_2^3 &= \mathbb{E}\left\|\int_{u_1}^{u_2} \mathcal{N}(t-s)H(s,Y(s))\mathrm{d}W(s)\right\|_{\mathbb{H}_2}^2 \\
&\leqslant \mathbb{E}\left(\int_{u_1}^{u_2}\|\mathcal{N}(t-s)\|_{\mathcal{L}(\mathbb{H}_2)}^2\|H(s,Y(s))\|_{\mathcal{L}_2(\mathbb{U}_0,\mathbb{H}_2)}^2\mathrm{d}s\right) \\
&\leqslant \int_{u_1}^{u_2}\|\mathcal{N}(t-s)\|_{\mathcal{L}(\mathbb{H}_2)}^2\mathbb{E}\|H(s,Y(s))\|_{\mathcal{L}_2(\mathbb{U}_0,\mathbb{H}_2)}^2\mathrm{d}s \qquad (19)\\
&\leqslant C_2\int_{u_1}^{u_2}\|\mathcal{N}(t-s)\|_{\mathcal{L}(\mathbb{H}_2)}^2\mathbb{E}\left(1+\|Y(s)\|_{\mathbb{H}_2}^p\right)\mathrm{d}s \\
&\leqslant C_2\left(\int_{u_1}^{u_2}\|\mathcal{N}(t-s)\|_{\mathcal{L}(\mathbb{H}_2)}^2\mathrm{d}s\right) + C_2\int_{u_1}^{u_2}\|\mathcal{N}(t-s)\|_{\mathcal{L}(\mathbb{H}_2)}^2\mathbb{E}\|Y(s)\|_{\mathbb{H}_2}^2\mathrm{d}s. \qquad (20)
\end{aligned}$$

Because,

$$\lim_{u_2\to-\infty}\int_{u_1}^{u_2}\left(\|\mathcal{N}_t(t-s)\|_{\mathcal{L}(\mathbb{H}_2)} + \|\mathcal{N}(t-s)\|_{\mathcal{L}(\mathbb{H}_2)} + \|\mathcal{N}(t-s)\|_{\mathcal{L}(\mathbb{H}_2)}^2\right)\mathrm{d}s = 0.$$

Then for $Y \in \mathcal{M}^p(\mathbb{R}, \mathbb{H}_2)$, we have,

$$\lim_{u_2\to-\infty}\mathbb{E}\|\Theta(u_2)-\Theta(u_1)\|_{\mathbb{H}_2}^p = 0.$$

Hence, the limit $\lim_{u\to-\infty}\Theta(u)$ exists, ensuring that Y in Equation (13) is well-defined at $-\infty$. Furthermore, we observe that,

$$\begin{aligned}
\mathcal{M}(t-u)Y(u)+\mathcal{N}(t-u)Y'(u) &= \int_{-\infty}^{u}\left(\mathcal{M}(t-u)\mathcal{N}_u(u-s)+\mathcal{N}(t-u)\mathcal{N}_{uu}(u-s)\right)F(s,Y(s))\mathrm{d}s \\
&+ \int_{-\infty}^{u}\left(\mathcal{M}(t-u)\mathcal{N}(u-s)+\mathcal{N}(t-u)\mathcal{N}_u(u-s)\right)G(s,Y(s))\mathrm{d}s \\
&+ \int_{-\infty}^{u}\left(\mathcal{M}(t-u)\mathcal{N}(u-s)+\mathcal{N}(t-u)\mathcal{N}_u(u-s)\right)H(s,Y(s))\mathrm{d}W(s) \\
&+ \mathcal{N}(t-u)F(u,Y(u)).
\end{aligned}$$

According to the definition 2.1 and its corresponding properties 6, we can derive the following equations, for $t \geqslant u \geqslant s$, $t \in \mathbb{R}$,

$$\mathcal{M}(t-u)\mathcal{N}(u-s)+\mathcal{N}(t-u)\mathcal{N}_u(u-s)=\mathcal{N}(t-s),$$

$$\mathcal{M}(t-u)\mathcal{N}_u(u-s)+\mathcal{N}(t-u)\mathcal{N}_{uu}(u-s)=\mathcal{N}_t(t-s).$$

This yields,

$$\begin{aligned}\mathcal{N}(t-u)Y(u)+\mathcal{N}(t-u)(Y'(u)-F(u,Y(u))) &= \int_{-\infty}^{u}\mathcal{N}_t(t-s)F(s,Y(s))\mathrm{d}s+\int_{-\infty}^{u}\mathcal{N}(t-s)G(s,Y(s))\mathrm{d}s\\ &+\int_{-\infty}^{u}\mathcal{N}(t-s)H(s,Y(s))\mathrm{d}W(s). \end{aligned}\tag{21}$$

By separating the integral in Equation (13) into two separate components and employing the equality stated in Equation (21), it follows that

$$\begin{aligned}Y(t) &= \int_{-\infty}^{u}\mathcal{N}_t(t-s)F(s,Y(s))\mathrm{d}s+\int_{-\infty}^{u}\mathcal{N}(t-s)G(s,Y(s))\mathrm{d}s+\int_{-\infty}^{u}\mathcal{N}(t-s)H(s,Y(s))\mathrm{d}W(s)\\ &+\int_{u}^{t}\mathcal{N}_t(t-s)F(s,Y(s))\mathrm{d}s+\int_{u}^{t}\mathcal{N}(t-s)G(s,Y(s))\mathrm{d}s+\int_{u}^{t}\mathcal{N}(t-s)H(s,Y(s))\mathrm{d}W(s)\\ &=\mathcal{M}(t-u)Y(u)+\mathcal{N}(t-u)(Y'(u)-F(u,Y(u)))+\int_{u}^{t}\mathcal{N}_t(t-s)F(s,Y(s))\mathrm{d}s\\ &+\int_{u}^{t}\mathcal{N}(t-s)G(s,Y(s))\mathrm{d}s+\int_{u}^{t}\mathcal{N}(t-s)H(s,Y(s))\mathrm{d}W(s).\end{aligned}$$

The first step is now complete.

Step 2: Existence and uniqueness.

Let's define the operator Γ as,

$$\Gamma Y(t):=\int_{-\infty}^{t}\mathcal{N}_t(t-s)F(s,Y(s))\mathrm{d}s+\int_{-\infty}^{t}\mathcal{N}(t-s)G(s,Y(s))\mathrm{d}s+\int_{-\infty}^{t}\mathcal{N}(t-s)H(s,Y(s))\mathrm{d}W(s).\tag{22}$$

Now, we aim to prove that if $Y\in\mathcal{M}^p(\mathbb{R},\mathbb{H}_2)$, then $\Gamma Y\in\mathcal{M}^p(\mathbb{R},\mathbb{H}_2)$. After the initial step, where $u_1\sim-\infty$ and $u_2:=t$, in the estimates (14)–(20), we have,

$$\mathbb{E}\|\Gamma Y(t)\|_{\mathbb{H}_2}^p\leqslant\left\{3^{p-1}C_pM_\delta^p\left(2\left(\frac{1}{\delta}\right)^p+K_{p/2}\left(\frac{1}{2\delta}\right)^{\frac{p}{2}}\right)\right\}\left(1+\sup_{s\in]-\infty,t]}\mathbb{E}\|Y(s)\|_{\mathbb{H}_2}^p\right).$$

Therefore, for $p\geqslant 2$,

$$\sup\nolimits_{t\in\mathbb{R}}\mathbb{E}\|\Gamma Y(t)\|^p\leqslant\beta_{1.p}(\delta)(1+\sup\nolimits_{t\in\mathbb{R}}\mathbb{E}\|Y(t))\|_{\mathbb{H}_2}^p).\tag{23}$$

If $Y\in\mathcal{M}^p(\mathbb{R},\mathbb{H}_2)$, we have $\Gamma Y\in\mathcal{M}^p(\mathbb{R},\mathbb{H}_2)$, $p\geqslant 2$. Given $Y^{(1)},Y^{(2)}\in\mathcal{M}^p(\mathbb{R},\mathbb{H}_2)$, according to the definition of Γ, we derive,

$$\begin{aligned}\mathbb{E}\|\Gamma Y^{(1)}(t)-\Gamma Y^{(2)}(t)\|^p &\leqslant 3^{p-1}\mathbb{E}\left\|\int_{-\infty}^{t}\mathcal{N}_t(t-s)(F(s,Y^{(1)}(s))-F(s,Y^{(2)}(s)))\mathrm{d}s\right\|_{\mathbb{H}_2}^p\\ &+3^{p-1}\mathbb{E}\left\|\int_{-\infty}^{t}\mathcal{N}(t-s)(G(s,Y^{(1)}(s))-G(s,Y^{(2)}(s)))\mathrm{d}s\right\|_{\mathbb{H}_2}^p\\ &+3^{p-1}\mathbb{E}\left\|\int_{-\infty}^{t}\mathcal{N}(t-s)(H(s,Y^{(1)}(s))-H(s,Y^{(2)}(s)))\mathrm{d}W(s)\right\|_{\mathbb{H}_2}^p\\ &:=3^{p-1}(\mathrm{J}_p^1+\mathrm{J}_p^2+\mathrm{J}_p^3).\end{aligned}\tag{24}$$

For $p \geqslant 2$, consider J_p^1,

$$\begin{aligned}
\mathrm{J}_p^1 &= \mathbb{E}\left\|\int_{-\infty}^t \mathcal{N}_t(t-s)(F(s,Y^{(1)}(s)) - F(s,Y^{(2)}(s)))\mathrm{d}s\right\|_{\mathbb{H}_2}^p \\
&\leqslant \mathbb{E}\left(\int_{-\infty}^t \left(\|\mathcal{N}_t(t-s)\|_{\mathcal{L}(\mathbb{H}_2)}^{1-p^{-1}}\right)\left(\|\mathcal{N}_t(t-s)\|_{\mathcal{L}(\mathbb{H}_2)}^{p^{-1}}\|F(s,Y^{(1)}(s)) - F(s,Y^{(2)}(s))\|_{\mathbb{H}_2}\right)\mathrm{d}s\right)^p \\
&\leqslant \left(\int_{-\infty}^t \|\mathcal{N}_t(t-s)\|_{\mathcal{L}(\mathbb{H}_2)}\mathrm{d}s\right)^{p-1}\int_{-\infty}^t \|\mathcal{N}_t(t-s)\|_{\mathcal{L}(\mathbb{H}_2)}\mathbb{E}\|F(s,Y^{(1)}(s)) - F(s,Y^{(2)}(s))\|_{\mathbb{H}_2}^p\mathrm{d}s \\
&\leqslant C_p\left(\int_{-\infty}^t \|\mathcal{N}_t(t-s)\|_{\mathcal{L}(\mathbb{H}_2)}\mathrm{d}s\right)^{p-1}\int_{-\infty}^t \|\mathcal{N}_t(t-s)\|_{\mathcal{L}(\mathbb{H}_2)}\mathbb{E}\|Y^{(1)}(s) - Y^{(2)}(s)\|_{\mathbb{H}_2}^p\mathrm{d}s \\
&\leqslant C_p\left(\int_{-\infty}^t \|\mathcal{N}_t(t-s)\|_{\mathcal{L}(\mathbb{H}_2)}\mathrm{d}s\right)^{p}\sup_{s\in]-\infty,t]}\mathbb{E}\|Y^{(1)}(s) - Y^{(2)}(s)\|_{\mathbb{H}_2}^p. \qquad (25)
\end{aligned}$$

Similarly, for J_p^2,

$$\begin{aligned}
\mathrm{J}_p^2 &= \mathbb{E}\left\|\int_{-\infty}^t \mathcal{N}(t-s)(G(s,Y^{(1)}(s)) - G(s,Y^{(2)}(s)))\mathrm{d}s\right\|_{\mathbb{H}_2}^p \\
&\leqslant \mathbb{E}\left(\int_{-\infty}^t \left(\|\mathcal{N}(t-s)\|_{\mathcal{L}(\mathbb{H}_2)}^{1-p^{-1}}\right)\left(\|\mathcal{N}(t-s)\|_{\mathcal{L}(\mathbb{H}_2)}^{p^{-1}}\|G(s,Y^{(1)}(s)) - F(s,Y^{(2)}(s))\|_{\mathbb{H}_2}\right)\mathrm{d}s\right)^p \\
&\leqslant \left(\int_{-\infty}^t \|\mathcal{N}(t-s)\|_{\mathcal{L}(\mathbb{H}_2)}\mathrm{d}s\right)^{p-1}\int_{-\infty}^t \|\mathcal{N}(t-s)\|_{\mathcal{L}(\mathbb{H}_2)}\mathbb{E}\|G(s,Y^{(1)}(s)) - G(s,Y^{(2)}(s))\|_{\mathbb{H}_2}^p\mathrm{d}s \\
&\leqslant C_p\left(\int_{-\infty}^t \|\mathcal{N}(t-s)\|_{\mathcal{L}(\mathbb{H}_2)}\mathrm{d}s\right)^{p-1}\int_{-\infty}^t \|\mathcal{N}(t-s)\|_{\mathcal{L}(\mathbb{H}_2)}\mathbb{E}\|Y^{(1)}(s) - Y^{(2)}(s)\|_{\mathbb{H}_2}^p\mathrm{d}s \\
&\leqslant C_p\left(\int_{-\infty}^t \|\mathcal{N}(t-s)\|_{\mathcal{L}(\mathbb{H}_2)}\mathrm{d}s\right)^{p}\sup_{s\in]-\infty,t]}\mathbb{E}\|Y^{(1)}(s) - Y^{(2)}(s)\|_{\mathbb{H}_2}^p. \qquad (26)
\end{aligned}$$

For $p > 2$, when considering J_p^3, we obtain,

$$\begin{aligned}
\mathrm{J}_p^3 &= \mathbb{E}\left\|\int_{-\infty}^t \mathcal{N}(t-s)(H(s,Y^{(1)}(s)) - H(s,Y^{(2)}(s)))\mathrm{d}W(s)\right\|_{\mathbb{H}_2}^p \\
&\leqslant K_{p/2}\mathbb{E}\left(\int_{-\infty}^t \|\mathcal{N}(t-s)\|_{\mathcal{L}(\mathbb{H}_2)}^2\|H(s,Y^{(1)}(s)) - H(s,Y^{(2)}(s))\|_{\mathcal{L}_2(\mathbb{U}_0,\mathbb{H}_2)}^2\mathrm{d}s\right)^{\frac{p}{2}} \\
&\leqslant K_{p/2}\left(\int_{-\infty}^t \||\mathcal{N}(t-s)\|_{\mathcal{L}(\mathbb{H}_2)}^2\mathrm{d}s\right)^{\frac{p}{2}-1}\int_{-\infty}^t \|\mathcal{N}(t-s)\|_{\mathcal{L}(\mathbb{H}_2)}^2\mathbb{E}\|H(s,Y^{(1)}(s)) - H(s,Y^{(2)}(s))\|_{\mathbb{H}_2}^p\mathrm{d}s \\
&\leqslant C_pK_{p/2}\left(\int_{-\infty}^t \|\mathcal{N}(t-s)\|_{\mathcal{L}(\mathbb{H}_2)}^2\mathrm{d}s\right)^{\frac{p}{2}-1}\int_{-\infty}^t \|\mathcal{N}(t-s)\|_{\mathcal{L}(\mathbb{H}_2)}^2\mathbb{E}\|Y^{(1)}(s) - Y^{(2)}(s)\|_{\mathbb{H}_2}^p\mathrm{d}s \\
&\leqslant C_pK_{p/2}\left(\int_{-\infty}^t \|\mathcal{N}(t-s)\|_{\mathcal{L}(\mathbb{H}_2)}^2\mathrm{d}s\right)^{\frac{p}{2}}\sup_{s\in]-\infty,t]}\mathbb{E}\|Y^{(1)}(s) - Y^{(2)}(s)\|_{\mathbb{H}_2}^p. \qquad (27)
\end{aligned}$$

Finally, for J_p^3, with $p = 2$,

$$\mathrm{J}_2^3 = \mathbb{E}\left\|\int_{-\infty}^t \mathcal{N}(t-s)(H(s,Y^{(1)}(s)) - H(s,Y^{(2)}(s)))\mathrm{d}W(s)\right\|_{\mathbb{H}_2}^2$$

$$\leqslant \mathbb{E}\left(\int_{-\infty}^{t} \|\mathcal{N}(t-s)\|^2_{\mathcal{L}(\mathbb{H}_2)} \|H(s, Y^{(1)}(s)) - H(s, Y^{(2)}(s))\|^2_{\mathcal{L}_2(\mathbb{U}_0,\mathbb{H}_2)} \mathrm{d}s\right)$$
$$\leqslant \int_{-\infty}^{t} \|\mathcal{N}(t-s)\|^2_{\mathcal{L}(\mathbb{H}_2)} \mathbb{E}\|H(s, Y^{(1)}(s)) - H(s, Y^{(2)}(s))\|^2_{\mathcal{L}_2(\mathbb{U}_0,\mathbb{H}_2)} \mathrm{d}s$$
$$\leqslant C_2 \int_{-\infty}^{t} \|\mathcal{N}(t-s)\|^2_{\mathcal{L}(\mathbb{H}_2)} \mathbb{E}\|Y^{(1)}(s) - Y^{(2)}(s)\|^2_{\mathbb{H}_2} \mathrm{d}s$$
$$\leqslant C_2 \left(\int_{-\infty}^{t} \|\mathcal{N}(t-s)\|^2_{\mathcal{L}(\mathbb{H}_2)} \mathrm{d}s\right) \sup_{s\in]-\infty,t]} \mathbb{E}\|Y^{(1)}(s) - Y^{(2)}(s)\|^2_{\mathbb{H}_2}. \tag{28}$$

Combining the inequalities from (24)−(27), we obtain, if $p > 2$,

$$\sup_{t\in\mathbb{R}} \mathbb{E}\|\Gamma Y^{(1)}(t) - \Gamma Y^{(2)}(t)\|^p_{\mathbb{H}_2} \leqslant \left\{3^{p-1} C_p M_\delta^p \left(2\left(\frac{1}{\delta}\right)^p + K_{p/2}\left(\frac{1}{2\delta}\right)^{\frac{p}{2}}\right)\right\} \sup_{t\in\mathbb{R}} \mathbb{E}\|Y^{(1)}(t) - Y^{(2)}(t)\|^p_{\mathbb{H}_2}.$$

When $p = 2$, by combining the inequalities (24)−(26) and (28) we have,

$$\sup_{t\in\mathbb{R}} \mathbb{E}\|\Gamma Y^{(1)}(t) - \Gamma Y^{(2)}(t)\|^2_{\mathbb{H}_2} \leqslant \left\{3C_2 M_\delta^2 \left(2\left(\frac{1}{\delta}\right)^2 + \left(\frac{1}{2\delta}\right)\right)\right\} \sup_{t\in\mathbb{R}} \mathbb{E}\|Y^{(1)}(t) - Y^{(2)}(t)\|^2_{\mathbb{H}_2}.$$

Therefore, Γ constitutes a contraction from $\mathcal{M}^p(\mathbb{R}, \mathbb{H}_2)$ to $\mathcal{M}^p(\mathbb{R}, \mathbb{H}_2)$ for $p \geqslant 2$. Assuming Y is a unique fixed point of Γ,

$$Y(t) = \int_{-\infty}^{t} \mathcal{N}_t(t-s) F(s, Y(s)) \mathrm{d}s + \int_{-\infty}^{t} \mathcal{N}(t-s) G(s, Y(s)) \mathrm{d}s + \int_{-\infty}^{t} \mathcal{N}(t-s) H(s, Y(s)) \mathrm{d}W(s).$$

Then,

$$\mathbb{E}\|Y(t)\|^p_{\mathbb{H}_2} \leqslant \left\{3^{p-1} C_p M_\delta^p \left(2\left(\frac{1}{\delta}\right)^p + K_{p/2}\left(\frac{1}{2\delta}\right)^{\frac{p}{2}}\right)\right\} + \left\{3^{p-1} C_p M_\delta^p \left(2\left(\frac{1}{\delta}\right)^{p-1}\right)\right\} \int_{-\infty}^{t} e^{-\delta(t-s)} \mathbb{E}\|Y(s)\|^p_{\mathbb{H}_2} \mathrm{d}s$$
$$+ \left\{3^{p-1} C_p M_\delta^p K_{p/2}\left(\frac{1}{2\delta}\right)^{\frac{p}{2}-1}\right\} \int_{-\infty}^{t} e^{-2\delta(t-s)} \mathbb{E}\|Y(s)\|^p_{\mathbb{H}_2} \mathrm{d}s.$$

We conclude based on the Lemma 3.2 that, if $\delta > \beta_{2.p}(\delta)$,

$$\mathbb{E}\,\|Y(t)\|^p \leqslant \frac{\delta}{\delta - \beta_{2.p}(\delta)} \beta_{1.p}(\delta) \leqslant \frac{\delta}{\delta - \left\{3^{p-1} C_p M_\delta^p \left(2\left(\frac{1}{\delta}\right)^{p-1} + K_{p/2}\left(\frac{1}{2\delta}\right)^{\frac{p}{2}-1}\right)\right\}}.$$

And,

$$\mathbb{E}\,\|Y(t)\|^2 \leqslant \frac{\delta}{\delta - \left\{3C_2 M_\delta^2 \left(2\left(\frac{1}{\delta}\right) + 1\right)\right\}}.$$

Additionally, according to [6, Theorem 7.2], almost all trajectories of this solution are continuous. The proof for Theorem 3.5 has concluded. □

Almost-Periodicity of the Solution

In this section, we aim to demonstrate that the solution to the previous equation exhibits almost-periodic behavior concerning the law. We introduce several assumptions to facilitate our analysis. Let $F^n, G^n : \mathbb{R}\times\mathbb{H}_2 \longrightarrow \mathbb{H}_2$ and $H^n : \mathbb{R}\times\mathbb{H}_2 \longrightarrow \mathcal{L}_2^0$ be sequences of mappings indexed by $n \in \bar{\mathbb{N}} := \{0, 1, \ldots, +\infty\}$.

Assumptions

(H4) For each $n \in \bar{\mathbb{N}}$, the mappings H^n , F^n, and G^n satisfy conditions (H1) through (H3).

Under the assumptions (H1) – (H4), let $(Y^n(t))_{t\in\mathbb{R}}$ denote the solution to the following equation:

$$\begin{cases} d(Y^{n\prime}(t) - F^n(t, Y(t))) = (AY^n(t) + BY'^n(t) + G^n(t, Y^n(t)))\mathrm{d}t + H^n(t, Y^n(t))\mathrm{d}W(t), \quad t \geqslant u, \\ Y(u) = \xi^n, \quad Y'(u) = \eta^n, \qquad t = u. \end{cases}$$

(H5) For all $(t, y) \in \mathbb{R} \times \mathbb{H}_2$, we assume that

$$\lim_{n\to\infty} F^n(t, y) := F^\infty(t, y), \quad \lim_{n\to\infty} G^n(t, y) := G^\infty(t, y), \quad \lim_{n\to\infty} H^n(t, Y(t)) := H^\infty(t, y).$$

(H6) In the space $(\mathcal{P}(\mathbb{H}_2 \times \mathrm{C}(\mathbb{R}, \mathbb{H}_1)), \mathrm{d}_{\mathrm{BL}})$, we assume that

$$\begin{cases} \lim_{n\to\infty} \mathrm{d}_{\mathrm{BL}} \left(\mathbb{P}_{(\xi^n,W)}, \mathbb{P}_{(\xi^\infty,W)}\right) = 0, \\ \lim_{n\to\infty} \mathrm{d}_{\mathrm{BL}} \left(\mathbb{P}_{(\eta^n,W)}, \mathbb{P}_{(\eta^\infty,W)}\right) = 0, \\ \lim_{n\to\infty} \mathrm{d}_{\mathrm{BL}} \left(\mathbb{P}_{(F^n(\cdot,\xi^n),W)}, \mathbb{P}_{(F^\infty(\cdot,\xi^\infty),W)}\right) = 0. \end{cases}$$

Key Results

We begin by revisiting the following proposition, initially outlined more comprehensively in [5]:

Proposition 3.6. *Under assumptions* (H1) – (H6)*, we have in* $(\mathcal{P}(\mathrm{C}([u, b], \mathbb{H}_2)), \mathrm{d}_{\mathrm{BL}})$*, for any* $b \geqslant u$*,*

$$\lim_{n\to\infty} \mathrm{d}_{\mathrm{BL}} (\mathbb{P}_{Y^n}, \mathbb{P}_{Y^\infty}) = 0.$$

We further establish the following Theorem:

Theorem 3.7. *Let assumptions* (H1) – (H6) *be fulfilled. Additionally, if*

$$\beta_{1,2}(\delta) := \left\{3C_2M_\delta^2\left(2\left(\frac{1}{\delta}\right)^2 + \left(\frac{1}{2\delta}\right)\right)\right\} < 1, \qquad \text{and} \qquad 2\beta_{2.2}(\delta) := \left\{6C_2M_\delta^2\left(2\left(\frac{1}{\delta}\right) + 1\right)\right\} < \delta,$$

then there exists a unique mild solution $Y \in \mathcal{M}^2(\mathbb{R}, \mathbb{H}_2)$ *that is almost periodic in distribution.*

Proof. The solution's existence and uniqueness were established via Theorem 3.5. Let us show that Y is almost periodic in distribution. We will utilize the definition provided in equation (4) for this purpose. Let $\{u'_n\} \subset \mathbb{R}$ and $\{v'_n\} \subset \mathbb{R}$. Our objective is to show the existence of subsequences $\{u_n\} \subset \{u'_n\}$ and $\{v_n\} \subset \{v'_n\}$ with the same indices, such that for every $t \in \mathbb{R}$, the limits

$$\lim_{n\to\infty} \lim_{m\to\infty} \pi(t + u_n + v_m), \quad \text{and} \quad \lim_{n\to\infty} \pi(t + u_n + v_n),$$

exist and are equal, where $\mu(t) := \mathrm{Law}(Y)(t) := \mathbb{P}_{Y(t)}$. Using (H3), we have

$$\begin{cases} \lim_{n\to\infty} \lim_{m\to\infty} F(t + u_n + v_m, y) = \lim_{n\to\infty} F(t + u_n + v_n, y) := F^0(t, y), \\ \lim_{n\to\infty} \lim_{m\to\infty} G(t + u_n + v_m, y) = \lim_{n\to\infty} G(t + u_n + v_n, y) := G^0(t, y), \\ \lim_{n\to\infty} \lim_{m\to\infty} H(t + u_n + v_m, y) = \lim_{n\to\infty} H(t + u_n + v_n, y) := H^0(t, y). \end{cases} \tag{29}$$

These limits exist pointwise with respect to $t \in \mathbb{R}$ and $y \in \mathbb{B}_2$. Consider now the sequences defined by $\{r_n := u_n + v_n\}$. For each integer n, we consider a semilinear stochastic differential equations,

$$d(Y_n'(t) - F(t + r_n, Y_n(t))) = (AY_n(t) + BY_n'(t) + G(t + r_n, Y_n(t)))\mathrm{d}t + H(t + r_n, Y_n(t))\,\mathrm{d}W(t).$$

The mild solution for $(Y_n(t))_{t\in\mathbb{R}}$ is given by,

$$Y_n(t) = \int_{-\infty}^{t} \mathcal{N}_t(t-s)F(s+r_n, Y_n(s))\mathrm{d}s + \int_{-\infty}^{t} \mathcal{N}(t-s)G(s+r_n, Y_n(s))\mathrm{d}s + \int_{-\infty}^{t} \mathcal{N}(t-s)H(s+r_n, Y_n(s))\mathrm{d}W(s).$$

Additionally, we have another stochastic differential equations,

$$d(Y^{0\prime}(t) - F^0(t, Y^0(t))) = (AY^0(t) + BY'^0(t) + G^0(t, Y^0(t)))\mathrm{d}t + H^0(t, Y^0(t))\mathrm{d}W(t).$$

With its mild solution,

$$Y^0(t) = \int_{-\infty}^{t} \mathcal{N}_t(t-s)F^0(s, Y^0(s))\mathrm{d}s + \int_{-\infty}^{t} \mathcal{N}(t-s)G^0(s, Y^0(s))\mathrm{d}s + \int_{-\infty}^{t} \mathcal{N}(t-s)H^0(s, Y^0(s))\mathrm{d}W(s).$$

Now, consider the process,

$$\begin{aligned} Y(t + r_n) = &\int_{-\infty}^{t+r_n} \mathcal{N}_t(t - (s - r_n))\,F(s, Y(s))\mathrm{d}s + \int_{-\infty}^{t+r_n} \mathcal{N}(t - (s - r_n))\,G(s, Y(s))\mathrm{d}s \\ &+ \int_{-\infty}^{t+r_n} \mathcal{N}(t - (s - r_n))\,H(s, Y(s))\mathrm{d}W(s). \end{aligned}$$

By making the change of variable $z := s - r_n$, we get,

$$\begin{aligned} Y(t + r_n) = &\int_{-\infty}^{t} \mathcal{N}_t(t-s)F(s + r_n, Y(s + r_n))\,\mathrm{d}s + \int_{-\infty}^{t} \mathcal{N}(t-s)G(s + r_n, Y(s + r_n))\,\mathrm{d}s \\ &+ \int_{-\infty}^{t} \mathcal{N}(t-s)H(s + r_n, Y(s + r_n))\,\mathrm{d}W_n(s), \end{aligned}$$

where,

$$W_n(t) := W(t + r_n) - W(r_n) \underset{\mathbb{P}_{W(t)}}{=} W(t).$$

Due to the independence of the increments of $W(t)$, we conclude that the process,

$$Y(t + r_n) \underset{\mathbb{P}_{Y(t)}}{=} Y_n(t).$$

Now, we aim to demonstrate that $Y_n(t)$ converges in $(p = 2)-$mean to $Y^0(t)$ for each fixed $t \in \mathbb{R}$, i.e.

$$\lim_{n\to\infty} \mathbb{E}\|Y_n(t) - Y^0(t)\|^2_{\mathbb{H}_2} = 0.$$

We start the analysis from,

$$\begin{aligned} \mathbb{E}\|Y_n(t) - Y^0(t)\|^2_{\mathbb{H}_2} \leqslant\; &3\mathbb{E}\left\|\int_{-\infty}^{t} \mathcal{N}_t(t-s)\left(F(s + r_n, Y_n(s)) - F^0(s, Y^0(s))\right)\mathrm{d}s\right\|^2_{\mathbb{H}_2} \\ &+ 3\mathbb{E}\left\|\int_{-\infty}^{t} \mathcal{N}(t-s)\left(G(s + r_n, Y_n(s)) - G^0(s, Y^0(s))\right)\mathrm{d}s\right\|^2_{\mathbb{H}_2} \end{aligned}$$

$$+ 3\mathbb{E}\left\|\int_{-\infty}^{t} \mathcal{N}(t-s)\left(H(s+r_n, Y_n(s)) - H^0(s+r_n, Y^0(s))\right)\mathrm{d}W(s)\right\|_{\mathbb{H}_2}^2$$
$$:= 3(\mathrm{I}^1 + \mathrm{I}^2 + \mathrm{I}^3). \tag{30}$$

For I^1, we get,

$$\begin{aligned}
\mathrm{I}^1 &= \mathbb{E}\left\|\int_{-\infty}^{t} \mathcal{N}_t(t-s)\left(F(s+r_n, Y_n(s)) - F^0(s, Y^0(s))\right)\mathrm{d}s\right\|_{\mathbb{H}_2}^2 \\
&\leqslant 2\mathbb{E}\left\|\int_{-\infty}^{t} \mathcal{N}_t(t-s)\left(F(s+r_n, Y_n(s)) - F(s+r_n, Y^0(s))\right)\mathrm{d}s\right\|_{\mathbb{H}_2}^2 \\
&+ 2\mathbb{E}\left\|\int_{-\infty}^{t} \mathcal{N}_t(t-s)\left(F(s+r_n, Y^0(s)) - F^0(s, Y^0(s))\right)\mathrm{d}s\right\|_{\mathbb{H}_2}^2 \\
&:= 2(\mathrm{J}^1 + \mathrm{J}^2).
\end{aligned} \tag{31}$$

Now, using $(2,2)-$Hölder's inequality, we obtain,

$$\begin{aligned}
\mathrm{J}^1 &= \mathbb{E}\left\|\int_{-\infty}^{t} \mathcal{N}_t(t-s)\left(F(s+r_n, Y_n(s)) - F(s+r_n, Y^0(s))\right)\mathrm{d}s\right\|_{\mathbb{H}_2}^2 \\
&\leqslant \mathbb{E}\left(\int_{-\infty}^{t}\left(\|\mathcal{N}_t(t-s)\|_{\mathcal{L}(\mathbb{H}_2)}^{2^{-1}}\right)\left(\|\mathcal{N}_t(t-s)\|_{\mathcal{L}(\mathbb{H}_2)}^{2^{-1}}\|F(s+r_n, Y_n(s)) - F(s+r_n, Y^0(s))\|_{\mathbb{H}_2}\right)\mathrm{d}s\right)^2 \\
&\leqslant \left(\int_{-\infty}^{t}\|\mathcal{N}_t(t-s)\|_{\mathcal{L}(\mathbb{H}_2)}\mathrm{d}s\right)\int_{-\infty}^{t}\|\mathcal{N}_t(t-s)\|_{\mathcal{L}(\mathbb{H}_2)}\mathbb{E}\|F(s+r_n, Y_n(s)) - F(s+r_n, Y^0(s))\|_{\mathbb{H}_2}^2\mathrm{d}s \\
&\leqslant C_2\left(\int_{-\infty}^{t}\|\mathcal{N}_t(t-s)\|_{\mathcal{L}(\mathbb{H}_2)}\mathrm{d}s\right)\int_{-\infty}^{t}\|\mathcal{N}_t(t-s)\|_{\mathcal{L}(\mathbb{H}_2)}\mathbb{E}\|Y_n(s) - Y^0(s)\|_{\mathbb{H}_2}^2\mathrm{d}s.
\end{aligned} \tag{32}$$

And for J^2, we have,

$$\begin{aligned}
\mathrm{J}^2 &= \mathbb{E}\left\|\int_{-\infty}^{t} \mathcal{N}_t(t-s)\left(F(s+r_n, Y_n(s)) - F^0(t, Y^0(s))\right)\mathrm{d}s\right\|_{\mathbb{H}_2}^2 \\
&\leqslant \mathbb{E}\left(\int_{-\infty}^{t}\left(\|\mathcal{N}_t(t-s)\|_{\mathcal{L}(\mathbb{H}_2)}^{2^{-1}}\right)\left(\|\mathcal{N}_t(t-s)\|_{\mathcal{L}(\mathbb{H}_2)}^{2^{-1}}\|F(s+r_n, Y_n(s)) - F^0(t, Y^0(s))\|_{\mathbb{H}_2}\right)\mathrm{d}s\right)^2 \\
&\leqslant \left(\int_{-\infty}^{t}\|\mathcal{N}_t(t-s)\|_{\mathcal{L}(\mathbb{H}_2)}\mathrm{d}s\right)\int_{-\infty}^{t}\|\mathcal{N}_t(t-s)\|_{\mathcal{L}(\mathbb{H}_2)}\mathbb{E}\|F(s+r_n, Y_n(s)) - F^0(s, Y^0(s))\|_{\mathbb{H}_2}^2\mathrm{d}s \\
&\leqslant \left(\int_{-\infty}^{t}\|\mathcal{N}_t(t-s)\|_{\mathcal{L}(\mathbb{H}_2)}\mathrm{d}s\right)^2 \sup_{s\in]-\infty,t]}\mathbb{E}\|F(s+r_n, Y_n(s)) - F^0(s, Y^0(s))\|_{\mathbb{H}_2}^2.
\end{aligned} \tag{33}$$

For I^2, we have,

$$\begin{aligned}
\mathrm{I}^2 &= \mathbb{E}\left\|\int_{-\infty}^{t} \mathcal{N}(t-s)\left(G(s+r_n, Y_n(s)) - G^0(s, Y^0(s))\right)\mathrm{d}s\right\|_{\mathbb{H}_2}^2 \\
&\leqslant 2\mathbb{E}\left\|\int_{-\infty}^{t} \mathcal{N}(t-s)\left(G(s+r_n, Y_n(s)) - G(s+r_n, Y^0(s))\right)\mathrm{d}s\right\|_{\mathbb{H}_2}^2 \\
&+ 2\mathbb{E}\left\|\int_{-\infty}^{t} \mathcal{N}(t-s)\left(G(s+r_n, Y^0(s)) - G^0(s, Y^0(s))\right)\mathrm{d}s\right\|_{\mathbb{H}_2}^2
\end{aligned}$$

$$:= 2(\mathrm{J}^3 + \mathrm{J}^4). \tag{34}$$

For J^3, we obtain that

$$\begin{aligned}
\mathrm{J}^3 &= \mathbb{E}\left\|\int_{-\infty}^{t} \mathcal{N}(t-s)\left(G(s+r_n, Y_n(s)) - G(s+r_n, Y^0(s))\right)\mathrm{d}s\right\|_{\mathbb{H}_2}^2 \\
&\leqslant \mathbb{E}\left(\int_{-\infty}^{t}\left(\|\mathcal{N}(t-s)\|_{\mathcal{L}(\mathbb{H}_2)}^{2^{-1}}\right)\left(\|\mathcal{N}(t-s)\|_{\mathcal{L}(\mathbb{H}_2)}^{2^{-1}}\|G(s+r_n, Y_n(s)) - G(s+r_n, Y^0(s))\|_{\mathbb{H}_2}\right)\mathrm{d}s\right)^2 \\
&\leqslant \left(\int_{-\infty}^{t}\|\mathcal{N}(t-s)\|_{\mathcal{L}(\mathbb{H}_2)}\mathrm{d}s\right)\int_{-\infty}^{t}\|\mathcal{N}(t-s)\|_{\mathcal{L}(\mathbb{H}_2)}\mathbb{E}\|G(s+r_n, Y_n(s)) - G(s+r_n, Y^0(s))\|_{\mathbb{H}_2}^2\mathrm{d}s \\
&\leqslant C_2\left(\int_{-\infty}^{t}\|\mathcal{N}(t-s)\|_{\mathcal{L}(\mathbb{H}_2)}\mathrm{d}s\right)\int_{-\infty}^{t}\|\mathcal{N}(t-s)\|_{\mathcal{L}(\mathbb{H}_2)}\mathbb{E}\|Y_n(s) - Y^0(s)\|_{\mathbb{H}_2}^2\mathrm{d}s.
\end{aligned} \tag{35}$$

Furthermore,

$$\begin{aligned}
\mathrm{J}^4 &= \mathbb{E}\left\|\int_{-\infty}^{t} \mathcal{N}(t-s)\left(G(s+r_n, Y^0(s)) - G^0(s, Y^0(s))\right)\mathrm{d}s\right\|_{\mathbb{H}_2}^2 \\
&\leqslant \mathbb{E}\left(\int_{-\infty}^{t}\left(\|\mathcal{N}(t-s)\|_{\mathcal{L}(\mathbb{H}_2)}^{2^{-1}}\right)\left(\|\mathcal{N}(t-s)\|_{\mathcal{L}(\mathbb{H}_2)}^{2^{-1}}\|G(s+r_n, Y_n(s)) - G^0(s, Y^0(s))\|_{\mathbb{H}_2}\right)\mathrm{d}s\right)^2 \\
&\leqslant \left(\int_{-\infty}^{t}\|\mathcal{N}(t-s)\|_{\mathcal{L}(\mathbb{H}_2)}\mathrm{d}s\right)\int_{-\infty}^{t}\|\mathcal{N}(t-s)\|_{\mathcal{L}(\mathbb{H}_2)}\mathbb{E}\|G(s+r_n, Y_n(s)) - G^0(s, Y^0(s))\|_{\mathbb{H}_2}^2\mathrm{d}s \\
&\leqslant \left(\int_{-\infty}^{t}\|\mathcal{N}(t-s)\|_{\mathcal{L}(\mathbb{H}_2)}\mathrm{d}s\right)^2 \sup_{s\in]-\infty,t]}\mathbb{E}\|G(s+r_n, Y_n(s)) - G^0(s, Y^0(s))\|_{\mathbb{H}_2}^2.
\end{aligned} \tag{36}$$

And for I^3 we have,

$$\begin{aligned}
\mathrm{I}^3 &= \mathbb{E}\left\|\int_{-\infty}^{t} \mathcal{N}(t-s)\left(H(s+r_n, Y_n(s)) - H^0(s+r_n, Y^0(s))\right)\mathrm{d}W(s)\right\|_{\mathbb{H}_2}^2 \\
&\leqslant 2\mathbb{E}\left\|\int_{-\infty}^{t} \mathcal{N}(t-s)\left(H(s+r_n, Y_n(s)) - H(s+r_n, Y^0(s))\right)\mathrm{d}W(s)\right\|_{\mathbb{H}_2}^2 \\
&+ 2\mathbb{E}\left\|\int_{-\infty}^{t} \mathcal{N}(t-s)\left(H(s+r_n, Y^0(s)) - H^0(s, Y^0(s))\right)\mathrm{d}W(s)\right\|_{\mathbb{H}_2}^2 \\
&:= 2(\mathrm{J}^5 + \mathrm{J}^6).
\end{aligned} \tag{37}$$

By applying Lemma 1.4, we derive,

$$\begin{aligned}
\mathrm{J}^5 &= \mathbb{E}\left\|\int_{-\infty}^{t} \mathcal{N}(t-s)\left(H(s+r_n, Y_n(s)) - H(s+r_n, Y^0(s))\right)\mathrm{d}W(s)\right\|_{\mathbb{H}_2}^2 \\
&\leqslant \mathbb{E}\left(\int_{-\infty}^{t}\|\mathcal{N}(t-s)\|_{\mathcal{L}(\mathbb{H}_2)}^2\|H(s+r_n, Y_n(s)) - H(s+r_n, Y^0(s))\|_{\mathcal{L}_2^0}^2\mathrm{d}s\right) \\
&\leqslant \int_{-\infty}^{t}\|\mathcal{N}(t-s)\|_{\mathcal{L}(\mathbb{H}_2)}^2\mathbb{E}\|H(s+r_n, Y_n(s)) - H(s+r_n, Y^0(s))\|_{\mathcal{L}_2^0}^2\mathrm{d}s \\
&\leqslant C_2\int_{-\infty}^{t}\|\mathcal{N}(t-s)\|_{\mathcal{L}(\mathbb{H}_2)}^2\mathbb{E}\|Y_n(s) - Y^0(s)\|_{\mathbb{H}_2}^2\mathrm{d}s.
\end{aligned} \tag{38}$$

In addition,

$$\begin{aligned} J^6 &= \mathbb{E}\left\|\int_{-\infty}^{t} \mathcal{N}(t-s)\left(H(s+r_n, Y_n(s)) - H^0(s, Y^0(s))\right)\mathrm{d}W(s)\right\|_{\mathbb{H}_2}^2 \\ &\leqslant \mathbb{E}\left(\int_{-\infty}^{t} \|\mathcal{N}(t-s)\|_{\mathcal{L}(\mathbb{H}_2)}^2 \|H(s+r_n, Y_n(s)) - H^0(s, Y^0(s))\|_{\mathcal{L}_2^0}^2 \mathrm{d}s\right) \\ &\leqslant \int_{-\infty}^{t} \|\mathcal{N}(t-s)\|_{\mathcal{L}(\mathbb{H}_2)}^2 \mathbb{E}\|H(s+r_n, Y_n(s)) - H^0(s, Y^0(s))\|_{\mathcal{L}_2^0}^2 \mathrm{d}s \\ &\leqslant \left(\int_{-\infty}^{t} \|\mathcal{N}(t-s)\|_{\mathcal{L}(\mathbb{H}_2)}^2 \mathrm{d}s\right) \sup_{s\in]-\infty,t]} \mathbb{E}\|H(s+r_n, Y_n(s)) - H^0(s, Y^0(s))\|_{\mathcal{L}_2^0}^2. \end{aligned} \tag{39}$$

Includes the inequalities from (30)−(39),

$$\begin{aligned} \mathbb{E}\|Y_n(t) - Y^0(t)\|_{\mathbb{H}_2}^2 \leqslant \left\{6C_2M_\delta^2\left(2\left(\frac{1}{\delta}\right)^2 + \left(\frac{1}{2\delta}\right)\right)\right\}\alpha_n(t) + 6C_2M_\delta^2\left(2\left(\frac{1}{\delta}\right)\right)\int_{-\infty}^{t} e^{-\delta(t-s)}\mathbb{E}\|Y_n(s) - Y^0(s)\|_{\mathbb{H}_2}^2 \mathrm{d}s \\ + 6C_2M_\delta^2 \int_{-\infty}^{t} e^{-2\delta(t-s)}\mathbb{E}\|Y_n(s) - Y^0(s)\|_{\mathbb{H}_2}^2 \mathrm{d}s. \end{aligned} \tag{40}$$

For a sequence,

$$\begin{aligned} &\alpha_n(t) := \sup_{s\in]-\infty,t]} \mathbb{E}\left\{\|F(s+r_n, Y_n(s)) - F^0(s, Y^0(s))\|_{\mathbb{H}_2}^2 + \|G(s+r_n, Y_n(s)) - G^0(s, Y^0(s))\|_{\mathbb{H}_2}^2 \right. \\ &\left. + \|H(s+r_n, Y_n(s)) - H^0(s, Y^0(s))\|_{\mathcal{L}_2^0}^2\right\}, \end{aligned}$$

such that which converges to 0 as $n \to \infty$ because $\sup_{t\in\mathbb{R}} \mathbb{E}\|Y^0(t)\|^2 < \infty$ which implies that $\left(Y^0(t)\right)_t$ is tight relatively to bounded sets. Since $\delta > \left\{6C_2M_\delta^2\left(2\left(\frac{1}{\delta}\right) + 1\right)\right\}$, we deduce from Lemma 3.2 that,

$$\lim_{n\to\infty} \mathbb{E}\|Y_n(t) - Y^0(t)\|_{\mathbb{H}_2}^2 = 0.$$

Hence $Y_n(t)$ converges in distribution to $Y^0(t)$. But, since the distribution of $Y_n(t)$ is the same as that of $Y(t + r_n)$, we deduce that $Y(t + r_n)$ converges in distribution to $Y^0(t)$, i.e.

$$\lim_{n\to\infty} \mu(t + u_n + v_n) := \mathrm{Law}(Y^0)(t).$$

By analogy and using (29) we can easily deduce that,

$$\lim_{n\to\infty} \lim_{m\to\infty} \mu(t + u_n + v_n) := \mathrm{Law}(Y^0)(t).$$

We have demonstrated the almost periodicity of Y in one-dimensionaldistributions. To establish the almost periodicity in multi-dimensional distributions for Y, we employ Proposition 3.6. For a fixed $u \in \mathbb{R}$, consider

$$\xi^n := Y(u + r_n), \qquad \eta^n := Y'(u + r_n),$$

and

$$F^n(t, y) := F(t + r_n, y), \qquad G^n(t, y) := G(t + r_n, y), \qquad H^n((t, y) := H(t + r_n, y).$$

From the previous discussions, we know that ξ^n (resp. η^n) converges in distribution to some variable $Y(u)$ (resp. $Y'(u)$) . Consequently, ξ^n (resp. η^n) is tight, and thus (ξ^n, W) (resp. (η^n, W)) is also tight. Therefore, we can select $Y(u)$ (resp. $Y'(u)$) such that (ξ^n, W) (resp. (η^n, W)) converges in distribution to $(Y(u), W)$ (resp.

$(Y'(u), W)$). Then, according to Proposition 3.6, for every $b \geqslant u$, $Y(\cdot + u_n)$ converges in distribution on $\mathcal{P}(C([u, b]; \mathbb{H}_2))$ to the (unique in distribution) solution to,

$$Y(t) := \mathcal{M}(t-u)Y(u) + \mathcal{N}(t-u)\left(Y'(u) - F(u, Y(u))\right) + \int_u^t \mathcal{N}_t(t-s)F(s, Y(s))\mathrm{d}s + \int_u^t \mathcal{N}(t-s)G(s, Y(s))\mathrm{d}s + \int_u^t \mathcal{N}(t-s)H(s, Y(s))\mathrm{d}W(s).$$

Observe that Y remains independent of the selected interval $[u, b]$, implying that the convergence occurs in $\mathcal{P}(C(\mathbb{R}; \mathbb{H}_2))$. Similarly, $Y^n := Y(\cdot + u_n)$ converges in distribution on $\mathcal{P}(C(\mathbb{R}; \mathbb{H}_2))$ to a continuous process X. For $t \geqslant u$, the expression for $X(t)$ is given by,

$$X(t) := \mathcal{M}(t-u)X(u) + \mathcal{N}(t-u)(Y'(u) - F(u, Y(u)) + \int_u^t \mathcal{N}(t-s)F(s, X(s))\mathrm{d}s + \int_u^t \mathcal{N}(t-s)G(s, X(s))\mathrm{d}s + \int_u^t \mathcal{N}(t-s)H(s, X(s))\mathrm{d}W(s).$$

However, based on (29), $Y(\cdot + r_n)$ converges in distribution to the identical process X. Consequently, we can assert that Y exhibits almost periodicity in multi-dimensional distributions. □

4. Special Case

In Theorems 3.5–3.7, within Condition (H1), a general condition was provided to streamline the analysis. Thus, we directly utilize Lemma 3.2 mentioned in [14] (particularly for the case where $m := 0$). Indeed, we will afford some leniency to this condition and underscore the significance of Lemma 3.2. Within condition (H1), we substitute point (1) with

(1)′ Assume the existence of an operator $\mathscr{A}$, defined as $A + \dfrac{B^2}{4}$, generating a strongly continuous cosine families $\{C(t) : t \in \mathbb{R}_+\}$ within $\mathbb{H}_2$. Furthermore, we posit that $B/2$ engenders a strongly continuous semi-group denoted by $\{\mathcal{T}(t) : t \in \mathbb{R}_+\}$. Additionally, we assert the presence of constants $M_{\delta_i} > 0, i = 1, 2, 3$, where $\delta_3 > \delta_2 > \delta_1 > 0$. These constants satisfy:

$$\|C(t-s)\|_{\mathcal{L}(\mathbb{H}_2)} \leqslant M_{\delta_1} e^{\delta_1(t-s)}, \quad \|\mathcal{S}(t-s)\|_{\mathcal{L}(\mathbb{H}_2)} \leqslant M_{\delta_1}(t-s)e^{\delta_1(t-s)}, \quad t \geqslant s,\ t \in \mathbb{R},$$

and

$$\|B\mathcal{S}(t-s)\|_{\mathcal{L}(\mathbb{H}_2)} \leqslant M_{\delta_2}(t-s)e^{\delta_2(t-s)}, \quad \|\mathcal{T}(t-s)\|_{\mathcal{L}(\mathbb{H}_2)} \leqslant M_{\delta_3} e^{-\delta_3(t-s)}, \quad t \geqslant s,\ t \in \mathbb{R}.$$

According to the definition provided in [17], we can infer the following relation:

$$\mathcal{T}(t)C(t) := \mathcal{E}C(t), \quad \text{and} \quad \mathcal{T}(t)\mathcal{S}(t) := \mathcal{E}\mathcal{S}(t).$$

Now, let's define some quantities:

$$\begin{cases} \hat{\beta}_{1.p}(\delta) := \left\{4^{p-1}C_p\hat{M}^p\left(\left(\frac{1}{\delta}\right)^p + 2\left(\frac{1}{\delta^2}\right)^p + K_{p/2}\left(\frac{1}{4\delta^3}\right)^{\frac{p}{2}}\right)\right\}, \\ \hat{\beta}_{2.p}(\delta) := \left\{4^{p-1}C_p\hat{M}^p\left(\left(\frac{1}{\delta}\right)^{p-1} + 2\left(\frac{1}{\delta^2}\right)^p + K_{p/2}\left(\frac{1}{4\delta^3}\right)^{\frac{p}{2}-1}\right)\right\}, \\ \hat{M} := M_{\delta_3}\max(M_{\delta_1}, M_{\delta_2}/2). \end{cases}$$

(Note that $\delta := \min(\delta_3 - \delta_1, \delta_3 - \delta_2, 2(\delta_3 - \delta_1), 2(\delta_3 - \delta_2)) = \delta_3 - \delta_2$).

Theorem 4.1. *Given the fulfillment of conditions* (H1)′−(H3) *and* $\hat{\beta}_{1.p}(\delta) < 1$ *for* $p \geqslant 2$*, it follows that there exists a unique mild solution* $Y \in \mathcal{M}^p(\mathbb{R}, \mathbb{H}_2)$ *to the equation* (8) *over* $\mathbb{R}$*. Furthermore, if* $\delta > 2 + \hat{\beta}_{2.p}(\delta)$ *for* $p \geqslant 2$*, the ensuing inequalities hold: For* $p > 2$*,*

$$\mathbb{E}\,\|Y(t)\|^p \leqslant \frac{\delta - 2}{\delta - \hat{\beta}_{2.p}(\delta) - 2}.$$

For $p = 2$*,*

$$\mathbb{E}\,\|Y(t)\|^2 \leqslant \frac{\delta - 2}{\delta - \hat{\beta}_{2.2}(\delta) - 2}.$$

The solution is expressed as:

$$Y(t) = \int_{-\infty}^{t} \mathcal{T}(t-s)\left(C(t-s) + (B/2)\mathcal{S}(t-s)\right)F(s, Y(s))\mathrm{d}s + \int_{-\infty}^{t} \mathcal{T}(t-s)\mathcal{S}(t-s)G(s, Y(s))\mathrm{d}s + \int_{-\infty}^{t} \mathcal{T}(t-s)\mathcal{S}(t-s)H(s, Y(s))\mathrm{d}W(s).$$

Proof. Using a method akin to the proof outlined in Theorem 3.5, we can establish the validity of step 1 by relying on the relation (see [17]), $\mathcal{N}(t) := \mathcal{E}S(t) := \mathcal{T}(t)\mathcal{S}(t)$ and,

$$\mathcal{M}(t) := \mathcal{E}C(t) - (B/2)\mathcal{E}S(t) := \mathcal{T}(t)C(t) - (B/2)\mathcal{T}(t)\mathcal{S}(t).$$

Additionally, we utilize the observation that if $u_1 < u_2$ and $\delta_3 > \delta_2 > \delta_1 > 0$, then,

$$\lim_{u_2 \to -\infty} \int_{u_1}^{u_2} (t-s)^j e^{-(\delta_3 - \delta_i)(t-s)}\mathrm{d}s = 0, \; i = 1, 2, \quad j = 0, 1, 2.$$

By employing a similar approach as in Theorem 3.5 and making use of Lemma 3.2, the proof can be concluded through the application of a contraction principle. □

Theorem 4.2. *Assuming the fulfillment of assumptions* (H1)′−(H6)*, and additionally, under the conditions where* $\hat{\beta}_{1.2}(\delta) < 1$ *and* $2\hat{\beta}_{2.2}(\delta) < \delta$*, it follows that there exists a unique mild solution* $Y \in \mathcal{M}^2(\mathbb{R}, \mathbb{H}_2)$ *to the equation* (8) *over* $\mathbb{R}$*, characterized by its almost-periodic in distribution.*

Proof. Using an analogous approach to that of Theorem 3.7 and incorporating Lemma 3.2 along with Theorem 4.1, the proof can be effectively concluded. □

5. Illustration

Let's consider the following boundary value problem (see [24, Example 5]),

$$\begin{cases} \frac{\partial^2}{\partial t^2} y(x,t) - \frac{\partial}{\partial t} f(t, y(x,t)) + b(x)y(x,t)\partial t = \frac{\partial^2}{\partial x^2} y(x,t)\partial t + g(t, y(x,t))\partial t + h(t, y(x,t))\mathrm{d}w(t), \\ \text{where, } t \in \mathbb{R}, \quad 0 \leqslant x < \pi, \\ \text{Boundary conditions: } y(0,t) = y(\pi,t) = 0, \quad t \in \mathbb{R}. \end{cases} \tag{41}$$

Here, $b : [0, \pi] \to]0, \infty[$ is a continuous function, and w represents a Q−Wiener process on a stochastic space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}}, \mathbb{P})$. We define the space $\mathbb{H} := L^2[0, \pi]$, and let $\mathbb{H}_1 := \mathbb{H}_2 := \mathbb{H}$ with inner product $(\cdot, \cdot)$. The operator $A : D(A) \longrightarrow \mathbb{H}_2$ is defined by $Ay := y''$ where the domain $D(A)$ is given by,

$$D(A) := \{y \in \mathbb{H} : y, y' \text{ are absolutely continuous}, y(0) = y'(\pi) = 0,\ y'' \in \mathbb{H}\}.$$

Moreover, A has a specific spectrum characterized by eigenvalues of the form $-n^2, n = 1, 2, 3....$ Hence, A can be represented as,

$$Ay := -\sum_{n=1}^{\infty} n^2 (y, e_n) e_n,$$

for $y \in D(A)$, where the corresponding eigenfunctions are given by $e_n(x) := \sqrt{2\pi^{-1}} \sin(\sqrt{n^2}x)$ for $0 \leqslant x \leqslant \pi$. We define $B : \mathbb{H} \to \mathbb{H}$ by $[By]x := b(x)y(x)$. Defining $\mathcal{A} := \frac{B^2}{4} + A$ we have,

$$[\mathcal{A}y](x) := \sum_{n=1}^{\infty} \left(\frac{B^2}{4} - n^2\right)(y, e_n) e_n(x),$$

and $\mathcal{A}$ generates the cosine family,

$$\begin{aligned}[C(t)y](x) := &\sum_{n=1}^{\infty} \cos\left(t\sqrt{n^2 - \frac{1}{4}b^2(x)}\right)(y, e_n) e_n(x) 1_{\left\{n^2 > \frac{b^2(x)}{4}\right\}} \\ &+ \sum_{n=1}^{\infty} \cosh\left(t\sqrt{\frac{1}{4}b^2(x) - n^2}\right)(y, e_n) e_n(x) 1_{\left\{n^2 < \frac{b^2(x)}{4}\right\}} + \sum_{n=1}^{\infty} (y, e_n) e_n(x) 1_{\left\{n^2 := \frac{b^2(x)}{4}\right\}},\end{aligned}$$

and its associated sine family can be expressed as,

$$\begin{aligned}[\mathcal{S}(t)y](x) := &\sum_{n=1}^{\infty} \frac{\sin\left(t\sqrt{n^2 - \frac{1}{4}b^2(x)}\right)}{\sqrt{n^2 - \frac{1}{4}b^2(x)}} (y, e_n) e_n(x) 1_{\left\{n^2 > \frac{b^2(x)}{4}\right\}} \\ &+ \sum_{n=1}^{\infty} \frac{\sinh\left(t\sqrt{\frac{1}{4}b^2(x) - n^2}\right)}{\sqrt{\frac{1}{4}b^2(x) - n^2}} (y, e_n) e_n(x) 1_{\left\{n^2 < \frac{b^2(x)}{4}\right\}} + \sum_{n=1}^{\infty} t\,(y, e_n) e_n(x) 1_{\left\{n^2 := \frac{b^2(x)}{4}\right\}}.\end{aligned}$$

Then there exist constants $M_{\delta_1}, \delta_1 > 0$ such that (see eg., [10]),

$$\|C(t-s)\|_{\mathcal{L}(\mathbb{H})} \leqslant M_{\delta_1} e^{\delta_1(t-s)}, \qquad \|\mathcal{S}(t-s)\|_{\mathcal{L}(\mathbb{H})} \leqslant M_{\delta_1}(t-s)e^{\delta_1(t-s)}, \quad t \geqslant s,\ t \in \mathbb{R}.$$

It is also noted that $-B/2$ generates the semi-group $\{\mathcal{T}(t) : t \in \mathbb{R}_+\}$ on $\mathbb{H}$ defined by,

$$[\mathcal{T}(t)y]x := e^{-\frac{1}{2}tb(x)} y(x), \qquad t \in \mathbb{R}_+.$$

and $D(B) := \mathbb{H}_2$. Let $\delta_3 := \min\{b(x)/2 : x \in [0, \pi]\}$. Then there exist constants M_{δ_i}, δ_i, $i := 2, 3$ such that,

$$\|B\mathcal{S}(t-s)\|_{\mathcal{L}(\mathbb{H})} \leqslant M_{\delta_2}(t-s)e^{\delta_2(t-s)}, \qquad \text{and} \qquad \|\mathcal{T}(t-s)\|_{\mathcal{L}(\mathbb{H}_2)} \leqslant M_{\delta_3} e^{-\delta_3 t}, \quad t \geqslant s, \quad t \in \mathbb{R}.$$

For $t \in \mathbb{R}$ and $y \in \mathbb{H}$, define, $F, G : \mathbb{R} \times \mathbb{H} \longrightarrow \mathbb{H}, H : \mathbb{R} \times \mathbb{H} \longrightarrow \mathcal{L}_2(\mathbb{U}_0, \mathbb{H})$, by,

$$G(t, y)(x) := g(t, y(x, t)), \quad F(t, y)(x) := f(t, y(x, t)), \qquad H(t, y)(x) := h(t, y(x, t)).$$

The problem (41) can then be expressed in the abstract form,

$$d(Y'(t) - F(t, Y(t)) = AY(t)\mathrm{d}t - BY'(t)\mathrm{d}t + G(t, Y(t))\mathrm{d}t + H(t, Y(t))\mathrm{d}W(t), \quad t \in \mathbb{R}.$$

Thus, under the conditions specified in Theorem 4.2, with $\delta := \delta_3 - \delta_2 > 0$, the semilinear stochastic equation boundary value problem (41) possesses a unique almost periodic solution in distribution.

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