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Quantum injectivity of continuous g-frames

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Abstract. A quantum injective frame is a frame that can be used to distinguish quantum systems by measuring density operators. The quantum detection problem aims to characterize all such frames. Recently, there has been significant attention on the problem of quantum detection in both finite and infinite dimensional Hilbert spaces, for both continuous and discrete frames. This paper aims to present various characterizations for injective continuous g-frames. The work solves the quantum detection problem in a more general context and provides characterizations for a broader class of injective continuous g-frames.

1. Introduction

In 1952, Duffin and Schaeffer [8] first introduced frames in Hilbert spaces to handle nonharmonic Fourier series. Since then, frames have been extensively studied, particularly after 1986, following the influential work of Daubechies, Grossmann, and Meyer [7]. A frame is an overcomplete coordinate system that contains more vectors than necessary to represent each vector in the Hilbert space. This concept can be extended from discrete to continuous by integrating over a measure space instead of summing over a countable set, resulting in what is known as continuous frames [2, 10]. In 2006, Sun [16] introduced g-frames, or generalized frames, while Abdollahpour and Faroughi [1] presented and investigated continuous and Riesz-type continuous g-frames.

In quantum theory, quantum state tomography [15] involves recovering a state (density operator) by observing the probability of outcomes from a series of measurements performed on the system in that state. Data from quantum systems is retrieved following quantum measurement theory [5, 15]. A positive operator-valued measure (POVM)[13] plays a vital role in this process. Recall that a POVM is informationally complete if it uniquely determines density operators [6, 9]. To solve the quantum detection problem, it is necessary to find POVMs that are informationally complete. The quantum detection problem with discrete frame coefficient measurements has been solved by Botelho-Andrade et al. for both finite and infinite dimensional Hilbert spaces in [3, 4]. They characterized the spanning properties of some

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derived sequences from the frame vectors. Studying this problem and its results for frame extensions is essential. Recently, Han et al. have explored the problem of quantum detection in its continuous frame version [12]. In this paper, we will provide characterizations for trace class and Hilbert-Schmidt injective continuous g-frames instead of continuous frames. Our research aims to deepen the understanding of quantum detection problems within a broader context. The results of this research expand the scope of solutions to the quantum detection problem and enhance related research. While some details may be technical, the results are valuable for analyzing injective continuous g-frames and hold significant intrinsic interest in quantum detection problems.

The rest of the paper is organized as follows. In Sect.2, we introduce continuous g-frames and POVMs and review the fundamentals of the quantum detection problem. In Sect.3, we introduce L_2-injective (respectively, L_1-injective) continuous g-frames and prove elementary facts. In Sect.4, we present multiple characterizations for injective continuous g-frames.

2. Preliminaries

Let $\mathcal{N}$ be a separable Hilbert space. The set of linear bounded operators on $\mathcal{N}$ is denoted by $\mathcal{B}(\mathcal{N})$, while the real linear space of self-adjoint bounded operators on $\mathcal{N}$ is denoted by $\mathcal{B}(\mathcal{N})_{sa}$. The real cone of positive operators on $\mathcal{N}$ is denoted by $\mathcal{B}(\mathcal{N})_+$. The space of trace-class operators on $\mathcal{N}$ is denoted by $L_1(\mathcal{N})$, and the Hilbert space of Hilbert-Schmidt operators with inner product $\langle A, B\rangle_2 = \mathrm{tr}(AB^*)$ is denoted by $L_2(\mathcal{N})$. The set of states or density operators on $\mathcal{N}$, consisting of ρ: $\rho \in L_1(\mathcal{N})$ such that $\rho \geq 0$ and $\mathrm{tr}(\rho) = 1$, is denoted by $\mathcal{S}(\mathcal{N})$. The index set, which can be countable, is denoted by $\mathbb{L}$. Finally, the notation "ONS" denotes an orthonormal basis.

For a given ONS $\{e_\ell\}_{\ell\in\mathbb{L}}$, an operator $K \in L_2(\mathcal{N})$ if

$$\|K\|_2 := \left(\sum_{\ell\in\mathbb{L}} \|Ke_\ell\|^2\right)^{\frac{1}{2}} < \infty,$$

and $K \in L_1(\mathcal{N})$ if

$$\|K\|_1 := \sum_{\ell\in\mathbb{L}} \langle |K|e_\ell, e_\ell\rangle < \infty.$$

In this case, the trace of K is given by $\mathrm{tr}(K) = \sum_{\ell\in\mathbb{L}} \langle Ke_\ell, e_\ell\rangle$, which is finite and independent of the ONS. Additionally, $L_2(\mathcal{N})$ and $L_1(\mathcal{N})$ are both ideals in $\mathcal{B}(\mathcal{N})$ [17].

In this paper, we consider a measure space (Δ, ν), where ν is a σ-finite positive measure. We assume that $\{\mathcal{M}_w\}_{w\in\Delta}$ is a family of Hilbert spaces.

We say that $H \in \prod_{w\in\Delta} \mathcal{M}_w$ is *strongly measurable* if H is a mapping of Δ to $\oplus_{w\in\Delta}\mathcal{M}_w$ and is measurable, where

$$\prod_{w\in\Delta} \mathcal{M}_w = \{m : \Delta \to \cup_{w\in\Delta}\mathcal{M}_w : m(w) \in \mathcal{M}_w\}.$$

We call $\mathcal{E} = \{\mathcal{E}_w \in \mathcal{B}(\mathcal{N}, \mathcal{M}_w) : w \in \Delta\}$ a *continuous g-frame for $\mathcal{N}$ with respect to $\{\mathcal{M}_w\}_{w\in\Delta}$ if*
(1) *for any $\zeta \in \mathcal{N}$, $\{\mathcal{E}_w\zeta : w \in \Delta\}$ is strongly measurable,*
(2) *there exist $\alpha, \beta > 0$ such that*

$$\alpha\|\zeta\|^2 \leq \int_\Delta \|\mathcal{E}_w(\zeta)\|^2 d\nu(w) \leq \beta\|\zeta\|^2, \quad \zeta \in \mathcal{N}. \tag{1}$$

$\mathcal{E}$ is called a Parseval continuous g-frame *if $\alpha = \beta = 1$. For simplicity, we use the term c-g-frame instead of a continuous g-frame for $\mathcal{N}$ with respect to $\{\mathcal{M}_w\}_{w\in\Delta}$ when no confusion arises.*

We consider the space $(\oplus_{w\in\Delta}\mathcal{M}_w, \nu)_{L^2}$ defined by

$$\left\{H \in \prod_{w\in\Delta} \mathcal{M}_w : H \text{ is strongly measurable}, \int_\Delta \|H(w)\|^2 d\nu(w) < \infty\right\}.$$

It is clear that $(\oplus_{w\in\Delta}\mathcal{M}_w, \nu)_{L^2}$ is a Hilbert space with the inner product given by

$$\langle H, G\rangle = \int_\Delta \langle H(w), G(w)\rangle d\nu(w).$$

Associated with each c-g-frame $\mathcal{E}$, there are three important linear bounded operators, i.e., *the analysis operator* $T_\mathcal{E}$, the *synthesis operator* $T^*_\mathcal{E}$ and the *frame operator* $S_\mathcal{E}$. They are defined as follows:

$$T_\mathcal{E} : \mathcal{N} \to (\oplus_{w\in\Delta}\mathcal{M}_w, \nu)_{L^2}, \;\; (T_\mathcal{E})(\zeta)(w) = \mathcal{E}_w\zeta, \; \forall\, w \in \Delta.$$

$T^*_\mathcal{E}$ is the adjoint of $T_\mathcal{E}$, so explicitly, $T^*_\mathcal{E} : (\oplus_{w\in\Delta}\mathcal{M}_w, \nu)_{L^2} \to \mathcal{N}$,

$$\langle T^*_\mathcal{E} H, \eta\rangle = \int_\Delta \langle \mathcal{E}^*_w H(w), \eta\rangle d\nu(w), \;\; H \in (\oplus_{w\in\Delta}\mathcal{M}_w, \nu)_{L^2}, \eta \in \mathcal{N}.$$

The frame operator $S_\mathcal{E} : \mathcal{N} \to \mathcal{N}$ is defined such that for each $\zeta, \eta \in \mathcal{N}$,

$$\langle S_\mathcal{E}\zeta, \eta\rangle = \int_\Delta \langle \zeta, \mathcal{E}^*_w\mathcal{E}_w\eta\rangle d\nu(w).$$

It is easy to see that $S_\mathcal{E}$ is invertible. Let $\mathcal{E} = \{\mathcal{E}_w \in \mathcal{B}(\mathcal{N}, \mathcal{M}_w) : w \in \Delta\}$ and $\mathcal{D} = \{\mathcal{D}_w \in \mathcal{B}(\mathcal{N}, \mathcal{M}_w) : w \in \Delta\}$ be c-g-frames such that

$$\langle \zeta, \eta\rangle = \int_\Delta \langle \zeta, \mathcal{D}^*_w\mathcal{E}_w\eta\rangle d\nu(w), \zeta, \eta \in \mathcal{N}.$$

Then $\mathcal{D}$ is called a *dual* of $\mathcal{E}$. Define $\widetilde{\mathcal{E}} = \{\widetilde{\mathcal{E}}_w = \mathcal{E}_w S^{-1}_\mathcal{E} : w \in \Delta\}$. Then $\widetilde{\mathcal{E}}$ is also a c-g-frame with frame operator $S^{-1}_\mathcal{E}$ and is a dual of $\mathcal{E}$. It is called the *canonical dual* of $\mathcal{E}$.

Two c-g-frames $\mathcal{E} = \{\mathcal{E}_w \in \mathcal{B}(\mathcal{N}, \mathcal{M}_w) : w \in \Delta\}$ and $\mathcal{D} = \{\mathcal{D}_w \in \mathcal{B}(\mathcal{M}, \mathcal{M}_w) : w \in \Delta\}$ for $\mathcal{N}$ and $\mathcal{M}$, respectively, are said to be *similar* if there is an invertible operator $S : \mathcal{N} \to \mathcal{M}$ such that $\mathcal{D}_w S = \mathcal{E}_w$ for all $w \in \Delta$.

We define operator-valued measures for a locally compact Hausdorff space Δ and its σ-algebra of Borel sets, denoted by Σ. This definition is based on the one provided in [11].

Definition 2.1. A mapping $\pi : \Sigma \to \mathcal{B}(\mathcal{N})$ is an *operator-valued measure (OVM)* if for every countable collection $\{\mathcal{F}_\ell\}_{\ell\in\mathbb{L}} \subseteq \Sigma$ with $\mathcal{F}_\ell \cap \mathcal{F}_\jmath = \emptyset$ for $\ell \neq \jmath$, we have

$$\pi\left(\bigcup_{\ell\in\mathbb{L}} \mathcal{F}_\ell\right) = \sum_{\ell\in\mathbb{L}} \pi(\mathcal{F}_\ell).$$

Additionally, we say π is a *positive operator-valued measure (POVM)* if it is *positive*, i.e., $\pi(\mathcal{F}) \in \mathcal{B}(\mathcal{N})_+$ for all $\mathcal{F} \in \Sigma$ and $\pi(\Delta) = I$.

If we have a state represented by the symbol τ, we can create a mapping denoted as $p : \Sigma \to \mathbb{R}$ based on the quantum measurement performed by a POVM denoted as π. This mapping is defined as follows:

$$p(\mathcal{F}) = \mathrm{tr}(\tau\pi(\mathcal{F})), \;\; \forall\, \mathcal{F} \in \Sigma. \tag{2}$$

The collection of bounded functions on Σ is symbolized as $\mathcal{B}(\Sigma, \mathbb{R})$. This set can be used in conjunction with a quantum system $\mathcal{N}$. The "quantum detection problem" aims to determine if there is a POVM π for a given mapping $\mathbb{P} : \mathcal{S}(\mathcal{N}) \to \mathcal{B}(\Sigma, \mathbb{R})$ such that $\mathbb{P}(\tau)(\mathcal{F}) = \mathrm{tr}(\tau\pi(\mathcal{F}))$ for all $\mathcal{F} \in \Sigma$. The main challenge is to find such a POVM and ensure that it is injective. Specifically, the question is whether a POVM π can distinguish between quantum states in a measurement. This means that if for $\tau_1, \tau_2 \in \mathcal{S}(\mathcal{N})$,

$$\mathrm{tr}(\tau_1\pi(\mathcal{F})) = \mathrm{tr}(\tau_2\pi(\mathcal{F})), \; \forall\, \mathcal{F} \in \Sigma,$$

it follows that $\tau_1 = \tau_2$.

For a c-g-frame $\mathcal{E} = \{\mathcal{E}_w \in \mathcal{B}(\mathcal{N}, \mathcal{M}_w) : w \in \Delta\}$, we define

$$\pi : \Sigma \to \mathcal{B}(\mathcal{N})_+, \quad \pi(\mathcal{F}) = \int_{\mathcal{F}} \mathcal{E}_w^* \mathcal{E}_w d\nu(w)$$

in the sense of

$$\langle \pi(\mathcal{F})\zeta, \eta \rangle = \int_{\mathcal{F}} \langle \mathcal{E}_w \zeta, \mathcal{E}_w \eta \rangle d\nu(w),$$

which naturally induces an OVM. In the case that $\mathcal{E}$ is a Parseval c-g-frame, then we also have $\pi(\Delta) = I$, which induces a POVM.

Let $\{\mathcal{M}_w\}_{w\in\Delta}$ ba a family of finite dimensional subspaces of $\mathcal{N}$, π be an OVM associated with a c-g-frame $\mathcal{E} = \{\mathcal{E}_w \in \mathcal{B}(\mathcal{N}, \mathcal{M}_w) : w \in \Delta\}$. Then the quantum measurement for a state $\tau \in \mathcal{S}(\mathcal{N})$ is given by the map $\mathbb{P} : \mathcal{S}(\mathcal{N}) \to \mathcal{B}(\Sigma, \mathbb{R})$

$$\mathbb{P}(\tau)(\mathcal{F}) = \mathrm{tr}(\tau\pi(\mathcal{F})) = \int_{\mathcal{F}} \mathrm{tr}(\mathcal{E}_w \tau \mathcal{E}_w^*) d\nu(w). \tag{3}$$

The quantum detection problem involves determining whether the mapping $\mathbb{P}$ is injective on the space $\mathcal{S}(\mathcal{N})$. If this condition holds, we say that $\mathcal{E}$ is *"quantum injective"* or simply *injective*. We are interested in defining injective c-g-frames.

3. Injective continuous g-frames

Let (Δ, Σ, ν) be a measure space, where Σ is a σ-algebra over Δ and ν is a σ-finite positive measure. In this section until the end of this paper, we consider a family of finite dimensional subspaces $\{\mathcal{M}_w\}_{w\in\Delta}$, each equipped with its own ONS $\{e_{wj} : j \in \mathbb{J}_w\}$ for all $w \in \Delta$.

Definition 3.1. Let $\mathcal{E} = \{\mathcal{E}_w \in \mathcal{B}(\mathcal{N}, \mathcal{M}_w) : w \in \Delta\}$ be a c-g-frame. We say that $\mathcal{E}$ is *L_2-injective (respectively, L_1-injective)* if whenever a self-adjoint Hilbert-Schmidt (respectively, self-adjoint trace-class) operator K satisfies

$$\mathrm{tr}(\mathcal{E}_w K \mathcal{E}_w^*) = 0, \ \ a.e.\ w \in \Delta,$$

then $K = 0$.

Remark 3.2. It is known that the quantum injectivity of a c-g-frame $\mathcal{E}$ is equivalent to the condition that if $\mathrm{tr}(\mathcal{E}_w K \mathcal{E}_w^*) = 0$, *a.e.* $w \in \Delta$ for a self-adjoint trace class operator K with trace zero, then $K = 0$. Thus, L_k-injectivity implies quantum injectivity for $k = 1, 2$.

The following elementary facts are helpful in L_k-injective continuous g-frames, $k = 1, 2$.

Proposition 3.3. Let $\mathcal{E} = \{\mathcal{E}_w \in \mathcal{B}(\mathcal{N}, \mathcal{M}_w) : w \in \Delta\}$ be a c-g-frame. If $S \in \mathcal{B}(\mathcal{N})$ is an invertible operator, then $\mathcal{E}S = \{\mathcal{E}_w S \in \mathcal{B}(\mathcal{N}, \mathcal{M}_w) : w \in \Delta\}$ is also a c-g-frame. Moreover, $\mathcal{E}$ is L_k-injective if and only if $\mathcal{E}S$ is L_k-injective, $k = 1, 2$.

Proof. Clearly, $\mathcal{E}S$ is a c-g-frame. We now show that $\mathcal{E}$ is L_k-injective if and only if $\mathcal{E}S$ is L_k-injective for $k = 1, 2$. Assume that $\mathcal{E}$ is L_k-injective for $k = 1, 2$. For any self-adjoint operator $K \in L_k(\mathcal{N}), k = 1, 2$, if

$$\mathrm{tr}(\mathcal{E}_w S K (\mathcal{E}_w S)^*) = \mathrm{tr}(\mathcal{E}_w (SKS^*) \mathcal{E}_w^*) = 0, \ a.e.\ w \in \Delta,$$

then $SKS^* = 0$ since $SKS^* \in L_k(\mathcal{N}), k = 1, 2$ and is self-adjoint. It follows that $K = 0$ because S is invertible.

To prove the converse direction, suppose that $\mathcal{E}S$ is injective. Let $K \in L_k(\mathcal{N}), k = 1, 2$ be any self-adjoint operator. If

$$\begin{aligned} \mathrm{tr}(\mathcal{E}_w K \mathcal{E}_w^*) &= \mathrm{tr}(\mathcal{E}_w S S^{-1} K (S^*)^{-1} S^* \mathcal{E}_w^*) \\ &= \mathrm{tr}(\mathcal{E}_w S (S^{-1} K (S^*)^{-1}) (\mathcal{E}_w S)^*) = 0, \ a.e.\ w \in \Delta, \end{aligned}$$

then $S^{-1}K(S^*)^{-1} = 0$, which implies that $K = 0$. □

More generally, the following theorem demonstrates that similar continuous g-frames preserve L_k-injectivity, where $k = 1, 2$.

Corollary 3.4. Let $\mathcal{E}$ and $\mathcal{D}$ be c-g-frames. If $\mathcal{E}$ and $\mathcal{D}$ are similar, then $\mathcal{E}$ is L_k-injective if and only if $\mathcal{D}$ is L_k-injective for $k = 1, 2$.

Corollary 3.5. Let $\mathcal{E}$ be a c-g-frame with frame operator $S_{\mathcal{E}}$. If $\mathcal{E}$ is L_k-injective, then the canonical Parseval c-g-frame $\mathcal{E}S_{\mathcal{E}}^{-\frac{1}{2}}$ is also L_k-injective for $k = 1, 2$.

Remark 3.6. The previous corollary implies that finding Parseval c-g-frame is unnecessary for solving the quantum detection problem. If a c-g-frame ensures injectivity, then its canonical Parseval c-g-frame will also guarantee injectivity.

As demonstrated by the following theorem, it is not necessary to use positive operators.

Theorem 3.7. Given a c-g-frame $\mathcal{E} = \{\mathcal{E}_w \in \mathcal{B}(\mathcal{N}, \mathcal{M}_w) : w \in \Delta\}$. For $k = 1$ or 2, the following are equivalent:
(1) If $K, S \in L_k(\mathcal{N})$ are positive operators, and

$$\operatorname{tr}(\mathcal{E}_w K \mathcal{E}_w^*) = \operatorname{tr}(\mathcal{E}_w S \mathcal{E}_w^*), \text{ for } a.e.\ w \in \Delta,$$

then $K = S$.
(2) If $K, S \in L_k(\mathcal{N})$ are self-adjoint, and

$$\operatorname{tr}(\mathcal{E}_w K \mathcal{E}_w^*) = \operatorname{tr}(\mathcal{E}_w S \mathcal{E}_w^*), \text{ for } a.e.\ w \in \Delta,$$

then $K = S$.
(3) $\mathcal{E}$ is L_k-injective.
(4) For any $K \in L_k(\mathcal{N})$, the condition $\operatorname{tr}(\mathcal{E}_w K \mathcal{E}_w^*) = 0 (a.e.\ w \in \Delta)$ implies that $K = 0$.

Proof. Clearly, we have (4) ⇒(3)⇒(2)⇒(1). For (1) ⇒ (4), let $K \in L_k(\mathcal{N})$ be such that $\operatorname{tr}(\mathcal{E}_w K \mathcal{E}_w^*) = 0 (a.e.\ w \in \Delta)$. Write

$$K = (K_1^+ - K_1^-) + i(K_2^+ - K_2^-),$$

where $K_1^+, K_1^-, K_2^+, K_2^-$ are positive operators in $L_k(\mathcal{N})$ [14]. Furthermore, the condition

$$\operatorname{tr}(\mathcal{E}_w K \mathcal{E}_w^*) = 0$$

implies that $\operatorname{tr}(\mathcal{E}_w (K_1^+ - K_1^-) \mathcal{E}_w^*) = 0$ and $\operatorname{tr}(\mathcal{E}_w (K_2^+ - K_2^-) \mathcal{E}_w^*) = 0$. Then $\operatorname{tr}(\mathcal{E}_w K_1^+ \mathcal{E}_w^*) = \operatorname{tr}(\mathcal{E}_w K_1^- \mathcal{E}_w^*)$ and $\operatorname{tr}(\mathcal{E}_w K_2^+ \mathcal{E}_w^*) = \operatorname{tr}(\mathcal{E}_w K_2^- \mathcal{E}_w^*)$. Thus, by (1), we get $K_1^+ = K_1^-, K_2^+ = K_2^-$ and so $K = 0$. □

After considering the trace condition, the proof of the following fact is the same as that of Theorem 3.7.

Corollary 3.8. Given a c-g-frame $\mathcal{E} = \{\mathcal{E}_w \in \mathcal{B}(\mathcal{N}, \mathcal{M}_w) : w \in \Delta\}$. The following are equivalent:
(1) $\mathcal{E}$ is quantum injective.
(2) If $K, S \in L_1(\mathcal{N})$ are self-adjoint and trace one, and

$$\operatorname{tr}(\mathcal{E}_w K \mathcal{E}_w^*) = \operatorname{tr}(\mathcal{E}_w S \mathcal{E}_w^*), \text{ for } a.e.\ w \in \Delta,$$

then $K = S$.
(3) If K is a self-adjoint trace class operator and trace zero, and

$$\operatorname{tr}(\mathcal{E}_w K \mathcal{E}_w^*) = 0, \text{ for } a.e.\ w \in \Delta,$$

then $K = 0$.
(4) If $K \in L_1(\mathcal{N})$ is trace zero, and

$$\operatorname{tr}(\mathcal{E}_w K \mathcal{E}_w^*) = 0, \text{ for } a.e.\ w \in \Delta,$$

then $K = 0$.

Clearly, we have "L_2-injectivity $\Rightarrow$ L_1-injectivity $\Rightarrow$ quantum injectivity". In the case that $\mathcal{E}$ is a Parseval c-g-frame, quantum injectivity also implies the L_1-injectivity.

Proposition 3.9. Let $\mathcal{E} = \{\mathcal{E}_w \in \mathcal{B}(\mathcal{N}, \mathcal{M}_w) : w \in \Delta\}$ be a Parseval c-g-frame. Then $\mathcal{E}$ is quantum injective if and only if it is L_1-injective.

Proof. Let $\mathcal{E}$ be a Parseval c-g-frame which is quantum injective. Therefore,

$$I = \int_\Delta \mathcal{E}_w^* \mathcal{E}_w d\nu(w),$$

and thus

$$\mathrm{tr}(K) = \int_\Delta \mathrm{tr}(\mathcal{E}_w K \mathcal{E}_w^*) d\nu(w).$$

Now suppose that

$$\mathrm{tr}(\mathcal{E}_w K \mathcal{E}_w^*) = 0, \ a.e. \ w \in \Delta,$$

for some self-adjoint trace-class operator K. Thus $\mathrm{tr}(K) = 0$. By Corollary 3.8, we get $K = 0$. Hence $\mathcal{E}$ is L_1-injective. □

4. Characterizations of injective continuous g-frames

The role of states or positive operators of trace one is vital in quantum theory. The following theorems provide a classification of injectivity for such states. We have achieved the ability to derive various characterizations of injective c-g-frames.

Denote by $\mathbb{K}$ the direct sum of the real Hilbert spaces l_2:

$$\mathbb{K} = \left(\sum_{\ell \in \mathbb{L}} \oplus l_2 \right)_{l_2}.$$

To avoid confusion with earlier notation, a vector in this direct sum will be written in the form: $X = (x_\ell)_{\ell \in \mathbb{L}}$, and we have

$$\langle X, Y \rangle_{\mathbb{K}} = \sum_{\ell \in \mathbb{L}} \langle x_\ell, y_\ell \rangle.$$

Theorem 4.1. Let $\mathcal{E} = \{\mathcal{E}_w \in \mathcal{B}(\mathcal{N}, \mathcal{M}_w) : w \in \Delta\}$ be a c-g-frame with analysis operator $T_{\mathcal{E}}$. Let $\iota_\ell = T_{\mathcal{E}} e_\ell, \forall \ell \in \mathbb{L}$ for some ONS $\{e_\ell\}_{\ell \in \mathbb{L}}$ of $\mathcal{N}$. Then the following are equivalent:
(1) $\mathcal{E}$ is L_2-injective.
(2)There exists a set $\mathcal{F} \subseteq \Delta$ with $\nu(\mathcal{F}) = 0$, if we set

$$\begin{aligned} \mathcal{S}(w) = (&\|\iota_1(w)\|^2, \mathrm{Re}\langle \iota_1(w), \iota_2(w)\rangle, \mathrm{Im}\langle \iota_1(w), \iota_2(w)\rangle \cdots, \\ &\|\iota_2(w)\|^2, \mathrm{Re}\langle \iota_2(w), \iota_3(w)\rangle, \mathrm{Im}\langle \iota_2(w), \iota_3(w)\rangle \cdots, \\ &\|\iota_3(w)\|^2, \mathrm{Re}\langle \iota_3(w), \iota_4(w)\rangle, \mathrm{Im}\langle \iota_3(w), \iota_4(w)\rangle \cdots, \cdots), \forall\, w \in \mathcal{F}^C, \end{aligned}$$

then $\mathrm{span}_{w \in \mathcal{F}^C}\{\mathcal{S}(w)\}$ is dense in $\mathbb{K}$.

Proof. Let $K \in L_2(\mathcal{N})$ be a self-adjoint operator and $\{e_\ell\}_{\ell \in \mathbb{L}}$ be the ONS of $\mathcal{N}$. We set $b_{\ell,\jmath} = \langle K e_\ell, e_\jmath \rangle$ and define

$$B = (B_1, B_2, B_3, \cdots, B_\ell, \cdots)_{\ell \in \mathbb{L}},$$

where

$$B_\ell = (b_{\ell,\ell}, 2\mathrm{Re}(b_{\ell,\ell+1}), -2\mathrm{Im}(b_{\ell,\ell+1}), 2\mathrm{Re}(b_{\ell,\ell+2}), -2\mathrm{Im}(b_{\ell,\ell+2}), \cdots).$$

Then $B \in \mathbb{K}$. In fact,

$$\|B_\ell\| \leq \left(\sum_{\jmath \geq \ell, \jmath \in \mathbb{L}} |2b_{\ell,\jmath}|^2 \right)^{\frac{1}{2}} = \left(\sum_{\jmath \geq \ell, \jmath \in \mathbb{L}} |2\langle Ke_\ell, e_\jmath \rangle|^2 \right)^{\frac{1}{2}} \leq 2\|Ke_\ell\|.$$

$$\|B\|_{\mathbb{K}} = \left(\sum_{\ell \in \mathbb{L}} \|B_\ell\|^2 \right)^{\frac{1}{2}} = \left(\sum_{\ell \in \mathbb{L}} (2\|Ke_\ell\|)^2 \right)^{\frac{1}{2}} = 2\|K\|_2.$$

Consider $\iota_\ell = T_{\mathcal{E}} e_\ell$ for all $\ell \in \mathbb{L}$. Then

$$\begin{aligned} \sum_{\ell \in \mathbb{L}} \|\iota_\ell(w)\|^2 &= \sum_{\ell \in \mathbb{L}} \|\mathcal{E}_w e_\ell\|^2 = \sum_{\ell \in \mathbb{L}} \left\| \sum_{\jmath \in \mathbb{J}_w} \langle \mathcal{E}_w e_\ell, e_{w\jmath} \rangle e_{w\jmath} \right\|^2 \\ &= \sum_{\jmath \in \mathbb{J}_w} \sum_{\ell \in \mathbb{L}} |\langle e_\ell, \mathcal{E}_w^* e_{w\jmath} \rangle|^2 = \sum_{\jmath \in \mathbb{J}_w} \|\mathcal{E}_w^* e_{w\jmath}\|^2 < \infty. \end{aligned}$$

This implies that there exists $\mathcal{F}$ such that $\nu(\mathcal{F}) = 0$ and $\sum_{\ell \in \mathbb{L}} \|\iota_\ell(w)\|^2 < \infty$ for all $w \in \mathcal{F}^C$. We set

$$\mathcal{S}(w) = (\mathcal{S}_1(w), \mathcal{S}_2(w), \cdots \mathcal{S}_\ell(w), \cdots)_{\ell \in \mathbb{L}},$$

where

$$\mathcal{S}_\ell(w) = (\|\iota_\ell(w)\|^2, \mathrm{Re}\langle \iota_\ell(w), \iota_{\ell+1}(w) \rangle, \mathrm{Im}\langle \iota_\ell(w), \iota_{\ell+1}(w) \rangle \cdots, \cdots), \ \forall\, w \in \mathcal{F}^C.$$

We have

$$\begin{aligned} \|\mathcal{S}(w)\|_{\mathbb{K}} &= \left(\sum_{\ell \in \mathbb{L}} \|\mathcal{S}_\ell(w)\|^2 \right)^{\frac{1}{2}} = \left(\sum_{\jmath \geq \ell, \ell \in \mathbb{L}} |\langle \iota_\ell(w), \iota_\jmath(w) \rangle|^2 \right)^{\frac{1}{2}} \\ &\leq \sum_{\ell \in \mathbb{L}} \|\iota_\ell(w)\|^2 < \infty. \end{aligned}$$

Then $\mathcal{S}(w)$ is in $\mathbb{K}$.

Now we have

$$\begin{aligned} &\mathrm{tr}(\mathcal{E}_w K \mathcal{E}_w^*) = \mathrm{tr}(\mathcal{E}_w^* \mathcal{E}_w K) = \sum_{\ell \in \mathbb{L}} \langle \mathcal{E}_w^* \mathcal{E}_w K e_\ell, e_\ell \rangle \\ &= \sum_{\ell \in \mathbb{L}} \langle \mathcal{E}_w K e_\ell, \mathcal{E}_w e_\ell \rangle = \sum_{\ell \in \mathbb{L}} \left\langle \sum_{\jmath \in \mathbb{L}} \langle K e_\ell, e_\jmath \rangle \mathcal{E}_w e_\jmath, \mathcal{E}_w e_\ell \right\rangle \\ &= \sum_{\ell \in \mathbb{L}} \sum_{\jmath \in \mathbb{L}} b_{\ell,\jmath} \langle T_{\mathcal{E}} e_\jmath(w), T_{\mathcal{E}} e_\ell(w) \rangle \\ &= \sum_{\ell \in \mathbb{L}} \sum_{\jmath \in \mathbb{L}} b_{\ell,\jmath} \langle \iota_\jmath(w), \iota_\ell(w) \rangle = \langle B, \mathcal{S}(w) \rangle_{\mathbb{K}}. \end{aligned}$$

Assume that $\mathcal{E}$ is L_2-injective. Then $\mathrm{tr}(\mathcal{E}_\omega K \mathcal{E}_w^*) = 0$, $\forall w \in \mathcal{F}^C$ implies that $K = 0$, which in turn implies that $B = 0$. Consequently, the orthogonal complement of $\mathrm{span}_{w \in \mathcal{F}^C}\{\mathcal{S}(w)\}$ is 0. Hence $\mathrm{span}_{w \in \mathcal{F}^C}\{\mathcal{S}(w)\}$ is dense in $\mathbb{K}$.

Conversely, if there exists a set $\mathcal{F}$ with $\nu(\mathcal{F}) = 0$ and for all $w \in \mathcal{F}^C$,

$$\mathrm{tr}(\mathcal{E}_w K \mathcal{E}_w^*) = \langle B, \mathcal{S}(w) \rangle_{\mathbb{K}} = 0.$$

Since $\mathrm{span}_{w \in F^C}\{\mathcal{S}(w)\}$ is dense in $\mathbb{K}$, we have $B = 0$, and hence $K = 0$. Thus $\mathcal{E}$ is L_2-injective. □

Theorem 4.2. Let $\mathcal{E} = \{\mathcal{E}_w \in \mathcal{B}(\mathcal{N}, \mathcal{M}_w) : w \in \Delta\}$ be a c-g-frame with analysis operator $T_{\mathcal{E}}$. Let $\iota_\ell = T_{\mathcal{E}} e_\ell, \forall \ell \in \mathbb{L}$ for some ONS $\{e_\ell\}_{\ell\in\mathbb{L}}$ of $\mathcal{N}$. Then the following are equivalent:
(1) $\mathcal{E}$ is L_2-injective.
(2) There exists a set $\mathcal{F} \subseteq \Delta$ with $\nu(\mathcal{F}) = 0$, if we set

$$\mathcal{M}(w) = (\langle \iota_\ell(w), \iota_\jmath(w)\rangle)_{\ell,\jmath}$$

for all $w \in \mathcal{F}^C$, then $\text{span}_{w\in\mathcal{F}^C}\{\mathcal{M}(w)\}$ is dense in $L_2(l_2(\mathbb{L}))_{sa}$.

Proof. Consider $\iota_\ell = T_{\mathcal{E}} e_\ell$ for all $\ell \in \mathbb{L}$. Then there exists $\mathcal{F}$ such that $\nu(\mathcal{F}) = 0$ and $\sum_{\ell\in\mathbb{L}} \|\iota_\ell(w)\|^2 < \infty$ for all $w \in \mathcal{F}^C$. We define the following matrix for all $w \in \mathcal{F}^C$ by

$$\mathcal{M}(w) = (\langle \iota_\ell(w), \iota_\jmath(w)\rangle)_{\ell,\jmath}.$$

Obviously, $\mathcal{M}(w)$ is a self-adjoint operator on $l_2(\mathbb{L})$. Since

$$\begin{aligned}\|(\langle \iota_\ell(w), \iota_\jmath(w)\rangle)_{\ell,\jmath}\|_2 = &\left(\sum_{\ell,\jmath\in\mathbb{L}} |\langle \iota_\ell(w), \iota_\jmath(w)\rangle|^2\right)^{\frac{1}{2}} \\ = \sum_{\ell} \|\iota_\ell(w)\|^2 < \infty,&\end{aligned}$$

we have $\mathcal{M}(w) \in L_2(l_2(\mathbb{L}))$ for all $w \in \mathcal{F}^C$. Let $K \in L_2(\mathcal{N})$ be a self-adjoint operator. We set $b_{\ell,\jmath} = \langle Ke_\ell, e_\jmath\rangle$ and define the matrix $(b_{\ell,\jmath})_{\ell,\jmath}$ on $l_2(\mathbb{L})$. Since K is self-adjoint, then

$$b_{\ell,\jmath} = \langle Ke_\ell, e_\jmath\rangle = \langle e_\ell, Ke_\jmath\rangle = \overline{\langle Ke_\ell, e_\jmath\rangle} = \overline{b_{\ell,\jmath}},$$

hence the matrix $(b_{\ell,\jmath})_{\ell,\jmath}$ is self-adjoint. Moreover,

$$\|(b_{\ell,\jmath})_{\ell,\jmath}\|_2 = \|K\|_2 < \infty,$$

which implies that $(b_{\ell,\jmath})_{\ell,\jmath}$ is a Hilbert-Schmidt operator. Now,

$$\begin{aligned}\text{tr}(\mathcal{E}_w K \mathcal{E}_w^*) &= \sum_{\ell\in\mathbb{L}} \langle \mathcal{E}_w K e_\ell, \mathcal{E}_w e_\ell\rangle \\ &= \sum_{\ell\in\mathbb{L}} \left\langle \sum_{\jmath\in\mathbb{L}} \langle Ke_\ell, e_\jmath\rangle \mathcal{E}_w e_\jmath, \mathcal{E}_w e_\ell \right\rangle \\ &= \sum_{\ell\in\mathbb{L}} \sum_{\jmath\in\mathbb{L}} b_{\ell,\jmath} \langle T_{\mathcal{E}} e_\ell, T_{\mathcal{E}} e_\jmath\rangle \\ &= \sum_{\ell\in\mathbb{L}} \sum_{\jmath\in\mathbb{L}} b_{\ell,\jmath} \langle \iota_\ell(w), \iota_\jmath(w)\rangle \\ &= \langle (\langle \iota_\ell(w), \iota_\jmath(w)\rangle)_{\ell,\jmath}, (\overline{b_{\ell,\jmath}})_{\ell,\jmath}\rangle_2.\end{aligned}$$

By orthogonality, we conclude that if $\mathcal{E}$ is L_2-injective, then the orthogonal complement space of $\text{span}_{w\in\mathcal{F}^C}\{(\langle \iota_\ell(w), \iota_\jmath(w)\rangle)_{\ell,\jmath}\}$ is 0. Hence $\text{span}_{w\in\mathcal{F}^C}\{\mathcal{M}(w)\}$ is dense in $L_2(l_2(\mathbb{L}))_{sa}$.

Conversely, for any self-adjoint $K \in L_2(\mathcal{N})$ and if there exists a set $\mathcal{F} \subseteq \Delta$ with $\nu(\mathcal{F}) = 0$, if $\text{tr}(\mathcal{E}_w K \mathcal{E}_w^*) = 0$ for all $w \in \mathcal{F}^C$, then

$$\left\langle \langle (\iota_\ell(w), \iota_\jmath(w)\rangle)_{\ell,\jmath}, (\overline{b_{\ell,\jmath}})_{\ell,\jmath} \right\rangle_2 = 0, \ \forall\ w \in \mathcal{F}^C.$$

Since $\text{span}_{w\in\mathcal{F}^C}\{\mathcal{M}(w)\}$ is dense in $L_2(l_2(\mathbb{L}))_{sa}$, we have $\overline{b_{\ell,\jmath}} = 0$, hence $b_{\ell,\jmath} = 0$ as well as $K = 0$. Thus $\mathcal{E}$ is L_2-injective. $\square$

Theorem 4.3. Let $\mathcal{E} = \{\mathcal{E}_w \in \mathcal{B}(\mathcal{N}, \mathcal{M}_w) : w \in \Delta\}$ be a c-g-frame with analysis operator $T_{\mathcal{E}}$. Let $\iota_\ell = T_{\mathcal{E}} e_\ell, \forall \ell \in \mathbb{L}$ for some ONS $\{e_\ell\}_{\ell \in \mathbb{L}}$ of $\mathcal{N}$. Then the following are equivalent:
(1) $\mathcal{E}$ is L_2-injective.
(2) There exists a set $\mathcal{F} \subseteq \Delta$ with $\nu(\mathcal{F}) = 0$, if we set

$$\phi(w) = (\|\iota_1(w)\|^2, \|\iota_2(w)\|^2, \ldots, \|\iota_\ell(w)\|^2, \cdots),\ \forall\, w \in \mathcal{F}^C,$$

then $\text{span}_{w \in \mathcal{F}^C}\{\phi(w)\}$ is dense in the real Hilbert space $l_2(\mathbb{L})$.

Proof. We prove the contrapositive. Suppose that (2) fails. Then there exists $\mathcal{F} \subseteq \Delta$ with $\nu(\mathcal{F}) = 0$ such that $\text{span}_{w \in \mathcal{F}^C}\{\phi(w)\}$ is not dense in the real Hilbert space $l_2(\mathbb{L})$. Consequently, there exists a non-zero vector $\vartheta = (\vartheta_1, \vartheta_2, \cdots \vartheta_\ell, \cdots) \in l_2(\mathbb{L})$ such that $\vartheta \perp \text{span}_{w \in \mathcal{F}^C}\{\phi(w)\}$. Define

$$K e_\ell = \vartheta_\ell e_\ell,\ \forall \ell \in \mathbb{L}.$$

Then K is non-zero self-adjoint operator and satisfies

$$\begin{aligned}
\text{tr}(\mathcal{E}_w K \mathcal{E}_w^*) &= \sum_{\ell \in \mathbb{L}} \langle \mathcal{E}_w K e_\ell, \mathcal{E}_\omega e_\ell \rangle \\
&= \sum_{\ell \in \mathbb{L}} \left\langle \sum_{\jmath \in \mathbb{L}} \langle K e_\ell, e_\jmath \rangle \mathcal{E}_w e_\jmath, \mathcal{E}_w e_\ell \right\rangle \\
&= \sum_{\ell \in \mathbb{L}} \sum_{\jmath \in \mathbb{L}} \langle K e_\ell, e_\jmath \rangle \langle \iota_\jmath(w), \iota_\ell(w) \rangle \\
&= \sum_{\ell \in \mathbb{L}} \vartheta_\ell \|\iota_\ell(w)\|^2 = 0,\ w \in \mathcal{F}^C.
\end{aligned}$$

This contradicts the L_2-injectivity of $\mathcal{E}$.

Conversely, let $K \in L_2(\mathcal{N})$ and hence compact operator. Then there exists an eigenbasis $\mathcal{E} = \{e_\ell\}_{\ell \in \mathbb{L}}$ for K with respect to the eigenvalues $\{\vartheta_\ell\}_{\ell \in \mathbb{L}}$. Note that $\vartheta_\ell \in \mathbb{R}$ since K is also self-adjoint. Meanwhile, since $K \in L_2(\mathcal{N})$, we know $\vartheta = (\vartheta_1, \vartheta_2, \cdots \vartheta_\ell, \cdots) \in l^2(\mathbb{L})$ because

$$\sum_{\ell \in \mathbb{L}} |\vartheta_\ell|^2 = \sum_{\ell \in \mathbb{L}} \|K e_\ell\|^2 < \infty.$$

If exists a set $\mathcal{F} \subseteq \Delta$ with $\nu(\mathcal{F}) = 0$, and let $K \in L_2(\mathcal{N})$ be a self-adjoint such that

$$\text{tr}(\mathcal{E}_w K \mathcal{E}_w^*) = 0,\ \ \forall\, w \in \mathcal{F}^C.$$

Then

$$\text{tr}(\mathcal{E}_w K \mathcal{E}_w^*) = \sum_{\ell \in \mathbb{L}} \vartheta_\ell \|\iota_\ell(w)\|^2 = \langle \vartheta, \phi(w) \rangle = 0,\ \forall\, w \in \mathcal{F}^C.$$

Since $\text{span}_{w \in \mathcal{F}^C}\{\phi(w)\}$ is dense in the real Hilbert space $l_2(\mathbb{L})$, we have $\vartheta = (\vartheta_1, \vartheta_2, \cdots \vartheta_\ell, \cdots) = 0$, and hence $K = 0$. So $\mathcal{E}$ is L_2-injective. □

If we replace the $l_2(\mathbb{L})$ by $c_o(\mathbb{L})$, then we obtain the following characterization for quantum injective frames.

Theorem 4.4. Let $\mathcal{E} = \{\mathcal{E}_w \in \mathcal{B}(\mathcal{N}, \mathcal{M}_w) : w \in \Delta\}$ be a c-g-frame with analysis operator $T_{\mathcal{E}}$. Let $\iota_\ell = T_{\mathcal{E}} e_\ell, \forall \ell \in \mathbb{L}$ for some ONS $\{e_\ell\}_{\ell \in \mathbb{L}}$ of $\mathcal{N}$. Then the following are equivalent:
(1) $\mathcal{E}$ is quantum injective.
(2) There exists a set $\mathcal{F} \subseteq \Delta$ with $\nu(\mathcal{F}) = 0$, if we set

$$\psi(w) = (\|\iota_1(w)\|^2, \|\iota_2(w)\|^2, \ldots, \|\iota_\ell(w)\|^2, \cdots),\ \forall\, w \in \mathcal{F}^C,$$

then $\text{span}_{w \in \mathcal{F}^C}\{\psi(w)\}$ is dense in the real Hilbert space $c_o(\mathbb{L})$.

Proof. By Corollary 3.8, we know that the quantum injectivity of $\mathcal{E}$ is equivalent to the injectivity for trace class self-adjoint operators of trace zero.

(1)⇒(2) Consider $\iota_\ell = T_{\mathcal{E}} e_\ell$ for all $\ell \in \mathbb{L}$. Then there exists a set $\mathcal{F}$ such that $\nu(\mathcal{F}) = 0$ and $\sum_{\ell\in\mathbb{L}} \|\iota_\ell(w)\|^2 < \infty$ for all $w \in \mathcal{F}^C$. If the span of $\{\psi(w)\}$ for $w \in \mathcal{F}^C$ is not dense in the real Hilbert space $c_o(\mathbb{L})$, then there exists a non-zero vector $\vartheta = (\vartheta_\ell)_{\ell\in\mathbb{L}} \in l_1(\mathbb{L})$ (as a linear functional on $c_o(\mathbb{L})$) because $c_o(\mathbb{L})^* = l_1(\mathbb{L})$ such that the span of $\{\psi(w)\}$ for $w \in \mathcal{F}^C$ is in the kernel of ϑ. Define $Ke_\ell = \vartheta_\ell e_\ell$. Then K is a non-zero self-adjoint trace class operator of trace zero and

$$\mathrm{tr}(\mathcal{E}_w K \mathcal{E}_w^*) = \sum_{\ell\in\mathbb{L}} \vartheta_\ell \|\iota_\ell(\omega)\|^2 = 0, \ \forall\, w \in \mathcal{F}^C.$$

This leads to a contradiction. Hence $\mathrm{span}_{w\in\mathcal{F}^C}\{\psi(w)\}$ is dense in the real Hilbert space $c_o(\mathbb{L})$.

(2)⇒(1) Let $K \in L_1(\mathcal{N})$ be a self-adjoint operator. Then from [17], there exists some ONS $\{e_\ell\}_{\ell\in\mathbb{L}}$ such that

$$K\xi = \sum_{\ell\in\mathbb{L}} \vartheta_\ell \langle \xi, e_\ell\rangle e_\ell, \ \forall \xi \in \mathcal{N},$$

where $\vartheta = (\vartheta_\ell)_{\ell\in\mathbb{L}} \in l_1(\mathbb{L})$. Then

$$\mathrm{tr}(\mathcal{E}_w K \mathcal{E}_\omega^*) = \sum_{\ell\in\mathbb{L}} \vartheta_\ell \|\iota_\ell(w)\|^2.$$

Thus $\mathrm{tr}(\mathcal{E}_w K \mathcal{E}_w^*) = 0, \ \forall\, w \in \mathcal{F}^C$ implies that $(\vartheta_\ell) = 0$ since $\mathrm{span}_{w\in\mathcal{F}^C}\{\psi(w)\}$ is dense in the real Hilbert space $c_o(\mathbb{L})$. Therefore, we get that $K = 0$. □

We define a subspace of the real space $l_1(\mathbb{L})$ as follows:

$$\mathcal{D} := \left\{ (\vartheta_1, \vartheta_2, \ldots) \in l_1(\mathbb{L}) : \sum_{\ell\in\mathbb{L}} \vartheta_\ell = 0 \right\}.$$

Theorem 4.5. *Let $\mathcal{E} = \{\mathcal{E}_w \in \mathcal{B}(\mathcal{N}, \mathcal{M}_w) : w \in \Delta\}$ be a c-g-frame with analysis operator $T_{\mathcal{E}}$. Let $\iota_\ell = T_{\mathcal{E}} e_\ell, \forall \ell \in \mathbb{L}$ for some ONS $\{e_\ell\}_{\ell\in\mathbb{L}}$ of $\mathcal{N}$. The following statements are equivalent:*
(1) If $K \in L_1(\mathcal{N})$ and $\mathrm{tr}(\mathcal{E}_w K \mathcal{E}_w^*) = 0$, a.e. $w \in \Delta$, then $K = 0$.
(2) There exists a set $\mathcal{F} \subseteq \Delta$ with $\nu(\mathcal{F}) = 0$, if $\vartheta \in \mathcal{D}$ such that $\sum_{\ell\in\mathbb{L}} \vartheta_\ell \|\iota_\ell(w)\|^2 = 0, \ \forall\, w \in \mathcal{F}^C$, then $\vartheta = 0$.

Proof. (1)⇒(2) Consider $\iota_\ell = T_{\mathcal{E}} e_\ell$ for all $\ell \in \mathbb{L}$. Then there exists $\mathcal{F}$ such that $\nu(\mathcal{F}) = 0$ and $\sum_{\ell\in\mathbb{L}} \|\iota_\ell(w)\|^2 < \infty$ for all $w \in \mathcal{F}^C$. Suppose that there exists nonzero vector $\vartheta \in \mathcal{D}$ such that $\sum_{\ell\in\mathbb{L}} \vartheta_\ell \|\iota_\ell(w)\|^2 = 0, \ \forall\, w \in \mathcal{F}^C$. Define

$$Ke_\ell = \vartheta_\ell e_\ell, \ \forall \ell \in \mathbb{L}.$$

Then K is non-zero and we have

$$\mathrm{tr}(K) = \sum_{\ell\in\mathbb{L}} \vartheta_\ell < \infty, \ \ \mathrm{tr}(K) = 0,$$

and

$$\mathrm{tr}(\mathcal{E}_w K \mathcal{E}_w^*) = \sum_{\ell\in\mathbb{L}} \vartheta_\ell \|\iota_\ell(w)\|^2 = 0 \Rightarrow \vartheta_\ell = 0, \ \forall\, w \in \mathcal{F}^C.$$

Thus we reach the contradiction that $\vartheta = 0$.

(2)⇒(1) Assume that $K \in L_1(\mathcal{N})$ and $\mathrm{tr}(\mathcal{E}_w K \mathcal{E}_w^*) = 0$, *a.e.* $w \in \Delta$. Then we observe that the operator K is compact. Therefore, by the spectral mapping theorem, there exists an ONS $\{e_\ell\}_{\ell\in\mathbb{L}}$ and an eigenbasis corresponding to $\{\vartheta_\ell\}_{\ell\in\mathbb{L}}$ such that $Ke_\ell = \vartheta_\ell e_\ell$. Consequently, we have

$$0 = \mathrm{tr}(\mathcal{E}_w K \mathcal{E}_w^*) = \sum_{\ell\in\mathbb{L}} \vartheta_\ell \|\iota_\ell(w)\|^2 = 0 \Rightarrow \vartheta_\ell = 0, \ a.e.\ w \in \Delta.$$

Hence $K = 0$. □

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