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Saturation numbers for linear forests

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Abstract. Given a graph H, we say a graph G is H-saturated if G does not contain H as a subgraph and the addition of any edge $e \notin E(G)$ results in H as a subgraph. The question of the minimum number of edges of an H-saturated graph on n vertices, known as the *saturation number*.

In this paper, we mainly research the saturation number for linear forests and characterize the extremal graphs.

1. Introduction

1.1. *Basic Definitions*

All graphs considered in this paper are simple, finite and undirected. Let $G = (V(G), E(G))$ be a nontrivial graph with the vertex set $V(G)$ and the edge set $E(G)$ where each edge $e \in E(G)$ is an unordered pair of distinct vertices $u,\ v \in V(G)$. We write $e = uv$ when $e = \{u,\ v\}$ and we say that u is adjacent to v in G. If u is adjacent to exactly k vertices in G, we say that u has degree k, and we write $d_G(u) = k$. The neighborhood of a vertex $u \in V(G)$ is the set of vertices which is adjacent to u, denoted by $N(u)$. And we write $N[u] = N(u) \cup \{u\}$. The minimum (maximum) degree of a graph G is denoted by $\delta(G)$ $(\Delta(G))$. A set of vertices $I \in V(G)$ is an independent set if for all pairs $u,\ v \in I$ the vertices u and v are not adjacent.

For $X \subseteq V(G)$, $G[X]$ denotes the subgraph of G induced by X. P_n, K_n denote the path on n vertices, the complete graph on n vertices.

Given two disjoint graphs G_1 and G_2, the *join* of G_1 and G_2, denoted by $G_1 \vee G_2$, is obtained from the vertex-disjoint copies of G_1 and G_2 by adding all edges between $V(G_1)$ and $V(G_2)$.

Let G and H be graphs. If $V(H) \subset V(G)$ and $E(H) \subset E(G)$, we say that H is a *subgraph* of G. If H is a subgraph of G such that $E(H) = \{uv \mid u, v \in V(H) \text{ and } uv \in E(G)\}$, we say that H is an induced subgraph of G and we may write $H = G[V(H)]$.

We say that two graphs G and H are *isomorphic* if there exists an adjacency-preserving bijection between their vertex sets and we write $G \cong H$. If there exists a subset $V' \subset V(G)$ and a subset $E' \subset E(G)$ such that H is isomorphic to the subgraph $H' = (V', E')$, we say that G *contains a copy* of H.

Note that by our definition above, when we say that G contains a copy of H, the subgraph H' of G such that $H \cong H'$ need not be an induced subgraph of G.

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If we obtain the graph G' from G by adding the edge $e \notin E(G)$, we have $E(G') = E(G) \cup \{e\}$ and we write $G' = G + e$. If $e = uv$, we may write $G' = G + uv$. For undefined notations and terminology, refer to [3]

1.2. Saturation Number

We say that G is *H-saturated* if G contains no copy of H and for all $e \in E(\bar{G})$ the graph $G + e$ does contain a copy of H. Denote by $SAT(n, H)$ the set of all H-saturated graphs of order n.

$$SAT(n, H) = \{G : \ G \text{ is } H - saturated, |V(G)| = n\}.$$

The maximum number of edges possible in a graph G on n vertices that is H-saturated is known as the *Turán* number [14] and is denoted $ex(n, H)$. We use the notation $ex(n, H)$ to indicate the extremal number for the target graph H with respect to H-saturated graphs of order n. That is

$$ex(n, H) = max\{|E(G)| : G \in SAT(n, H)\}.$$

For a family of graphs $\mathscr{F}$, $ex(n, \mathscr{F})$ is the maximum number of edges in an H-saturated graph of order n for any $H \in \mathscr{F}$. The set of all H-saturated graphs of order n having size $ex(n, H)$ is denoted $EX(n, H)$.

$$EX(n, H) = \{G \in SAT(n, H) : |E(G)| = ex(n, H)\}.$$

The *saturation number* of a target graph H with respect to host graphs of order n, denoted $sat(n, H)$, is the minimum number of edges in an H-saturated graph on n vertices. That is,

$$sat(n, H) = min\{|E(G)| : G \in SAT(n, H)\}.$$

For a family of graphs $\mathscr{F}$, $sat(n, \mathscr{F})$ is the minimum number of edges in an $\mathscr{F}$-saturated graph G of order n, that is G is H-saturated for every $H \in \mathscr{F}$. The set of all H-saturated graphs of order n having size $sat(n, H)$ is denoted $Sat(n, H)$.

$$Sat(n, H) = \{G \in SAT(n, H) : |E(G)| = sat(n, H)\}.$$

The original paper established $sat(n, K_k)$ and the uniqueness of the graph in $Sat(n, K_k)$ by *Erdös*, Hajnal and Moon in [7].

Theorem 1.1. *([7]) If $2 \le k \le n$, then $sat(n, K_k) = (k-2)(n-k+2) + \binom{k-2}{2} = \binom{n}{2} - \binom{n-k+2}{2}$ and $Sat(n, K_k)$ contains only one graph, $K_{k-2} + \overline{K_{n-k+2}}$.*

In 1986 *Kászonyi* and Tuza [11] found the best known general upper bound for $sat(n, F)$, where F is a class of forbidden graphs. Since then, $sat(n, F)$ and $SAT(n, F)$ have been investigated for a range of graphs F, including unions of cliques [2], nearly complete graphs[10], tripartite graphs [13], and cycles [4]. For a summary of known results see [6].

Chen et al. [5] focused on the saturation numbers for the linear forests and obtained a series of interesting results and proposed a few conjectures, one of which is about the saturation number for linear forests $P_k \cup tP_2$. In [8], [12] they also discussed the saturation numbers for the linear forests.

2. Main Results

In this work, we will focus on the saturation number of the linear forests $P_6 \cup tP_2$. Our main result is as follows.

Theorem 2.1. *Let n and t be two positive integers. And $n = 10q + r \ge \frac{10}{3}t + 10 - [\frac{r}{3}]/3$. Then*
(i) $sat(n, P_6 \cup tP_2) = min\{n - [\frac{n}{10}], 3t + 18\}$, and
(ii) $Sat(n, P_6 \cup tP_2) = \{K_7 \cup (t-1)K_3 \cup \overline{K_{n-3t-4}}\}$ for $n > \frac{10}{3}t + 20$ and $n = 10q + r$, $1 \le r \le 9$ or $n \ge \frac{10}{3}t + 20$ and $n = 10q$.

2.1. Preliminaries

Lemma 2.2. (The Berge-Tutte Formula, [1])
For a graph G,

$$\alpha'(G) = \frac{1}{2}min\{|G| + |X| - o(G - X) : X \subseteq V(G)\},$$

where $o(G)$ denotes the number of odd components of G and $\alpha'(G)$ denotes the number of edges in a maximum matching of G.

Lemma 2.3. *Let G be a $(P_6 \cup tP_2)$-saturated graph, where t is a positive integer. Then G has the following properties.*
(i) *If $N_G(x) = \{u,\ v\}$, then $uv \in E(G)$.*[5]
(ii) *If $|V_0(G)| > 0$, then $V_1(G) = \emptyset$, $V_i(G)$ is the set of vertices of G with degree i.*
(iii) *If $|V_0(G)| > 0$, for any $x \in V(G) - V_0(G)$, we have $N_G(x) \subseteq V(F)$ where F is some copy of $P_6 \cup tP_2$ in $G + xy$ and $y \in V_0(G)$.*

Proof. (ii) If $V_1(G) \neq \emptyset$, let x_1 be one vertex in $V_1(G)$ and x_1x_2 be the edge incident to x_1 ($d_G(x_2) \geq 1$). Then the graph $G + x_2x_3$ contains a copy F of $P_6 \cup tP_2$ where $x_3 \in V_0(G)$ (obviously $x_1 \notin V(F)$). By replacing the edge x_2x_3 with x_1x_2, we get a copy of $P_6 \cup tP_2$ in G, a contradiction. Therefore, $V_1(G) = \emptyset$.
(iii) If there is one vertex $x \in V(G) - V_0(G)$ satisfying $N_G(x) \nsubseteq V(F)$ where F is any copy of $P_6 \cup tP_2$ in $G + xy$ and $y \in V_0(G)$, then there exists a vertex $x' \in N_G(x)$ such that $x' \in N_G(x) - V(F)$. Since G is a $(P_6 \cup tP_2)$-saturated graph, $G + xy$ contains a copy of $P_6 \cup tP_2$, say F. Thus $xy \in E(F)$. Then by replacing the edge xy with xx' in F, we obtain a copy of $P_6 \cup tP_2$ in G, a contradiction. □

Lemma 2.4. *Let G be a $(P_6 \cup tP_2)$-saturated graph with $|V_0(G)| \geq 2$ and H be the graph spanned by all the nontrivial components $H_1, \ldots, H_k$ of G, where k is the number of components of G. If $|V(H)| \geq 2t + 6$, $\delta(H) \geq 2$ and $|V(H_i)| \geq 5$ for $1 \leq i \leq k$, then*
(i) *G is a $(P_4 \cup (t + 1)P_2)$-saturated graph, and*
(ii) $|E(G)| > 3t + 18$.

Proof. (i) Since G is a $(P_6 \cup tP_2)$-saturated graph, the additional edge $e \in E(\overline{G})$ will result in a $(P_6 \cup tP_2)$ in $G + e$. Hence, for any edge $e \in E(\overline{G})$, $G + e$ contains a copy of $P_4 \cup (t + 1)P_2$.

If G is not a $(P_4 \cup (t + 1)P_2)$-saturated graph, then G contains a copy of a $P_4 \cup (t + 1)P_2$. Let M be a copy of $P_4 \cup sP_2$ in G such that s is maximum. Then, $s \geq t + 1$.

Since $\delta(H) \geq 2$ and $|V(H_i)| \geq 5$ for $1 \leq i \leq k$, every component H_i of G contains a copy of P_4. P_4 contains a copy of $2P_2$. We consider two cases depending on $|E(H_i) \cap M|$.

Case 1. $\forall i,\ i \in [k]$, $|E(H_i) \cap M| = 2$.

H_i contains a copy of P_4. Assume that $P_4 = x_1x_2x_3x_4$. Since $|V_0(G)| \geq 2$ and G is $(P_6 \cup tP_2)$-saturated, there exists one component H_j ($j \in [k]$) such that $|V(H_j)| > 5$. By the choice of M, $V(H_j) - V(P_4)$ is independent. $\forall x \in V(H_j) - V(P_4)$, if x is adjacent to one end of P_4, it is easily verified that $H_j[\{x, x_1, x_2, x_3, x_4, y\}]$ contains a copy of P_6 for $y \in V(H_j) - V(P_4),\ y \neq x$, which together with the edges in $M - E(H_j[\{x, x_1, x_2, x_3, x_4, y\}])$ forms a copy of $(P_6 \cup tP_2)$, a contradiction. Therefore, $N(x) = \{x_2,\ x_3\}$, $\forall x \in V(H_j) - V(P_4)$. Then $G + e$ will not contain P_6 for the additional edge $e = x_2\omega$ with $\omega \in V_0(G)$, a contradiction.

Case 2. There exists one component H_i such that $|E(H_i) \cap M| > 2$ for $i \in [k]$.

Since every component of G contains a copy of P_4, H_i contains a copy of $P_4 \cup rP_2$. If $r = 0$, then $|E(H_i) \cap M| = 3$. If there exists one component H_j such that $|V(H_j)| > 5$ and $|E(H_j) \cap M| = 3$, we can obtain a contradiction through discussions similar to Case 1. Hence $|V(H_i)| = 5$ for $i \in [k]$. Then $G + e$ will not contain P_6 for the additional edge $e = \omega_1\omega_2$ with $\omega_1,\ \omega_2 \in V_0(G)$, a contradiction. Therefore we just discuss $r > 1$. Assume that $P_4 = x_1x_2x_3x_4$. It is obvious that x_1 is not incident with any edge $e = u_jv_j \in rP_2$.

Subcase 2.1. $V(H_i) - V(M) = \emptyset$.

Since x_1 is not incident with any edge $e \in rP_2$, x_2 (or x_3) is incident with one edge $e = u_jv_j \in rP_2$. If $x_1x_4 \notin E(H_i)$, then $N(x_1) = N(x_4) = \{x_2, x_3\}$ for $\delta(H) \geq 2$. And $H_i[\{x_1, x_2, x_3, x_4, u_j, v_j\}]$ contains a copy of P_6, which together with the edges in $M - E(P_4) - e$ forms a copy of $(P_6 \cup tP_2)$, a contradiction. If $x_1x_4 \in E(H_i)$, we also obtain a contradiction by the similar arguments.

Subcase 2.2. $V(H_i) - V(M) \neq \emptyset$.

By the choice of M, $V(H_i)-V(M)$ is independent. We assume $x \in V(H_i)-V(M)$. By the above arguments, x_i is not incident with any edge in rP_2 for $i = 1, 2, 3, 4$. Therefore, x is incident with one edge $e = u_jv_j \in rP_2$ and x is adjacent to x_i for some $i \in \{1, 2, 3, 4\}$. Then $H_i[\{x, x_1, x_2, x_3, x_4, u_j, v_j\}]$ contains a copy of P_6, which together with the edges in $M - E(P_4) - e$ forms a copy of $(P_6 \cup tP_2)$, a contradiction.

(ii) Assume that $|E(G)| \leq 3t + 18$.

It follows from (i) that H is $(P_4 \cup (t+1)P_2)$-saturated. And every component of H contains a copy of P_4, $\alpha'(H) = t + 2$. By the Berge-Tutte Formula, we choose a subset S of $V(H)$ such that

$$t + 2 = \frac{1}{2}min\{|H| + |X| - o(H - X) : X \subseteq V(H)\} = \frac{1}{2}\{|H| + |S| - o(H - S)\}$$

.

Let $H'_1, H'_2, \cdots, H'_l$ be the components of $H - S$.

Claim 1. $H[S \cup V(H'_i)]$ is a clique for $i \in [l]$.

Proof. If not, there exist $x,\ y \in S \cup V(H'_i)$ such that $xy \notin E(H)$. Set $H' = H + xy$. Then H' contains a copy of $(P_4 \cup (t+1)P_2)$ and $\alpha(H') \geq t + 3$. Then $o(H' - S) = o(H - S)$. Therefore $\frac{1}{2}\{|H| + |S| - o(H' - S)\} = \frac{1}{2}\{|H| + |S| - o(H - S)\} = t + 2$. By the Berge-Tutte Formula, $\alpha(H') \leq t + 2$, a contradiction. □

Claim 2. $S \neq \emptyset$.

Proof. If $S = \emptyset$, H_i is a complete graph by Claim 1 for $1 \leq i \leq k$. Since $|V_0(G)| \geq 2$, there exists one component H_i with $|V(H_i)| \geq 6$. Without loss of generality, we assume $|V(H_1)| \geq 6$. Then

$$\begin{aligned} 2|E(H)| &= \sum_{u \in V(H)} d_H(u) = \sum_{i=1}^{k} |V(H_i)|(|V(H_i)| - 1) \\ &\geq 4|V(H)| + |V(H_1)|(|V(H_1)| - 5) + \sum_{|(H_i)| \geq 6,\ H_i \neq H_1} |V(H_i)|(|V(H_i)| - 5), \end{aligned}$$

This together with $|V(H)| \geq 2t + 6$ and $|E(H)| = |E(G)| \leq 3t + 18$ implies that $|V(H_1)| = 6$, $|V(H_i)| = 5$ for $i \neq 1$ and $t \leq 3$. Hence $|V(H)| = 5k + 1 \geq 2t + 6$. It implies that $k \geq 2$ and if $k = 2$, then $t = 2$. $|E(H)| = 10k + 5 \leq 3t + 18$. implies that $k \leq 2$ and if $k = 2$, then $t = 3$, a contradiction. □

Combining Claim 1 with Claim 2, $N_H(x) = V(H) - \{x\}$ for $x \in S$. For $y \in V_0(G)$, we have $\{x,\ y\} \cup N_G(x) \subseteq V(F)$ by Lemma 2.3(iii), where F is a copy of $P_6 \cup tP_2$ in $G + xy$. And $2t + 7 \leq |H| + 1 = |\{x,\ y\} \cup N_H(x)| \leq |V(F)| = 2t + 6$, a contradiction. □

Lemma 2.5. *Let $G \in Sat(n, P_6 \cup tP_2)$ with $|V_0(G)| \geq 2$, where $n \geq 3t + 6$ and $t \geq 1$. If $|E(G)| \leq 3t + 18$, then $|E(G)| = 3t + 18$ and $G \cong K_7 \cup (t-1)K_3 \cup (n - 3t - 4)K_1$.*

Proof. It follows from Lemma 2.3 that $V_1(G) = \emptyset$. It is easily verified that the components of order 3 in G are complete. If there exists one component H of order 4 in G, it is clear that H is complete. $G + xy$ does not contain a copy of $P_6 \cup tP_2$ for $x \in V(H)$, $y \in V_0(G)$. Therefore the components of order 4 in G are empty.

Set $G' = G - t_3K_3$, where t_3 is the number of components of G with order 3. That is, $G \cong G' \cup t_3K_3$. Since $V_0(G') = V_0(G)$, $G + xy$ contains a copy of $P_6 \cup tP_2$ where $x,\ y \in V_0(G)$, G' contains a copy of P_6. Then $t_3 \leq t - 1$. That is, $t - t_3 \geq 1$.

Set $t' = t - t_3$. Then $G' \in Sat(n', P_6 \cup t'P_2)$ where $n' = n - 3t_3$. Let H' be the graph spanned by all nontrivial components of G'. It follows from Lemma 2.3 that $V_1(G') = \emptyset$. That is, $\delta(H') \geq 2$. This together with $|V_0(G')| \geq 2$ and $|E(G')| = |E(G)| - 3t_3 \leq 3t' + 18$ implies that $|H'| \leq 2t + 5$ by Lemma 2.4. It is obvious that $H' \cong K_{2t'+5}$. Since $|E(H')| = |E(G')| \leq 3t' + 18$, it follows that $t' = 1$ and $H' \cong K_7$. Therefore $G' \cong K_7 \cup (n' - 7)K_1$ and $t_3 = t - 1$. Then $G \cong G' \cup t_3K_3 \cong K_7 \cup (t-1)K_3 \cup (n - 3t - 4)K_1$. □

2.2. Proof of Theorem 2.1

In this section, we prove Theorem 2.1. For a graph F, let $SAT^*(n,F)$ be the set of F-saturated graphs G of order n with $V_0(G) = \emptyset$. The minimum number of edges in a graph in $SAT^*(n,F)$ is denoted by $sat^*(n,F)$.

Recall that a perfect degree three tree is a tree such that every vertex has degree 3 or degree 1, and all vertices of degree 1 are the same distance from the center. For $k \geq 2$, we will let T_k denote the perfect degree three tree whose longest path contains k vertices.

Lemma 2.6. *Let G be a $(P_6 \cup tP_2)$-saturated graph and H_1, H_2 two nontrivial components of G. If both H_1 and H_2 are trees, then each tree component contains a copy of T_5. Hence, $|V(H_1)| \geq 10$ and $|V(H_2)| \geq 10$.*

Proof. For $i = 1,\ 2$, let v_i be a pendant vertex of H_i with $N_G(v_i) = \{\omega_i\}$. Then $G + \omega_1\omega_2$ contains a copy of $P_6 \cup tP_2$, denoted by F. Then $\omega_1\omega_2 \in E(F)$. If $\omega_1\omega_2$ is not in the P_6 of F, we obtain a copy of $P_6 \cup tP_2$ in G by replacing the edge $\omega_1\omega_2$ with $\omega_1 v_1$, a contradiction. Thus $\omega_1\omega_2$ is in the copy of P_6 of F. Without loss of generality, we may assume that H_1 contains a copy of $P_4 = \omega_1 u_2 u_3 u_4$ starting from ω_1. Then $P = v_1\omega_1 u_2 u_3 u_4$ is a copy of P_5 in H_1. If $N_G(\omega_1) = \{v_1,\ u_2\}$, then by Lemma 2.3(i) $v_1 u_2 \in E(H_1)$ and H_1 contains a triangle, a contradiction. Thus, $d_G(\omega_1) \geq 3$. Similarly, $d_G(u_2) \geq 3$ and $d_G(u_3) \geq 3$. We assume that $\{v_1,\ u_2,\ x\} \subseteq N_G(\omega_1)$, $\{\omega_1,\ u_3,\ y\} \subseteq N_G(u_2)$ and $\{u_2,\ u_4,\ z\} \subseteq N_G(u_3)$ with $x \neq y \neq z$. If $d_G(y) = 1$, then $G + \omega_1 u_3$ contains a copy of $P_6 \cup tP_2$ and $\omega_1 u_3$ is in the P_6. We consider the following cases.

Case 1. u_2 is a vertex of P_6.

Then there exists a subpath P' such that $u_2\omega_1 u_3 \subseteq P' \subseteq P_6$ (or $u_2 u_3 \omega_1 \subseteq P' \subseteq P_6$). And we obtain a copy of $P_6 \cup tP_2$ in G by replacing P' with $v_1\omega_1 u_2 u_3$ or $\omega_1 u_2 u_3 u_4$, a contradiction.

Case 2. P_6 starts from ω_1 (or P_6 starts from u_3).

Then we get a copy of $P_6 \cup tP_2$ in G by replacing $\omega_1 u_3$ with $u_2\omega_1$ (or $u_2 u_3$), a contradiction.

Case 3. P_6 starts from v_1.

By replacing $v_1\omega_1 u_3$ with $y u_2 u_3$, we obtain a copy of $P_6 \cup tP_2$ in G, a contradiction.

Case 4. P_6 starts from x (or P_6 starts from z or P_6 starts from u_4).

If P_6 starts from x, we assume that $P_6 = x\omega_1 u_3 z z_1 z_2$ (or $P_6 = x\omega_1 u_3 u_4 t_1 t_2$) where $z_1 \in N_G(z)$ and $z_2 \in N_G(z_1)$ (or $t_1 \in N_G(u_4)$ and $t_2 \in N_G(t_1)$). It follows from Lemma 2.3(i) that $d_G(z) \geq 3$ (or $d_G(u_4) \geq 3$). We assume that $\{u_3,\ z_1,\ z_1'\} \subseteq N_G(z)$ (or $\{u_3,\ t_1,\ t_1'\} \subseteq N_G(u_4)$). We replace P_6 by $v_1\omega_1 u_2 u_3 z z_1$ (or $v_1\omega_1 u_2 u_3 u_4 t_1$), then we have a copy of $P_6 \cup tP_2$ in G, a contradiction. If P_6 starts from z or P_6 starts from u_4, we obtain a copy of $P_6 \cup tP_2$ in G by similar arguments, a contradiction.

Case 5. P_6 starts from x_1 where $x_1 \in N_G(x)$ (or P_6 starts from z_1 where $z_1 \in N_G(z)$ or P_6 starts from t_1 where $t_1 \in N_G(u_4)$).

If P_6 starts from x_1 where $x_1 \in N_G(x)$, we assume that $P_6 = x_1 x\omega_1 u_3 z z_1$ where $z_1 \in N_G(z)$ or $P_6 = x_1 x\omega_1 u_3 u_4 t_1$ where $t_1 \in N_G(u_4)$. Then G contains a copy of $P_6 \cup tP_2$ by replacing P_6 by $x\omega_1 u_2 u_3 z z_1$ (or $x\omega_1 u_2 u_3 u_4 t_1$), a contradiction. If P_6 starts from z_1 where $z_1 \in N_G(z)$ or P_6 starts from t_1 where $t_1 \in N_G(u_4)$, we also obtain contradictions by similar arguments.

Therefore, $d_G(y) \neq 1$. By Lemma 2.3(i), $d_G(y) \geq 3$. And we assume that $\{\ u_2,\ y_1,\ y_2\} \subseteq N_G(y)$. Then $H_1[\{v_1,\ x,\ \omega_1,\ u_2,\ u_3,\ u_4,\ z,\ y,\ y_1,\ y_2\}]$ is an induced T_5 in H_1. This completes the proof of Lemma 2.6. □

Theorem 2.7. *For $n = 10q + r \geq \frac{10}{3}t + 10 - [\frac{r}{3}]/3$, $sat^*(n, P_6 \cup tP_2) = n - [\frac{n}{10}]$.*

Proof. Denotes $n = 10q + r$, where $q = [\frac{n}{10}]$, $0 \leq r \leq 9$.

If $0 \leq r \leq 2$, consider the graph $G_n = (q-1)T_5 \cup T_5^*$ where T_5^* denotes the graph obtained from by attaching r leaves to T_5 and maintaining the degree of the center 3. And T_5^* has r leaves more than T_5.

If $3 \leq r \leq 5$, consider the graph $G_n = (q-1)T_5 \cup T_5^{**}$ where T_5^{**} denotes the graph obtained from by attaching one vertex to the center of T_5 and attaching the remain $r-1$ leaves to the previous vertex.

If $6 \leq r \leq 8$, consider the graph $G_n = (q-1)T_5 \cup T_5^{***}$ where T_5^{***} denotes the graph obtained from by attaching two vertices to the center of T_5 and attaching two leaves to one of the previous two vertices and attaching the remain $r-4$ leaves to the other one of the previous two vertices.

If $r = 9$, consider the graph $G_n = (q-1)T_5 \cup T_5^{****}$ where T_5^{****} denotes the graph obtained from by attaching three vertices to the center of T_5 and attaching two leaves to each of the previous three vertices.

Obviously, it contains no copy of P_6, but the addition of any edge $e \in E(\overline{G_n})$ results in a copy of $P_6 \cup (3(q-1) + [\frac{r}{3}])P_2$.

If $0 \le r \le 2$, we have $n = 10q + r \ge \frac{10}{3}t + 10$, $t \le 3(q-1) = 3(q-1) + [\frac{r}{3}]$.

If $3 \le r \le 5$, we have $n = 10q + r \ge \frac{10}{3}t + \frac{29}{3}$, $t \le 3(q-1) + 1 = 3(q-1) + [\frac{r}{3}]$.

If $6 \le r \le 8$, we have $n = 10q + r \ge \frac{10}{3}t + \frac{28}{3}$, $t \le 3(q-1) + 2 = 3(q-1) + [\frac{r}{3}]$.

If $r = 9$, we have $n = 10q + 9 \ge \frac{10}{3}t + \frac{27}{3} = \frac{10}{3}t + 9$, $t \le 3(q-1) + 3 = 3(q-1) + [\frac{r}{3}]$.

Hence G_n is $P_6 \cup tP_2$-saturated. Since $V_0(G_n) = \emptyset$, $G_n \in SAT^*(n, P_6 \cup tP_2)$, $sat^*(n, P_6 \cup tP_2) \le |E(G_n)| = n - q = n - [\frac{n}{10}]$.

If $sat^*(n, P_6 \cup tP_2) < n - [\frac{n}{10}]$, then there is a graph $G \in SAT^*(n, P_6 \cup tP_2)$ with $|E(G)| < n - [\frac{n}{10}]$. Let $G_0 = G - (H_1 \cup H_2 \cup \cdots \cup H_k)$, where $H_1, H_2, \cdots, H_k$ are all the tree components of G. By Lemma 2.6, $|V(H_1)| \ge 10$, $|V(H_2)| \ge 10, \cdots$ and $|V(H_k)| \ge 10$. Hence $n \ge 10k$, $k \le \frac{n}{10}$. And

$$|E(G)| = |E(G_0)| + \sum |E(H_i)| \ge |G_0| + \sum (|H_i| - 1) = n - k.$$

this together with $|E(G)| < n - [\frac{n}{10}]$ implies that $k > [\frac{n}{10}]$, a contradiction. □

Proof of Theorem 2.1.

(i) By Lemma 2.5 and Theorem 2.7, we obtain that $sat(n, P_6 \cup tP_2) \le min\{n - [\frac{n}{10}], 3t + 18\}$.

Assume there exists a graph $G \in Sat(n, P_6 \cup tP_2)$ with $|E(G)| < min\{n - [\frac{n}{10}], 3t + 18\}$. By Lemma 2.5 and Theorem 2.7, it follows that $|V_0(G)| = 1$. By Lemma 2.3, $V_1(G) = \emptyset$. Hence $|E(G)| \ge (|G| - 1)$. Then $n \le |E(G)| + 1 < n - [\frac{n}{10}]$, a contradiction. Thus, Theorem 2.1(i) is true.

(ii) By $n \ge \frac{10}{3}t + 20$ when $n = 10q$ and $n > \frac{10}{3}t + 20$ when $n = 10q + r$, $1 \le r \le 9$, we have $n - [\frac{n}{10}] > 3t + 18$. Thus $sat(n, P_6 \cup tP_2) = 3t + 18$. If $G \in Sat(n, P_6 \cup tP_2)$ with $|E(G)| = 3t + 18$, by Theorem 2.7, $G \notin Sat^*(n, P_6 \cup tP_2)$. Thus $V_0(G) \ne \emptyset$.

If $|V_0(G)| = 1$, then $|E(G)| \ge (|G| - 1) \ge \frac{10}{3}t + 19 > 3t + 18$, a contradiction.

If $|V_0(G)| \ge 2$, $Sat(n, P_6 \cup tP_2) = K_7 \cup (t-1)K_3 \cup (n - 3t - 4)K_1$ by Lemma 2.5.

Therefore, Theorem 2.1(ii) is true.

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