Filomat 39:17 (2025), 5825–5832
https://doi.org/10.2298/FIL2517825S

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Soft sets whose soft measure is zero

Nenad Stojanović[a]

[a]*Faculty of Science, University of Kragujevac, Radoja Domanovića 12, 34000 Kragujevac, Serbia*

Abstract. The concept of soft sets is widely used in reasoning when dealing with vague, incomplete, and imprecise information. Since the introduction of the soft set concept by Molodtsov, the theory of soft sets has become the subject of interest for many researchers who have defined new concepts such as soft measure, soft σ-algebra, and other concepts that are defined in classical measure theory. In the work [5], it was shown that with certain extensions, a soft measure can be constructed on the obtained soft σ-algebra based on soft premeasures on a soft semiring. This study demonstrates that the same conclusion can be reached using a simpler and more natural construction that does not involve the use of a soft exterior measure but rather considers soft sets whose soft measure is zero.

1. Introduction

The field of soft set theory has developed and found applications in all sciences dealing with vague, incomplete, and imprecise data. The concept of soft sets was introduced by Molodtsov in 1999 (see [10] or [11]) as an alternative approach to dealing with uncertainty and imprecision, which differed from all previously known approaches. Since 1999, the study and development of this mathematical tool have become more frequent and have progressed rapidly. Since its introduction, research in soft set theory has been expanded by numerous researchers in various disciplines. After defining the concept of soft sets as a mathematical object, it is logical to define certain operations among such sets. The most well-known operations with soft sets, as well as their properties, are defined in works such as [1] and [7]. To fill gaps and overcome shortcomings in defining operations on soft sets, works such as [1], [2], [13], [17], [18], [20] also contribute to soft set theory in this context and significantly enrich the overall theory of soft sets.

If we consider the historical development and the number of works and researchers studying structures based on soft sets, we can conclude that soft set theory is rapidly evolving and growing, with new developments and applications regularly explored and proposed. It remains an important and active area of research with the potential for significant impact in many disciplines, especially when integrating the concept of soft sets with another concept. In the work [9], Samanta and Majumdar introduced the concept of soft mappings, and in the work [14], Riaz and Naim studied measurable soft mappings and some possible applications of soft set theory. The concept of soft σ-algebras and their basic properties have been studied

2020 *Mathematics Subject Classification*. Primary 06D72.
Keywords. soft set, soft σ-algebra, soft measure, $\widetilde{\mu}$-null soft set, soft completion
Received: 08 April 2024; Accepted: 04 April 2025
Communicated by Hari M. Srivastava
This work was supported by the Serbian Ministry of Education, Science and Technological Development (Agreement No. 451-03-137/2025-03/200122).
Email address: nenad.s@kg.ac.rs (Nenad Stojanović)
ORCID iD: https://orcid.org/0000-0003-3837-4043 (Nenad Stojanović)

in works such as [14] and [4]. The subject of study in the work [15] is soft measures and soft outer measures, while Stojanović and Boričić Joksimović defined soft content and soft premeasure in the work [19]. In the work [5], it was shown that a soft measure on a soft σ-algebra, with certain extensions, can be obtained by upgrading a mapping that is not a soft measure on a structure that is not a soft σ-algebra. In other words, it was demonstrated that starting from soft premeasures defined on a soft semiring, a soft measure can be constructed on the resulting soft σ-algebra.

In addition to introductory discussions and mentioning basic concepts and significant results from soft set theory, this paper demonstrates that with certain extensions, a soft measure can be constructed on the obtained soft σ-algebra based on soft premeasures given on a soft semiring. Section 3 outlines a simpler and more natural construction that does not involve the use of a soft exterior measure but rather considers soft sets whose soft measure is zero.

2. Preliminaries

Let X be an initial universe set and E_X be the set of all possible parameters under consideration with respect to X. The power set of X is denoted by $\mathcal{P}(X)$ and A is a subset of E. Usually, parameters are attributes, characteristics, or properties of objects in X. In what follows, E_X (simply denoted by E) always means the universe set of parameters with respect to X, unless otherwise specified.

Definition 2.1. *[10] A pair (F, A) is called a soft set over X where $A \subseteq E$ and $F : A \to \mathcal{P}(X)$ is a set valued mapping. In other words, a soft set over X is a parameterized family of subsets of the universe X. For all $e \in A$, $F(e)$ may be considered as the set of e-approximate elements of the soft set (F, A). It is worth noting that $F(e)$ may be arbitrary. Some of them may be empty, and some may have nonempty intersection.*

Definition 2.2. *[8] A soft set F_A on the universe X is defined by the set of ordered pairs $F_A = \{(e, f_A(e)) \mid e \in E, f_A(e) \in \mathcal{P}(X)\}$, where $f_A : E \to \mathcal{P}(X)$, such that $f_A(e) \neq \emptyset$ if $e \in A \subseteq E$ and $f_A(e) = \emptyset$, if $e \notin A$. Here, f_A is called an approximate function of the soft set F_A. The value of $f_A(e)$ may be arbitrary.*

Note that the set of all soft sets over X will be denoted by $\mathcal{S}(X, E)$.

Definition 2.3. *[2] Let $F_A \in \mathcal{S}(X, E)$. If $f_A(e) = \emptyset$, for all $e \in E$, then F_A is called an empty soft set, denoted by F_Φ or Φ. $f_A(e) = \emptyset$ means that there is no element in X related to the parameter $e \in E$. Therefore, we do not display such elements in the soft sets, as it is meaningless to consider such parameters.*

Definition 2.4. *[2] Let $F_A \in \mathcal{S}(X, E)$. If $f_A(e) = X$, for all $e \in A$, then F_A is called an A-universal soft set, denoted by $F_{\widetilde{A}} = \widetilde{A}$. If $A = E$, then the A-universal soft set is called a universal soft set, denoted by $F_{\widetilde{E}} = \widetilde{E}$.*

Definition 2.5. *[16] Let Y be a nonempty subset of X, then $\widetilde{Y}$ denotes the soft set Y_E over X for which $Y(e) = Y$, for all $e \in E$. In particular, X_E will be denoted by $\widetilde{X}$.*

Two soft sets can be compared in the following way.

Definition 2.6. *[2] Let $F_A, G_B \in \mathcal{S}(X, E)$. Then,*

1. *F_A is a soft subset of G_B, denoted by $F_A \sqsubseteq G_B$, if $f_A(e) \subseteq g_B(e)$, for all $e \in E$.*
2. *F_A and G_B are soft equal, denoted by $F_A = G_B$, if and only if $f_A(e) = g_B(e)$, for all $e \in E$.*

Operations defined on soft sets, as well as their properties, have been the subject of study by many researchers.

Definition 2.7. *[2] Let $F_A, G_B \in \mathcal{S}(X, E)$. Then,*

1. *the soft union $F_A \sqcup G_B$ of F_A and G_B is defined by the approximate function $h_{A\cup B}(e) = f_A(e) \cup g_B(e)$, for all $e \in E$.*

2. *the soft intersection* $F_A \sqcap G_B$ *of* F_A *and* G_B *is defined by the approximate function* $h_{A\cap B}(e) = f_A(e) \cap g_B(e)$, *for all* $e \in E$.
3. *the soft difference* $F_A \backslash G_B$ *of* F_A *and* G_B *is defined by the approximate function* $h_{A\backslash B}(e) = f_A(e)\backslash g_B(e)$, *for all* $e \in E$.
4. *the soft complement* F_A^c *is defined by the approximate function* $f_{A^c}(e) = f_A^c(e)$, *where* $f_A^c(e)$ *is the complement of the set* $f_A(e)$, *i.e.* $f_A^c(e) = X\backslash f_A(e)$, *for all* $e \in E$.

Definition 2.8. *[22] Let I be an arbitrary index set and let* $\{(F_A)_i\}_{i\in I}$ *be a subfamily of* $\mathcal{S}(X,E)$.

- *The union of these soft sets is the soft set* G_C, *where* $g_C(e) = \cup_{i\in I}(F_A)_i(e)$, *for all* $e \in E$. *We write* $G_C = \sqcup_{i\in I}(F_A)_i$.
- *The intersection of these soft sets is the soft set* H_D, *where* $h_D(e) = \cap_{i\in I}(F_A)_i(e)$, *for all* $e \in E$. *We write* $H_D = \sqcap_{i\in I}(F_A)_i$.

As in the classical set theory, so in the soft set theory, a significant place takes the study of special collections of sets, i.e. a collection of sets with specific properties. For studing soft measure it is necessary to define collections of soft sets like soft algebra, soft σ-algebra, etc.

Definition 2.9. *[4] A collection* $\widetilde{\mathcal{A}}$ *of soft subsets of* $\widetilde{X}$ *is called a soft* σ*-algebra on* $\widetilde{X}$ *if and only if it satisfies the following conditions*

- $\Phi \in \widetilde{\mathcal{A}}$,
- *if* $F_A \in \widetilde{\mathcal{A}}$, *then* $F_A^c = \widetilde{X}\backslash F_A \in \widetilde{\mathcal{A}}$,
- *if* $(F_A)_1, (F_A)_2, (F_A)_3 \ldots$ *is a countable collection of soft sets in* $\widetilde{\mathcal{A}}$, *then* $\bigsqcup_{i=1}^{\infty}(F_A)_i \in \widetilde{\mathcal{A}}$.

  The pair $(\widetilde{X}, \widetilde{\mathcal{A}})$ *is called a soft measurable space and* $(F_A)_i \in \widetilde{\mathcal{A}}$ *is called a measurable soft set.*

As well as the classical set theory, the soft set theory defines the notion of measure [15], called soft measure.

Definition 2.10. *[15] Let* $\widetilde{\mathcal{A}}$ *be a soft* σ*-algebra of soft subsets of* $\widetilde{X}$ *and* $\widetilde{\mu}$ *be an extended soft real-valued mapping on* $\widetilde{\mathcal{A}}$. *Then* $\widetilde{\mu}$ *is called a soft measure on* $\widetilde{\mathcal{A}}$, *if*

- $\widetilde{\mu}(\Phi) = 0$,
- $\widetilde{\mu}(F_A) \geqslant 0$ *for each* $F_A \in \widetilde{\mathcal{A}}$,
- $\widetilde{\mu}$ *is countably soft additive, i.e.*

$$\widetilde{\mu}\left(\bigsqcup_{i=1}^{\infty}(F_A)_i\right) = \sum_{i=1}^{\infty}\widetilde{\mu}((F_A)_i),$$

  $(F_A)_i$*'s being pairwise soft disjoint.*

  If $\widetilde{\mu}$ *is a soft measure on a soft* σ*-algebra* $\widetilde{\mathcal{A}}$, *then the triplet* $(\widetilde{X}, \widetilde{\mathcal{A}}, \widetilde{\mu})$ *is called a soft measure space.*

Definitions of terms such as soft semiring, soft premeasures, soft content, soft outer measures, and so on can be found in articles [4], [5], [8], [9], [14], [15], [19].

3. Soft completion soft measure

In the article [19], it is shown how, starting from a soft outer measure, we can obtain a soft measure on the corresponding soft σ-algebra, i.e., Theorem 3.1. and Lemma 3.2. are proven.

Theorem 3.1. *[19] Let $\widetilde{\mu}^*$ be a soft outer measure. Define*

$$\mathcal{A}_{\widetilde{\mu}^*} = \{F_A \sqsubseteq \widetilde{X} \mid F_A \text{ is soft } \widetilde{\mu}^*\text{-measurable}\}.$$

Then $\mathcal{A}_{\widetilde{\mu}^*}$ is a soft σ-algebra and $\widetilde{\mu}^* \restriction_{\mathcal{A}_{\widetilde{\mu}^*}}$ is a soft measure.

Lemma 3.2. *[19] Let $\widetilde{\mu}^*$ be a soft outer measure and $F_A \in \mathcal{P}(\widetilde{X})$.*

1. *If $\widetilde{\mu}^*(F_A) = 0$ or $\widetilde{\mu}^*(F_A^c) = 0$, then F_A is soft $\widetilde{\mu}^*$-measurable.*
2. *F_A is soft $\widetilde{\mu}^*$-measurable if, and only if,*

$$(\forall Q_P \in \mathcal{P}(\widetilde{X}))\widetilde{\mu}^*(Q_P) = \widetilde{\mu}^*(Q_P \sqcap F_A) + \widetilde{\mu}^*(Q_P \sqcap F_A^c).$$

The main idea in the article [5] was to construct a soft measure by extending soft premeasures, and the main results in that work are Theorem 3.3. and Theorem 3.4. The mentioned extension of soft premeasures is done using a mapping called soft outer measure.

Theorem 3.3. *[5] Let $\widetilde{X}$ be a soft set, $\widetilde{\mathcal{S}} \sqsubseteq \mathcal{P}(\widetilde{X})$ with $\Phi \in \widetilde{\mathcal{S}}$, and $\widetilde{\mu} : \widetilde{\mathcal{S}} \to [0, \infty]$ with $\widetilde{\mu}(\Phi) = 0$. Then $\widetilde{\mu}^* : \mathcal{P}(\widetilde{X}) \to [0, \infty]$,*

$$\widetilde{\mu}^*(F_A) = \inf\left\{\sum_{i=1}^{\infty} \widetilde{\mu}((F_A)_i) \mid ((F_A)_i)_{i\in\mathbb{N}} \text{ sequence in } \widetilde{\mathcal{S}}, F_A \sqsubseteq \bigsqcup_{i=1}^{\infty}(F_A)_i\right\},\ \inf\emptyset = \infty \tag{1}$$

defines a soft outer measure on $\widetilde{X}$. This soft outer measure is called the soft outer measure induced by $\widetilde{\mu}$.

Theorem 3.4. *[5] (Soft Carathéodory extension Theorem) Let $\widetilde{\mathcal{S}}$ be a soft semiring on the soft set $\widetilde{X}$ and let $\widetilde{\mu} : \widetilde{\mathcal{S}} \to [0, \infty]$ be a soft content. If $\widetilde{\mu}^*$ is defined by (2), then the following holds:*

(a) $\widetilde{\mu}^*$ is a soft outer measure and each $F_A \in \widetilde{\mathcal{S}}$ is soft $\widetilde{\mu}^*$-measurable (i.e. $\widetilde{\mathcal{S}} \sqsubseteq \sigma(\widetilde{\mathcal{S}}) \sqsubseteq \mathcal{A}_{\widetilde{\mu}^*}$);

(b) If $\widetilde{\mu}$ is a soft premeasure, then $\widetilde{\mu}^* \restriction_{\widetilde{\mathcal{S}}} = \widetilde{\mu}$ (thus, in this case, $\widetilde{\mu}^* \restriction_{\mathcal{A}_{\widetilde{\mu}^*}}$ is a soft measure that extends the soft premeasure $\widetilde{\mu}$ to a soft σ-algebra containing $\sigma(\widetilde{\mathcal{S}})$);

(c) If $\widetilde{\mu}$ is not a soft premeasure, then $(\exists F_A \in \widetilde{\mathcal{S}})\widetilde{\mu}^*(F_A) < \widetilde{\mu}(F_A)$.

Soft sets of soft measure zero play a special role in soft measure theory, giving rise to the following definitions.

Definition 3.5. *Let $(\widetilde{X}, \widetilde{\mathcal{A}})$ be a soft measurable space and let $\widetilde{\mu} : \widetilde{\mathcal{A}} \to [0, \infty]$ be a soft measure.*

(a) A soft set N_O is called a $\widetilde{\mu}$-null soft set (or just a null soft set if $\widetilde{\mu}$ is understood) if, and only if, $\widetilde{\mu}(N_O) = 0$.

(b) $(\widetilde{X}, \widetilde{\mathcal{A}})$ and $\widetilde{\mu}$ are called soft complete if, and only if, each soft subset of a $\widetilde{\mu}$-null soft set is soft $\widetilde{\mu}$-measurable, i.e. if, and only if,

$$(\forall F_A \in \widetilde{\mathcal{A}})(\forall G_B \sqsubseteq F_A)(\widetilde{\mu}(F_A) = 0 \Rightarrow G_B \in \widetilde{\mathcal{A}}).$$

Let $(\widetilde{X}, \widetilde{\mathcal{A}})$ be a soft measurable space, and let $\widetilde{\mu} : \widetilde{\mathcal{A}} \to [0, \infty]$ be a soft measure. We can use the soft measure $\widetilde{\mu}$ in the conditions of Theorem 3.4. as soft content to obtain the extension $\widetilde{\mu}^* \upharpoonright_{\mathcal{A}_{\widetilde{\mu}^*}}$. In accordance with Lemma 3.2., we have that this extension $\widetilde{\mu}^* \upharpoonright \mathcal{A}_{\widetilde{\mu}^*}$ is soft complete. In this way, we can extend any soft measure to a soft complete soft measure.

The same conclusion can be obtained using a simpler and more natural construction that does not involve the use of a soft outer measure, in accordance with proven Theorem 3.6. Namely, if $\widetilde{\mu}$ is soft σ-finite, then both mentioned constructions yield the same soft complete extension (the so-called soft completion of $\widetilde{\mu}$), and this can be verified by considering Theorem 3.8. below.

Theorem 3.6. *Let $(\widetilde{X}, \widetilde{\mathcal{A}})$ be a soft measurable space and let $\widetilde{\mu} : \widetilde{\mathcal{A}} \to [0, \infty]$ be a soft measure. Define $\mathcal{N}$ to be the set consisting of all soft subsets of $\widetilde{\mu}$-null soft sets, i.e.*

$$\mathcal{N} = \left\{ N_O \sqsubseteq \widetilde{X} \mid (\exists F_A \in \widetilde{\mathcal{A}})(N_O \sqsubseteq F_A \Rightarrow (\widetilde{\mu}) = 0) \right\}.$$

Also define

$$\widetilde{\mathcal{A}}' = \{F_A \sqcup N_O \mid F_A \in \widetilde{\mathcal{A}}, N_O \in \mathcal{N}\}$$

$$\widetilde{\mu}' : \widetilde{\mathcal{A}}' \to [0, \infty], \;\; \widetilde{\mu}'(F_A \sqcup N_O) = \widetilde{\mu}(F_A) \;\; for \;\; F_A \in \widetilde{\mathcal{A}}, N_O \in \mathcal{N}.$$

Then the following holds:

(a) $\widetilde{\mathcal{A}}'$ is a soft σ-algebra, $\widetilde{\mu}'$ is well-defined, and $(\widetilde{X}, \widetilde{\mathcal{A}}'$ and $\widetilde{\mu}'$ are soft complete. Moreover, $\widetilde{\mu}'$ is the only soft content on $\widetilde{\mathcal{A}}'$ that is an extension of $\widetilde{\mu}$.

(b) Each soft complete soft measure $\widetilde{\nu}$ that extends $\widetilde{\mu}$ must also extend $\widetilde{\mu}'$ (i.e. $\widetilde{\mu}'$ is the "smallest" soft complete extension of $\widetilde{\mu}$).

Proof. (a) We verify $\widetilde{\mathcal{A}}'$ to be a soft σ-algebra: $\Phi = \Phi \sqcup \Phi \in \widetilde{\mathcal{A}}'$; if $F_A \in \widetilde{\mathcal{A}}$, $N_O \in \mathcal{N}$, then there exists $G_B \in \widetilde{\mathcal{A}}$ such that $N_O \sqsubseteq G_B$, $\widetilde{\mu}(G_B) = 0$. Then we compute

$$(F_A \sqcup N_O)^c = F_A^c \sqcap N_O^c = F_A^c \sqcap (H_C^c \sqcup (H_C \sqcap N_O^c)) = \underbrace{(F_A^c \sqcap H_C^c)}_{\in \widetilde{\mathcal{A}}} \sqcup \underbrace{(F_A^c \sqcap H_C \sqcap N_O^c)}_{\sqsubseteq H_C} \in \widetilde{\mathcal{A}}',$$

showing $(F_A \sqcup N_O)^c \in \widetilde{\mathcal{A}}'$.

If $(F_A)_n$, $n \in \mathbb{N}$ is a sequence in $\widetilde{\mathcal{A}}$ and $(N_O)_n$, $n \in \mathbb{N}$ is a sequence in $\mathcal{N}$, then, for each $n \in \mathbb{N}$, there exists $(G_B)_n \in \widetilde{\mathcal{A}}$ such that $(N_O)_n \sqsubseteq (G_B)_n$, $\widetilde{\mu}((G_B)_n) = 0$. Then $F_A = \bigsqcup_{n=1}^{+\infty} (F_A)_n \in \widetilde{\mathcal{A}}$, $G_B = \bigsqcup_{n=1}^{+\infty} (G_B)_n \in \widetilde{\mathcal{A}}$, $\widetilde{\mu}(G_B) = 0$, and $N_O = \bigsqcup_{n=1}^{+\infty} (N_O)_n \sqsubseteq G_B$ showing $\bigsqcup_{n=1}^{+\infty} ((F_A)_n \sqcup (N_O)_n) = F_A \sqcup N_O \in \widetilde{\mathcal{A}}$, proving $\widetilde{\mathcal{A}}'$ to be a soft σ-algebra.

Next, we show $\widetilde{\mu}'$ to be well-defined: If $(F_A)_1 \sqcup (N_O)_1 = (F_A)_2 \sqcup (N_O)_2$ with $(F_A)_1, (F_A)_2 \in \widetilde{\mathcal{A}}$ and $(N_O)_1, (N_O)_2 \in \mathcal{N}$, then there exists $G_B \in \widetilde{\mathcal{A}}$ with $(N_O)_2 \sqsubseteq G_B$, $\widetilde{\mu}(G_B) = 0$, implying $(F_A)_1 \sqsubseteq (F_A)_1 \sqcup (N_O)_1 \sqsubseteq (F_A)_2 \sqcup G_B$ and $\widetilde{\mu}((F_A)_1) \leqslant \widetilde{\mu}((F_A)_2) + \widetilde{\mu}(G_B) = \widetilde{\mu}((F_A)_2)$. Analogously, one obtains $\widetilde{\mu}((F_A)_2) \leqslant \widetilde{\mu}((F_A)_1)$, i.e. $\widetilde{\mu}((F_A)_1) = \widetilde{\mu}((F_A)_2)$ and $\widetilde{\mu}'$ is well-defined.

Choosing $N_O = \Phi$ proves $\widetilde{\mu}'$ to be an extension of $\widetilde{\mu}$. To verify soft σ-additivity of $\widetilde{\mu}'$, let $(F_A)_n$, $n \in \mathbb{N}$ be a sequence of disjoint soft sets in $\widetilde{\mathcal{A}}$ and $(N_O)_n$, $n \in \mathbb{N}$ a sequence of disjoint soft sets in $\mathcal{N}$. Then

$$\widetilde{\mu}' \left(\bigsqcup_{n=1}^{+\infty} ((F_A)_n \sqcup (N_O)_n) \right) = \widetilde{\mu}' \left(\left(\bigsqcup_{n=1}^{+\infty} (F_A)_n \right) \sqcup \left(\bigsqcup_{n=1}^{+\infty} (N_O)_n \right) \right) = \widetilde{\mu} \left(\bigsqcup_{n=1}^{+\infty} (F_A)_n \right) = \sum_{n=1}^{+\infty} \widetilde{\mu}((F_A)_n) = \sum_{n=1}^{+\infty} \widetilde{\mu}((F_A)_n \sqcup (N_O)_n),$$

i.e. $\widetilde{\mu}'$ is soft σ-additive and, thus, a soft measure.

To see that $\widetilde{\mu}'$ is soft complete, let $H_C \sqsubseteq F_A \sqcup N_O$ with $F_A \in \widetilde{\mathcal{A}}$, $N_O \in \mathcal{N}$, $\widetilde{\mu}(F_A) = 0$. Then there exits $G_B \in \widetilde{\mathcal{A}}$ such that $\widetilde{\mu}(G_B) = 0$. Then $H_C \sqsubseteq F_A \sqcup G_B \in \widetilde{\mathcal{A}}$ with $\widetilde{\mu}(F_A \sqcup G_B) = 0$. Thus, $H_C \in \mathcal{N} \sqsubseteq \widetilde{\mathcal{A}}'$, proving $\widetilde{\mu}'$ to be soft complete.

Let $\widetilde{\rho}$ be a soft content on $\widetilde{\mathcal{A}}'$ that is an extension of $\widetilde{\mu}$. Then $\widetilde{\rho}(F_A) = \widetilde{\mu}(F_A)$ for each $F_A \in \widetilde{\mathcal{A}}$ and, thus, $\widetilde{\rho}(N_O) = 0$ for each $N_O \in \mathcal{N}$. Thus, if $F_A \in \widetilde{\mathcal{A}}$ and $N_O \in \mathcal{N}$, then, by soft monotonicity and soft subadditivity of $\widetilde{\rho}$,

$$\widetilde{\mu}(F_A) = \widetilde{\rho}(F_A) \leqslant \widetilde{\rho}(F_A \sqcup N_O) \leqslant \widetilde{\rho}(F_A) \sqcup \widetilde{\rho}(N_O) = \widetilde{\mu}(F_A),$$

showing $\widetilde{\rho}(F_A \sqcup N_O) = \widetilde{\mu}(F_A)$ and $\widetilde{\rho} = \widetilde{\mu}'$ as claimed.

(b) Let $(\widetilde{X}, \widetilde{\mathcal{B}}, \widetilde{\nu})$ a soft complete soft measure space such that $\widetilde{\mathcal{A}} \subseteq \widetilde{\mathcal{B}}$ and $\widetilde{\nu} \upharpoonright_{\widetilde{\mathcal{A}}} = \widetilde{\mu}$. As $\widetilde{\nu}$ is soft complete, one has $\mathcal{N} \subseteq \widetilde{\mathcal{B}}$. As $\widetilde{\nu}$ extends$\widetilde{\mu}$, one has $\widetilde{\nu}(N_O) = 0$ for each $N_O \in \mathcal{N}$. Then $\widetilde{\mathcal{A}}' \subseteq \widetilde{\mathcal{B}}$ and $\widetilde{\nu} \upharpoonright_{\widetilde{\mathcal{A}}'} = \widetilde{\mu}'$ follows from the last part of (a). □

Definition 3.7. *Let $(\widetilde{X}, \widetilde{\mathcal{A}})$ be a soft measurable space and let $\widetilde{\mu} : \widetilde{\mathcal{A}} \to [0, \infty]$ be a soft measure. Then the soft measure space $(\widetilde{X}, \widetilde{\mathcal{A}}', \widetilde{\mu}')$, where $\widetilde{\mathcal{A}}'$ and $\widetilde{\mu}'$ are defined as in Theorem 3.6, is called the soft completion of $(\widetilde{X}, \widetilde{\mathcal{A}}, \widetilde{\mu})$; $\widetilde{\mu}'$ is called the soft completion of $\widetilde{\mu}$. According to Theorem 3.6. (a),(b), the soft completion of $\widetilde{\mu}$ is the smallest soft complete soft measure that extends $\widetilde{\mu}$.*

Theorem 3.8. *Let $\mathcal{S}$ be a soft semiring on the set $\widetilde{X}$, let $\widetilde{\mu} : \mathcal{S} \to [0, \infty]$ be a σ-finite soft premeasure and $\widetilde{\mu}^* : \mathcal{P}(\widetilde{X}) \to [0, \infty]$ the induced soft outer measure defined by (1) in Theorem 3.3. Then $\widetilde{\mu}^* \upharpoonright_{\widetilde{\mathcal{A}}_{\widetilde{\mu}^*}}$ is the soft completion of $\widetilde{\mu}^* \upharpoonright_{\sigma(\mathcal{S})}$ and, in particular, this is the unique extension of $\widetilde{\mu}$ to a soft measure on $\widetilde{\mathcal{A}}_{\widetilde{\mu}^*}$.*

Proof. As we know $\widetilde{\mu}^* \upharpoonright_{\widetilde{\mathcal{A}}_{\widetilde{\mu}^*}}$ to be soft complete, it remains to prove $\widetilde{\mathcal{A}}_{\widetilde{\mu}^*} \subseteq \widetilde{\sigma(\mathcal{S})}$. To this end, let $G_B \in \widetilde{\mathcal{A}}_{\widetilde{\mu}^*}$. First, we assume $\widetilde{\mu}^*(G_B) < +\infty$. Then, for each $k \in \mathbb{N}$, there exists a sequence $(F_A)_{n_k}$ of soft sets in $\mathcal{S}$ such that $G_B \sqsubseteq \bigsqcup_{n=1}^{+\infty}(F_A)_{n_k}$ and $\sum_{n=1}^{+\infty} \widetilde{\mu}((F_A)_{n_k}) \leqslant \widetilde{\mu}^*(G_B) + \frac{1}{k}$. Defining

$$F_A = \bigsqcap_{k=1}^{+\infty} \bigsqcup_{n=1}^{+\infty} (F_A)_{n_k},$$

we obtain $F_A \in \sigma(\mathcal{S})$, $G_B \sqsubseteq F_A$, $(\forall n \in \mathbb{N})\widetilde{\mu}^*(F_A) \leqslant \widetilde{\mu}^*(G_B) + \frac{1}{n}$.

Thus,$\widetilde{\mu}^*(F_A) = \widetilde{\mu}^*(G_B)$. We can now apply what we have just shown to $F_A \setminus G_B$ instead of G_B, proving the existence of $H_C \in \sigma(\mathcal{S})$ such that $F_A \setminus G_B \sqsubseteq H_C$ and $\widetilde{\mu}^*(H_C) = \widetilde{\mu}^*(F_A \setminus G_B) = \widetilde{\mu}^*(F_A) - \widetilde{\mu}^*(G_B) = 0$. Then $G_B = (F_A \setminus H_C) \sqcup (G_B \sqcap H_C) \in \widetilde{\sigma(\mathcal{S})}$, since $F_A \setminus H_C \in \sigma(\mathcal{S})$ and $G_B \sqcap H_C$ is a soft subset of the $(\widetilde{\mu}^* \upharpoonright_{\sigma(\mathcal{S})})$-null soft set H_C.

It remains to consider the case $G_B \in \widetilde{\mathcal{A}}_{\widetilde{\mu}^*}$ with $\widetilde{\mu}^*(G_B) = +\infty$. Since $\widetilde{\mu}$ is assumed to be soft σ-finite, there exists a sequence $(I_D)_n$, $n \in \mathbb{N}$ in $\mathcal{S}$ such that $\widetilde{X} = \bigsqcup_{n=1}^{+\infty}(I_D)_n$ and $\widetilde{\mu}((I_D)_n) < +\infty$ for each $n \in \mathbb{N}$. Since, for each $n \in \mathbb{N}$, $\widetilde{\mu}^*(G_B \sqcap (I_D)_n) < +\infty$ implies $G_B \sqcap (I_D)_n \in \widetilde{\sigma(\mathcal{S})}$, we obtain $G_B = \bigsqcup_{n=1}^{+\infty}(G_B \sqcap (I_D)_n) \in \widetilde{\sigma(\mathcal{S})}$ as needed.

According to Theorem 3.6. (a), $\widetilde{\mu}^* \upharpoonright_{\widetilde{\mathcal{A}}_{\widetilde{\mu}^*}}$ is the only soft measure on $\widetilde{\mathcal{A}}_{\widetilde{\mu}^*}$ that extends $\widetilde{\mu}^* \upharpoonright_{\sigma(\mathcal{S})}$. Since, by Theorem 3.14. from [12], $\widetilde{\mu}^* \upharpoonright_{\sigma(\mathcal{S})}$ is the only soft measure on $\sigma(\mathcal{S})$ that extends $\widetilde{\mu}$, $\widetilde{\mu}^* \upharpoonright_{\widetilde{\mathcal{A}}_{\widetilde{\mu}^*}}$ is the only soft measure on $\widetilde{\mathcal{A}}_{\widetilde{\mu}^*}$ that extends $\widetilde{\mu}$. □

Example 3.9. *Let $X = \{h_1, h_2, h_3\}$ and $E = \{e_1, e_2\}$. Collection $\mathcal{S} = \{(F_A)_1, (F_A)_2, (F_A)_3, (F_A)_4\}$, where*
$(F_A)_1 = \Phi$,
$(F_A)_2 = \{(e_1, \{h_1\}), (e_2, \{h_2\})\}$,
$(F_A)_3 = \{(e_1, \{h_2\}), (e_2, \{h_1\})\}$ i
$(F_A)_4 = \{(e_1, \{h_3\}), (e_2, \{h_3\})\}$,
is one soft semiring on $\widetilde{X}$.
Notice the mapping $\widetilde{\mu} : \mathcal{S} \to [0, \infty]$ be defined as

$$\widetilde{\mu}(F_A) = \begin{cases} 0, & F_A = \Phi, \\ 1, & F_A \neq \Phi. \end{cases}$$

It is clear that the mapping $\widetilde{\mu}$ is a soft content, and also a soft premeasure on soft semiring $\mathcal{S}$. Now, the soft semirnig $\mathcal{S}$ can be extanded to a minimal soft σ-algebra

$$\{(F_A)_1, (F_A)_2, (F_A)_3, (F_A)_4, (F_A)_5, (F_A)_6, (F_A)_7, (F_A)_8\},$$

where
$(F_A)_1 = \Phi$,
$(F_A)_2 = \{(e_1, \{h_1\}), (e_2, \{h_2\})\}$,
$(F_A)_3 = \{(e_1, \{h_2\}), (e_2, \{h_1\})\}$,
$(F_A)_4 = \{(e_1, \{h_3\}), (e_2, \{h_3\})\}$,
$(F_A)_5 = \{(e_1, \{h_2, h_3\}), (e_2, \{h_1, h_3\})\}$,
$(F_A)_6 = \{(e_1, \{h_1, h_3\}), (e_2, \{h_2, h_3\})\}$,
$(F_A)_7 = \{(e_1, \{h_1, h_2\}), (e_2, \{h_1, h_2\})\}$ *and*
$(F_A)_8 = \widetilde{X}$.
Let $\widetilde{\mu}^*$ be a mapping defined as in Theorem 3.3. It is clear, that it is
$\widetilde{\mu}^*((F_A)_1) = 0$, $\widetilde{\mu}^*((F_A)_2) = 1$, $\widetilde{\mu}^*((F_A)_3) = 1$, $\widetilde{\mu}^*((F_A)_4) = 1$,
$\widetilde{\mu}^*((F_A)_5) = 2$, $\widetilde{\mu}^*((F_A)_6) = 2$, $\widetilde{\mu}^*((F_A)_7) = 2$ *i* $\widetilde{\mu}^*((F_A)_8) = 3$.
Hence, $\widetilde{\mu}^* \restriction_{\mathcal{A}_{\widetilde{\mu}^*}}$ is the soft completion of $\widetilde{\mu}^* \restriction_{\sigma(\mathcal{S})}$.

4. Conclusion

In addition to introducing the concept of soft sets, Molodtsov also presented several possible applications of that theory [10] in his research. One of the many aspects of applying soft set theory is introducing the concept of soft measures, and accordingly, this paper represents a continuation of research in the field of soft measure theory. Another way to obtain a soft measure from a mapping that is not a soft measure through certain extensions is demonstrated in this paper. New concepts such as $\widetilde{\mu}$-null soft set and soft completion of measures are introduced in line with the mentioned problematics.

References

[1] M. I. Ali, F. Feng, X. Y. Liu, W. K. Min, M. Shabir, On some new operations in soft set theory, Computers and Mathematics with Applications 57 (9) (2009) 1547–1553.
[2] N. Çağman, S. Enginoglu, Soft set theory and uni-int decision making, Eur. J. Oper. Res. 207 (2010) 848–855.
[3] D. Chen, E. C. C. Tsang, D. S. Yueng, X. Wang, The parametrization reduction of soft sets and its applications, Computers and Mathematics with Applications 49 (2005) 757–763.
[4] A. Z. Khameneh, A. Kilicman, On soft σ–algebras, Malaysian Journal of Mathematical Science 7 (1) (2013) 17–29.
[5] M. Laković, N. Vučićević, N. Stojanović, Soft Carathéodory extension Theorem and Soft Outer Measure, Filomat 37 (17) (2023) 5581–5589.
[6] P. K. Maji, R. Biswas, A. R. Roy, An application of soft sets in decision making problem, Computer and Mathematics with Applications 44 (2002) 1077–1083.
[7] P. K. Maji, R. Biswas, A. R. Roy, Soft set theory, Computers and Mathematics with Applications 45 (2003) 555–562.
[8] P. Majumdar, S.K. Samanta, Similarity measure of soft set. New Math. Nat. Comput. 4 (1) (2008) 1–12.
[9] P. Majumdar, S. K. Samanta, On soft mappings, Computer and Mathematics with Applications 60 (2010) 2666–2672.
[10] D. Molodtsov, Soft set theory-first results, Computers and Mathematics with Applications 37 (1999) 19–31.
[11] D. Molodtsov, The theory of soft sets, URSS Publishers, Moscow (in Russian) (2004).
[12] Lj. Mudrić-Staniškovski, M. Laković, N. Stojanović, A soft Dynkin system, Filomat 38 (1) (2024) 217–226.
[13] D. Pei, D. Miao, From soft sets to information systems, in: X. Hu, Q. Liu, A. Skowron, T.Y. Lin, R.R. Yager, B. Zhang (Eds.), Proceedings of Granular Computing, IEEE (2) (2005) 617–621.
[14] M. Riaz, K. Naeem, Measurable Soft Mappings, Punjab University Journal of mathematics 48 (2) (2016) 19–34.
[15] M. Riaz, K. Naeem, M. Ozair. Ahmad, Novel concepts of soft sets with applications, Annals of Fuzzy Mathematics and Informatics 13 (2) (2017) 239–251.
[16] M. Shabir, M. Naz, On soft topological spaces, Computers and Mathematics with Applications 61 (2011) 1786–1799.
[17] A. Sezgin, S. Ahmad, A. Mehmood, A New Operation on Soft Sets: Extended Difference of Soft Sets, Journal of New Theory 27 (2019) 33–42.
[18] A. Sezgin, A. O. Atagun, On operations of soft sets, Comput. Math. Appl. 61 (2011) 1457–1467.
[19] N. Stojanović, M. Boričić Joksimović, Soft Outer Measure and Soft Premeasure, Filomat 36 (6) (2022) 2129–2141.
[20] N. Stojanović, A new operation on soft sets: extended symmetric difference of soft sets, Vojnotehnički glasnik/Military Technical Courier 69 (4) (2021) 779–791.

[21] C.F. Yang, A note on soft set theory, Comput. Math. Appl. 56 (2008) 1899–1900. [Comput. Math. Appl. 45 (4-5) (2003) 555–562].
[22] I. Zorlutuna, M. Akdag, W.K. Min, S. Atmaca, Remarks on soft topological spaces. Ann. Fuzzy Math. Inf. 3 (2) (2012) 171–185.