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Approximate integrals and approximate variational measures

Surinder Pal Singh Kainth[a], **Narinder Singh**[b,*]

[a]*Department of Mathematics, Panjab University, Chandigarh, India*
[b]*Department of Mathematics, Lovely Professional University, Phagwara, Punjab, India*

Abstract. We establish that the approximate McShane integral is equivalent to the McShane integral and hence the Lebesgue integral for real functions. Then we consider a function V_F^{ap} analogous to V_F, for which some generalizations of the Hake's theorem can be established. We also provide a measure theoretic characterization for the approximate Henstock integral, in terms of V_F^{ap}. We prove that $V_F^{ap}(E) = V_F(E)$, for all measurable subsets E of the domain of F, if F is the primitive of an HK-integrable function. Some examples, when these two differ, are also provided.

1. Introduction

The notion of approximate derivative has been first considered by A. Khintchine in 1916. In [4], the approximately continuous Perron integral (AP-integral) was introduced, in terms of ap-major and ap-minor functions. Analogously we have approximate Denjoy and approximate Henstock integrals, which are known to be equivalent (see [13, 14, 17]). For the descriptive definitions and convergence theorems for approximate integrals, see [3, 5, 7, 10, 11, 17]. In [14, Theorem 2.8.], an analogue of the Hake's theorem for the approximate Denjoy integral is presented. A survey on approximately continuous integrals appeared in [18], which also presents some open problems in this direction.

In this paper, we discuss the approximate McShane integral, analogous to the approximate Henstock-Kurzweil (HK) integral, and establish its equivalence to the McShane integral and, consequently, to the Lebesgue integral for real functions.

The variational measure V_F is used for characterizations of primitives of HK-integrable functions. A function $F : [a, b] \longrightarrow \mathbb{R}$ is primitive of some HK-integrable function if and only if V_F generated by F is absolutely continuous with respect to the Lebesgue measure [2, Theorem 3]. The variational measure V_F and variational measure with respect to measurable gauges (V_F^m) are equal for primitives of HK-integrable functions (see [12]). Motivated by this, we consider a function V_F^{ap} analogous to V_F, for which some generalizations of Hake's theorem can be established. We also provide a measure-theoretic characterization for the approximate Henstock integral in terms of V_F^{ap}. We prove that $V_F^{ap}(E) = V_F(E)$, for all measurable subsets E of the domain of F, if F is the primitive of an HK-integrable function. An example when these two differ, is also provided.

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* Corresponding author: Narinder Singh
Email addresses: sps@pu.ac.in (Surinder Pal Singh Kainth), narinder2467singh@gmail.com (Narinder Singh)
ORCID iDs: https://orcid.org/0000-0002-2469-265X (Surinder Pal Singh Kainth),
https://orcid.org/0000-0002-0032-9514 (Narinder Singh)

2. Preliminaries

We follow the notations of [6, Chapter 16]. Let $\mathbb{R}$ denote the set of real numbers, μ be the Lebesgue measure on $\mathbb{R}$, $[a,b]$ be a compact real interval, E be a Lebesgue measurable subset of $[a,b]$, and A be an arbitrary subset of $[a,b]$, A real $x \in \mathbb{R}$ is said to be the *density point* of E if

$$\lim_{h\to 0^+} \frac{\mu(E \cap (x-h, x+h))}{2h} = 1.$$

Let E^d denote the set of density points of E which belong to E.

A collection Δ of point-interval pairs (t, I) such that $t \in [a,b]$ and $I \subset [a,b]$ is said to be an *approximate full cover* of $A \subset [a,b]$ if for every $x \in A$, there exists a measurable set $S_x \subset [a,b]$ such that

$$x \in S_x^d, \text{ and } (x, [c,d]) \in \Delta \text{ if and only if } c, d \in S_x.$$

The collection $\{S_x : x \in A\}$ will be termed as the *selection* on A, generated by Δ. If $B \subset A$, we write $\Delta_B := \{(t,I) \in \Delta : t \in B\}$.

A finite family $\mathcal{P} := \{(x_i, I_i) : 1 \le i \le m\}$ of point interval pairs is said to be

1. a *partial M-division* in $[a,b]$ if I_i's are pairwise non-overlapping subintervals of $[a,b]$ such that $\bigcup_{i=1}^m I_i \subset [a,b]$ and $x_i \in [a,b]$ for all $1 \le i \le m$.
2. a *partial division* in $[a,b]$ if it is an partial M-division and $x_i \in I_i$ for all $1 \le i \le m$.
3. a *division (an M-division)* of $[a,b]$ if it is a partial division (partial M-division) such that $\bigcup_{i=1}^m I_i = [a,b]$.
4. *A-anchored* if $x_i \in A$, for each i.

Throughout this paper, let $\mathcal{P}$ denote the above collection and $f : [a,b] \longrightarrow \mathbb{R}$. Define

$$S(\mathcal{P}, f) := \sum_{i=1}^m f(x_i)\mu(I_i).$$

The function f is said to be *approximate Henstock integrable* (or simply *AH*-integrable) if there is some $A \in \mathbb{R}$ such that for every $\epsilon > 0$ there exists an approximate full cover Δ of $[a,b]$ such that $\left|S(\mathcal{P}, f) - A\right| < \epsilon$, for every division $\mathcal{P}$ of $[a,b]$ such that $\mathcal{P} \subset \Delta$.

In this case, the number A is called the approximate Henstock-Kurzweil integral of f over $[a,b]$ and will be denoted by $(AH)\int_a^b f d\mu$. Further, a function F on subintervals of $[a,b]$ will be called the *AH-primitive* of f if

$$F(J) = (AH)\int_J f d\mu, \text{ for every interval } J \subset [a,b].$$

Note that the AH-integral is well-defined only if for every approximate full cover Δ of $[a,b]$, there exists a division $\mathcal{P}$ of $[a,b]$ such that $\mathcal{P} \subset \Delta$. This is indeed true (see [6, Lemma 16.3, p.246]).

It is immediate that every Henstock integrable function is AH-integrable. However, the converse is not true (see [14, Example 2.5]). Various results for the Henstock integral, including the Saks-Henstock lemma, remain valid even for the AH-integral. For details, the reader is referred to [6, Theorem 16.18] and [17, 18]. The descriptive approach to the AH-integral requires the notion of approximate continuity and approximate differentiability. A function $f : [a,b] \longrightarrow \mathbb{R}$ is said to be

1. *approximately continuous* at $c \in [a,b]$ if there exists a measurable set $E \subset [a,b]$ such that $c \in E^d$ and $f|_E$ is continuous at c.
2. *approximately differentiable* at $c \in [a,b]$ if there exists a measurable set $E \subset [a,b]$ such that $c \in E^d$ and the limit
$$\lim_{x\in E, x\to c} \frac{f(x) - f(c)}{x - c}$$
exists. In this case, the above limit is denoted by $f'_{ap}(c)$.

Unless specified, for $F : [a, b] \longrightarrow \mathbb{R}$ and $[c, d] \subset [a, b]$, we write $F([c, d]) := F(d) - F(c)$. Given $E \subset [a, b]$ and an approximate full cover Δ on E, we define

$$V^{ap}(F, E, \Delta) := \sup_{\mathcal{P}} \sum_{i=1}^{n} |F(J_i)|,$$

where the supremum is taken over all E-anchored partial divisions $\mathcal{P} := \{(x_i, J_i) : 1 \leq i \leq n\}$ such that $\mathcal{P} \subset \Delta$. The *approximate variational measure* V_F^{ap} is defined as

$$V_F^{ap}(E) = \inf\{V^{ap}(F, E, \Delta) : \Delta \text{ is an approximate full cover on } E\}.$$

The function V_F^{ap} will be called *absolutely continuous* if $V_F^{ap}(E) = 0$, whenever $\mu(E) = 0$. In this case, we write $V_F^{ap} \ll \mu$. As in [16], it can be shown that for any $F : [a, b] \longrightarrow \mathbb{R}$, V_F^{ap} is an outer measure on $[a, b]$.

3. Approximate McShane Integral

Replacing divisions with M-divisions, in the definition of the approximate Henstock integral, we define *the approximate McShane integral (or AM-integral).* In this section, we show that the latter is equivalent to the Lebesgue integral for real functions.

First we establish that the approximately McShane integral is absolute. Our proof is based upon the idea in [15, Theorem 3.6.9]

Theorem 3.1. *If $f : [a, b] \longrightarrow \mathbb{R}$ is approximately McShane integrable, then so is $|f|$.*

Proof. Let $(AM)\int_a^b f$ denote the approximately McShane integral of f, $\epsilon > 0$ be given and Δ be an approximate full cover of $[a, b]$ for this ϵ, as per the requirement of McShane integrability. Let $Q_1 = \{(I_i, t_i) : i = 1, \ldots, m\}$ and $Q_2 = \{(J_j, s_j) : j = 1, \ldots, n\}$ be M-divisions of $[a, b]$ such that $Q_1, Q_2 \subset \Delta$. Define M-divisions P_1 and P_2 of $[a, b]$ as follows:

$$P_1 := \left\{ \begin{array}{l} (I_i \cap J_j, t_i) : \text{ if } f(t_i) \geq f(s_j) \\ (I_i \cap J_j, s_j) : \text{if } f(t_i) < f(s_j) \end{array} \; 1 \leq i \leq m, 1 \leq j \leq n \right\}$$
$$P_2 := \left\{ \begin{array}{l} (I_i \cap J_j, s_j) : \text{ if } f(t_i) \geq f(s_j) \\ (I_i \cap J_j, t_i) : \text{if } f(t_i) < f(s_j) \end{array} \; 1 \leq i \leq m, 1 \leq j \leq n \right\}$$

Using our hypothesis, it can be verified that both P_1 and P_2 are M-divisions of $[a, b]$ and $P_1, P_2 \subset \Delta$. Therefore

$$\begin{aligned} \sum_{i=1}^{m} \sum_{j=1}^{n} |f(t_i) - f(s_j)| \mu(I_i \cap J_j) &= |S(P_1, f) - S(P_2, f)| \\ &\leq \left| S(P_1, f) - (AM)\int_a^b f \right| + \left| (AM)\int_a^b f - S(P_2, f) \right| < 2\epsilon. \end{aligned}$$

Hence we observe that

$$\begin{aligned} \Big| S(Q_1, |f|) - S(Q_2, |f|) \Big| &= \Big| \sum_{i=1}^{m} |f(t_i)|\, \mu(I_i) - \sum_{j=1}^{n} |f(s_j)|\, \mu(J_j) \Big| \\ &= \Big| \sum_{i,j} (|f(t_i)| - |f(s_j)|) \mu(I_i \cap J_j) \Big| \leq \sum_{i,j} \Big| |f(t_i)| - |f(s_j)| \Big| \mu(I_i \cap J_j) \\ &\leq \sum_{i,j} \Big| f(t_i) - f(s_j) \Big| \mu(I_i \cap J_j) < 2\epsilon \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, the result is established using Cauchy criterion. □

Theorem 3.2. *Let $f : [a,b] \longrightarrow \mathbb{R}$. Then f is approximately McShane integrable if and only if f is Lebesgue integrable.*

Proof. The converse is immediate, as every Lebesgue integral is equivalent to the McShane integral, and every McShane integrable function is approximate McShane integrable.

For the necessity part, assume that f is approximately McShane integrable. By Theorem 3.1, $|f|$ is also AM-integrable, and hence AH-integrable. Now by [6, Theorem 16.14], f is Lebesgue measurable. Further applying [6, Theorem 16.15], $|f|$ is Lebesgue integrable. Hence f is Lebesgue integrable. □

4. A Characterization in terms of Variational Measures

In this section, we provide a characterization of AH-integrable functions in terms of the variational measure V_F^{ap}.

Theorem 4.1. *If $f : [a,b] \longrightarrow \mathbb{R}$ is AH-integrable function with primitive F, then $V_F^{ap} \ll \mu$.*

Proof. Pick any $E \subset [a,b]$ such that $\mu(E) = 0$. Let $g := f\chi_{[a,b]\setminus E}$. Clearly, g is also an AH-integrable function with primitive F.

Let $\epsilon > 0$ be given. We choose an approximate full cover Δ on $[a,b]$, by Saks-Henstock lemma, see [6, Lemma 16.9]. That is, for every partial division $\mathcal{P} := \{(t_j, J_j) : 1 \leq j \leq n\} \subset \Delta$, we have

$$\sum_{j=1}^{n} |F(J_j) - g(t_j)\mu(J_j)| < \epsilon.$$

In particular, if the above division $\mathcal{P}$ is anchored in E, we obtain $\sum_{j=1}^{n} |F(J_j)| < \epsilon$. Since $\epsilon > 0$ is arbitrary, we conclude $V_F^{ap}(E) = 0$. Hence the result. □

Next we provide a version of the fundamental theorem of calculus, in our setting.

Theorem 4.2. *Let F be approximately differentiable almost everywhere such that $V_F^{ap} \ll \mu$. Then F' is AH-integrable with primitive F.*

Proof. Let $E := \{x \in [a,b] : F'_{ap}$ does not exist at $x\}$ and $f := F'_{ap}\chi_{[a,b]\setminus E}$. Since $\mu(E) = 0$, it is enough to prove that f is AH-integrable with primitive F.

Let $\epsilon > 0$ be given. Since $V_F^{ap} \ll \mu$, there exists an approximate full cover Δ_0 of E such that $\sum_{i=1}^{m} |F(J_i)| < \epsilon/2$, for every E-anchored partial division $\{(y_i, J_i) : 1 \leq i \leq m\} \subset \Delta_0$. Now pick any $x \in [a,b] \setminus E$. Since $F'_{ap}(x) = f(x)$, there exists a measurable set A_x such that $x \in A_x^d$ and

$$\lim_{y \in A_x, y \to x} \frac{F(y) - F(x)}{y - x} = f(x).$$

Hence there exists a $\delta_x > 0$ such that for all $y \in A_x \cap (x - \delta_x, x + \delta_x)$, we have

$$|(F(y) - F(x)) - f(x)(y - x)| < \frac{\epsilon|y - x|}{2(b - a)}.$$

Now we define an approximate full cover Δ on $[a,b]$ as

$$\Delta := \Delta_0 \cup \{(x, [c,d]) : x \in [a,b] \setminus E; c, d \in A_x \cap (x - \delta_x, x + \delta_x)\}.$$

Let $\{(x_j, J_j) : 1 \leq j \leq n\} \subset \Delta$ be a division of $[a,b]$. Writing $\wedge := \{j : x_j \in E\}$, we obtain

$$\begin{aligned}\sum_{j=1}^{n} |F(J_j) - f(x_j)\mu(J_j)| &\leq \sum_{j \in \wedge} |F(J_j)| + \sum_{j \notin \wedge} |F(J_j) - f(x_j)\mu(J_j)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2\mu([a,b])} \sum_{j \notin \wedge} \mu(J_j) \leq \epsilon.\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, the result follows. □

Definition 4.3. *Let $F : [a, b] \longrightarrow \mathbb{R}$ and $A \subset [a, b]$. Then F is said to be*

1. *AC^{ap} on E if for every $\epsilon > 0$ there exists an approximate full cover Δ of E and $\eta > 0$ such that $\sum_{j=1}^{n} |F(I_j)| < \epsilon$, for every E-anchored partial division $\{(t_j, I_j) : 1 \leq j \leq n\}$ inside Δ satisfying $\sum_{j=1}^{n} \mu(I_j) < \eta$.*
2. *ACG^{ap} on $[a, b]$, if $[a, b] = \cup_{n=1}^{\infty} E_n$, for a sequence of measurable sets $\{E_n\}$ such that F is AC^{ap} on E_n, for all $n \in \mathbb{N}$.*

Theorem 4.4 (Theorem 16.18, [6]). *A function $f : [a, b] \longrightarrow \mathbb{R}$ is AH-integrable on $[a, b]$ if and only if there exists a function $F : [a, b] \longrightarrow \mathbb{R}$ such that $F'_{ap} = f$, almost everywhere on $[a, b]$ and F is ACG^{ap} on $[a, b]$.*

Theorem 4.5. *A function $f : [a, b] \longrightarrow \mathbb{R}$ is AH-integrable on $[a, b]$ if and only if there exists a function $F : [a, b] \longrightarrow \mathbb{R}$ such that $F'_{ap} = f$, almost everywhere on $[a, b]$ and $V_F^{ap} \ll \mu$ on $[a, b]$.*

Proof. Assume that f is *AH*-integrable on $[a, b]$ with primitive F. Applying Theorems 4.1 and 4.4, $V_F^{ap} \ll \mu$ and $F'_{ap} = f$, almost everywhere on $[a, b]$.

Conversely, suppose that $F : [a, b] \longrightarrow \mathbb{R}$ is a function such that $V_F^{ap} \ll \mu$ on $[a, b]$ and $F'_{ap} = f$, almost everywhere on $[a, b]$. By Theorem 4.2, f is *AH*-integrable on $[a, b]$ with primitive F. Hence the result. □

In [14, Theorem 2.8.], an analogue of the Hake's theorem for the approximate Denjoy integral (and thence its equivalent AH-integral) is presented. We have already provided some further generalizations of this theorem, in terms of the Henstock variational measure V_F (see [19, 20]). It can be verified that a result analogous to [20] can be obtained for the AH-integral too.

If V_F denote the standard Henstock variational measure, it is easy to see that $V_F^{ap}(E) \leq V_F(E)$. A strict inequality may hold here (see Examples 4.7-4.8). Readers interested in V_F are referred to chapter 5 of [9]. Below we recall a generalization of [1, p.104, Theorem 7.5], as in [8, Theorem 3.9].

Theorem 4.6. *If $f : [a, b] \longrightarrow \mathbb{R}$ is HK-integrable with primitive F, then*

$$V_F(E) = \int_E |f| d\mu \text{ for every measurable set } E \subset [a, b],$$

even if the right hand side is infinity. Here the integral on the right denotes the Lebesgue integral.

Analogously, one can conclude that if $f : [a, b] \longrightarrow \mathbb{R}$ is *AH*-integrable with primitive F, then $V_F^{ap}(E) = \int_E |f| d\mu$ for every measurable set $E \subset [a, b]$.

Consequently, if F is primitive of some *HK* integrable function, then $V_F^{ap}(E) = V_F(E)$ for every measurable set $E \subset [a, b]$.

In general, V_F qnd V_F^{ap} may differ. Consider the following example.

Example 4.7. *Let F denote the Dirichlet function on $[0, 1]$, defined as*

$$F(x) := \begin{cases} 1, & \text{if } x \in [0, 1] \cap \mathbb{Q} \\ 0, & \text{if } x \in [0, 1] \setminus \mathbb{Q}. \end{cases}$$

Write $E := [0, 1] \setminus \mathbb{Q}$. It is clear that for corresponding interval function $V_F(E) = \infty$. Consider the selection $\{S_x : x \in [0, 1] \setminus \mathbb{Q}\}$ defined as $S_x := E$ for all $x \in E$. Let Δ be an approximate full cover of E, w.r.t. this selection. Then $V^{ap}(F, E, \Delta) = 0$ which implies $V_F^{ap}(E) = 0$. Hence $V_F^{ap}(E) < V_F(E)$.

Further, every point of E is its density point and $\mu(E) = 1$. Therefore F is approximately continuous on E. Also note that $F'_{ap}(c) = 0$ for all $c \in E$. While F is not the primitive of its almost everywhere approximate derivative. Hence V_F^{ap} is not absolutely continuous. □

Note that the function in the above example is not a primitive of an AH-integrable function. Here we provide another example.

Example 4.8. *Let f be the AH-integrable function of [14, Example 2.5], which is not HK-integrable, and F be the AH-primitive of f. Then $V_F^{ap} \ll \mu$ and hence $V_F^{ap}(\{a\}) = 0$. However, F is not continuous at a and $V_F(\{a\}) = 1$. Hence $V_F^{ap}(\{a\}) < V_F(\{a\})$.* □

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