Filomat 39:17 (2025), 5839–5863
https://doi.org/10.2298/FIL2517839H

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

An affine scaling interior-point adaptive cubic regularization algorithm with line search filter technique for derivative-free nonlinear optimization subject to bounds

Lingyun He[a], Jueyu Wang[b,*], Detong Zhu[c]

[a]School of Software, Henan University of Science and Technology, Luoyang 471000, China
[b]School of Statistics and Mathematics, Shanghai Lixin University of Accounting and Finance, Shanghai 201209, China
[c]Mathematics and Science College, Shanghai Normal University, Shanghai 200234, China

Abstract. In this paper, we propose an adaptive cubic regularization method with line search filter technique for solving derivative-free bound constrained optimization using an interior affine scaling approach. The affine scaling interior-point cubic model is based on the quadratic interpolation model of the objective function. The new iteration is obtained by solving the adaptive cubic regularization algorithm with line search filter technique. The global convergence and local superlinear convergence rate of the proposed algorithm are established under some mild conditions. Finally, the numerical results are detailed to show the effectiveness of the proposed algorithm.

1. Introduction

In this paper, we consider the following minimization problem with the bound constraints:

$$\min_{x\in\mathbb{R}^n} \; f(x), \qquad \text{s.t.} \quad l \le x \le u, \tag{1}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a nonlinear function, sufficiently smooth, but its derivative information is unavailable or unreliable, $l \in \{\mathbb{R} \cup \{-\infty\}\}^n$, $u \in \{\mathbb{R} \cup \{+\infty\}\}^n$, $l < u$. We define the feasible set $\Omega = \{x : l \le x \le u\}$ and the strict interior $\mathrm{int}(\Omega) = \{x : l < x < u\}$.

Minimization problems for derivative-free nonlinear optimization with simple bound constraints form an important and common class in various circumstances. There are many useful and successful algorithms

2020 *Mathematics Subject Classification*. Primary 49M37; Secondary 65K05, 90C30, 90C55
Keywords. Cubic regularization, Affine scaling, Derivative-free optimization, Lagrange interpolation model, Line search filter, Convergence theory
Received: 07 November 2020; Revised: 30 March 2025; Accepted: 13 July 2025
Communicated by Predrag Stanimirović
Research supported by the National Natural Science Foundation Grant(11371253) of China, the Frontier Exploration Projects of Longmen Laboratory (No. LMQYTSKT034), Doctoral Scientific Research Foundation of Henan University of Science and Technology (Grant No.13480006).
* Corresponding author: Jueyu Wang
Email addresses: hly@haust.edu.cn (Lingyun He), shnu201005@hotmail.com (Jueyu Wang), dtzhu@shnu.edu.cn (Detong Zhu)
ORCID iDs: https://orcid.org/0000-0002-0475-7903 (Lingyun He), https://orcid.org/0000-0002-9872-9443 (Jueyu Wang)

for these type of optimizations (e.g.[10, 14, 22, 28]). These authors use different tools to solve this kind of optimizations. In order to increase the tools available for solving the derivative-free bound constrained optimization problems, we extent the cubic regularization algorithm with line search filter technique to solve the bound constrained optimization without derivatives based on a polynomial interpolation approach.

Recently, Cartis, et al. [2, 3] proposed an adaptive cubic regularization (ARC) method for solving the unconstrained optimization which has roots in earlier algorithmic proposals by Griewank in [20], Nesterov and Polyak [25] and Weiser et al. [30]. The excellent global and local convergence properties were obtained, and the numerical experiments with small-scale test problems from the CUTEr set showed encouraging performance of the ARC procedures when compared with a basic trust region methods for solving small scale problems. At each iteration, the objective function is locally replaced by a cubic approximation, in which third order Taylor's expansion is replaced by a cubic regularization term with an adaptive regularization parameter whose role is related to the local Lipschitz constant of the objective's Hessian. In [4], Cartis, et al. extended the cubic algorithms for unconstrained optimizations to finite-difference versions and yielded complexity bounds for first-order and derivative-free methods applied on the same problems class. In [5], the authors extended the bound of [2, 3] to nonlinear problems with convex constraints. Gould et al. [16] presented a new updating strategy for the adaptive regularization parameter and provided numerical experiments on large nonlinear least-squares problems. Huang and Zhu in [22] proposed an affine scaling cubic regularization algorithm for derivative-free bound constrained optimization using backtracking line search technique to obtain the step size. In order to conquer the lack of bound constraints information in the framework of ARC method, we turn to an affine scaling technique in [6]. Affine scaling techniques usually combine with trust-region methods for solving bound constrained optimization problems (e.g. [6, 32]). However, for various reasons, there were many examples in computational science and engineering, their (at least some) derivatives were unavailable or unreliable. But they may still be desirable to get the optimizations. Such situations motivated researchers to go after techniques for derivative-free optimization. Recently, there were many papers proposing various different methods for the derivative-free optimization problems. Algorithms for bounded constrained optimizations with global convergence results using the trust region derivative-free methods were presented in [7, 12, 19]. Powell in [26, 27] proposed NEWUOA algorithm, which employs a quadratic polynomial interpolation of the objective function, with good practical performance for the unconstrained or simple bound constrained optimizations. In [8], Conn et al. gave trust-region methods for derivative-free optimization, maintaining linear or quadratic models which were based only on the objective function values computed at sample points. The corresponding models were constructed by means of polynomial interpolation or regression or by any other approximation techniques. Wild and Shoemaker [31] extended the work of Conn et al. in [8] to fully linear models which included a nonlinear term and analyze global convergence of derivative-free trust region algorithms relying on radial basis function interpolation models. Fletcher and Leyffer first introduced the filter technique for constrained nonlinear optimization in [13]. The underlying concept of filter is that trial points are accepted if they improve the objective function or improve the constrained violation instead of a combination of those two measures defined by a merit function. Recently, Gould et al. in [15, 18] extended to the work to nonlinear feasibility problems, including nonlinear equations and nonlinear least-squares, to minimize the norm of the violations of a set of constraints. Then, Gould, et al. extended the filter techniques further to general unconstrained optimization problems in [17]. They showed that the procedure was global convergence to at least one second-order critical point and numerical experiments indicated encouraging performance of the filter-trust-region method when compared to the classical trust-region algorithms. Li and Zhu in [23] proposed an affine trust-region method with line search filter technique, using a backtracking relevance condition, for bound constrained optimization problems to obtain the global convergence.

Stimulated by the progress of these ideas, we will try our best to introduce an affine scaling interior-point adaptive cubic regularization method with line search filter technique, which does not depend on external restoration phase, for solving the bound constrained derivative-free optimization (1) under some mild assumptions in this paper. In absence of derivatives, we use Lagrange polynomials interpolation to build models of the objective function based on sample function values. There are two advantages of our proposed algorithm. One is that we only need to solve the subproblem once at each iteration if the interpolating radius meets the desirable condition, while the cubic regularization derivative-based algorithms need to solve the subproblem repeatedly if the trial step is not accepted, in which solving the subproblem is computationally more expensive than using line search to get the trial step. The other advantage is that the proposed algorithm uses a line search technique such that the new iteration is strictly feasible. Global convergence and local convergence results are also retained and numerical results show that the proposed algorithm is effective.

This paper is organized as follows. In Section 2, we review the basic concepts needed in this paper and introduce our algorithm for problem (1). The corresponding analysis of global convergence is investigated in Section 3 and local convergence is reported in Section 4. In Section 5, we draw the numerical experiments in details.

In this paper, unless otherwise noted, we all write $\|\cdot\| = \|\cdot\|_2$ for brevity.

2. Development of the algorithm

In this section, we first describe the components of our algorithm and then formally state the overall algorithm in detail.

2.1. *Affine-scaling technique*

There are many algorithms for minimization problems with upper and/or lower bounds. But almost all of these methods for problem (1) were " active set" methods. Coleman and Li [6] proposed a new trust region approach with an affine scaling technique for solving (1). The scaling matrix was motivated by examining the optimality conditions for (1).

Let x_* be a local minimum point for (1), then the first-order necessary conditions for (1) at x_* are

$$[\nabla f(x_*)]_i \begin{cases} = 0, & \text{if } l_i < [x_*]_i < u_i, \\ \leq 0, & \text{if } [x_*]_i = u_i, \\ \geq 0, & \text{if } [x_*]_i = l_i. \end{cases} \tag{2}$$

where $[\nabla f(x_*)]_i$, $[x_*]_i$, l_i, u_i are the ith component of $\nabla f(x_*)$, x_*, l, u, respectively.

Following an observation by Coleman and Li [6], the first order optimality conditions of (1) are equivalent to the nonlinear system of equations

$$D(x_*)^{-2}\nabla f(x_*) = 0,$$

where x_* is a local minimizer,

$$D(x) = \text{diag}\{|v(x)|^{-\frac{1}{2}}\}, \tag{3}$$

and

$$[v(x)]_i = \begin{cases} x_i - u_i, & \text{if } [\nabla f(x)]_i < 0 \text{ and } u_i < +\infty, \\ x_i - l_i, & \text{if } [\nabla f(x)]_i \geq 0 \text{ and } l_i > -\infty, \\ -1, & \text{if } [\nabla f(x)]_i < 0 \text{ and } u_i = +\infty, \\ 1, & \text{if } [\nabla f(x)]_i \geq 0 \text{ and } l_i = -\infty. \end{cases} \tag{4}$$

It is clear that the scaling matrix $D(x)$ depends on the distance of x to the bounds and the first-order derivative $\nabla f(x)$. And at the same time, we can observe that $D(x_*)^{-2}\nabla f(x_*) = 0$ is equivalent to $D(x_*)^{-1}\nabla f(x_*) = 0$ from the definition of $D(x)$.

Besides above, the following definition in [6] is also important.

Definition 2.1. *A point $x \in \Omega$ is nondegenerate if, each index i,*

$$\nabla f(x) = 0 \quad \Rightarrow \quad l_i < x_i < u_i. \tag{5}$$

Similar to the view expressed in [6], we consider the following diagonal system :

$$D(x)^{-2}\nabla f(x) = 0. \tag{6}$$

It is easy to see that system (6) is continuous but not everywhere differentiable. Nondifferentiability occurs when $v_i = 0$, but we can avoid such points by instricting $x_k \in \text{int}(\Omega)$. Discontinuity of v_i may also occur when $[\nabla f(x)]_i = 0$, however $D(x)^{-2}\nabla f(x)$ is continuous at such points. Assume that $x_k \in \text{int}(\Omega)$, a Newton step d_k for (6) satisfies

$$\left\{D_k^{-2}\nabla^2 f(x_k) + \text{diag}\{\nabla f(x_k)\}J_k\right\} d_k = -D_k^{-2}\nabla f(x_k), \tag{7}$$

where $D_k = D(x_k)$, $J_k \in \mathbb{R}^{n\times n}$ is the Jacobian matrix of $|v(x_k)|$ when $|v(x)|$ is differentiable at x_k.

Define

$$M_k = \nabla^2 f(x_k) + C_k,\;\; C_k = D_k\text{diag}\{\nabla f(x_k)\}J_k D_k,\;\; \hat{M}_k = D_k^{-1}M_k D_k^{-1},\;\; \widehat{\nabla f}(x_k) = D_k^{-1}\nabla f(x_k).$$

Lemma 2.3 in [6] gave the following results:

Lemma 2.2. *Assume that $x_* \in \Omega$ and $D_*^{-1} = D(x_*)^{-1}$. Then*
(a) $\widehat{\nabla f}(x_*) = 0$ if and only if (2) is satisfied;
(b) $\hat{M}_*$ is positive definite and $\widehat{\nabla f}(x_*) = 0$ if and only if the second-order sufficiency conditions are satisfied at x_*;
(c) $\hat{M}_*$ is positive semidefinite and $\widehat{\nabla f}(x_*) = 0$ if and only if the second-order necessary conditions are satisfied at x_*.

Following the suggestion in [6], (7) can be rewritten as the following equations

$$\left\{D_k^{-1}\nabla^2 f(x_k)D_k^{-1} + \text{diag}\{\nabla f(x_k)\}J_k\right\}(D_k d_k) = -D_k^{-1}\nabla f(x_k). \tag{8}$$

2.2. *Adaptive Cubic Regularization algorithm (ARC).*

Following the notations proposed by the predecessors in the introductions, Cartis et al. [2, 3] gave the ARC model following

$$\psi_k(d) = f(x_k) + \nabla f(x_k)^T d + \frac{1}{2}d^T B_k d + \frac{1}{3}\sigma_k\|d\|^3, \tag{9}$$

where B_k is a symmetric approximation to the Hessian $\nabla^2 f(x_k)$, and σ_k is a dynamic positive parameter. Here, the parameter σ_k performs a double task. One is account for the discrepancy between the objective function and its second order Taylor expansion, the other is for the difference between the exact and the approximate Hessian. The rules for updating σ_k are analogy to the trust-region methods. But σ_k might be regarded as the reciprocal of the trust-region radius and it can adjust automatically under some criterions. If good agreements between model and function are observed, there may be benefits in decreasing σ_k. By contrast, the only recourse available is to increase σ_k prior to reducing the size of the step to the next iteration.

By employing the affine scaling technique, we construct the adaptive cubic regularization method for bound constrained optimization. The adaptive cubic regularization method for box constrained optimization of (1) at the kth iteration is defined as follows

$$\bar{\psi}_k(d) = \left[D_k^{-1}\nabla f(x_k)\right]^T(D_k d) + \frac{1}{2}(D_k d)^T\left[D_k^{-1}\nabla^2 f(x_k)D_k^{-1} + \text{diag}\{\nabla f(x_k)\}J_k\right](D_k d) + \frac{1}{3}\sigma_k\|D_k d\|^3. \tag{10}$$

2.3. *Derivative-free model*

There are many approaches for derivative-free optimization. Here, we are interest in the interpolation model in [9]. Powell [27] proposed the quadratic interpolation model which approximated to the objective function that were highly useful for obtaining a fast rate of convergence in iterative algorithms for simple bound constrained optimizations. In this paper, in order to guarantee the quadratic interpolation model is fully quadratic, we choose $(n+1)(n+2)/2$ sampling points to construct a quadratic interpolation model. With quadratic approximation, the model has to belong to a fully quadratic class. This concept requires the following assumption [9].

Assumption 2.1. *Suppose a set $S \subseteq \mathbb{R}^n$ and give a radius $\Delta_{\max}$. Assume that f is twice continuously differentiable with Lipschitz continuous Hessian in an appropriate open domain containing the $\Delta_{\max}$ neighborhood $\bigcup_{x\in S} B(x;\Delta_{\max})$ of the set S.*

Definition 2.3. *Give a function f that satisfies Assumption2.1. A set of model functions $\mathcal{M} = \left\{m : \mathbb{R}^n \to \mathbb{R}, m \in C^2\right\}$ is called a fully quadratic class of models if the following hold:*

1. *There exist positive constants κ_{ef}, κ_{eg}, κ_{eh}, and κ_{blh}, such that for any $x \in S$, and $\Delta \in (0, \Delta_{\max}]$ there exists a model function $m(x+s)$ in $\mathcal{M}$, with Lipschitz continuous Hessian and corresponding Lipschitz constant bounded by κ_{blh}, and such that*
   - *the error between the Hessian of the model and the Hessian of the function satisfies*

$$\|\nabla^2 f(x+s) - \nabla^2 m(x+s)\| \le \kappa_{eh}\Delta \quad \forall\, x+s \in B(x,\Delta), \tag{11}$$

   - *the error between the gradient of the model and the gradient of the function satisfies*

$$\|\nabla f(x+s) - \nabla m(x+s)\| \le \kappa_{eg}\Delta^2 \quad \forall\, x+s \in B(x,\Delta), \tag{12}$$

   - *the error between the model and the function satisfies*

$$\|f(x+s) - m(x+s)\| \le \kappa_{ef}\Delta^3 \quad \forall\, x+s \in B(x,\Delta). \tag{13}$$

From [9], we know that if a model is fully quadratic on $B(x,\bar{\Delta})$ with respect to some (large enough) constants $\kappa_{ef}, \kappa_{eg}, \kappa_{eh}$ and for some $\bar{\Delta} \in (0, \tilde{\Delta}]$, then it is also fully quadratic on $B(x,\Delta)$ for any $\Delta \in [\bar{\Delta}, \tilde{\Delta}]$.

Now we expand (10) to the derivative-free version. Using the polynomial interpolation mentioned above, the derivative-free adaptive cubic regularization method for box constrained optimization of (1) at the kth iteration is defined as follows,

$$\varphi_k(d) = \left[\bar{D}_k^{-1} g_k\right]^T (\bar{D}_k d) + \frac{1}{2}(\bar{D}_k d)^T \left[\bar{D}_k^{-1} H_k \bar{D}_k^{-1} + \mathrm{diag}\{g_k\}\bar{J}_k\right](\bar{D}_k d) + \frac{1}{3}\sigma_k \|\bar{D}_k d\|^3, \tag{14}$$

where $g_k = \nabla m(x_k)$, $H_k = \nabla^2 m(x_k)$. In the derivative-free case, $g_k \neq \nabla f(x_k)$, $H_k \neq \nabla^2 f(x_k)$. $\bar{D}_k$ and $\bar{J}_k$ are the corresponding terms of $\bar{D}(x)$ and $\bar{J}(x)$ at x_k, where $\bar{D}(x)$ and $\bar{J}(x)$ are defined following:

$$\bar{D}(x) = \mathrm{diag}\left\{|\bar{v}(x)|^{-\frac{1}{2}}\right\},$$

and

$$[\bar{v}(x)]_i = \begin{cases} x_i - u_i, & \text{if } [\nabla m(x)]_i < 0 \text{ and } u_i < +\infty, \\ x_i - l_i, & \text{if } [\nabla m(x)]_i \ge 0 \text{ and } l_i > -\infty, \\ -1, & \text{if } [\nabla m(x)]_i < 0 \text{ and } u_i = +\infty, \\ 1, & \text{if } [\nabla m(x)]_i \ge 0 \text{ and } l_i = -\infty, \end{cases}$$

and $\bar{J}(x)$ is the Jacobian matrix of $|\bar{v}(x)|$.

By employing the notations $\hat{H}_k = \bar{D}_k^{-1} H_k \bar{D}_k^{-1} + \mathrm{diag}\{g_k\}\bar{J}_k$, $\hat{g}_k = \bar{D}_k^{-1} g_k$ and $\hat{d} = \bar{D}_k d$, we rewrite (14) as follows

$$\varphi_k(d) = \hat{g}_k^T \hat{d} + \frac{1}{2}\hat{d}^T \hat{H}_k \hat{d} + \frac{1}{3}\sigma_k \left\|\hat{d}\right\|^3. \tag{15}$$

2.4. Line search filter technique

Filter technique was proposed firstly by Fletcher and Leyffer [13] for constrained nonlinear optimization. The notion of filter in the half-plane $\{(\|h\|, f) \in \mathbb{R}^2\}$ is based on that of dominance. The needed definitions are defined as follows:

Definition 2.4. *A pair $(\|h(x_k)\|, f(x_k))$ dominates another pair $(\|h(x_l)\|, f(x_l))$ if both $\|h(x_k)\| \le \|h(x_l)\|$ and $f(x_k) \le f(x_l)$.*

Definition 2.5. *A filter $\mathcal{F}$ is a list of pairs $(\|h(x_l)\|, f(x_l))$ such that no pair dominate another.*

Definition 2.6. *A pair $(\|h(x_k)\|, f(x_k))$ is said to be acceptable for the filter $\mathcal{F}$ when it is not dominated by any pair in the filter $\mathcal{F}$.*

The idea of using filter is to interpret the system of (1) as a biobjective optimization problem with two goals: minimizing the objective function $f(x)$ and minimizing the constraint violation $\|h(x)\|$, where $h(x) = \bar{D}(x)^{-1}g(x)$ and $g(x) = \nabla m(x)$. In order to ensure f or $\|h\|$ decreasing sufficiently, we gave such a strategy:

$$\|h(x_k(\alpha_{k,l}))\| \le (1 - \gamma_h)\|h(x_k)\|, \tag{16}$$

or

$$f(x_k(\alpha_{k,l})) \le f(x_k) - \gamma_f \alpha_{k,l}\|h(x_k)\|^2, \tag{17}$$

where $\gamma_h,\ \gamma_f \in (0, 1)$ are small positive constants and

$$x_k(\alpha_{k,l}) = x_k + \alpha_{k,l} d_k. \tag{18}$$

Now we consider using a filter mechanism to potentially accepted $x_k(\alpha_{k,l})$ as a new iterate. If (16) or (17) holds, we say that the trial point $x_k(\alpha_{k,l})$ to be acceptable to the current filter $\mathcal{F}$. If $x_k(\alpha_{k,l})$ satisfies

$$(\|h(x_k(\alpha_{k,l}))\|, f(x_k(\alpha_{k,l}))) \in \mathcal{F}, \tag{19}$$

we say that the trial point $x_k(\alpha_{k,l})$ is not acceptable to the current filter, that is, neither (16) nor (17) holds. The filter is augmented as follows:

$$\mathcal{F}_{k+1} = \mathcal{F}_k \cup \left\{(\|h\|, f) \in \mathbb{R}^n : \|h\| \ge (1 - \gamma_h)\|h(x_k)\| \text{ and } f \ge f(x_k) - \gamma_f \alpha_k \|h(x_k)\|^2\right\}, \tag{20}$$

after the new iterate has been accepted. Otherwise, the filter remains unchanged. At the beginning, the filter is initialized to

$$\mathcal{F}_0 = \{(\|h\|, f) : \|h\| \ge \|h\|_{\max}\}, \tag{21}$$

for some $\|h\|_{\max} > \|h(x_0)\|$, so the algorithm will never allow trial points to be accepted that have a gradient norm large than $\|h\|_{\max}$. In this way, this procedure ensures that the algorithm cannot cycle.

2.5. Algorithm (DFFARC)

Now we describe an affine scaling interior-point adaptive cubic regularization algorithm with line search filter technique for solving derivative-free optimization subject to bounds below:

Initialization step:
An initial point $x_0 \in \text{int}(\Omega) = \{x : l < x < u\}$, a positive parameter $\sigma_0 > 0$ and a radius $1 \ge \Delta_0 > 0$ are given. The constants $\eta_1,\ \eta_2,\ \gamma_1,\ \gamma_2,\ \tau_1,\ \tau_2$ are also given and satisfy that

$$0 < \eta_1 \le \eta_2 < 1, \quad 0 < \gamma_1 < 1 \le \gamma_2 < 2, \quad 0 < \tau_1 \le \tau_2 < 1.$$

Initialize the filter $\mathcal{F}_0$. Choose $\epsilon \in (0, 1),\ \gamma_f, \gamma_h \in (0, 1),\ \gamma_3 \in (0, 1)$. Set $k \leftarrow 0$, and go to main step.

Main Step:

Step 1. Choose a set Y_k of interpolation points with $x_k \in Y_k \subseteq B(x_k, \Delta_k)$, then applying Algorithms 6.1 and 6.3 in [9] to construct the corresponding model $m_k = m(x_k)$ which is fully quadratic on $B(x_k, \Delta_k)$. Set $g_k = \nabla m_k,\ H_k = \nabla^2 m_k$. Compute $\bar{D}_k$ and $h(x_k) \overset{\text{def}}{=} \bar{D}_k^{-1} g_k$.

Step 2. If $\|h(x_k)\| \le \varepsilon$, stop with the solution x_k, else go to next.

Step 3. Compute a finite search direction d_k for which

$$\varphi_k(d_k) \le \varphi_k(d_k^C), \tag{22}$$

where

$$d_k^C = -\zeta_k^C \bar{D}_k^{-2} g_k, \quad \zeta_k^C = \arg\min_{\zeta \in \mathbb{R}_+} \varphi_k(-\zeta \bar{D}_k^{-2} g_k), \tag{23}$$

and $\varphi_k(d)$ is defined in (15).

Step 4. Backtracking line search.

Step 4.1 Set $\alpha_{k,0} = 1$ and $l \leftarrow 0$.

Step 4.2 Compute $x_k(\alpha_{k,l}) = x_k + \alpha_{k,l} d_k$.

Step 4.3 If $(\|h(x_k(\alpha_{k,l}))\|, f(x_k(\alpha_{k,l}))) \in \mathcal{F}_k$, reject the trial step size $\alpha_{k,l}$ and go to Step 4.6, else go to Step 4.4.

Step 4.4 If

$$\|h(x_k(\alpha_{k,l}))\| \le (1 - \gamma_h)\|h(x_k)\|,$$

with

$$x_k + \alpha_{k,l} d_k \in \Omega$$

holds, accept the trial step $\alpha_{k,l}$ and go to Step 4.7, else go to Step 4.5.

Step 4.5 If

$$f(x_k(\alpha_{k,l})) \le f(x_k) - \gamma_f \alpha_{k,l} \|h(x_k)\|^2,$$

with

$$x_k + \alpha_{k,l} d_k \in \Omega$$

holds, accept the trial step $\alpha_{k,l}$ and go to Step 4.7, else go to Step 4.6.

Step 4.6 Set $\alpha_{k,l+1} \in [\tau_1 \alpha_{k,l}, \tau_2 \alpha_{k,l}]$, set $l \leftarrow l + 1$ and go to Step 4.2.

Step 4.7 Set $\alpha_k = \alpha_{k,l}$ and

$$s_k = \begin{cases} \alpha_k d_k, & \text{if} \quad x_k + \alpha_k d_k \in \operatorname{int}(\Omega), \\ \theta_k \alpha_k d_k, & \text{if} \quad \text{otherwise}, \end{cases}$$

where $\theta_k \in (\theta_0, 1]$, for some $0 < \theta_0 < 1$ and $\theta_k - 1 = O(\|d_k\|)$.

Step 5. Let $\tilde{\Delta}_k = \Delta_k$.

If $\tilde{\Delta}_k > \min(\|h(x_k)\|, \|\hat{s}_k\|)$,

set $\tilde{\Delta}_k \leftarrow \gamma_3 \tilde{\Delta}_k$ and applying Algorithm 6.1, 6.3 in [9] to construct model $\tilde{m}_k$ which is fully quadratic on $B(x_k, \tilde{\Delta}_k)$. Set $\tilde{g}_k = \nabla \tilde{m}_k,\ \tilde{H}_k = \nabla^2 \tilde{m}_k$. Go to Step 3 to compute $\tilde{d}_k$ that satisfies (22), (23).

Until $\tilde{\Delta}_k \le \min(\|\tilde{h}(x_k)\|, \|\tilde{\hat{s}}_k\|)$.

Update $m_k = \tilde{m}_k,\ g_k = \nabla \tilde{m}_k,\ H_k = \nabla^2 \tilde{m}_k,\ \Delta_k = \tilde{\Delta}_k$ and $s_k = \tilde{s}_k$, compute $\varphi_k(s_k)$, then go to Step 6.

Step 6. Compute $x_{k+1} = x_k + s_k,\ f(x_{k+1}) = m_{k+1} = m(x_{k+1}),\ g_{k+1} = g(x_{k+1}) = \nabla m(x_{k+1})$ and $H_{k+1} = \nabla^2 m(x_{k+1})$.

Step 7. Compute the following ratio ρ_k,

$$\rho_k = \frac{f(x_k) - f(x_{k+1})}{-\varphi_k(s_k)}.$$

Set

$$\sigma_{k+1} = \begin{cases} \gamma_1 \sigma_k, & \text{if } \rho_k > \eta_2, \\ \sigma_k, & \text{if } \eta_1 \le \rho_k \le \eta_2, \\ \gamma_2 \sigma_k, & \text{if } \rho_k < \eta_1. \end{cases}$$

Step 8. Augment the filter if necessary. If neither (16) nor (17) holds, augment the filter using (20), else the filter remains unchanged.

Step 9. Set $y_0^{k+1} = x_{k+1}$, $\Delta_{k+1} = \Delta_k$. Choose q interpolation points in region $B(x_{k+1}, \Delta_{k+1})$ to construct the sample set $Y_{k+1} = \{y_0^{k+1}, \cdots, y_q^{k+1}\}$. Determine the corresponding interpolation model $m_{k+1}(x_{k+1})$ on the sample set Y_{k+1}. Compute g_{k+1} and H_{k+1}. Set $k \leftarrow k+1$ and go to Step 1.

Remark 2.7. *The scalar α_k given in (17) of Step 4 denotes the step size along the direction d_k to the boundary on the variables $l \le x_k + \alpha_k d_k \le u$, that is,*

$$\alpha_k \overset{\text{def}}{=} \min\left\{\max\left\{\frac{l_i - [x_k]_i}{[d_k]_i}, \frac{u_i - [x_k]_i}{[d_k]_i}\right\}, i = 1, 2, \dots, n\right\}, \tag{24}$$

where l_i, u_i, $[x_k]_i$ and $[d_k]_i$ are the ith components of l, u, x_k and d_k, respectively.

3. Global convergence analysis

In this section, we present some lemmas which provide some basic properties of the proposed algorithm. And these properties will also be referred to next on global convergence and local convergence analysis. Throughout this paper, we denote $\mathcal{A}$ to the set of indices which the filter has been augmented, i.e. $k \in \mathcal{A} \Leftrightarrow \mathcal{F}_k \subset \mathcal{F}_{k+1}$. Before we study the properties of the algorithm, we make the following assumptions.

Assumption 3.1. *Assume that $f : \mathbb{R}^n \to \mathbb{R}$ is twice continuously differentiable and bounded from below, its gradient is Lipschitz continuous on the paths of iterates with Lipschitz constants L_g.*

Give an initial point $x_0 \in \text{int}(\Omega)$ and denote the level set of f by $\mathcal{L}(x_0) = \{x \in \mathbb{R}^n \mid f(x) \le f(x_0),\ x \in \Omega\}$.

Assumption 3.2. *The iterates x_k remain in $\mathcal{L}(x_0)$, which is compact on $\mathbb{R}^n$.*

Assumption 3.3. *There exist some positive constants κ_D, κ_g and κ_h such that $\|\bar{D}(x)^{-1}\| \le \kappa_D$, $\|D(x)^{-1}\| \le \kappa_D$, $\|\nabla f(x)\| \le \kappa_g$, $\|\nabla^2 f(x)\| \le \kappa_h$ for all $x \in \mathcal{L}(x_0)$, respectively.*

Consequently, from Assumptions 3.1-3.3, (12) and the choice of Δ_k in Step 5, we can get that

$$\kappa_{lf} \le f(x) \le \kappa_{uf}, \quad \|g_k\| \le \|\nabla f(x_k)\| + \|\nabla f(x_k) - g_k\| \le \kappa_g + \kappa_{eg}\Delta_0^2 \overset{\text{def}}{=} \kappa_{ng},$$

and

$$\|H_k\| \le \kappa_h + \kappa_{eh}\Delta_0 \overset{\text{def}}{=} \kappa_{nh}, \quad \|\hat{H}_k\| \le \kappa_D^2 \kappa_{nh} + \kappa_{ng} \overset{\text{def}}{=} \kappa_H.$$

Firstly, we give the global optimality result for the cubic model directly from [2].

Lemma 3.1. *A step d_k is a solution to (14) if and only if d_k satisfies the following equation*

$$\left[\hat{H}_k + \sigma_k\|\hat{d}_k\|I\right]\hat{d}_k \overset{\text{def}}{=} \left[\bar{D}_k^{-1}H_k\bar{D}_k^{-1} + \text{diag}\{g_k\}\bar{J}_k + \sigma_k\|\bar{D}_k d_k\|I\right](\bar{D}_k d_k) = -\bar{D}_k^{-1}g_k \overset{\text{def}}{=} -\hat{g}_k, \tag{25}$$

where $\hat{H}_k + \sigma_k\|\hat{d}_k\|I \overset{\text{def}}{=} \bar{D}_k^{-1}H_k\bar{D}_k^{-1} + \text{diag}\{g_k\}\bar{J}_k + \sigma_k\|\bar{D}_k d_k\|I$ is positive semidefinite.

The predicted reduction $-\varphi_k(d_k)$ satisfies a guaranteed low bound, which makes sure the global convergence of the proposed algorithm. The following lemma is from [22].

Lemma 3.2. *Suppose that d_k is the solution in Step 3. Then for $k \ge 0$, we have the result following:*

$$-\varphi_k(d_k) \ge \frac{1}{6}\|\hat{g}_k\| \min\left\{\frac{\|\hat{g}_k\|}{\|\hat{H}_k\|}, \frac{\sqrt{3}}{4}\sqrt{\frac{\|\hat{g}_k\|}{\sigma_k}}\right\}. \tag{26}$$

Assumption 3.4. *Suppose that $\hat{H}_k$ is positive semidefinite for all $x_k \in \mathcal{L}(x_0)$.*

From Assumption 3.4, we have that

$$-\varphi_k(\alpha_k d_k) \geq \frac{\alpha_k}{6}\|\hat{g}_k\| \min\left\{\frac{\|\hat{g}_k\|}{\|\hat{H}_k\|}, \frac{\sqrt{3}}{4}\sqrt{\frac{\|\hat{g}_k\|}{\sigma_k}}\right\}. \tag{27}$$

The following lemma shows that the direction of the trial step is a sufficiently descent direction. This lemma is from [22].

Lemma 3.3. *Suppose that Assumption 3.4 holds. If d_k is a solution in Step 3, then*

$$g_k^T d_k \leq -\frac{\|\hat{g}_k\|}{6} \min\left\{\frac{\|\hat{g}_k\|}{\|\hat{H}_k\|}, \frac{\sqrt{3}}{4}\sqrt{\frac{\|\hat{g}_k\|}{\sigma_k}}\right\}. \tag{28}$$

The following lemma from [22] gives a useful bound on the step d_k.

Lemma 3.4. *If d_k is a solution in Step 3 and Assumption 3.4 holds, then*

$$\|\bar{D}_k d_k\| = \|\hat{d}_k\| \leq 3\sqrt{\frac{\|\hat{g}_k\|}{\sigma_k}}. \tag{29}$$

Lemmas 3.3-3.4 and Assumption 3.3 imply that

$$g_k^T d_k \leq -\frac{\|\hat{g}_k\|}{6} \min\left\{\frac{\|\hat{g}_k\|}{\kappa_H}, \frac{\sqrt{3}}{12}\|\hat{d}_k\|\right\}. \tag{30}$$

Assumption 3.5. *Suppose that $\|\bar{D}_k^{-2}\nabla_d \varphi_k(d_k)\| \leq \kappa_\theta \min\left\{1, \|\hat{d}_k\|\right\} \|\bar{D}_k^{-2} g_k\|$, where $\kappa_\theta \in (0,1)$.*

Now, we show that the algorithm will not loop infinitely in the Step 5 unless the current iterate is a first-order stationary point.

Lemma 3.5. *If $\|D_k^{-1}\nabla f(x_k)\| \neq 0$ and Assumption 3.5 holds, the Step 5 of the algorithm will terminate in a finite number.*

Proof. The proof is very similar to Lemma 5.1 in [8], but we repeat the details of Lemma 5.1 in [8] here in order to maintain the integrity of the paper and emphasize the importance of this lemma. In order to obtain a contradiction, we suppose that there are infinitely many improvement steps.

In the beginning, according to the algorithm, we know that, if we implement the improvement algorithm, either m_k is not fully quadratic on $B(x_k, \Delta_k)$ or $\Delta_k > \min\{\|h(x_k)\|, \|\hat{s}_k\|\}$. Then we let $g_k^{(0)} = \nabla m_k$ and improve the model until it is fully quadratic on $B(x_k, \gamma_3^0 \Delta_k)$. If $\gamma_3^0 \Delta_k \leq \min(\|h(x_k)^{(1)}\|, \|\hat{s}_k^{(1)}\|)$, the procedure terminates with $\tilde{\Delta}_k = \gamma_3^0 \Delta_k \leq \min(\|h(x_k)^{(1)}\|, \|\hat{s}_k^{(1)}\|)$. Otherwise, that is, if $\min(\|h(x_k)^{(1)}\|, \|\hat{s}_k^{(1)}\|) < \gamma_3^0 \Delta_k$, we improve the model until it is fully quadratic on $B(x_k, \gamma_3 \Delta_k)$. Then, either the procedure terminates or multiplies γ_3 to $\tilde{\Delta}_k$ again, and so on.

This procedure will be infinite only when the following inequality always holds

$$\min(\|h(x_k)^{(l)}\|, \|\hat{s}_k^{(l)}\|) < \gamma_3^{l-1}\Delta_k \quad \text{for all } l \geq 1, \tag{31}$$

where $g_k^{(l)} = \nabla m_k^{(l)}$, $h(x_k)^{(l)} = \bar{D}_k^{-1} g_k^{(l)}$. We will get that $\lim_{l \to +\infty} \gamma_3^{l-1}\Delta_k = 0$ since that $0 < \gamma_3 < 1$. Then from the above inequality, we will get that $\min(\|h(x_k)^{(l)}\|, \|\hat{s}_k^{(l)}\|) \to 0 (l \to +\infty)$.

Since each model $m_k^{(l)}$ is fully quadratic on $B(x_k, \gamma_3^{l-1}\Delta_k)$, then (12) with $s = 0$ and $x = x_k$ provides

$$\|\nabla f(x_k) - g_k^{(l)}\| \le \kappa_{eg}(\gamma_3^{l-1}\Delta_k)^2 \quad \text{for all } l \ge 1.$$

Thus, $D_k = \bar{D}_k$ from the definition of $D(x)$ and $\bar{D}(x)$ for sufficiently large l.

From the choice of α_k, if $\|\hat{s}_k\| \le \|\hat{s}_k^{(l)}\|$, then $\|h(x_k)^{(l)}\| = \min\left\{\|h(x_k)^{(l)}\|, \|\hat{s}_k\|\right\} \le \min\left\{\|h(x_k)^{(l)}\|, \|\hat{s}_k^{(l)}\|\right\} \to 0$, as $l \to \infty$. Using the triangle inequality, we have that,

$$\begin{aligned}
\|D_k^{-1}\nabla f(x_k)\| &\le \|D_k^{-1}\| \cdot \|\nabla f(x_k) - g_k^{(l)}\| + \|h(x_k)^{(l)}\| \\
&\le \kappa_D\|\nabla f(x_k) - g_k^{(l)}\| + \|h(x_k)^{(l)}\| \\
&\le \kappa_D\kappa_{eg}(\gamma_3^{l-1}\Delta_k)^2 + \gamma_3^{l-1}\Delta_k \\
&\le \kappa_D(\kappa_{eg}\Delta_k + 1)\gamma_3^{l-1}\Delta_k,
\end{aligned}$$

where $\gamma_3 \in (0, 1)$. Then we infer that

$$\|D_k^{-1}\nabla f(x_k)\| = 0,$$

which contradicts $\|D_k^{-1}\nabla f(x_k)\| \ne 0$, hence the conclusion holds.

If $\|\hat{s}_k\| > \|\hat{s}_k^{(l)}\|$, i.e. $\alpha_k > \alpha_{k,l}$, then (17) also hold for α_k at the trial step $x_k(\alpha_{k,l}) = x_k + \alpha_{k,l}d_k$. And from (12), we have

$$\|D_k^{-1}\nabla f(x_k) - h(x_k)^{(l)}\| = \|D_k^{-1}\nabla f(x_k) - \bar{D}_k^{-1}g_k^{(l)}\| \le \kappa_D\kappa_{eg}(\gamma_3^{l-1}\Delta_k)^2 \quad \text{for all } l \ge 1.$$

Then, $\|h(x_k)^{(l)}\| \ne 0$ from $\|D_k^{-1}\nabla f(x_k)\| \ne 0$ for sufficiently large l. Thus there exists a $\varepsilon > 0$, such that $\|h(x_k)^{(l)}\| > \varepsilon$ for large l. Similar to prove Lemma 4.1 in [4], we can have that

$$\|\hat{s}_k\| \ge \frac{\alpha_k\theta_0(1-\kappa_\theta)\varepsilon}{\kappa_D(\kappa_H + 3\sqrt{\sigma_k\kappa_D\kappa_{ng}})} \stackrel{\text{def}}{=} \mu(\alpha_k, \sigma_k)\varepsilon,$$

from Assumption 3.5. Since $\gamma_3 \in (0, 1)$, there exists a $K = \left[\frac{\log\left[\min\{\varepsilon, \mu(\alpha_k,\sigma_k)\varepsilon\}\right]}{\log\gamma_3}\right]_+$, such that for $l > K + 1$,

$$\gamma_3^{l-1}\Delta_k < \gamma_3^{l-1} \le \min\{\varepsilon, \mu(\alpha_k, \sigma_k)\varepsilon\} \le \min\left\{\|h(x_k)^{(l)}\|, \|\hat{s}_k\|\right\},$$

which is contradiction to (31). Thus, the algorithm terminates in a finite number. □

Lemma 3.6. *Suppose that Assumptions 3.1-3.2 hold. Then*

$$\Theta_k := \min\{\|h\| : (\|h\|, f) \in \mathcal{F}_k\} > 0 \tag{32}$$

for all k.

Proof. By induction, it is clear from initialization step of the Algorithm that the claim is true for $k = 0$ since $\|h\|_{\max} > 0$. Suppose the claim holds for k. Then, we prove the claim is right for $k + 1$. If the algorithm proposed has not terminated, $\|h(x_k)\| > 0$. Without loss of generality, we assume the filter is augmented in iteration k. It is clear from the update rule (20) that $\Theta_{k+1} > 0$, since $\gamma_h \in (0, 1)$. □

Lemma 3.7. *Suppose that Assumptions 3.1-3.4 hold. If m_k is fully quadratic on $B(x_k; \Delta_k)$ and*

$$\sqrt{\sigma_k\|\hat{g}_k\|} \ge \frac{216}{1-\eta_1}\max\left\{\kappa_D(L_g\kappa_D + \kappa_{eg}\Delta_0), \kappa_H\right\} \stackrel{\text{def}}{=} \kappa_{HB}, \tag{33}$$

then $\rho_k \ge \eta_1$ and

$$\sigma_{k+1} \le \sigma_k. \tag{34}$$

Proof. The proof is very similar to Lemma 3.11 in [22], but we repeat the details of Lemma 3.11 in [22] here in order to maintain the integrity of the paper and emphasize the importance of this lemma. From (33), we will get that $\|\hat{g}_k\| \neq 0$, because otherwise, the algorithm would have terminated, then $\|\hat{g}_k\| = 0$ will conflict with (33). However,

$$\rho_k - \eta_1 = \frac{f(x_k + s_k) - f(x_k) - \eta_1 \varphi_k(s_k)}{\varphi_k(s_k)} = \frac{f(x_k + s_k) - f(x_k) - \varphi_k(s_k) + (1-\eta_1)\varphi_k(s_k)}{\varphi_k(s_k)}.$$

Firstly, using a Taylor expansion, for some $\xi_k \in (0,1)$, the first term of the numerator in the fraction above becomes that

$$\begin{aligned}
& f(x_k + s_k) - f(x_k) - \varphi_k(s_k) \\
= \;& f(x_k) + \nabla f(x_k + \xi_k s_k)^T s_k - f(x_k) - \hat{g}_k^T \hat{s}_k - \frac{1}{2}\hat{s}_k^T \hat{H}_k \hat{s}_k - \frac{\sigma_k}{3}\|\hat{s}_k\|^3 \\
= \;& \left(\bar{D}_k^{-1}\nabla f(x_k + \xi_k s_k) - \hat{g}_k\right)^T \hat{s}_k - \frac{1}{2}\hat{s}_k^T \hat{H}_k \hat{s}_k - \frac{\sigma_k}{3}\|\hat{s}_k\|^3 \\
\leq \;& \left(\bar{D}_k^{-1}\nabla f(x_k + \xi_k s_k) - \hat{g}_k\right)^T \hat{s}_k \\
\leq \;& \kappa_D \left(\|\nabla f(x_k + \xi_k s_k) - \nabla f(x_k)\| + \|\nabla f(x_k) - g_k\|\right) \cdot \|\hat{s}_k\| \\
\leq \;& \theta_k^2 \alpha_k^2 \kappa_D \left[L_g \kappa_D + \kappa_{eg}\Delta_0\right] \cdot \|\hat{d}_k\|^2 \\
\leq \;& \theta_k^2 \alpha_k^2 \max\{\kappa_D(L_g \kappa_D + \kappa_{eg}\Delta_0), \kappa_H\} \cdot \|\hat{d}_k\|^2 \\
\leq \;& \theta_k \alpha_k^2 \max\{\kappa_D(L_g \kappa_D + \kappa_{eg}\Delta_0), \kappa_H\} \cdot \frac{9\|\hat{g}_k\|}{\sigma_k},
\end{aligned} \tag{35}$$

where we used Lemma 3.4, Assumption 3.1, (12) and the triangle inequality.

Next, we consider the second term of the numerator in the fraction.

$$\begin{aligned}
(1-\eta_1)\varphi_k(s_k) \;\leq\;& -\frac{(1-\eta_1)\theta_k\alpha_k}{6}\|\hat{g}_k\| \cdot \min\left\{\frac{\|\hat{g}_k\|}{\|\hat{H}_k\|}, \frac{\sqrt{3}}{4}\sqrt{\frac{\|\hat{g}_k\|}{\sigma_k}}\right\} \\
\leq\;& \frac{(\eta_1 - 1)\theta_k\alpha_k}{6} \cdot \|\hat{g}_k\|^2 \cdot \min\left\{\frac{1}{\kappa_H}, \frac{\sqrt{3}}{4}\sqrt{\frac{1}{\|\hat{g}_k\|\sigma_k}}\right\} \\
\overset{(33)}{=}\;& \frac{\sqrt{3}(\eta_1 - 1)\theta_k\alpha_k}{24} \cdot \|\hat{g}_k\|^2 \cdot \sqrt{\frac{1}{\|\hat{g}_k\|\sigma_k}} \\
=\;& \frac{\sqrt{3}\theta_k\alpha_k(\eta_1 - 1)}{24} \cdot \frac{\|\hat{g}_k\|^{\frac{3}{2}}}{\sqrt{\sigma_k}} \\
\leq\;& \frac{\theta_k\alpha_k^2(\eta_1 - 1)}{24}\frac{\|\hat{g}_k\|^{\frac{3}{2}}}{\sqrt{\sigma_k}},
\end{aligned} \tag{36}$$

where we used (27), Assumption 3.3 and $0 < \alpha_k \leq 1$.

However, the denominator $\varphi_k(s_k) < 0$ is from (27).

Thus, following with (33), (35) and (36), we get that,

$$\begin{aligned}
\rho_k - \eta_1 \;\geq\;& \frac{\theta_k\alpha_k^2 \max\left\{\kappa_D(L_g\kappa_D + \kappa_{eg}\Delta_0), \kappa_H\right\}\frac{9\|\hat{g}_k\|}{\sigma_k} + \frac{\alpha_k^2(\eta_1 - 1)}{24}\frac{\|\hat{g}_k\|^{\frac{3}{2}}}{\sqrt{\sigma_k}}}{\varphi_k(s_k)} \\
=\;& \frac{\theta_k\alpha_k^2\|\hat{g}_k\| \cdot \left[216\max\left\{\kappa_D(L_g\kappa_D + \kappa_{eg}\Delta_0), \kappa_H\right\} + (\eta_1 - 1)\sqrt{\sigma_k\|\hat{g}_k\|}\right]}{24\sigma_k\varphi_k(s_k)}
\end{aligned}$$

$$= \frac{\theta_k \alpha_k^2 \|\hat{g}_k\| \cdot \left[\frac{216}{1-\eta_1} \max\left\{\kappa_D (L_g \kappa_D + \kappa_{eg}\Delta_0), \kappa_H\right\} - \sqrt{\sigma_k \|\hat{g}_k\|}\right]}{24\sigma_k \varphi_k(s_k)}$$

$$\stackrel{(33)}{\geq} 0. \tag{37}$$

Thus, $\rho_k \geq \eta_1$ and (34) follows from the updating rule in Step 7. □

Assumption 3.6. *Suppose that* $\sigma_k \geq \sigma_{\min} > 0$, *then* $\lim_{k\to\infty} \sigma_k = \infty$ *as* $\alpha_k \to 0$.

The Assumption 3.6, which is reasonable when we use Armijo backing line search or Wolfe condition (ii) for global convergence, plays an important role to promote global convergence.

If $x_k + \alpha_k d_k$ satisfies Armijo condition, $f(x_k + \alpha_k d_k) \leq f(x_k) + c_1 \alpha_k \nabla f(x_k)^T d_k$, where $c_1 \in (0,1)$, without loss of generality, we assume that $\alpha_k = \alpha_{k,l}$, where l is the first integer such that $\alpha_{k,l}$ satisfies Armijo condition, we have that

$$f(x_k + \frac{\alpha_k}{\tau_1} d_k) > f(x_k) + \frac{\alpha_k}{\tau_1} c_1 \nabla f(x_k)^T d_k. \tag{38}$$

From the mean value theorem, we have that

$$f(x_k + \frac{\alpha_k}{\tau_1} d_k) = f(x_k) + \frac{\alpha_k}{\tau_1} \nabla f(x_k + \xi_k \frac{\alpha_k}{\tau_1} d_k)^T d_k. \tag{39}$$

where $\xi_k \in (0,1)$. Combining (38) and (39), we can get the following inequality,

$$\frac{\alpha_k}{\tau_1}(\nabla f(x_k + \xi_k \frac{\alpha_k}{\tau_1} d_k) - \nabla f(x_k))^T d_k + \frac{\alpha_k}{\tau_1}(1 - c_1)(\nabla f(x_k) - g_k)^T d_k + \frac{\alpha_k}{\tau_1} c_1 g_k^T d_k > 0. \tag{40}$$

From Assumption 3.1 and (12), then,

$$(\frac{\alpha_k}{\tau_1})^2 L_g \|d_k\|^2 + \frac{\alpha_k^3}{\tau_1}(1 - c_1)\kappa_{eg}\kappa_D^2 \|d_k\|^3 + \frac{\alpha_k}{\tau_1} c_1 g_k^T d_k > 0. \tag{41}$$

Dividing (41) by $\frac{\alpha_k}{\tau_1}$ and applying Lemma 3.3, we have that

$$\frac{\alpha_k}{\tau_1} L_g \|d_k\|^2 + \alpha_k^2 (1 - c_1)\kappa_{eg}\kappa_D^2 \|d_k\|^3 > c_1 \frac{\|\hat{g}_k\|}{6} \min\left\{\frac{\|\hat{g}_k\|}{\|\hat{H}_k\|}, \frac{\sqrt{3}}{4}\sqrt{\frac{\|\hat{g}_k\|}{\sigma_k}}\right\}. \tag{42}$$

If $\|\hat{g}_k\| > \varepsilon$, using Assumption 3.3 and the fact that $\|d_k\|$ is bounded, which is from Lemma 3.4, $\|d_k\| \leq \kappa_D \|\hat{d}_k\| \leq 3\kappa_D \sqrt{\frac{\|\hat{g}_k\|}{\sigma_k}} \leq 3\kappa_D^{\frac{3}{2}} \sqrt{\frac{\kappa_{ng}}{\sigma_{\min}}}$, we can infer that $\sigma_k \to \infty$ as $\alpha_k \to 0$ from (42).

Next we show that Assumption 3.6 can be also achieved when $x_k(\alpha_k) = x_k + \alpha_k d_k$ satisfies the Wolfe condition (ii) $\nabla f(x_k + \alpha_k d_k)^T d_k \geq c_2 \nabla f(x_k)^T d_k$, where $c_2 \in (0,1)$. We rewrite it, then

$$(\nabla f(x_k + \alpha_k d_k) - \nabla f(x_k))^T d_k + (1 - c_2)(\nabla f(x_k) - g_k)^T d_k + (1 - c_2) g_k^T d_k > 0.$$

From Assumption 3.1 and (12) again, we have that,

$$L_g \alpha_k \|d_k\|^2 + (1 - c_2)\alpha_k^2 \kappa_{eg}\kappa_D^2 \|d_k\|^3 + (1 - c_2) g_k^T d_k > 0.$$

Using Lemma 3.3 and Assumption 3.3, the following inequality

$$L_g \alpha_k \|d_k\|^2 + (1 - c_2)\alpha_k^2 \kappa_{eg}\kappa_D^2 \|d_k\|^3 > (1 - c_2)\frac{\|\hat{g}_k\|}{6} \min\left\{\frac{\|\hat{g}_k\|}{\kappa_H}, \frac{\sqrt{3}}{4}\sqrt{\frac{\|\hat{g}_k\|}{\sigma_k}}\right\},$$

holds. Consequently, $\sigma_k \to \infty$ as $\alpha_k \to 0$ if $\|\hat{g}_k\| > \varepsilon$.

Lemma 3.8. *Suppose that Assumptions 3.1-3.6 hold and the filter is augmented only a finite number of times, i.e. $|\mathcal{A}| < \infty$. If any limit point x_* of $\{x_k\}$ is nondegenerate, then*

$$\lim_{k\to\infty} \|h(x_k)\| = 0. \tag{43}$$

Proof. Choose K such that for all iteration $k \geq K$ the filter is not augmented in iteration k, i.e. $k \notin \mathcal{A}$ for all $k \geq K$. From Step 4 of Algorithm DFFARC, we have that either (16) or (17) holds. If (16) holds for $k \geq K$, since $\gamma_h \in (0, 1)$, we have that

$$\|h(x_{k+1})\| \leq (1-\gamma_h)\|h(x_k)\| \leq (1-\gamma_h)^2\|h(x_{k-1})\| \leq \cdots \leq (1-\gamma_h)^{k-K}\|h(x_{K+1})\|.$$

This implies that (43) holds. If (16) does not hold for some k sufficiently large, the mechanism of Algorithm DFFARC then implies that (17) holds. This implies

$$f(x_{k+1}) - f(x_k) \leq -\gamma_f \alpha_k \|h(x_k)\|^2, \tag{44}$$

for all $k \geq K$. Hence, for all $j = 1, 2, \ldots,$

$$f(x_{K+j}) = f(x_K) + \sum_{k=K}^{K+j-1} (f(x_{k+1}) - f(x_k)) \leq f(x_K) - \sum_{k=K}^{K+j-1} \gamma_f \alpha_k \|h(x_k)\|^2.$$

In order to get a contradiction, we suppose that $\|h(x_k)\| > \epsilon$. Thus,

$$\sum_{k=K}^{K+j-1} \gamma_f \alpha_k \epsilon^2 \leq \sum_{k=K}^{K+j-1} \gamma_f \alpha_k \|h(x_k)\|^2 \leq f(x_K) - f(x_{K+j}).$$

Since $f(x_{K+j})$ is bounded below as $j \to \infty$, the series on the left-hand side above is bounded as $j \to \infty$, then $\alpha_k \to 0$ as $k \to \infty$.

Assume that α_k given by (24) in Step 4 is the step size to the boundary of box constraints Ω along d_k. If $D_*^{-1}\nabla f(x_*) = 0$ for any i with $[\nu_*]_i = 0$, without loss of generality, assume $[x_*]_i = l_i$ for some i, we get $[\nabla f(x_*)]_i > 0$ since x_* is nondegenerate. As a consequence, $[\nabla f(x_k)]_i > \frac{1}{2}[\nabla f(x_*)]_i > 0$ for k sufficiently large since the smoothness of $f(x)$. From (12), we get that $[g_k]_i > 0$ for k sufficiently large, then $\alpha_k = \frac{|[\bar{v}_k]_i|}{|[d_k]_i|}$. Left multiplying $\bar{D}_k^{-1}$ at the side of (25), we have

$$(\mathrm{diag}\{g_k\}\bar{J}_k + \sigma_k\|\hat{d}_k\|I)d_k = -\bar{D}_k^{-2}(g_k + H_k d_k).$$

Since $\mathrm{diag}\{g_k\}\bar{J}_k + \sigma_k\|\hat{d}_k\|I$ is positive semidefinite, we have

$$\alpha_k = \frac{|[g_k]_i| + \sigma_k\|\hat{d}_k\|}{|[g_k]_i + [H_k d_k]_i|} \geq \frac{|[g_k]_i| + \sigma_k\|\hat{d}_k\|}{\|g_k + H_k d_k\|_\infty}.$$

If $D_*^{-1}\nabla f(x_*) = 0$ with $[\nabla f(x_*)]_i = 0$ for any i, we obtain that $l_i < [x_*]_i < u_i$ from that x_* is nondegenerate, then $l_i < [x_k]_i < u_i$ for k sufficiently large. Obviously, we get $\alpha_k \nrightarrow 0$ as $k \to \infty$ from (24).

Furthermore, if α_k is obtained from Step 4, we have that $\sigma_k \to \infty$ from Assumption 3.6. Recall that $\|\hat{g}_k\| = \|h(x_k)\| > \epsilon$, we can have that $\sqrt{\sigma_k\|\hat{g}_k\|} > \sqrt{\sigma_k \varepsilon} > \kappa_{HB}$ for sufficiently large k, then $\sigma_{k+1} \leq \sigma_k$ from Lemma 3.7. According to the updating role in σ_k, if $\sigma_{k+1} = \gamma_2\sigma_k > \sigma_k$, then $\sigma_k \leq \frac{\kappa_{HB}^2}{\varepsilon} < \gamma_2\frac{\kappa_{HB}^2}{\varepsilon}$ from Lemma 3.7. By induction, $\sigma_0 \leq \max\{\sigma_0, \gamma_2\frac{\kappa_{HB}^2}{\varepsilon}\}$. Suppose the claim that $\sigma_k \leq \max\{\sigma_0, \gamma_2\frac{\kappa_{HB}^2}{\varepsilon}\}$ holds. It is obvious that $\sigma_{k+1} \leq \max\{\sigma_0, \gamma_2\frac{\kappa_{HB}^2}{\varepsilon}\}$ if $\sigma_{k+1} \leq \sigma_k$. If $\sigma_{k+1} > \sigma_k$, then $\sigma_k \leq \frac{\kappa_{HB}^2}{\varepsilon}$, consequently, $\sigma_{k+1} \leq \max\{\sigma_0, \gamma_2\frac{\kappa_{HB}^2}{\varepsilon}\}$ since $\sigma_{k+1} = \gamma_2\sigma_k (\leq \gamma_2\frac{\kappa_{HB}^2}{\epsilon})$. The claim $\sigma_k \leq \sigma_{\max} = \max\{\sigma_0, \frac{\gamma_2\kappa_{HB}^2}{\epsilon}\}$, which contradicts $\sigma_k \to \infty$. Hence, $\alpha_k \nrightarrow 0$, our hypothesis $\|h(x_k)\| > \epsilon$ is impossible. Thus, the conclusion holds. □

Lemma 3.9. *Suppose that Assumptions 3.1-3.6 hold and* $|\mathcal{A}| = \infty$. *If any limit point* x_* *of* $\{x_k\}$ *is nondegenerate, then*

$$\lim_{k\to\infty, k\in\mathcal{A}} \|h(x_k)\| = 0. \tag{45}$$

Proof. In order to obtain a contradiction, we suppose that there exists an infinite subsequence $\{k_i\} \subseteq \mathcal{A}$, such that

$$\|h(x_{k_i})\| \geq \epsilon, \tag{46}$$

for all large k_i and some $\epsilon > 0$. By the definition of k_i, the pair of $(\|h(x_{k_i})\|, f(x_{k_i}))$ is added to the filter. This implies that the filter is not augmented in the square $[\|h(x_{k_i})\| - \gamma_h\epsilon, \|h(x_{k_i})\|] \times [f(x_{k_i}) - \gamma_f\alpha_{k_i}\epsilon^2, f(x_{k_i})]$ or with the intersection of this square with $\mathcal{A}_0$. Observe that the area of each of these squares is $\gamma_h\gamma_f\alpha_{k_i}\epsilon^3$.

Similarly as in the proof in Lemma 3.8, we have $\alpha_{k_i} \nrightarrow 0$ as $k_i \to \infty$. Thus, there exists an infinite subsequence $\{k_{i_j}\} \subseteq \{k_i\}$, such that $\alpha_{k_{i_j}} \geq \epsilon$ as $j \to \infty$, the set $\mathcal{A}_0 \cap \{(\|h\|, f) | f \leq \kappa_f\}$ is completely covered by at most a finite number of such squares, for any choice of $\kappa_f \geq \kappa_{lf}$. Since the filter is keeping on being added, $f(x_{k_{i_j}})$ tends to infinity as $i \to \infty$. Without loss of generality, we assume that $f(x_{k_{i_{(j+1)}}}) \geq f(x_{k_{i_j}})$ for all j sufficiently large, which means that (17) does not hold and then (16) holds. However, (16) implies that

$$\|h(x_{k_{i_{j+1}}})\| \leq (1-\gamma_h)\|h(x_{k_{i_j}})\|,$$

and as a consequence, $\|h(x_{k_{i_j}})\|$ converges to zero since $0 < \gamma_h < 1$ and $\|h(x_{k_{i_j}})\| \leq \kappa_D\kappa_{ng}$, which contradicts (46). Hence our hypothesis is impossible and (45) holds. □

Lemma 3.10. *Suppose that Assumptions 3.1-3.6 hold. If any limit point* x_* *of* $\{x_k\}$ *is nondegenerate, then*

$$\liminf_{k\to\infty} \|h(x_k)\| = 0, \tag{47}$$

furthermore,

$$\liminf_{k\to\infty} \|D_k^{-1}\nabla f(x_k)\| = 0. \tag{48}$$

Proof. (47) is directly from Lemmas 3.8-3.9. We only need to prove (48). Since $|[g_k]_i - [\nabla f(x_k)]_i| \leq \|g_k - \nabla f(x_k)\| \leq \kappa_{eg}\min\{\|\hat{s}_k\|^2, \|h(x_k)\|^2\} \leq \kappa_{eg}\|h(x_k)\|^2$, we can deduce that $[g_k]_i$ and $[\nabla f(x_k)]_i$ have the same sign for k sufficiently large from (47). Consequently, we get that $D_k = \bar{D}_k$ for k sufficiently large. Moreover, we have that

$$\begin{aligned} \|D_k^{-1}\nabla f(x_k)\| &\leq \|[\bar{D}_k^{-1}\nabla f(x_k) - h(x_k)\| + \|h(x_k)\| \\ &\leq \kappa_D\kappa_{eg}\|h(x_k)\|^2 + \|h(x_k)\|, \end{aligned}$$

thus, combining with (47) again, we infer that (48) holds. □

Theorem 3.11. *Suppose that Assumptions 3.1-3.6 hold. If any limit point* x_* *of* $\{x_k\}$ *is nondegenerate, then*

$$\lim_{k\to\infty} \|h(x_k)\| = 0. \tag{49}$$

Proof. If the filter is augmented only a finite number of times, Lemma 3.8 implies the conclusion. If in the other extreme there exists some $K \in \mathbb{N}$, such that the filter is updated by (20) in all iterations $k \geq K$, then the conclusion follows from Lemma 3.9. It remain to consider the case where for all $K \in \mathbb{N}$, there exist $k_1, k_2 \geq K$ with $k_1 \in \mathcal{A}$ and $k_2 \notin \mathcal{A}$. Assume that (49) does not hold. Then there exists a subsequence $\{x_{k_i}\}$ such that $\|h(x_{k_i})\| \geq 2\varepsilon$ and $k_i \notin \mathcal{A}$. From Lemma 3.9, we also have that $\|h(x_{l_i})\| < \varepsilon$ for each k_i, the iterate x_{l_i} is the first iterate after x_{k_i} such that $(\|h(x_{l_i})\|, f(x_{l_i}))$ is included in the filter, that is, $l_i \in \mathcal{A}$. Thus, we have that

$$\|h(x_k)\| \geq \varepsilon, \quad \text{for all } i \text{ with } k_i \leq k < l_i. \tag{50}$$

Obviously, $\mathcal{K} = \{k \in \mathbb{N} \mid k_i \le k < l_i\}$ is infinite, where k_i and l_i are defined as above.

For all $k = k_i, \ldots, l_i - 1 \notin \mathcal{A}$, we have that either (16) or (17) holds. If (16) holds, then,

$$\|h(x_{k+1})\| \le (1-\gamma_h)\|h(x_k)\|.$$

Since $0 < \gamma_h < 1$ and $\|h(x_k)\| \le \kappa_D \kappa_{ng}$, $\|h(x_{k+1})\| \to 0$, which contradicts (50). The DFFARC algorithm implies that (17) holds. Thus,

$$f(x_{k+1}) \le f(x_k) - \gamma_f \alpha_k \|h(x_k)\|^2 < f(x_k) - \gamma_f \alpha_k \varepsilon^2. \tag{51}$$

Furthermore, $\{f(x_k)\}$ is monotonically decreasing and $f(x_{l_i}) \le f(x_{k_i+1})$. Consequently, for all i,

$$f(x_{l_i}) \le f(x_{k_i+1}) < f(x_{k_i}) - \gamma_f \alpha_{k_i} \|h(x_{k_i})\|^2 < f(x_{k_i}) - \gamma_f \alpha_{k_i} \varepsilon^2.$$

This ensures that, for all $K \in \mathbb{N}$, there exists some $i \ge K$ with

$$f(x_{k_{(i+1)}}) \ge f(x_{l_i}), \tag{52}$$

because otherwise the inequality above would imply

$$f(x_{k_{(i+1)}}) < f(x_{l_i}) \le f(x_{k_i+1}) \le f(x_{k_i}) - \gamma_f \alpha_{k_i} \|h(x_{k_i})\|^2 < f(x_{k_i}) - \gamma_f \alpha_{k_i} \varepsilon^2, \tag{53}$$

for all i.

Similar to prove Lemma 3.8, we can to prove that $\alpha_{k \in \mathcal{K}_1} \nrightarrow 0$, where $\mathcal{K}_1 \subseteq \mathbb{N}$ and α_k is the step size to the boundary of box constraints along d_k. Furthermore, from Assumption 3.6, we can get $\sigma_k \to \infty$ as $\alpha_k \to 0$. Recall that $\|\hat{g}_k\| \le \kappa_D \kappa_{ng}$ and $\|\hat{d}_k\| \le 3\sqrt{\frac{\|\hat{g}_k\|}{\sigma_k}}$, thus $\|\hat{d}_k\| \to 0$, consequently, $\|d_k\| \le \kappa_D \|\hat{d}_k\| \to 0$, furthermore, $\theta_k \to 1$ where θ_k is defined in Step 4, then, $s_k = \alpha_k d_k$, $\|\hat{s}_k\| = \alpha_k \|\hat{d}_k\| \to 0$. However,

$$\begin{aligned}
-\bar{\varphi}_k(s_k) &= -\bar{\varphi}(\alpha_k d_k) \\
&\ge \frac{\alpha_k}{6}\|\hat{g}_k\| \min\left\{ \frac{\|\hat{g}_k\|}{\|\hat{H}_k\|}, \frac{\sqrt{3}}{4}\sqrt{\frac{\|\hat{g}_k\|}{\sigma_k}} \right\} \\
&\ge \frac{\alpha_k \varepsilon}{6} \min\left\{ \frac{\varepsilon}{\kappa_H}, \frac{\sqrt{3}}{12}\|\hat{d}_k\| \right\} \\
&= \frac{\sqrt{3}}{72}\varepsilon \|\hat{s}_k\|,
\end{aligned} \tag{54}$$

and

$$\begin{aligned}
f(x_k+s_k) - f(x_k) - \varphi_k(s_k) &= \nabla f(x_k + \xi_k s_k)^T s_k - \hat{g}_k^T \hat{s}_k - \frac{1}{2}\hat{s}_k^T \hat{H}_k \hat{s}_k - \frac{1}{3}\sigma_k \|\hat{s}_k\|^3 \\
&\le \nabla f(x_k + \xi_k s_k)^T s_k - \hat{g}_k^T \hat{s}_k \\
&= \left[\nabla f(x_k + \xi_k d_k) - \nabla f(x_k)\right]^T s_k + \left[\bar{D}_k^{-1} \nabla f(x_k) - \hat{g}_k\right]^T \hat{s}_k \\
&\le \|\nabla f(x_k + \xi_k s_k) - \nabla f(x_k)\| \, \|s_k\| + \kappa_D \|\nabla f(x_k) - g_k\| \, \|\hat{s}_k\| \\
&\le L_g \kappa_D^2 \|\hat{s}_k\|^2 + \kappa_D \kappa_{eg} \|\hat{s}_k\|^3.
\end{aligned}$$

Thus, combining (54) and above, we can have that

$$\rho_k = 1 - \frac{f(x_k+s_k) - f(x_k) - \varphi_k(s_k)}{-\varphi_k(s_k)} \ge 1 - \frac{L_g \kappa_D^2 \|\hat{s}_k\|^2 + \kappa_D \kappa_{eg} \|\hat{s}_k\|^3}{\frac{\sqrt{3}}{72}\varepsilon \|\hat{s}_k\|} \ge \eta_2,$$

for sufficiently large k. Then,

$$f(x_k) - f(x_{k+1}) = f(x_k) - f(x_k + s_k) \ge \eta_2[-\varphi_k(s_k)] \ge \frac{\sqrt{3}}{72}\eta_2 \varepsilon \|\hat{s}_k\|. \tag{55}$$

Consequently,

$$\|x_{k_i}-x_{l_i}\| \le \sum_{k=k_i}^{l_i-1}\|x_k-x_{k+1}\| \le \sum_{k=k_i}^{l_i-1}\kappa_D\|\hat{s}_k\| \le \sum_{k=k_i}^{l_i-1}\frac{72\kappa_D}{\sqrt{3}\eta_2\varepsilon}\left[f(x_k)-f(x_{k+1})\right] \le \frac{72\kappa_D}{\sqrt{3}\eta_2\varepsilon}[f(x_{k_i})-f(x_{l_i})]. \tag{56}$$

Since $f(x_k)$ is convergent, we can get $\|x_{k_i}-x_{l_i}\| \to 0$ from (56). However, $\|h(x_{k_i})-h(x_{l_i})\| = \|\bar{D}_{k_i}^{-1}g_{k_i}-\bar{D}_{l_i}^{-1}g_{l_i}\| \le \kappa_D\|g_{k_i}-g_{l_i}\| + \|\bar{D}_{k_i}^{-1}-\bar{D}_{l_i}^{-1}\|\cdot\|g_{l_i}\|$ and $|[\bar{v}_{k_i}]_j-[\bar{v}_{l_i}]_j| \le |[x_{k_i}]_j-[x_{l_i}]_j| \le \|x_{k_i}-x_{l_i}\|$ implies $\|\bar{D}_{k_i}^{-1}-\bar{D}_{l_i}^{-1}\| \to 0$, consequently, $\|h(x_{k_i})-h(x_{l_i})\| \to 0$ from the fact that $\|g_{l_i}\| \le \kappa_{ng}$, which contradicts $\|h(x_{k_i})-h(x_{l_i})\| \ge \|h(x_{k_i})\| - \|h(x_{l_i})\| \ge \varepsilon$. Hence $\alpha_k \nrightarrow 0$.

Combining (53) with $\alpha_{k_i} \nrightarrow 0$, we get that $f(x_{k_i}) \to -\infty$, which is in contradiction to the fact that $\{f(x_k)\}$ is bounded below. Thus, from (52), there exists a subsequence $\{i_j\}$ of $\{i\}$ such that

$$f(x_{k_{(i_j+1)}}) \ge f(x_{l_{i_j}}). \tag{57}$$

Since $k_{(i_j+1)} \notin \mathcal{A}$ for $k_{(i_j+1)} \in \{k_i\}$ and $\mathcal{F}_{l_{i_j}} \subseteq \mathcal{F}_{k_{(i_j+1)}}$ for $k_{(i_j+1)} > l_{i_j} > k_{i_j}$, otherwise, $k_{(i_j+1)} \le l_{i_j}$ would contradict with the choice of l_{i_j}, it follows from (57) and (20) that

$$\|h(x_{k_{(i_j+1)}})\| \le (1-\gamma_h)\|h(x_{l_{i_j}})\|. \tag{58}$$

Since $l_{i_j} \in \{l_i\} \subseteq \mathcal{A}$ for all j, Lemma 3.9 implies $\lim_{j\to\infty}\|h(x_{l_{i_j}})\| = 0$. Consequently, from (58), we get that $\lim_{j\to\infty}\|h(x_{k_{(i_j+1)}})\| = 0$ which contradicts the fact that $\|h(x_{k_{(i_{j+1})}})\| \ge 2\varepsilon$ since $k_{(i_{j+1})} \notin \mathcal{A}$ for the definition of k_i. Thus, the claim holds. □

4. Properties of the local convergence

Theorem 3.11 shows that any limit point x_* of (1) is a stationary point. In this section, we investigate the local convergence properties of the proposed algorithm. It needs the following assumptions.

Assumption 4.1. $\lim_{k\to\infty}\frac{\|(D_k^{-1}\nabla^2 f(x_k)D_k^{-1}-\bar{D}_k^{-1}H_k\bar{D}_k^{-1})\hat{d}_k\|}{\|\hat{d}_k\|} = 0$ *as* $\lim_{k\to\infty}\|h(x_k)\| = 0$.

Assumption 4.2. *Assume that x_* satisfies the strong second-order condition, that is,*

$$\exists\kappa_c > 0 \quad such \quad that \;\; d^T D(x_*)^{-1}\nabla^2 f(x_*)D(x_*)^{-1}d \ge 2\kappa_c\|d\|^2, \quad \forall d \ne 0.$$

Assumption 4.3. *[22] Assume that*

$$\operatorname{sign}([\nabla f(x)]_i) = \operatorname{sign}([g(x)]_i), \;\; \forall x \in B(x_*,\delta), \text{ when } [\nabla f(x_*)]_i = 0, \;\; i = 1,2,\ldots,n.$$

Noting that $D(x_*)^{-2}\nabla f(x_*) = 0$ is equivalent to $D(x_*)^{-1}\nabla f(x_*) = 0$ from the definition of $D(x)$, we have that the fact $\|D_k^{-2}\nabla f(x_k)\| \to 0$ is equivalent to the fact $\|D_k^{-1}\nabla f(x_k)\| \to 0$. Let $h_1(x) = \bar{D}(x)^{-2}g(x)$. From (12), the fact $\|h_1(x_k)\| \to 0$ is equivalent to the fact $\|h(x_k)\| \to 0$, as $k \to \infty$. In order to illustrate the properties of the local convergence, we substitute $h(x_k)$ for $h_1(x_k)$ in Step 4.4. That is, we accept the trial step $\alpha_{k,l}$ if

$$\|h_1(x_k(\alpha_{k,l}))\| \le (1-\gamma_h)\|h_1(x_k)\|$$

with

$$x_k + \alpha_{k,l}d_k \in \Omega$$

holds in Step 4.4 in the following sections.

Theorem 4.1. *Suppose that Assumptions 3.1-3.6, 4.1-4.3 hold. If any limit point x_* of $\{x_k\}$ is nondegenerate, then $\|d_k\| \to 0, \alpha_k \equiv 1$ for sufficiently large k and all iterations eventually satisfy $\rho_k > \eta_2, \sigma_k$ is bounded from above as $k \to \infty$.*

Proof. Under the Assumptions in Theorem 4.1, Theorem 3.11 provides that $\|h(x_k)\| \to 0$ as $k \to \infty$. Consequently, Assumption 4.1 implies that

$$\lim_{k\to\infty} \frac{\|(D_k^{-1}\nabla^2 f(x_k)D_k^{-1} - \bar{D}_k^{-1}H_k\bar{D}_k^{-1})\hat{d}_k\|}{\|\hat{d}_k\|} = 0. \tag{59}$$

Thus, for sufficiently large k,

$$\hat{d}_k^T \left(D_k^{-1}\nabla^2 f(x_k)D_k^{-1} - \bar{D}_k^{-1}H_k\bar{D}_k^{-1}\right)\hat{d}_k \le \kappa_c\|\hat{d}_k\|^2, \tag{60}$$

where κ_c is defined in Assumption 4.2. And from Assumptions 3.1, 4.2, we also have that,

$$\hat{d}_k^T D_k^{-1}\nabla^2 f(x_k)D_k^{-1}\hat{d}_k \ge 2\kappa_c\|\hat{d}_k\|^2,$$

for sufficiently large k. Combining (60) and above, we can infer that,

$$2\kappa_c\|\hat{d}_k\|^2 \le \hat{d}_k^T\left[D_k^{-1}\nabla^2 f(x_k)D_k^{-1} - \bar{D}_k^{-1}H_k\bar{D}_k^{-1}\right]\hat{d}_k + \hat{d}_k^T\bar{D}_k^{-1}H_k\bar{D}_k^{-1}\hat{d}_k \le \kappa_c\|\hat{d}_k\|^2 + \hat{d}_k^T\bar{D}_k^{-1}H_k\bar{D}_k^{-1}\hat{d}_k.$$

As a consequence, $\hat{d}_k^T\bar{D}_k^{-1}H_k\bar{D}_k^{-1}\hat{d}_k \ge \kappa_c\|\hat{d}_k\|^2$. So $\bar{D}_k^{-1}H_k\bar{D}_k^{-1}$ and $\hat{H}_k$ are positive definite. And the following relations are from (25) and the Cauchy-Schwarz inequality,

$$\kappa_c\|\hat{d}_k\|^2 \le \hat{d}_k^T\bar{D}_k^{-1}H_k\bar{D}_k^{-1}\hat{d}_k \le \hat{d}_k^T[\hat{H}_k + \sigma_k\|\hat{d}_k\|I]\hat{d}_k \overset{(25)}{=} -\hat{g}_k^T\hat{d}_k \le \|\hat{g}_k\|\|\hat{d}_k\|.$$

The first and last terms above give

$$\|\hat{d}_k\| \le \frac{\|\hat{g}_k\|}{\kappa_c} = \frac{\|h(x_k)\|}{\kappa_c}, \tag{61}$$

since $\hat{d}_k \ne 0$. Otherwise, $-\varphi_k(d_k) = 0$ from Lemma 3.1. This, however, contradicts $-\varphi_k(d_k) > 0$ since $\hat{g}_k \ne 0$. Thus, (61) gives that $\|\hat{d}_k\| \to 0$ from the fact that $\|h(x_k)\| \to 0$. Furthermore, $\|d_k\| \le \kappa_D\|\hat{d}_k\| \to 0$ and $\theta_k \to 1$, where θ_k is defined in Step 4. Then $s_k = \alpha_k d_k$, $\hat{s}_k = \alpha_k\hat{d}_k$ for sufficiently large k.

Assumption 4.3 provides that $[g_k]_i$ and $[\nabla f(x_k)]_i$ have the same sign for sufficiently large k if $\nabla[f(x_*)]_i = 0$. On the other side, (12) gives that $[g_k]_i$ and $[\nabla f(x_k)]_i$ have the same sign for sufficiently large k if $[\nabla f(x_*)]_i \ne 0$ for some i. Thus, $D_k = \bar{D}_k$ for sufficiently large k.

Now we consider the tendency of σ_k. From the definition of ρ_k in Step 7, we have that

$$\rho_k = 1 - \frac{f(x_k + s_k) - f(x_k) - \varphi_k(s_k)}{-\varphi_k(s_k)}. \tag{62}$$

However, using a Taylor expansion, for some $\xi_k \in (0, 1)$,

$$\begin{aligned} f(x_k + s_k) - f(x_k) - \varphi_k(s_k) &= \nabla f(x_k)^T s_k + \frac{1}{2}s_k^T\nabla^2 f(x_k + \xi_k s_k)s_k - g_k^T s_k - \frac{1}{2}\hat{s}_k^T\hat{H}_k\hat{s}_k - \frac{1}{3}\sigma_k\|\hat{s}_k\|^3 \\ &\le (\nabla f(x_k) - g_k)^T s_k + \frac{1}{2}s_k^T\nabla^2 f(x_k + \xi_k s_k)s_k - \frac{1}{2}s_k^T H_k s_k \\ &\le \kappa_D\kappa_{eg}\|\hat{g}_k\|\|\hat{s}_k\|^2 + \frac{1}{2}\kappa_D^2\|\hat{s}_k\|^2\|\nabla^2 f(x_k + \xi_k s_k) - \nabla^2 f(x_k)\| \\ &\quad + \frac{1}{2}\|\hat{s}_k\|^2\frac{\left\|(D_k^{-1}\nabla^2 f(x_k)D_k^{-1} - \bar{D}_k^{-1}H_k\bar{D}_k^{-1})\hat{s}_k\right\|}{\|\hat{s}_k\|}, \end{aligned} \tag{63}$$

where the first inequality is from the fact that $\mathrm{diag}\{g_k\}\bar{J}_k$ is positive semidefinite.
And

$$\varphi_k(s_k) = \hat{g}_k^T\hat{s}_k + \frac{1}{2}\hat{s}_k^T\hat{H}_k\hat{s}_k + \frac{1}{3}\sigma_k\|\hat{s}_k\|^3$$

$$\begin{aligned}
&\overset{(25)}{=} && -\left[\hat{s}_k^T(\hat{H}_k+\sigma_k\|\hat{s}_k\|I)\hat{s}_k\right]+\frac{1}{2}\hat{s}_k^T\hat{H}_k\hat{s}_k+\frac{1}{3}\sigma_k\|\hat{s}_k\|^3\\
&= && -\frac{1}{2}\hat{s}_k^T\hat{H}_k\hat{s}_k-\frac{2}{3}\sigma_k\|\hat{s}_k\|^3\\
&\le && -\frac{1}{2}\hat{s}_k^T\bar{D}_k^{-1}H_k\bar{D}_k^{-1}\hat{s}_k\\
&\le && -\frac{\kappa_c}{2}\|\hat{s}_k\|^2. \qquad (64)
\end{aligned}$$

Together (62), (63) with (64), we obtain that

$$\rho_k\ge 1-\frac{\kappa_D\kappa_{eg}\|\hat{g}_k\|+\frac{1}{2}\kappa_D^2\|\nabla^2 f(x_k+\xi_k s_k)-\nabla^2 f(x_k)\|+\frac{1}{2}\frac{\|(D_k^{-1}\nabla^2 f(x_k)D_k^{-1}-\bar{D}_k^{-1}H_k\bar{D}_k^{-1})\hat{s}_k\|}{\|\hat{s}_k\|}}{\frac{\kappa_c}{2}}>\eta_2$$

for sufficiently large k, where the last inequality is from that $\|h(x_k)\|=\|\hat{g}_k\|\to 0$ and Assumptions 3.1, 4.1. Recall the updating role in σ_k in Step 7, we get that

$$\sigma_{k+1}=\gamma_1\sigma_k<\sigma_k,\ \ 0<\gamma_1<1,\ \ k \text{ sufficiently large.}$$

Thus, σ_k is bounded from above and $\sigma_k\|\hat{d}_k\|\to 0$. Clearly, $\|h_1(x_k)\|=\|\bar{D}_k^{-2}g_k\|\ne 0$. Set

$$\omega_k=\frac{1-\gamma_h}{2\delta_1}\|h_1(x_k)\|,$$

where δ_1 is sufficiently small such that

$$\|h_1(x_k)+\nabla h_1(x_k)d_k\|=\|\bar{D}_k^{-2}g_k+(\bar{D}_k^{-2}H_k+\operatorname{diag}\{g_k\}\bar{J}_k)d_k\|\overset{(25)}{=}\sigma_k\|\hat{d}_k\|\|d_k\|\le\omega_k\|d_k\|, \qquad (65)$$

whenever $\|d_k\|\le\delta_1$. Let

$$\varepsilon_k=\frac{1-\gamma_h}{2\delta_2}\|h_1(x_k)\|,$$

where δ_2 is sufficiently small such that

$$\|h_1(x_k+d_k)-h_1(x_k)-\nabla h_1(x_k)d_k\|\le\varepsilon_k\|d_k\|, \qquad (66)$$

whenever $\|d_k\|\le\delta_2$. Such a δ_2 exists by Lemma 1.2 in [11].

Then, it follows that

$$\begin{aligned}
\|h_1(x_k+d_k)\| &\le \|h_1(x_k+d_k)-h_1(x_k)-\nabla h_1(x_k)d_k\|+\|h_1(x_k)+\nabla h_1(x_k)d_k\|\\
&\le \varepsilon_k\|d_k\|+\omega_k\|d_k\|\\
&\le \frac{(1-\gamma_h)}{\delta}\|h_1(x_k)\|\|d_k\|\\
&\le (1-\gamma_h)\|h_1(x_k)\|, \qquad (67)
\end{aligned}$$

whenever $\|d_k\|\le\delta$, where $\delta=\min\{\delta_1,\ \delta_2\}$.

Let α_k be given in (24) in Step 4 denotes the step size along the direction d_k to the boundary. Similar to prove Lemma 3.8, we can also get that $\alpha_k=\frac{|[g_k]_i|+\sigma_k\|\hat{d}_k\|}{|[g_k]_i+[H_kd_k]_i|}$ from the nondegenerate property at x_*. So $\alpha_k\ge 1$ for sufficiently large k since $|[H_kd_k]_i|\le\|H_kd_k\|\le\kappa_{nh}\|d_k\|\to 0$. Therefore, $\alpha_k\equiv 1$ for sufficiently large k. □

Corollary 4.2. *Under the conditions of Theorem 4.1, then*

$$\frac{\|D_{k+1}^{-2}\nabla f(x_{k+1})\|}{\|D_k^{-2}\nabla f(x_k)\|}\to 0,\ \ \textit{as}\ \ k\to\infty, \qquad (68)$$

and

$$\frac{\|x_{k+1}-x_*\|}{\|x_k-x_*\|}\to 0,\ \ \textit{as}\ \ k\to\infty. \qquad (69)$$

Proof. Recalling (12) and the fact that $D_k = \bar{D}_k$ for sufficiently large k, we obtain the following inequalities,

$$\|D_{k+1}^{-1}\nabla f(x_{k+1})\| \leq \|D_{k+1}^{-1}\nabla f(x_{k+1}) - h(x_{k+1})\| + \|h(x_{k+1})\| \leq \left(\kappa_D\kappa_{eg}\|h(x_{k+1})\| + 1\right) \cdot \|h(x_{k+1})\|, \tag{70}$$

and

$$\|D_k^{-1}\nabla f(x_k)\| \geq \|h(x_k)\| - \|h(x_k) - D_k^{-1}\nabla f(x_k)\| \geq (1 - \kappa_D\kappa_{eg}\|h(x_k)\|) \cdot \|h(x_k)\|. \tag{71}$$

Hence,

$$\frac{\|D_{k+1}^{-1}\nabla f(x_{k+1})\|}{\|D_k^{-1}\nabla f(x_k)\|} \leq \frac{(\kappa_D\kappa_{eg}\|h(x_{k+1})\| + 1) \cdot \|h(x_{k+1})\|}{(1 - \kappa_D\kappa_{eg}\|h(x_k)\|) \cdot \|h(x_k)\|} \to 0, \tag{72}$$

as $k \to \infty$. The limit in (72) is from (67). In fact, (67) gives that $\|h_1(x_{k+1})\| = o(\|h_1(x_k)\|)$ since $\|d_k\| \to 0$. Thus, $\|h(x_{k+1})\| = o(\|h(x_k)\|)$. Noting that $\|D_k^{-2}\nabla f(x_k)\| \to 0$ is equivalently to $\|D_k^{-1}\nabla f(x_k)\| \to 0$, we have that

$$\frac{\|D_{k+1}^{-2}\nabla f(x_{k+1})\|}{\|D_k^{-2}\nabla f(x_k)\|} \to 0, \text{ as } k \to \infty.$$

Since x_* is the limit point of $\{x_k\}$, there exist positive scales $\beta_1, \beta_2, \beta_3, \beta_4$, such that

$$\beta_1\|x_{k+1} - x_*\| \leq \|D_{k+1}^{-2}\nabla f(x_{k+1})\| \leq \beta_2\|x_k - x_*\|,$$

and

$$\beta_3\|x_{k+1} - x_*\| \leq \|D_k^{-2}\nabla f(x_k)\| \leq \beta_4\|x_k - x_*\|.$$

Consequently, (69) holds from the inequalities above and (68). □

5. Numerical results

In this section, we present some typical examples to analyze the feasibility and effectiveness of the proposed algorithm. The program is implemented using MATLAB with double precision and runs under MATLAB Version 7.10.0.499 (R2010a) in a notebook I5, 1.8GHz, 8GB of RAM. The selected parameters of our proposed algorithm (called DFFARC) are: $\Delta_0 = 1$, $\epsilon = 10^{-4}$, $\gamma_1 = 0.3$, $\gamma_2 = 1.5$, $\gamma_3 = 0.5$, $\sigma_0 = 0.5$, $\eta_1 = 0.01$, $\eta_2 = 0.9$, $\gamma_h = 10^{-5}$ $\gamma_f = 10^{-5}$, $\tau_1 = 0.25$, $\tau_2 = 0.75$. The computation of DFFARC terminates if $\|\hat{g}_k\| \leq \varepsilon$ is satisfied and $\Delta_k \leq \min(\|\hat{g}_k\|, \|\hat{s}_k\|)$ and the interpolation model is fully quadratic on $B(x_k, \Delta_k)$. We claim that the method of DFFARC fails, when the following condition holds:

(I) the number of iterations is greater than or equal to 1000.
(II) the execution time of the algorithm over 2 hours.

Any run exceeding this is flagged as a failure, which we use the symbol (Failed).

In Tables 1-3, n is the dimension of the problem, NIT means the number of iterations, T expresses the CPU Time in seconds, NF and NG stand for the number of function and gradient evaluations, respectively. The test problems are selected from [1], [21] and [29] with the dimensions from 2 up to 25. We use $(n + 1)(n + 2)/2$ interpolation points to construct quadratic interpolation model for the derivative-free optimization. If the examples are unconstrained, we took the precaution of including the simple bounds $-100 \leq [x_k]_i \leq 100, \ 1 \leq i \leq n$. And for examples from [21] and [29], we use the starting point supplied by the problems. Preliminary results are presented in Tables 1-2 and they show that the proposed algorithm is effective.

Table 1: Numerical results

Problem	n	NIT	$f(x_*)$	$\|\hat{g}(x_*)\|$	NF	NG	T
ARWHEAD	15	10	4.405364e-013	5.042711e-005	157	16	272.772756
Badly scaled augmented Powell's	6	41	0.003207	4.534216e-005	237	73	7.757471
BIGGSB1 (CUTE)	20	3	2.246682e-011	7.612936e-005	32	3	795.998078
	25	3	3.148723e-010	7.711032e-005	357	3	1379.017105
BRKMCC	2	10	0.169042	9.680970e-005	32	11	0.791369
Broyden Tridiagonal	5	10	1.092810e-012	6.631958e-005	46	12	6.076346
	10	21	1.283415e-012	8.810823e-005	126	21	45.759181
	20	17	1.152413e-012	8.504238e-005	279	17	604.257868
Chandrasekhar's H-equation	5	5	5.222661e-012	3.189531e-005	33	5	5.594066
	10	5	3.560478e-013	8.119432e-006	78	5	49.691945
	20	5	9.6210131e-014	2.920578e-006	243	5	674.493165
Complementary	4	7	4.483507e-012	5.011525e-005	32	8	5.225067
DENSCHNA	2	7	1.211647e-013	5.333000e-006	20	11	0.820089
DENSCHNC	2	10	7.037155e-013	2.221761e-005	26	17	0.745808
DENSCHND	3	22	5.313245e-014	4.360934e-006	62	33	1.176189
DENSCHNE	3	33	8.001662e-013	1.760822e-005	95	44	1.525745
DENSCHNF	2	11	1.255654e-013	9.122356e-005	34	13	0.762065
Discrete boundary value	5	4	1.506474e-012	4.079295e-005	29	5	5.729851
	10	4	2.116612e-012	5.260158e-005	74	5	48.584642
	20	3	7.378956e-012	9.808574e-005	236	4	602.704476
DIXON3DQ	10	5	2.811364e-010	7.184837e-005	78	5	42.071922
DQDRTIC	10	11	6.427845e-013	0	93	14	98.860231
	20	6	1.589138e-011	0	243	9	3796.685038
ENGVAL1	2	9	3.677146e-013	2.982002e-005	24	15	0.873645
Exponential function 1	5	10	1.307281e-008	4.601213e-005	42	16	5.992573
	10	9	1.336812e-008	5.451432e-005	85	14	41.939545
	20	8	2.033617e-008	8.827135e-005	248	12	625.532067
Exponential function 2	5	5	8.689089e-011	1.345969e-005	31	7	5.542916
	10	13	0.005203	8.413612e-005	91	24	49.559337
	20	23	4.646012e-010	9.720014e-005	294	26	641.525970
Exponential function 3	5	3	2.791074e-008	5.134633e-005	27	3	6.519383
	10	2	2.262812e-008	8.164931e-005	69	2	55.126244
	20	2	6.893817e-009	3.598932e-005	234	2	645.017193
Ferraris-Tronconi (1986)	2	8	3.901203e-010	1.835368e-005	21	14	0.729169
Function 15	5	13	1.333302e-010	9.449993e-005	56	14	5.651515
	10	218	2.544360e-011	9.935652e-005	717	218	56.510921
Hanbook function	5	6	5.306477e-012	0	34	8	11.690561
Himmelblau function	2	66	7.555989e-014	4.278545e-005	219	90	1.003909
	4	57	1.095483e-012	2.524473e-005	201	77	4.107451
GULF	3	1	32.835000	4.258427e-009	10	1	11.068458
HS3	2	7	3.486251e-007	5.102719e-005	24	7	1.032979
HS4	2	5	2.666667	1.026215e-005	18	5	0.657579

Table 2: Numerical results

Problem	n	NIT	$f(x_*)$	$\|\hat{g}(x_*)\|$	NF	NG	T
Minimal	5	3	1.145281e-014	1.145281e-014	26	4	8.415264
NASTY	2	2	0.500000	0	9	2	0.454205
Penalty I	5	27	3.372686e-007	6.139816e-005	91	37	6.505059
Powell singual	4	2	2.733561e-013	7.362638e-005	18	2	4.186743
	12	3	1.458409e-014	1.161427e-006	97	3	96.617548
	20	3	2.425262e-014	1.511478e-006	237	3	664.935606
Powell (CUTE)	12	5	1.5179e-012	4.4180e-005	103	5	86.614451
	20	5	2.665854e-012	8.595065e-005	240	4	793.190653
Power	10	11	7.158913e-012	0	91	16	98.107606
	20	20	3.700532e-011	0	285	23	1939.458521
Extended PSC1 function	4	11	1.546398	9.208528e-005	36	20	3.465759
QUARTC	5	13	1.546124e-008	3.727713e-005	43	23	5.067744
	10	12	3.142712e-008	5.336147e-005	88	23	35.064146
	20	13	8.129413e-008	3.021472e-005	255	25	509.965435
Raydan 1	5	6	1.500000	1.580859e-005	32	10	5.631576
	10	6	5.500000	1.894513e-005	77	10	40.597998
	20	6	21.000000	8.228471e-005	241	11	490.496471
S201	2	4	2.996375e-014	6.548827e-006	12	7	0.642334
S206	2	11	7.870869e-013	1.774314e-005	35	12	0.597334
S207	2	8	1.330297e-011	6.541827e-005	27	8	0.581048
S261	4	13	2.327938e-008	9.942237e-005	51	13	2.861911
S271	6	6	2.866745e-012	0	40	9	13.431779
S273	6	14	3.317621e-012	0	67	14	4.744452
S283	10	30	1.604658e-009	7.120493e-005	153	30	28.972625
S290	2	14	1.438378e-008	6.331972e-005	32	27	0.761901
S308	2	23	0.773214	7.947815e-005	58	37	0.563101
S311	2	66	7.648325e-014	4.167234e-005	219	90	0.936182
S314	2	10	0.169043	9.6842e-005	32	11	0.608592
S328	2	16	1.744219	1.426438e-007	50	17	0.598852
Singular function	4	13	4.619432e-008	4.114478e-005	39	25	2.885978
	5	13	2.779684e-008	7.171909e-005	45	25	5.565229
	10	16	5.213718e-008	3.464832e-005	100	27	41.841570
SISSER	2	12	9.943707e-009	3.983812e-005	28	23	1.074144
Strictly convex 1	5	7	3.882245e-011	8.809467e-005	37	13	5.181868
	10	8	6.446012e-015	1.134537e-006	82	13	47.690457
Strictly convex 2	5	8	1.917355e-011	1.110369e-005	36	14	5.589002
	10	8	1.819143e-010	7.278062e-005	81	14	47.111603
Tridiagonal exponential	5	4	7.377892e-012	4.923239e-005	28	6	5.262962
	10	4	3.775757e-012	3.128624e-005	73	6	37.796012
Trigonometric	5	8	7.179389e-015	1.695222e-006	37	13	5.886580
	10	9	8.246573e-014	5.730712e-006	83	16	41.248582
Troesch	5	8	1.917036e-012	6.458804e-005	39	11	5.142342
	10	10	1.558365e-011	8.212104e-005	90	13	38.278590
Wood	4	14	2.816335e-009	7.543268e-006	50	22	2.063650
Variable dimensioned	5	9	1.232562e-013	5.241241e-006	45	9	6.698718

From Tables 1-2, we notice that the proposed algorithm provides efficient method for solving the bounded constrained optimization. We also observe that the NG is larger than the NIT. The reason is that we use the filter mechanism to potentially accept the trial point $x_k(\alpha_{k,l})$ as the new iterate. If $x_k(\alpha_{k,l})$ is not acceptable for the filter, it need to evaluate gradient again in the main step. On the other hand, we also observe that DFFARC for the derivative-free optimization needs more function evaluations than the case of gradient evaluations, which since NF in DFFARC includes two parts evaluation, the evaluations in main step, which is equal to NG, and the evaluations of the quadratic interpolation model, which may need the evaluation up to $(n+1)(n+1)/2$.

In the next experiments, we compare the proposed algorithm DFFARC with the existing algorithms: BC-DFO, NEWUOA and SID-PSM in [19]. The numerical results are presented in Table 3. In Table 3, we compare the number of function evaluations needed by each solver to achieve the desired accuracy in the objective function value. We use two different levels of accuracy: 2 and 4 significant figures in $f(x_*)$. We notice that four methods have their own advantages from Table 3.

Table 3: Comparison between DFFARC, BC-DFO, NEWUOA and SID-PIM

		nf DFFRAC		nf BC-DFO		nf NEWUOA		nf SID-PSM	
Problem	n	2fig	4fig	2fig	4fig	2fig	4fig	2fig	4fig
ARWHEAD	15	146	157	16	16	513	579	33	33
BDQRTIC	10	88	Failed	347	435	181	236	180	358
BIGGSB1	25	357	357	35	35	144	341	33	62
BRKMCC	2	12	32	7	13	7	7	8	23
DENSCHNA	3	17	20	Failed	Failed	Failed	Failed	Failed	Failed
DENSCHNC	3	23	26	Failed	Failed	Failed	Failed	Failed	Failed
DENSCHND	3	15	62	58	78	45	45	3	3
DENSCHNE	3	59	95	67	76	87	92	32	32
DENSCHNF	2	23	34	15	18	23	25	58	77
DIXON3DQ	10	72	78	31	31	72	72	124	124
DQDRTIC	10	72	93	44	44	71	71	3	3
ENGVAL1	2	18	24	4	4	17	23	15	52
GULF	3	10	10	197	307	187	336	646	1435
HELIX	3	22	27	57	66	66	75	78	98
HS3	2	18	24	3	8	6	9	6	6
HS4	2	15	18	5	575	7	5	5	Failed
MEXHAT	2	13	39	263	508	64	65	122	3786
NASTY	2	9	9	3	3	Failed	Failed	7	7
POWER	10	70	91	358	704	218	289	3	3
QURTC	20	249	255	Failed	Failed	Failed	Failed	Failed	Failed
S202	2	27	Failed	Failed	Failed	Failed	Failed	Failed	Failed
S203	2	25	Failed	Failed	Failed	Failed	Failed	Failed	Failed
SINEVAL	2	133	Failed	171	177	203	217	414	437
SINGULAR	4	31	39	67	99	60	82	73	113
SISSER	2	22	28	3	8	16	27	5	13
TROESCH	10	72	90	Failed	Failed	Failed	Failed	Failed	Failed
WOOD	4	50	50	Failed	Failed	Failed	Failed	Failed	Failed

To have comprehensive comparisons among the reported results of Table 3, we use the performance profiles as described in [24]. The performance profiles are defined in terms of a performance measure $t_{p,s} > 0$ obtained for each $p \in P$ and $s \in S$, where P and S are the problems set and the solvers set. In this paper, our profiles are based on the numbers of function evaluations. For any pair (p, s) of problem p and solver s, the performance evaluations ratio is defined by

$$r_{p,s} = \frac{t_{p,s}}{\min\{t_{p,s} : s \in S\}}.$$

Note that the best solver for a particular problem attains the lower bound $r_{p,s} = 1$. The convention $r_{p,s} = \infty$ is used when solver s fails to satisfy the convergence test on problem p. The performance profile of a solver $s \in S$ is defined as the fraction of problems where the performance ratio is at most τ, that is,

$$\rho_s(\tau) = \frac{1}{n_p} size\{p \in P : r_{p,s} \leq \tau\}.$$

Figures 1-2 give the performance profiles of these four algorithms for NF with 2 digits accuracy and 4 digits accuracy, respectively. From the performance profiles of these four algorithms, we an observe that DFFARC is better than the other existing algorithms, which shows that our proposed algorithm DFFARC is competitive, especially for 2fig in NF.

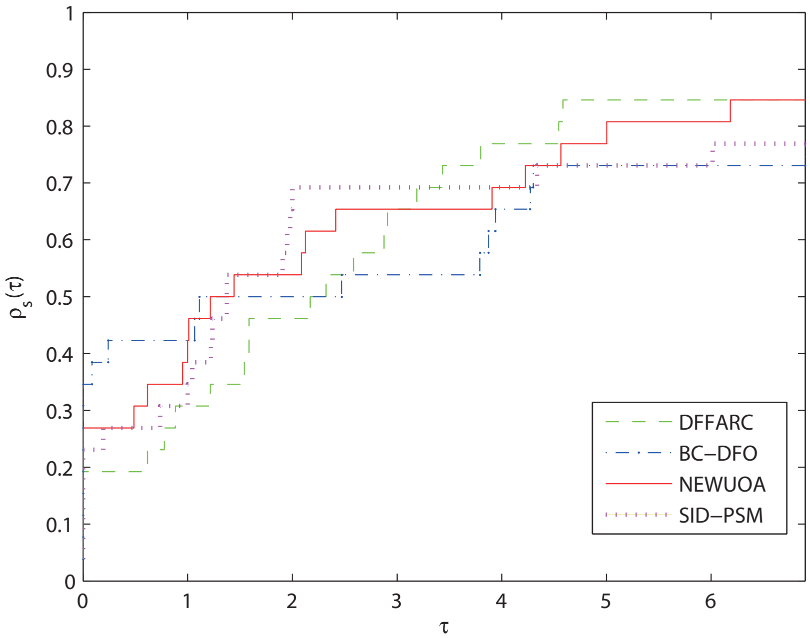

Figure 1: Performance profile for NF of DFFARC, BC-DFO, NEWUOA and SID-PSM algorithms (2fig)

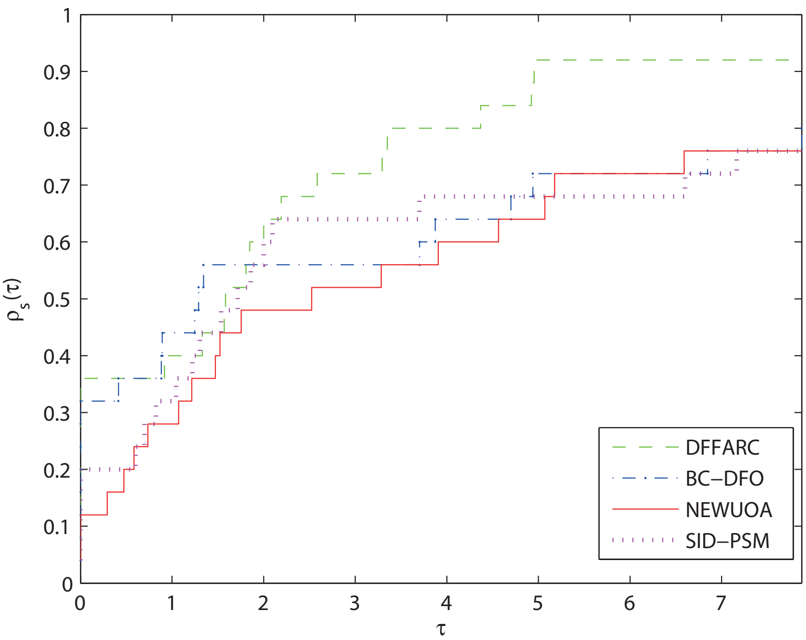

Figure 2: Performance profile for NF of DFFARC, BC-DFO, NEWUOA and SID-PSM algorithms (4fig)

6. Conclusions

In this paper, we propose an adaptive cubic regularization method with line search filter technique for solving derivative-free bound constrained optimization using an interior affine scaling approach. In constrained optimization, the combination of filter and line search techniques proposed by us provides several key advantages, utilizing the strengths of these two methods to improve robustness, efficiency, and convergence. Classical methods (e.g., penalty or barrier functions) require careful tuning of penalty parameters to balance objective minimization and constraint satisfaction. Poor choices can lead to numerical instability or slow convergence. The filter technique directly manages constraints by maintaining a set of non-dominated solutions (balancing objective improvement vs. constraint violation), and line search technique ensures sufficient progress in either the objective or constraint violation at each step.

References

[1] N. Andrei, *An unconstrained optimization test functions collection*, Adv. Model. Optim. 10(2008) 147–161.
[2] C. Cartis, N.I.M. Gould, Ph.L. Toint, *Adaptive cubic regularisation methods for unconstrained optimization. Part I: motivation, convergence and numerical results*, Math. Program. 127(2011) 245–295.
[3] C. Cartis, N.I.M. Gould, Ph.L. Toint, *Adaptive cubic regularisation methods for unconstrained optimization. Part II: worst-case function- and derivative-evaluation complexity*, Math. Program. 130(2011) 295–319.
[4] C. Cartis, N.I.M. Gould, Ph.L. Toint, *On the oracle complexity of first-order and derivative-free algorithms for smooth nonconvex minimization*, SIAM J. Optim. 22(2012) 66–86.
[5] C. Cartis, N.I.M. Gould, Ph.L. Toint, *An adaptive cubic regularization algorithm for nonconvex optimization with convex constraints and its function-evalution complexity*, IMA J. Numer. Anal. 32(2012) 1662–1695.
[6] Th.F. Coleman, Y. Li, *An interior trust region approach for nonlinear minimization subject to bounds*, SIAM J. Optim. 6(1996) 418–445.
[7] A.R. Conn, N.I.M. Gould, Ph.L. Toint, *Trust region methods*, SIAM, Philadelphia, 2000.
[8] A.R. Conn, K. Scheinberg, L.N. Vicente, *Global convergence of general derivative-free trust-region algorithms to first- and second-order critical points*, SIAM J. Optim. 20(2009) 387–415.
[9] A.R. Conn, K. Scheinberg, L.N. Vicente, *Introduction to derivative-free optimization*, SIAM, Philadelphia, 2009.

[10] N. Echebest, M.L. Schuverdt, R.P. Vignau, *A derivative-free method for solving box-constrained underdetermined nonlinear systems of equations*, Appl. Math. Comput. 219(2012) 3198–3208.
[11] S.C. Eisenstat, H.F. Walker, *Globally convergent inexact Newton methods*, SIAM J. Optim. 4(1994) 393–422.
[12] G. Fasano, J.L. Morales, J. Nocedal, *On the geometry phase in model-based algorithms for derivative-free optimization*, Optim. Methods Softw. 24(2009) 145–154.
[13] R. Fletcher, S. Leyffer, *Nonlinear programming without a penalty function*, Math. Program. 91(2002) 239–269.
[14] U.M. García-Palomares, I.J. Garci ía-Urrea, P.S. Rodríguez-Hernández, *On sequential and parallel non-monotone derivative-free algorithms for box constrained optimization*, Optim. Methods Softw. 28(2013), 1233–1261.
[15] N.I.M. Gould, S. Leyffer, Ph.L. Toint, *A multidimensional filter algorithm for nonlinear equations and nonlinear least-squares*, SIAM J. Optim. 15(2004) 17–38.
[16] N.I.M. Gould, M. Porcelli, Ph.L. Toint, *Updating the regularization parameter in the adaptive cubic regularization algorethm*, Comput. Optim. Appl. 53(2012) 1–22.
[17] N.I.M. Gould, C. Sainvitu, Ph.L. Toint, *A filter-trust-region method for unconstrained optimization*, SIAM J. Optim. 16(2005) 341–357.
[18] N.I.M. Gould, Ph. L. Toint, *FILTRANE, a Fortran 95 Filter-Trust-Region Package for Solving Systems of Nonlinear Equalities, Nonlinear Inequalities and Nonlinear Least-Squares Problems*, Technical report 03/15, Rutherford Appleton Laboratory, Chilton, Oxforddhire, UK, 2003.
[19] S. Gratton, Ph. L. Toint, A. Tröltzsch, *An active-set trust-region method for derivative-free nonlinear bound-constrained optimization*, Optim. Methods Softw. 26(2011) 873–894.
[20] A. Griewank, *The modification of Newton's method for unconstrained optimization by bounding cubic terms*, Technical Report NA/12(1981), Departemnt of Applied Mathematics and Therotical physics, University of Cambridge (1981).
[21] W. Hock, K. Schittkowski, *Test examples for nonlinear programming codes*, Lecture Notes in Economics and Mathematics System, 187. Springer (1981).
[22] X. Huang, D. Zhu, *An interior affine cubic regularization algorithm for derivative-free optimization subject to bound constraints*, J. Comput. Appl. Math. 321(2017) 108–127.
[23] D. Li, D. Zhu, *An affine scaling interior trust-region method combining with line search filter technique for optimization subject to bounds on variables*, Numer. Algorithms 77 (2018) 1159–1182.
[24] J.J. Moré, S.M. Wild, *Benchmarking derivative-free optimization algorithms*, SIAM J. Optim. 20(2009) 172–191.
[25] Y. Nesterov, B.T. Polyak, *Cubic regularization of Newton's method and its global performance*, Math. Program. 108(2006) 177–205.
[26] M.J.D. Powell, *The NEWUOA software for unconstrained optimization without derivatives*, Technical Report DAMTP NA2004/08, Department of Applied Mathematics and Theoretical Physics, Cambridge University, Cambridge CB3 9EW, UK, 2004.
[27] M.J.D. Powell, *Developments of NEWUOA for minimization without derivatives*, IMA J. Numer. Anal. 28(2008) 649–664.
[28] M.J.D. Powell, *The BOBYQA algorithm for bound constrained optimization without derivatives*, Technical report, Depattment of Applied Mathematics and Theoretical Physics, University of Cambridge (2009).
[29] K. Schittkowski, *MORE test examples for nonlinear programming codes*, Lecture Notes in Economics and Mathematics System, 187. Springer (1981).
[30] M. Weiser, P. Deuflhard, B. Erdmann, *Affine conjugate adaptive Newton methods for nonlinear elastomechanics*, Optim. Methods Softw. 22(2007) 413–431.
[31] S.M. Wild, C. Shoemaker, *Global convergence of radial basis function trust region derivative-free algorithms*, SIAM J. Optim. 21(2011) 761–781.
[32] D.T. Zhu, *An affine scaling trust-region algorithm with interior backtracking technique for soling bound-constrained nonlinear systems*, J. Comput. Appl. Math. 184(2005) 343–361.