Filomat 39:17 (2025), 5817–5824
https://doi.org/10.2298/FIL2517817G

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Extended two-variable Fubini-type polynomials and their properties

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Abstract. This paper introduces a novel family of two-variable Fubini-type polynomials utilizing the two-parameter Mittag-Leffler function. The proposed approach also leads to the introduction of a new type of Stirling numbers of the second kind. The paper systematically explores various intriguing properties associated with the introduced polynomials and numbers. The analytical properties, including differential formulas, summation formulas, and connections to well-known polynomials and numbers, are thoroughly investigated and presented.

1. Introduction and Preliminaries

Special functions play a vital role in mathematical physics, especially in approximating integrals arising in applications such as the propagation of flattened Gaussian beams [1–3, 6, 17, 19]. Recently, truncated polynomials and numbers have attracted growing interest due to their usefulness as generating functions. These include truncated Bernoulli [7], Euler [13], Fubini-type [22, 24], Appell [10], Laguerre-type [3], Apostol-type [21], Frobenius-Euler [14], Sheffer sequences [23], and Mittag-Leffler polynomials [26], among others. For a detailed overview of the two-parameter Mittag-Leffler function and its functional extensions, we refer the reader to the recent survey in [19].

This study introduces a new class of truncated polynomials and numbers, focusing on their connections with classical families and potential applications in mathematical and physical contexts. We begin by reviewing known truncated polynomials and their relations to classical ones. Throughout, we use the notations $\mathbb{N}$, $\mathbb{R}$, $\mathbb{C}$ for natural, real, and complex numbers, and define $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For any $m \in \mathbb{N}$, the rising factorial is given by

$$(z)^{(m)} = z(z+1)\cdots(z+m-1).$$

2020 *Mathematics Subject Classification*. Primary 11B83; Secondary 11B68, 33E12, 05A15.
Keywords. Fubini polynomials, Truncated Fubini polynomials, Truncated Bernoulli polynomials, Truncated Euler polynomials, Stirling numbers of the second kind, Mittag-Leffler function, Two-parameter Mittag-Leffler function.
Received: 06 April 2024; Revised: 11 April 2025; Accepted: 26 April 2025
Communicated by Hari M. Srivastava
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Also, for a negative integer $-m$, we use the binomial expansion

$$(z+v)^{-m}=\sum_{r=0}^{\infty}(-1)^r\binom{m+r-1}{r}z^r v^{-(m+r)},\quad |z|<v. \tag{1}$$

The truncated Bernoulli polynomials $\mathbb{B}_{m,p}(u)$, where $m\in\mathbb{N}_0$, are defined as follows (see [7]):

$$\frac{\frac{t^m}{m!}}{e^t-\sum\limits_{j=0}^{m-1}\frac{t^j}{j!}}e^{ut}=\sum_{p=0}^{\infty}\mathbb{B}_{m,p}(u)\frac{t^p}{p!}. \tag{2}$$

For the case $m=1$, $\mathbb{B}_{m,p}(u)$ yields the classical Bernoulli polynomials $\mathbb{B}_p(u)$, given by (see [5, 18, 20])

$$\frac{t}{e^t-1}e^{ut}=\sum_{p=0}^{\infty}\mathbb{B}_p(u)\frac{t^p}{p!}\qquad (|t|<2\pi).$$

The truncated Euler polynomials $\mathbb{E}_{m,p}(u)$ for $m\in\mathbb{N}_0$ are defined as follows (see [13]):

$$\frac{2\frac{t^m}{m!}}{e^t+1-\sum\limits_{j=0}^{m-1}\frac{t^j}{j!}}e^{ut}=\sum_{p=0}^{\infty}\mathbb{E}_{m,p}(u)\frac{t^p}{p!}. \tag{3}$$

For the case $m=0$, $\mathbb{E}_{m,p}(u)$ yields the classical Euler polynomials $\mathbb{E}_p(u)$, expressed as follows (see [18, 20]):

$$\frac{2}{e^t+1}e^{ut}=\sum_{p=0}^{\infty}\mathbb{E}_p(u)\frac{t^p}{p!}.$$

It is evident that by setting $u=0$ in (2) and (3), we obtain the truncated Bernoulli and truncated Euler numbers, respectively, i.e.,

$$\mathbb{B}_{m,p}(0)=\mathbb{B}_{m,p}\ \text{ and }\ \mathbb{E}_{m,p}(0)=\mathbb{E}_{m,p}.$$

In 2019, Duran and Acikgoz [4] introduced a novel class of truncated polynomials called the truncated Fubini polynomials $\mathbb{F}_{m,p}(u,v)$, defined by the expression

$$\frac{\frac{t^m}{m!}}{1-v\Big(e^t-1-\sum\limits_{j=0}^{m-1}\frac{t^j}{j!}\Big)}e^{ut}=\sum_{p=0}^{\infty}\mathbb{F}_{m,p}(u,v)\frac{t^p}{p!}.$$

For the special case $m=0$, the truncated Fubini polynomials reduce to the classical Fubini polynomials of two-variable $\mathbb{F}_p(u,v)$, given as follows (see [9, 11, 12, 25]):

$$\frac{1}{1-v(e^t-1)}e^{ut}=\sum_{p=0}^{\infty}\mathbb{F}_p(u,v)\frac{t^p}{p!}. \tag{4}$$

Setting $u=0$ in (4), we obtain the usual Fubini polynomials $\mathbb{F}_p(v)$, expressed as follows (see [9, 11, 12, 25]):

$$\frac{1}{1-v(e^t-1)}=\sum_{p=0}^{\infty}\mathbb{F}_p(v)\frac{t^p}{p!}. \tag{5}$$

Furthermore, the case $v=1$ in (5) yields the familiar Fubini numbers $\mathbb{F}_p$ as follows:

$$\frac{1}{2-e^t}=\sum_{p=0}^{\infty}\mathbb{F}_p\frac{t^p}{p!}.$$

Additionally, Duran and Acikgoz [4] proposed the truncated Stirling numbers of the second kind $\mathbb{S}_{2,m}(p,q)$ defined by the expression

$$\frac{\left(e^t - 1 - \sum_{j=0}^{m-1} \frac{t^j}{j!}\right)^q}{q!} = \sum_{p=0}^{\infty} \mathbb{S}_{2,m}(p,q)\frac{t^p}{p!}. \tag{6}$$

Setting $m = 0$ in (6), we obtain the classical Stirling numbers of the second kind $\mathbb{S}_2(p,q)$ as follows (see [15]):

$$\frac{(e^t-1)^q}{q!} = \sum_{p=0}^{\infty} \mathbb{S}_2(p,q)\frac{t^p}{p!}.$$

The main objective of this study is to identify a new class of two-variable Fubini-type polynomials and a new type of Stirling numbers of the second kind by utilizing the two-parameter Mittag-Leffler function [27]. The two-parameter Mittag-Leffler function $E_{\alpha,\beta}(t)$, which is a generalization of the standard Mittag-Leffler function, is defined as (see [27], see also [19])

$$E_{\alpha,\beta}(t) = \sum_{p=0}^{\infty} \frac{t^p}{\Gamma(\beta+\alpha p)} \qquad (t \in \mathbb{C}, \Re(\alpha) > 0 \text{ and } \Re(\beta) > 0). \tag{7}$$

Setting $\beta = 1$ in (7), we obtain the classical Mittag-Leffler function $E_\alpha(t)$ [16] given by

$$E_{\alpha}(t) = \sum_{p=0}^{\infty} \frac{t^p}{\Gamma(1+\alpha p)} \qquad (t \in \mathbb{C} \text{ and } \Re(\alpha) > 0). \tag{8}$$

Several special cases of $E_{\alpha,\beta}(t)$ are given below [8]:

$$E_{1,2}(t) = \frac{e^t-1}{t},\ E_{1,1}(t) = e^t,\ E_{2,1}(t^2) = \cosh t \text{ and } E_{2,1}(-t^2) = \cos t.$$

2. Extension of the two-variable Fubini Polynomials

In this section, we introduce a further extension of the two-variable Fubini polynomials and the Stirling numbers of the second kind by employing the two-parameter Mittag-Leffler function [27].

Definition 2.1. *For $m, p \in \mathbb{N}_0$, with $\Re(\alpha) > 0$, $\Re(\beta) > 0$, and $v(t^m E_{\alpha,\beta}(t) - 1) \neq 1$, we define the extended Fubini-type polynomials $\mathbb{F}_{m,p}^{(\alpha,\beta)}(u,v)$ as follows:*

$$\frac{\frac{t^m}{m!}}{1 - v\left(t^m E_{\alpha,\beta}(t) - 1\right)} e^{ut} = \sum_{p=0}^{\infty} \mathbb{F}_{m,p}^{(\alpha,\beta)}(u,v)\frac{t^p}{p!}. \tag{9}$$

Setting $u = 0$ in (9), we obtain a new family of two-variable Fubini-type polynomials denoted as $\mathbb{F}_{m,p}^{(\alpha,\beta)}(v)$, given by

$$\frac{\frac{t^m}{m!}}{1 - v\left(t^m E_{\alpha,\beta}(t) - 1\right)} = \sum_{p=0}^{\infty} \mathbb{F}_{m,p}^{(\alpha,\beta)}(v)\frac{t^p}{p!}. \tag{10}$$

Furthermore, by setting $v = 1$ in (10), we arrive at the new extended Fubini-type numbers $\mathbb{F}_{m,p}^{(\alpha,\beta)}$ as follows:

$$\frac{\frac{t^m}{m!}}{2 - t^m E_{\alpha,\beta}(t)} = \sum_{p=0}^{\infty} \mathbb{F}_{m,p}^{(\alpha,\beta)}\frac{t^p}{p!}.$$

We observe that from (9), it is clearly shown that

$$\mathbb{F}_{0,p}^{(1,1)}\left(u,-\frac{1}{2}\right)=\mathbb{E}_p(u) \text{ and } \mathbb{F}_{0,p}^{(1,1)}\left(0,-\frac{1}{2}\right)=\mathbb{E}_p,$$

where $\mathbb{E}_p(u)$ and $\mathbb{E}_p$ are the classical Euler polynomials and numbers, respectively [18].

Remark 2.2. *Setting $\alpha = 1$ and $\beta = m+1$ ($m \in \mathbb{N}_0$) in (9) and using the fact*

$$E_{1,m+1}(t)=\frac{e^t-\sum\limits_{j=0}^{m-1}\frac{t^j}{j!}}{t^m}, \tag{11}$$

we obtain the two-variable truncated Fubini polynomials $\mathbb{F}_{m,p}(u,v)$ as introduced by Duran and Acikgoz [4]. Furthermore, for $m = 0$, the expression yields the two-variable Fubini polynomials $\mathbb{F}_p(u,v)$ given in (4). Moreover, by setting $m = 0$, $\alpha = 1$, and $\beta = 1$ in (9), we obtain the traditional Fubini polynomials $\mathbb{F}_p(u,v)$ as provided in (4).

Definition 2.3. *Let $m, p \in \mathbb{N}_0$, with $\Re(\alpha) > 0$ and $\Re(\beta) > 0$. Then, the extended Stirling numbers of the second kind $\mathbb{S}_{2,m}^{(\alpha,\beta)}(p,q)$ are described by*

$$\frac{(t^m E_{\alpha,\beta}(t)-1)^q}{q!}=\sum_{p=0}^{\infty}\mathbb{S}_{2,m}^{(\alpha,\beta)}(p,q)\frac{t^p}{p!}. \tag{12}$$

The case $\alpha = 1$ and $\beta = m+1$ ($m \in \mathbb{N}_0$) in (12) corresponds to the truncated Stirling numbers of the second kind introduced by Duran and Acikgoz [4]. Furthermore, for $m = 0$, we obtain the standard Stirling numbers of the second kind as provided in (6).

Now, we discuss some noteworthy properties of the proposed polynomials and numbers.

Theorem 2.4. *Let $m, p \in \mathbb{N}_0$, $\Re(\alpha) > 0$, $\Re(\beta) > 0$ and $w \in \mathbb{R}$. Then we have*

$$\mathbb{F}_{m,p}^{(\alpha,\beta)}(u+w,v)=\sum_{q=0}^{p}\binom{p}{q}w^q\,\mathbb{F}_{m,p-q}^{(\alpha,\beta)}(u,v), \tag{13}$$

$$\mathbb{F}_{m,p}^{(\alpha,\beta)}(u+w,v)=\sum_{q=0}^{p}\binom{p}{q}(u+w)^q\,\mathbb{F}_{m,p-q}^{(\alpha,\beta)}(v), \tag{14}$$

and

$$\mathbb{F}_{m,p}^{(\alpha,\beta)}(u,v)=\sum_{q=0}^{p}\binom{p}{q}u^q\,\mathbb{F}_{m,p-q}^{(\alpha,\beta)}(v). \tag{15}$$

Proof. Starting with the expression (9), we have

$$\begin{aligned}\sum_{p=0}^{\infty}\mathbb{F}_{m,p}^{(\alpha,\beta)}(u+w,v)\frac{t^p}{p!} &= \frac{\frac{t^m}{m!}}{1-v(t^m E_{\alpha,\beta}(t)-1)}e^{(u+w)t}\\ &= \frac{\frac{t^m}{m!}}{1-v(t^m E_{\alpha,\beta}(t)-1)}e^{ut}\,e^{wt}\\ &= \sum_{p=0}^{\infty}\mathbb{F}_{m,p}^{(\alpha,\beta)}(u,v)\frac{t^p}{p!}\sum_{q=0}^{\infty}w^q\,\frac{t^q}{q!}\\ &= \sum_{p=0}^{\infty}\left(\sum_{q=0}^{p}\binom{p}{q}w^q\,\mathbb{F}_{m,p-q}^{(\alpha,\beta)}(u,v)\right)\frac{t^p}{p!}.\end{aligned}$$

By comparing coefficients of the same powers of t, we obtain the identity (13). Similarly, the identity (14) can be derived by utilizing (10). Finally, setting $w = 0$ in (14) establishes identity (15). □

Remark 2.5. *If we compare the identities* (13) *and* (14)*, then we easily get the following interesting result:*

$$\sum_{q=0}^{p}\binom{p}{q}w^q\,\mathbb{F}_{m,p-q}^{(\alpha,\beta)}(u,v)=\sum_{q=0}^{p}\binom{p}{q}(u+w)^q\,\mathbb{F}_{m,p-q}^{(\alpha,\beta)}(v)\quad(\Re(\alpha)>0,\ \Re(\beta)>0).$$

Theorem 2.6. *For $m,p\in\mathbb{N}_0$, $\Re(\alpha)>0$ and $\Re(\beta)>0$, we have*

$$\mathbb{S}_{2,m}^{(\alpha,\beta)}(p,q+\sigma)=\frac{q!\sigma!}{(q+\sigma)!}\sum_{r=0}^{p}\binom{p}{r}\mathbb{S}_{2,m}^{(\alpha,\beta)}(p-r,q)\,\mathbb{S}_{2,m}^{(\alpha,\beta)}(r,\sigma).\tag{16}$$

Proof. By applying (12) to the left-hand side of (16) and performing a brief simplification, we obtain the desired identity. □

Theorem 2.7. *For $p,q\in\mathbb{N}_0$, the following relation holds true:*

$$\mathbb{S}_{2,1}^{(1,2)}(p,q)=2^q\ \mathbb{S}_2(p,q:\frac{1}{2}),\tag{17}$$

where $\mathbb{S}_2(p,q:\frac{1}{2})$ are the Apostol-type Stirling numbers of the second kind given by (see [15])

$$\sum_{p=0}^{\infty}\mathbb{S}_2(p,q:\nu)\,\frac{t^p}{p!}=\frac{(\nu e^t-1)^q}{q!}\quad(\nu\in\mathbb{C}).$$

Proof. This result follows directly by applying (12) to the left-hand side of (17). □

Theorem 2.8. *For $m,p\in\mathbb{N}_0$ with $p\geq m$, $\Re(\alpha)>0$ and $\Re(\beta)>0$, we have*

$$\mathbb{F}_{m,p}^{(\alpha,\beta)}(v)=\sum_{q=0}^{\infty}\binom{p}{m}v^q\,q!\,\mathbb{S}_{2,m}^{(\alpha,\beta)}(p-m,q).\tag{18}$$

Proof. From (10) and (12), we have

$$\sum_{p=0}^{\infty}\mathbb{F}_{m,p}^{(\alpha,\beta)}(v)\frac{t^p}{p!}=\frac{\frac{t^m}{m!}}{1-v(t^mE_{\alpha,\beta}(t)-1)}=\frac{t^m}{m!}\sum_{q=0}^{\infty}v^q\left(t^mE_{\alpha,\beta}(t)-1\right)^q=\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}v^q\,q!\,\mathbb{S}_{2,m}^{(\alpha,\beta)}(p,q)\,\frac{t^{p+m}}{p!m!},$$

which, upon comparing the coefficients of t^p, provides our needed result (18). □

Theorem 2.9. *For $m\in\mathbb{N}_0$, $p\geq1$, $\Re(\alpha)>0$ and $\Re(\beta)>0$, the following derivative formula holds true:*

$$\frac{\partial}{\partial u}\,\mathbb{F}_{m,p}^{(\alpha,\beta)}(u,v)=p\,\mathbb{F}_{m,p-1}^{(\alpha,\beta)}(u,v).\tag{19}$$

Proof. From (9), we observe that

$$\sum_{p=1}^{\infty}\frac{\partial}{\partial u}\,\mathbb{F}_{m,p}^{(\alpha,\beta)}(u,v)\,\frac{t^p}{p!}=t\,\frac{\frac{t^m}{m!}}{1-v(t^mE_{\alpha,\beta}(t)-1)}e^{ut}=\sum_{p=1}^{\infty}\mathbb{F}_{m,p-1}^{(\alpha,\beta)}(u,v)\,\frac{t^p}{(p-1)!}.$$

By comparing the coefficients of t^p, we obtain the desired result (19). □

Theorem 2.10. *For $p, m \in \mathbb{N}_0$, $\Re(\alpha) > 0$ and $\Re(\beta) > 0$, we have*

$$\mathbb{F}_{m,p}^{(\alpha,\beta)}(u,v) = \sum_{q=0}^{\infty}\sum_{r=0}^{p} (u)^{(q)}\binom{p}{r}\mathbb{F}_{m,p-r}^{(\alpha,\beta)}(-q,v)\,\mathbb{S}_2(r,q). \tag{20}$$

Proof. By using (9) and (1), we have

$$\begin{aligned}\sum_{p=0}^{\infty}\mathbb{F}_{m,p}^{(\alpha,\beta)}(u,v)\frac{t^p}{p!} &= \frac{\frac{t^m}{m!}}{1-v(t^m E_{\alpha,\beta}(t)-1)}(e^{-t})^{-u}\\ &= \frac{\frac{t^m}{m!}}{1-v(t^m E_{\alpha,\beta}(t)-1)}\sum_{q=0}^{\infty}\binom{u+q-1}{q}(1-e^{-t})^q\\ &= \frac{\frac{t^m}{m!}}{1-v(t^m E_{\alpha,\beta}(t)-1)}\sum_{q=0}^{\infty}(u)^{(q)}\frac{(e^t-1)^q}{q!}e^{-qt}\\ &= \sum_{q=0}^{\infty}(u)^{(q)}\sum_{p=0}^{\infty}\mathbb{F}_{m,p}^{(\alpha,\beta)}(-q,v)\frac{t^p}{p!}\sum_{r=0}^{\infty}\mathbb{S}_2(r,q)\frac{t^r}{r!}\\ &= \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{r=0}^{p}(u)^{(q)}\mathbb{F}_{m,p-r}^{(\alpha,\beta)}(-q,v)\,\mathbb{S}_2(r,q)\frac{t^p}{(p-r)!\,r!},\end{aligned}$$

which yields our needed result (20). □

Theorem 2.11. *For $p, m \in \mathbb{N}_0$, the following relations holds true:*

$$\mathbb{F}_{m,p}^{(1,m+1)}(u,-\frac{1}{2}) = \mathbb{E}_{m,p}(u) \tag{21}$$

and

$$\mathbb{F}_{m,p}^{(1,m+1)}(-\frac{1}{2}) = \mathbb{E}_{m,p}, \tag{22}$$

where $\mathbb{E}_{m,p}(u)$ and $\mathbb{E}_{m,p}$ denote the truncated Euler polynomials and numbers, respectively.

Proof. By applying (9) and (11) to the left-hand side of (21), we readily obtain the claimed result. Furthermore, setting $u = 0$ in (21) yields the result (22). □

Theorem 2.12. *Let $p, m, q, j \in \mathbb{N}_0$ such that $p \geq j+q$, $\Re(\alpha) > 0$ and $\Re(\beta) > 0$. Then the following relations hold true:*

$$\mathbb{F}_{m,p}^{(\alpha,\beta)}(u,v) = \frac{m!p!}{2(p+m)!}\sum_{q=0}^{p+m}\binom{p+m}{q}\mathbb{F}_{m,p+m-q}^{(\alpha,\beta)}(v)\,\mathbb{E}_{m,q}(u) + \frac{p!m!}{2}\sum_{q=0}^{p}\sum_{j=0}^{p}\frac{\mathbb{F}_{m,p-j-q}^{(\alpha,\beta)}(v)\,\mathbb{E}_{m,q}(u)}{q!(p-j-q)!(j+m)!} \tag{23}$$

and

$$\mathbb{F}_{m,p}^{(\alpha,\beta)}(u,v) = p!m!\sum_{q=0}^{p}\sum_{j=0}^{p}\frac{\mathbb{F}_{m,p-j-q}^{(\alpha,\beta)}(v)\,\mathbb{B}_{m,q}(u)}{q!(p-j-q)!(j+m)!}, \tag{24}$$

where $\mathbb{E}_{m,q}(u)$ and $\mathbb{B}_{m,q}(u)$ represent the truncated Bernoulli and truncated Euler polynomials described in (3) *and* (2), *respectively.*

Proof. From (9), we can write

$$\sum_{p=0}^{\infty} \mathbb{F}_{m,p}^{(\alpha,\beta)}(u,v)\frac{t^p}{p!} = \frac{\frac{t^m}{m!}\, e^{ut}}{1-v(t^m E_{\alpha,\beta}(t)-1)} \frac{2\frac{t^m}{m!}}{e^t+1-\sum\limits_{j=0}^{m-1}\frac{t^j}{j!}} \frac{e^t+1-\sum\limits_{j=0}^{m-1}\frac{t^j}{j!}}{\frac{2t^m}{m!}}$$

$$= \frac{m!}{2t^m}\sum_{p=0}^{\infty}\mathbb{F}_{m,p}^{(\alpha,\beta)}(v)\frac{t^p}{p!}\sum_{q=0}^{\infty}\mathbb{E}_{m,q}(u)\,\frac{t^q}{q!}\Big(\sum_{j=m}^{\infty}\frac{t^j}{j!}+1\Big)$$

$$= \frac{m!}{2}\sum_{p=0}^{\infty}\sum_{q=0}^{p}\mathbb{F}_{m,p-q}^{(\alpha,\beta)}(v)\,\mathbb{E}_{m,q}(u)\,\frac{t^{p-m}}{q!(p-q)!}\Big(\sum_{j=0}^{\infty}\frac{t^{j+m}}{(j+m)!}+1\Big)$$

$$= \frac{m!}{2}\sum_{p=0}^{\infty}\sum_{q=0}^{p}\mathbb{F}_{m,p-q}^{(\alpha,\beta)}(v)\,\mathbb{E}_{m,q}(u)\,\frac{t^{p-m}}{q!(p-q)!}$$

$$+\frac{m!}{2}\sum_{p=0}^{\infty}\sum_{q=0}^{p}\sum_{j=0}^{p}\mathbb{F}_{m,p-j-q}^{(\alpha,\beta)}(v)\,\mathbb{E}_{m,q}(u)\,\frac{t^p}{q!(p-j-q)!(j+m)!}.$$

By comparing the coefficients of t^p, we have the result (23). Similarly, the second result (24) can be established using the same approach. □

3. Concluding remarks

In this article, we have introduced a novel family of two-variable Fubini polynomials and Stirling numbers of the second kind by employing the two-parameter Mittag-Leffler function in the generating function. We have thoroughly investigated various analytical properties of these proposed polynomials and numbers, including differential formulas, summation formulas, and their relationships with other well-known polynomials and numbers.

In the final discussion, we briefly explore the variations in the generating functions of the introduced polynomials $\mathbb{F}_{m,p}^{(\alpha,\beta)}(u,v)$ and numbers $\mathbb{S}_{2,m}^{(\alpha,\beta)}(p,q)$. We have the following connections of two-parameter Mittag-Leffler function $E_{\alpha,\beta}(t)$ with the Wright hypergeometric function ${}_p\Psi_q$ and the Fox H-function $H_{r,s}^{m,n}$. Specifically, the relations [8] are given by

$$E_{\alpha,\beta}(t) = {}_1\Psi_1\left[\begin{array}{c|c}(1,1) & \\ (\beta,\alpha) & t\end{array}\right] \text{ and } E_{\alpha,\beta}(t) = H_{1,2}^{1,1}\left[\begin{array}{c|c} & (0,1) \\ -t & (0,1),(1-\beta,\alpha)\end{array}\right].$$

Using the above relations, we express the variations in the generating functions of our introduced polynomials and numbers as follows:

$$\frac{\frac{t^m}{m!}\, e^{ut}}{1-v\left(t^m{}_1\Psi_1\left[\begin{array}{c|c}(1,1) & \\ (\beta,\alpha) & t\end{array}\right]-1\right)} = \sum_{p=0}^{\infty}\mathbb{F}_{m,p}^{(\alpha,\beta)}(u,v)\frac{t^p}{p!},$$

$$\frac{\frac{t^m}{m!}\, e^{ut}}{1-v\left(t^m H_{1,2}^{1,1}\left[\begin{array}{c|c} & (0,1) \\ -t & (0,1),(1-\beta,\alpha)\end{array}\right]-1\right)} = \sum_{p=0}^{\infty}\mathbb{F}_{m,p}^{(\alpha,\beta)}(u,v)\frac{t^p}{p!},$$

$$\frac{\left(t^m{}_1\Psi_1\left[\begin{array}{c|c}(1,1) & \\ (\beta,\alpha) & t\end{array}\right]-1\right)^q}{q!} = \sum_{p=0}^{\infty}\mathbb{S}_{2,m}^{(\alpha,\beta)}(p,q)\frac{t^p}{p!},$$

and

$$\frac{\left(t^m H_{1,2}^{1,1}\left[\ -t \ \middle|\ \begin{array}{c}(0,1)\\(0,1),(1-\beta,\alpha)\end{array}\right]-1\right)^q}{q!}=\sum_{p=0}^{\infty}\mathbb{S}_{2,m}^{(\alpha,\beta)}(p,q)\frac{t^p}{p!}.$$

These relations reveal the essential connections between the newly introduced polynomials and numbers and the well-established functions ${}_p\Psi_q$ and $H_{r,s}^{m,n}$.

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