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Some estimates for multilinear Calderón-Zygmund operators with fractional kernels of Dini's type and their commutators on generalized fractional mixed Morrey spaces

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Abstract. In this paper, we mainly study the boundedness of multilinear Calderón-Zygmund operator with fractional kernel of type $h(t)$ and its commutators on generalized fractional mixed Morrey space $L^{\vec{q},\eta,\psi}(\mathbb{R}^n)$. Firstly, with the help of the extrapolation theorem, the monotone convergence theorem and the boundedness of fractional integral operator I_ζ on mixed Lebesgue space $L^{\vec{q}}(\mathbb{R}^n)$, we obtain the boundedness of T on space $L^{\vec{q}}(\mathbb{R}^n)$. Secondly, the boundedness of T on space $L^{\vec{q},\eta,\psi}(\mathbb{R}^n)$ is derived by applying the boundedness of T on space $L^{\vec{q}}(\mathbb{R}^n)$. Thirdly, we prove that the commutators $T_{\prod \vec{b}}$ and $T_{\sum \vec{b}}$ are bounded from $L^{\vec{p_1},\eta_1,\psi}(\mathbb{R}^n) \times \cdots \times L^{\vec{p_m},\eta_m,\psi}(\mathbb{R}^n)$ to $L^{\vec{q},\eta,\psi}(\mathbb{R}^n)$. Finally, as applications, the boundedness for the multilinear fractional new maximal operator $\mathcal{M}_{\varphi,\beta}$ and its commutators $\mathcal{M}_{\vec{b},\varphi,\beta}$ and $[\vec{b}, \mathcal{M}_{\varphi,\beta}]$ on space $L^{\vec{q},\eta,\psi}(\mathbb{R}^n)$ is presented.

1. Introduction

In 1961, the $L^{\vec{p}}(\mathbb{R}^n)$ space with mixed norm was introduced by Benedek and Panzone in [2] as follows

$$L^{\vec{p}}(\mathbb{R}^n) = \left\{ f \in \mathcal{U}(\mathbb{R}^n) : \|f\|_{L^{\vec{p}}(\mathbb{R}^n)} < \infty \right\},$$

where $\mathcal{U}(\mathbb{R}^n)$ denotes the set of Lebesgue measurable functions on $\mathbb{R}^n$, $\vec{p} = (p_1, \cdots, p_n) \in (0, \infty]^n$ and the mixed Lebesgue norm $\|\cdot\|_{L^{\vec{p}}(\mathbb{R}^n)}$ is defined by

$$\|f\|_{L^{\vec{p}}(\mathbb{R}^n)} = \left(\int_{\mathbb{R}} \cdots \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(x_1, x_2, \cdots, x_n)|^{p_1} \mathrm{d}x_1 \right)^{\frac{p_2}{p_1}} \mathrm{d}x_2 \right)^{\frac{p_3}{p_2}} \cdots \mathrm{d}x_n \right)^{\frac{1}{p_n}}.$$

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Since then, the boundedness of some classical operators and their commutators on mixed Lebesgue space $L^{\vec{p}}(\mathbb{R}^n)$ was investigated, for example see [1, 7, 11, 12, 17, 28]. In 2019, Nogayama [24] introduced the mixed Morrey space $\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)$, which coincides with the mixed Lebesgue space $L^{\vec{q}}(\mathbb{R}^n)$ [2] and the classical Morrey space $\mathcal{M}_{q,\lambda}(\mathbb{R}^n)$ [22] and $\mathcal{M}^p_q(\mathbb{R}^n)$ [24] when some indexes take special values. The mixed Morrey space $\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)$ is as follows

$$\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n) = \left\{ f \in \mathcal{U}(\mathbb{R}^n) : \|f\|_{\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)} < \infty \right\},$$

where $\vec{q} = (q_1, \cdots, q_n) \in (0, \infty]^n$, $p \in (0, \infty]$, $\sum_{j=1}^n \frac{1}{q_j} \geq \frac{n}{p}$ and the mixed Morrey norm $\|\cdot\|_{\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)}$ is defined by

$$\|f\|_{\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)} = \sup_{x \in \mathbb{R}^n, r>0} |B(x,r)|^{\frac{1}{p} - \frac{1}{n}\sum_{j=1}^n \frac{1}{q_j}} \|f\chi_{B(x,r)}\|_{L^{\vec{q}}(\mathbb{R}^n)}.$$

In [24], the author also obtained some basic properties of $\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)$ and showed the boundedness of fractional integral operators and singular integral operators on $\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)$. In [6], Guliyev et al. gave the generalized Morrey space $\mathcal{M}^u_q(\mathbb{R}^n)$ and proved the sublinear operators H generated by Calderón-Zygmund operators, $H_\alpha (0 < \alpha < n)$ generated by Riesz potential operators and their commutators $H_{b,\alpha} (0 \leq \alpha < n)$ are bounded on $\mathcal{M}^u_q(\mathbb{R}^n)$, where the sufficient condition on the pair (u_1, u_2) ensures the boundedness of $H_{b,\alpha}$ from $\mathcal{M}^{u_1}_{q_1}(\mathbb{R}^n)$ to $\mathcal{M}^{u_2}_{q_2}(\mathbb{R}^n)$ with $\frac{1}{q_1} - \frac{1}{q_2} = \frac{\alpha}{n}$. In 2022, Wei [31] extended the mixed Morrey space $\mathcal{M}^p_{\vec{q}}(\mathbb{R}^n)$ and the generalized Morrey space $\mathcal{M}^u_q(\mathbb{R}^n)$ to the generalized mixed Morrey space $\mathcal{M}^u_{\vec{q}}(\mathbb{R}^n)$ and demonstrated the sublinear operators H generated by Calderón-Zygmund operators, $H_\alpha (0 < \alpha < n)$ generated by fractional integral operators, fractional maximal operators, Hardy-Littlewood maximal operator and their commutators are bounded on $\mathcal{M}^u_{\vec{q}}(\mathbb{R}^n)$. Let $u(x,r) : \mathbb{R}^n \times (0,\infty) \to (0,\infty)$ be a Lebesgue measurable function and $1 \leq \vec{q} < \infty$. Then a function $f \in \mathcal{U}(\mathbb{R}^n)$ belongs to $\mathcal{M}^u_{\vec{q}}(\mathbb{R}^n)$ if

$$\|f\|_{\mathcal{M}^u_{\vec{q}}(\mathbb{R}^n)} = \sup_{x \in \mathbb{R}^n, r>0} \frac{\|f\chi_{B(x,r)}\|_{L^{\vec{q}}(\mathbb{R}^n)}}{u(x,r)\|\chi_{B(x,r)}\|_{L^{\vec{q}}(\mathbb{R}^n)}} < \infty.$$

There are a lot of researches on the function spaces with mixed norm, such as see [32] for the generalized mixed Hardy-Morrey space, [10] for the anisotropic mixed norm Campanato-type space and anisotropic mixed norm Hardy space, [35] for the global and local mixed Morrey-type spaces, [21] for the Lorentz-Marcinkiewicz space with mixed norm, [20] for the mixed λ-central Morrey space, [4, 33] for the Lorentz space with mixed norm. In 2017, Tan and Liu [29] introduced the following generalized fractional Morrey space $L^{q,\eta,\psi}(\mathbb{R}^n)$ and established the boundedness of multilinear operator associated to singular integral operator with variable Calderón-Zygmund kernel on $L^{q,\eta,\psi}(\mathbb{R}^n)$. Let ψ be a positive, increasing function on $(0,\infty)$, $0 < \eta < n$ and $1 \leq q < \frac{n}{\eta}$. Then $f \in \mathcal{U}(\mathbb{R}^n)$ belongs to $L^{q,\eta,\psi}(\mathbb{R}^n)$ if

$$\|f\|_{L^{q,\eta,\psi}(\mathbb{R}^n)} = \sup_{x \in \mathbb{R}^n, r>0} \frac{\|f\chi_{B(x,r)}\|_{L^q(\mathbb{R}^n)}}{[\psi(r)]^{\frac{1}{q} - \frac{\eta}{n}}} < \infty,$$

where ψ satisfies $\psi(2t) \leq C\psi(t)$ for $t > 0$ and some constant $C > 0$. Noting that as $\eta = 0$, then $1 \leq q < \infty$.

In this article, we mainly study the boundedness of multilinear Calderón-Zygmund operator with fractional kernel of Dini's type T and its commutators on generalized fractional mixed Morrey space $L^{\vec{q},\eta,\psi}(\mathbb{R}^n)$. Next, we recall the definitions of the generalized fractional mixed Morrey space $L^{\vec{q},\eta,\psi}(\mathbb{R}^n)$ in [18], space $\mathrm{BMO}_{\varphi,\beta}$, T and the commutators of T in [15] and Lipschitz function in [25].

Definition 1.1. Suppose that ψ is a positive, increasing function on $(0,\infty)$ and there is a constant $C_\psi > 0$ such that

$$\psi(2t) \leq C_\psi \psi(t), \quad \text{for any } t > 0,$$

here the best possible constant C_ψ is called the doubling constant for ψ. Assume that $\eta \in [0, n)$, $\vec{q} = (q_1, \cdots, q_n) \in (0, \infty]^n$ and $\sum_{j=1}^n \frac{1}{q_j} > \eta$. Then the generalized fractional mixed Morrey space is defined by

$$L^{\vec{q},\eta,\psi}(\mathbb{R}^n) = \left\{ f \in \mathscr{U}(\mathbb{R}^n) : \|f\|_{L^{\vec{q},\eta,\psi}(\mathbb{R}^n)} < \infty \right\},$$

where

$$\|f\|_{L^{\vec{q},\eta,\psi}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n, r>0} \frac{\|f\chi_{B(x,r)}\|_{L^{\vec{q}}(\mathbb{R}^n)}}{[\psi(r)]^{\frac{1}{n}\sum_{j=1}^n \frac{1}{q_j} - \frac{\eta}{n}}}.$$

Throughout this paper, we always assume that $h(t) : [0, \infty) \to [0, \infty)$ is a nondecreasing function with $0 < h(1) < \infty$. For $a > 0$, we say that $h \in \mathrm{Dini}(a)$, if

$$|h|_{\mathrm{Dini}(a)} = \int_0^1 \frac{h^a(t)}{t} \mathrm{d}t < \infty.$$

Clearly, as $a_2 > a_1 > 0$, $\mathrm{Dini}(a_2) \supset \mathrm{Dini}(a_1)$. In what follows, $A \lesssim B$ denotes there is a constant $C > 0$ such that $A \le CB$ and $A \approx B$ means there exist the positive constants C_1 and C_2 such that $C_1 B \le A \le C_2 B$. Set $\vec{q} = (q_1, \cdots, q_n) \in (0, \infty]^n$, $1 < \vec{q} < \infty$ means $1 < q_j < \infty (j = 1, \cdots, n)$, $\frac{1}{\vec{q}} = (\frac{1}{q_1}, \cdots, \frac{1}{q_n})$ and $\vec{q}\,' = (q_1', \cdots, q_n')$, where q_j' is the conjugate exponent of q_j. Let $\varphi(s) = (1 + s)^\gamma$ for $s \ge 0$ and $0 \le \gamma < \infty$.

Definition 1.2. Let $0 \le \beta < 1$ and $K_\beta(x, y_1, \cdots, y_m)$ be a locally integrable function defined away from the diagonal $x = y_1 = \cdots = y_m$ in $(\mathbb{R}^n)^{m+1}$. Suppose that $T : \mathcal{S}(\mathbb{R}^n) \times \cdots \times \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ (from the product of Schwarz spaces to the space of tempered distributions) is bounded from $L^{p_1}(\mathbb{R}^n) \times \cdots \times L^{p_m}(\mathbb{R}^n)$ into $L^{p,\infty}(\mathbb{R}^n)$ for $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$ with $1 \le p_1, \cdots, p_m < \infty$. Then T is called the multilinear Calderón-Zygmund operator with fractional kernel of type $h(t)$ if there exist the constant $N \ge 0$ and $K_\beta(x, y_1, \cdots, y_m)$ that satisfies (1.1)-(1.3) such that for $x \notin \bigcap_{i=1}^m \mathrm{supp} f_i$, there holds

$$T(f_1, \cdots, f_m)(x) = \int_{(\mathbb{R}^n)^m} K_\beta(x, y_1, \cdots, y_m) \prod_{i=1}^m f_i(y_i) \mathrm{d}\vec{y},$$

where (1.1)-(1.3) are given as follows
(i) for all $(x, y_1, \cdots, y_m) \in (\mathbb{R}^n)^{m+1}$ with $x \ne y_i (i = 1, \cdots, m)$, there holds

$$|K_\beta(x, y_1, \cdots, y_m)| \lesssim \frac{1}{(\sum_{i=1}^m |x - y_i|)^{mn(1-\beta)} (1 + (\sum_{i=1}^m |x - y_i|)^n)^N}; \tag{1.1}$$

(ii) if $|x - x'| \le \frac{1}{2} \max_{1\le i\le m} |x - y_i|$, then

$$\begin{aligned} &|K_\beta(x, y_1, \cdots, y_m) - K_\beta(x', y_1, \cdots, y_m)| \\ &\lesssim \frac{1}{(\sum_{i=1}^m |x - y_i|)^{mn(1-\beta)} (1 + (\sum_{i=1}^m |x - y_i|)^n)^N} h\left(\frac{|x - x'|}{\sum_{i=1}^m |x - y_i|} \right); \end{aligned} \tag{1.2}$$

(iii) if $|y_i - y_i'| \le \frac{1}{2} \max_{1\le i\le m} |x - y_i|$, then

$$\begin{aligned} &|K_\beta(x, y_1, \cdots, y_i, \cdots, y_m) - K_\beta(x, y_1, \cdots, y_i', \cdots, y_m)| \\ &\lesssim \frac{1}{(\sum_{i=1}^m |x - y_i|)^{mn(1-\beta)} (1 + (\sum_{i=1}^m |x - y_i|)^n)^N} h\left(\frac{|y_i - y_i'|}{\sum_{i=1}^m |x - y_i|} \right). \end{aligned} \tag{1.3}$$

Definition 1.3. Let b be a locally integrable function and $0 \leq \beta < 1$. We say that $b \in \mathrm{BMO}_{\varphi,\beta}$ if

$$\|b\|_{\mathrm{BMO}_{\varphi,\beta}} = \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{(\varphi(|B(x,r)|)|B(x,r)|)^{1-\beta}} \int_{B(x,r)} |b(y) - b_{B(x,r)}| \mathrm{d}y < \infty,$$

where $B = B(x,r)$ represents an open ball centered at $x \in \mathbb{R}^n$ with radius $r > 0$, $b_{B(x,r)}$ denotes the mean value of b over ball $B(x,r)$ and $\varphi(|B(x,r)|) = (1 + |B(x,r)|)^\gamma$ for $0 \leq \gamma < \infty$. As $\gamma = 0$ and $\beta = 0$, $\mathrm{BMO}_{\varphi,\beta}$ space will reduce to $\mathrm{BMO}(\mathbb{R}^n)$ space. Clearly, $\mathrm{BMO}(\mathbb{R}^n) \subset \mathrm{BMO}_{\varphi,\beta}$ when $0 \leq \beta \leq \frac{\gamma}{1+\gamma}$, here the fact $\frac{|B(x,r)|}{(\varphi(|B(x,r)|)|B(x,r)|)^{1-\beta}} = \frac{|B(x,r)|^\beta}{(1+|B(x,r)|)^{\gamma(1-\beta)}} \leq \left(\frac{|B(x,r)|}{1+|B(x,r)|}\right)^\beta \leq 1$ is used.

Definition 1.4. Let $0 < \alpha < 1$. Then for any $b \in L^1_{loc}(\mathbb{R}^n)$ and $x, y \in \mathbb{R}^n$, b is said to belong to the Lipschitz space $\mathrm{Lip}_\alpha(\mathbb{R}^n)$ if there is some constant $C > 0$ such that

$$|b(x) - b(y)| \leq C|x - y|^\alpha,$$

where the smallest positive constant C is called the $\mathrm{Lip}_\alpha(\mathbb{R}^n)$ norm of b and denoted by $\|b\|_{\mathrm{Lip}_\alpha(\mathbb{R}^n)}$.

Definition 1.5. Let $0 \leq \beta < 1$, $\vec{f} = (f_1, \cdots, f_m)$ and $\vec{b} = (b_1, \cdots, b_m)$. Assume that T is the multilinear Calderón-Zygmund operator with fractional kernel of type $h(t)$. Then the m-linear iterative commutator generated by T and $\vec{b}$ is defined by

$$T_{\prod \vec{b}}(\vec{f})(x) = \int_{(\mathbb{R}^n)^m} \prod_{i=1}^m (b_i(x) - b_i(y_i)) K_\beta(x, y_1, \cdots, y_m) f_1(y_1) \cdots f_m(y_m) \mathrm{d}\vec{y},$$

and the multilinear commutator generated by T and $\vec{b}$ is defined by

$$T_{\sum \vec{b}}(\vec{f})(x) = \sum_{i=1}^m T^i_{b_i}(\vec{f})(x),$$

where

$$T^i_{b_i}(\vec{f})(x) = b_i(x) T(f_1, \cdots, f_i, \cdots, f_m)(x) - T(f_1, \cdots, b_i f_i, \cdots, f_m)(x), \text{ for } i = 1, \cdots, m.$$

In 2003, the commutators of multilinear Calderón-Zygmund operators were introduced by Pérez and Torres in [27], and they also gave the strong and weak type estimates for the commutators of multilinear Calderón-Zygmund operators. From this time on, many researchers investigated the related contents. In [5, 13], the authors studied the multilinear Calderón-Zygmund operators with standard kernels (i.e. $\beta = 0$, $N = 0$ and $h(t) = t^\varepsilon (\varepsilon > 0)$ in (1.1)-(1.3)) and their commutators. In [19], Lu and Zhang considered the m-linear operators with m-linear Calderón-Zygmund kernels of type $h(t)$ (i.e. $\beta = 0$ and $N = 0$ in (1.1)-(1.3)) and their commutators. In [26], Pan and Tang established the certain classes of multilinear operators (i.e. $\beta = 0$, $(1 + (\sum_{i=1}^m |x - y_i|)^n)^N = (1 + \sum_{i=1}^m |x - y_i|)^N$ and $h(t) = t^\varepsilon (\varepsilon > 0)$ in (1.1)-(1.3)) and their iterated commutators. In [36], Zhao and Zhou considered the certain multilinear Calderón-Zygmund operators with kernels of Dini's type (i.e. $\beta = 0$ and $(1 + (\sum_{i=1}^m |x - y_i|)^n)^N = (1 + \sum_{i=1}^m |x - y_i|)^N$ in (1.1)-(1.3)) and their commutators. In [16, 34], the authors studied the multilinear singular integral operators with generalized kernels and their commutators.

The paper is organized as follows. In Section 2, we give some important lemmas. In Section 3, the boundedness of T on generalized fractional mixed Morrey space $L^{\vec{q},\eta,\psi}(\mathbb{R}^n)$ will be showed. In Section 4, we will prove that $T_{\prod \vec{b}}$ and $T_{\sum \vec{b}}$ associated with $\mathrm{BMO}_{\varphi,\beta}$ functions and Lipschitz functions are bounded on space $L^{\vec{q},\eta,\psi}(\mathbb{R}^n)$. Finally, the estimates for the multilinear fractional new maximal operator and its commutators on space $L^{\vec{q},\eta,\psi}(\mathbb{R}^n)$ shall be presented in Section 5.

2. Some preliminaries

The following extrapolation theorem and the boundedness of I_ζ on mixed Lebesgue space $L^{\vec{q}}(\mathbb{R}^n)$ have been proved in [31, 35]. Before stating Lemma 2.1, we first recall the definitions of $A_p(\mathbb{R}^n)$ weight, $A_1(\mathbb{R}^n)$ weight and $L^p(\omega)$ in [13, 23]. Let ω be a locally integrable function on $\mathbb{R}^n$ which takes values in $(0,\infty)$ at almost everywhere, $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Then $\omega \in A_p(\mathbb{R}^n)$ if there exists a constant $C > 0$ such that for all balls B,

$$\left(\frac{1}{|B|}\int_B \omega(x)\mathrm{d}x\right)^{\frac{1}{p}}\left(\frac{1}{|B|}\int_B \omega(x)^{1-p'}\mathrm{d}x\right)^{\frac{1}{p'}} \leq C,$$

where $B = B(x,r)$ denotes an open ball centered at $x \in \mathbb{R}^n$ with radius $r > 0$. In particular, this is called that ω belongs to $A_1(\mathbb{R}^n)$ when $p = 1$, if there is a constant $C > 0$ such that

$$\frac{1}{|B|}\int_B \omega(x)\mathrm{d}x \leq C\inf_B \omega,$$

where the infimum of these constants C is called the $A_1(\mathbb{R}^n)$ constant of ω. And the weighted Lebesgue space $L^p(\omega)(0 < p < \infty)$ consists of all $f \in \mathcal{U}(\mathbb{R}^n)$ such that

$$\|f\|_{L^p(\omega)} = \left(\int_{\mathbb{R}^n} |f(x)|^p \omega(x)\mathrm{d}x\right)^{\frac{1}{p}} < \infty.$$

Lemma 2.1. *[31] Let $\mathcal{G}$ be a family of ordered pairs of nonnegative measurable functions, $\vec{q} = (q_1, \cdots, q_n) \in (0,\infty)^n$ and $0 < q_0 < \infty$ with $q_0 < \vec{q}$. If*

$$\|f\|_{L^{q_0}(\omega)} \lesssim \|g\|_{L^{q_0}(\omega)}$$

for $\omega \in A_1(\mathbb{R}^n)$ and $(f,g) \in \mathcal{G}$, then the inequality

$$\|f\|_{L^{\vec{q}}(\mathbb{R}^n)} \lesssim \|g\|_{L^{\vec{q}}(\mathbb{R}^n)}$$

holds for $(f,g) \in \mathcal{G}$ such that the left-hand side is finite.

Lemma 2.2. *[35] Suppose that $1 < \vec{p} \leq \vec{q} < \infty$, $\sum_{j=1}^n \frac{1}{q_j} = \sum_{j=1}^n \frac{1}{p_j} - \zeta$ and $0 \leq \zeta < n$. Then*

$$\|I_\zeta(f)\|_{L^{\vec{q}}(\mathbb{R}^n)} \lesssim \|f\|_{L^{\vec{p}}(\mathbb{R}^n)},$$

where I_ζ is the fractional integral operator defined by

$$I_\zeta(f)(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\zeta}}\mathrm{d}y.$$

From [2], we know that Hölder's inequality and Minkowski inequality still hold on space $L^{\vec{q}}(\mathbb{R}^n)$.

Lemma 2.3. *Assume that $1 \leq \vec{q} \leq \infty$ and $\frac{1}{\vec{q}} + \frac{1}{\vec{q}'} = 1$. Then for any $f \in L^{\vec{q}}(\mathbb{R}^n)$ and $g \in L^{\vec{q}'}(\mathbb{R}^n)$, we have*

$$\int_{\mathbb{R}^n} |f(x)g(x)|\mathrm{d}x \leq \|f\|_{L^{\vec{q}}(\mathbb{R}^n)}\|g\|_{L^{\vec{q}'}(\mathbb{R}^n)}.$$

Lemma 2.4. *Let $1 \leq \vec{q} \leq \infty$. Then for any $f, g \in L^{\vec{q}}(\mathbb{R}^n)$, there holds*

$$\|f+g\|_{L^{\vec{q}}(\mathbb{R}^n)} \leq \|f\|_{L^{\vec{q}}(\mathbb{R}^n)} + \|g\|_{L^{\vec{q}}(\mathbb{R}^n)}.$$

Next, we give the following relationship between the dyadic fractional maximal operator $M^d_{\delta,\varphi,\beta}$ and the dyadic fractional sharp maximal function $M^{\sharp,d}_{\delta,\varphi,\beta}$ and the inequality for the dyadic fractional sharp maximal function of T in [15]. Before stating the lemmas, we need to provide some necessary concepts. Let $0 \le \beta < 1$ and $0 < \delta < \infty$. Then $M^d_{\delta,\varphi,\beta}$ and $M^{\sharp,d}_{\delta,\varphi,\beta}$ are defined as follows

$$M^d_{\delta,\varphi,\beta}(f) = (M^d_{\varphi,\beta}(|f|^\delta))^{\frac{1}{\delta}} \quad \text{and} \quad M^{\sharp,d}_{\delta,\varphi,\beta}(f) = (M^{\sharp,d}_{\varphi,\beta}(|f|^\delta))^{\frac{1}{\delta}},$$

where

$$M^d_{\varphi,\beta}f(x) = \sup_{x\in B(\text{dyadic cube})} \frac{1}{(\varphi(|B|)|B|)^{1-\beta}} \int_B |f(y)|\mathrm{d}y,$$

and

$$\begin{aligned} M^{\sharp,d}_{\varphi,\beta}(f)(x) &= \sup_{x\in B(\text{dyadic cube})} \inf_c \frac{1}{(\varphi(|B|)|B|)^{1-\beta}} \int_B |f(y) - c|\mathrm{d}y \\ &\approx \sup_{x\in B(\text{dyadic cube})} \frac{1}{(\varphi(|B|)|B|)^{1-\beta}} \int_B |f(y) - f_B|\mathrm{d}y. \end{aligned}$$

Let $\vec{f} = (f_1, \cdots, f_m)$ be a collection of locally integrable functions and $0 \le \beta < 1$. Then the multilinear fractional new maximal operator $\mathcal{M}_{\varphi,\beta}$ was given in [14] as follows

$$\mathcal{M}_{\varphi,\beta}(\vec{f})(x) = \sup_{x\in\mathbb{R}^n, r>0} \prod_{i=1}^m \frac{1}{(\varphi(|B(x,r)|)|B(x,r)|)^{1-\beta}} \int_{B(x,r)} |f_i(y_i)|\mathrm{d}y_i.$$

Here we recall the definition of the weight $A_{p,\beta}(\varphi)$ in [9], which included $A_p(\varphi)(\beta = 0)$ weight in [30] and the classical Muckenhoupt weight $A_p(\mathbb{R}^n)(\beta, \gamma = 0)$ in [23]. Let $1 < p < \infty$, $0 \le \beta < 1$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Then $\omega \in A_{p,\beta}(\varphi)$ if there is a constant $C > 0$ such that for all balls B,

$$\left(\frac{1}{(\varphi(|B|)|B|)^{1-\beta}} \int_B \omega(x)\mathrm{d}x\right)^{\frac{1}{p}} \left(\frac{1}{(\varphi(|B|)|B|)^{1-\beta}} \int_B \omega(x)^{1-p'}\mathrm{d}x\right)^{\frac{1}{p'}} \le C,$$

where $B = B(x, r)$ denotes an open ball centered at $x \in \mathbb{R}^n$ with radius $r > 0$. Especially, this is called that ω belongs to $A_{1,\beta}(\varphi)$ when $p = 1$, if there exists a constant $C > 0$ such that

$$\frac{1}{(\varphi(|B|)|B|)^{1-\beta}} \int_B \omega(x)\mathrm{d}x \le C \inf_B \omega,$$

where the infimum of these constants C is called the $A_{1,\beta}(\varphi)$ constant of ω. Noticing that $A_{\infty,\beta}(\varphi) = \bigcup_{p\ge1} A_{p,\beta}(\varphi)$.

Lemma 2.5. *[15] Assume that $0 < p < \infty$, $0 < \delta < \infty$ and $\omega \in A_{\infty,\beta}(\varphi)$ with $0 \le \beta < 1$. Then for $f \in L^p(\omega)$, there holds*

$$\|f\|_{L^p(\omega)} \le \|M^d_{\delta,\varphi,\beta}(f)\|_{L^p(\omega)} \lesssim \|M^{\sharp,d}_{\delta,\varphi,\beta}(f)\|_{L^p(\omega)}.$$

Lemma 2.6. *[15] Suppose that T is the multilinear Calderón-Zygmund operator with fractional kernel of type $h(t)$, $0 \le \beta \le \frac{\gamma}{1+\gamma}$ with $0 \le \gamma < \infty$ and $0 < \delta < \frac{1}{m}$. If $h \in \mathrm{Dini}(1)$, then for all bounded measurable functions $\vec{f} = (f_1, \cdots, f_m)$ with compact support, we have*

$$M^{\sharp,d}_{\delta,\varphi,\beta}(T(\vec{f}))(x) \lesssim \mathcal{M}_{\varphi,\beta}(\vec{f})(x).$$

Lemma 2.5 together with Lemma 2.6 implies that the following lemma is true.

Lemma 2.7. *Let T be the multilinear Calderón-Zygmund operator with fractional kernel of type $h(t)$, $0 \le \beta \le \frac{\gamma}{1+\gamma}$ with $0 \le \gamma < \infty$ and $0 < p < \infty$. If $h \in \mathrm{Dini}(1)$ and $\omega \in A_{\infty,\beta}(\varphi)$, then*

$$\|T(\vec{f})\|_{L^p(\omega)} \lesssim \|\mathcal{M}_{\varphi,\beta}(\vec{f})\|_{L^p(\omega)}.$$

3. T on space $L^{\vec{q},\eta,\psi}(\mathbb{R}^n)$

The main result of this section is stated as follows.

Theorem 3.1. *Let T be the multilinear Calderón-Zygmund operator with fractional kernel of type $h(t)$, $h \in \mathrm{Dini}(1)$, $0 \le \gamma \le m-1$, $0 \le \beta \le \frac{\gamma}{1+\gamma}$ and $0 < C_\psi < 2^n$. If $n\beta \le \eta_i < \frac{n}{m} + n\beta$ $(i = 1, \cdots, m)$, $\eta = \sum_{i=1}^m \eta_i - mn\beta$, $\sum_{j=1}^n \frac{1}{q_j} = \sum_{j=1}^n \sum_{i=1}^m \frac{1}{p_{ij}} - mn\beta$ with $\sum_{j=1}^n \frac{1}{p_{ij}} > \eta_i$, $m \ge 2$ and $1 < \vec{q}, \vec{p_i} < \infty$, then T is bounded from $L^{\vec{p_1},\eta_1,\psi}(\mathbb{R}^n)\times\cdots\times L^{\vec{p_m},\eta_m,\psi}(\mathbb{R}^n)$ to $L^{\vec{q},\eta,\psi}(\mathbb{R}^n)$, i.e.*

$$\|T(\vec{f})\|_{L^{\vec{q},\eta,\psi}(\mathbb{R}^n)} \lesssim \prod_{i=1}^m \|f_i\|_{L^{\vec{p_i},\eta_i,\psi}(\mathbb{R}^n)}.$$

To prove Theorem 3.1, we need the following boundedness of T on space $L^{\vec{q}}(\mathbb{R}^n)$.

Lemma 3.2. *Suppose that T is a multilinear Calderón-Zygmund operator with fractional kernel of type $h(t)$, $m \ge 2$, $h \in \mathrm{Dini}(1)$ and $0 \le \beta \le \frac{\gamma}{1+\gamma}$ with $0 \le \gamma < \infty$. If $\sum_{j=1}^n \frac{1}{q_j} = \sum_{j=1}^n \sum_{i=1}^m \frac{1}{p_{ij}} - mn\beta$ and $1 < \vec{q}, \vec{p_i} < \infty$, then T is bounded from $L^{\vec{p_1}}(\mathbb{R}^n) \times \cdots \times L^{\vec{p_m}}(\mathbb{R}^n)$ to $L^{\vec{q}}(\mathbb{R}^n)$, i.e.*

$$\|T(\vec{f})\|_{L^{\vec{q}}(\mathbb{R}^n)} \lesssim \prod_{i=1}^m \|f_i\|_{L^{\vec{p_i}}(\mathbb{R}^n)}.$$

Proof. Let $\vec{f} \in L^{\vec{p_1}}(\mathbb{R}^n) \times \cdots \times L^{\vec{p_m}}(\mathbb{R}^n)$. We only need to prove the case $f_i \in L_c^\infty(\mathbb{R}^n)(i = 1, \cdots, m)$ because the class of bounded functions with compact support are dense in $L^{\vec{p_i}}(\mathbb{R}^n)$ (see [2]). Set

$$T^k(\vec{f})(x) = \min\{k, T(\vec{f})\}\chi_{B(0,k)}(x),$$

and

$$\mathcal{G} = \{(T^k(\vec{f}), \mathcal{M}_{\varphi,\beta}(\vec{f})) : \vec{f} \in (L_c^\infty(\mathbb{R}^n))^m, k \in \mathbb{N}^+\},$$

where $\mathbb{N}^+$ represents the set of the positive integers. Then for every pair $(T^k(\vec{f}), \mathcal{M}_{\varphi,\beta}(\vec{f})) \in \mathcal{G}$, any $0 < q_0 < \infty$ with $q_0 < \vec{q}$, and $\omega \in A_1(\mathbb{R}^n) \subset A_{1,\beta}(\varphi) \subset A_{\infty,\beta}(\varphi)$, by applying Lemma 2.7, there holds

$$\|T(\vec{f})\|_{L^{q_0}(\omega)} \lesssim \|\mathcal{M}_{\varphi,\beta}(\vec{f})\|_{L^{q_0}(\omega)}.$$

This inequality together with the definition of T^k implies that

$$\|T^k(\vec{f})\|_{L^{q_0}(\omega)} \lesssim \|\mathcal{M}_{\varphi,\beta}(\vec{f})\|_{L^{q_0}(\omega)}.$$

And because $\|\chi_{B(0,k)}\|_{L^{\vec{q}}(\mathbb{R}^n)} < \infty$, then $\|T^k(\vec{f})\|_{L^{\vec{q}}(\mathbb{R}^n)} < \infty$ for each $\vec{f} \in (L_c^\infty(\mathbb{R}^n))^m$. We may choose $\vec{q_i}$ such that $\frac{1}{\vec{q}} = \sum_{i=1}^m \frac{1}{\vec{q_i}}$ and $\sum_{j=1}^n \frac{1}{q_{ij}} = \sum_{j=1}^n \frac{1}{p_{ij}} - \beta_i$, then $\beta = \frac{1}{mn}\sum_{i=1}^m \beta_i$. Thus, using Hölder's inequality, Lemma 2.1 and Lemma 2.2 to each pair in $\mathcal{G}$, we have

$$\begin{aligned}
\|T^k(\vec{f})\|_{L^{\vec{q}}(\mathbb{R}^n)} \lesssim \|\mathcal{M}_{\varphi,\beta}(\vec{f})\|_{L^{\vec{q}}(\mathbb{R}^n)} &\le \|\prod_{i=1}^m M_{\beta_i}(f_i)\|_{L^{\vec{q}}(\mathbb{R}^n)} \\
&\lesssim \prod_{i=1}^m \|I_{\beta_i}(|f_i|)\|_{L^{\vec{q_i}}(\mathbb{R}^n)} \\
&\lesssim \prod_{i=1}^m \|f_i\|_{L^{\vec{p_i}}(\mathbb{R}^n)},
\end{aligned}$$

here M_{β_i} is the fractional maximal operator defined by

$$M_{\beta_i}(f_i)(x) = \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{|B(x,r)|^{1-\frac{\beta_i}{n}}} \int_{B(x,r)} |f_i(y_i)|\mathrm{d}y_i,$$

and we use the following estimate

$$\begin{aligned}\mathcal{M}_{\varphi,\beta}(\vec{f})(x) &= \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{(\varphi(|B(x,r)|)|B(x,r)|)^{m(1-\beta)}} \prod_{i=1}^{m} \int_{B(x,r)} |f_i(y_i)|\mathrm{d}y_i \\ &\leq \sup_{x\in\mathbb{R}^n, r>0} \prod_{i=1}^{m} \frac{1}{|B(x,r)|^{1-\frac{\beta_i}{n}}} \int_{B(x,r)} |f_i(y_i)|\mathrm{d}y_i \\ &\leq \prod_{i=1}^{m} M_{\beta_i}(f_i)(x). \end{aligned} \tag{3.1}$$

Since $T^k(\vec{f})(x)$ increases almost everywhere to $T(\vec{f})(x)$ for each $\vec{f} \in (L_c^\infty(\mathbb{R}^n))^m$, then from the monotone convergence theorem on mixed Lebesgue space in [2], it follows that

$$\|T(\vec{f})\|_{L^{\vec{q}}(\mathbb{R}^n)} = \lim_{k\to\infty} \|T^k(\vec{f})\|_{L^{\vec{q}}(\mathbb{R}^n)} \lesssim \prod_{i=1}^{m} \|f_i\|_{L^{\vec{p}_i}(\mathbb{R}^n)}.$$

The proof is finished. □

Proof. [The proof of Theorem 3.1] It is sufficient to prove the case $m = 2$ because the similar arguments also work for the other cases. Let $\vec{f} = (f_1, f_2) \in L^{\vec{p_1},\eta_1,\psi}(\mathbb{R}^n) \times L^{\vec{p_2},\eta_2,\psi}(\mathbb{R}^n)$, $B =: B(x,r)$ for $r > 0$, $2B(x,r) =: B(x,2r)$ and $(2B(x,r))^c =: \mathbb{R}^n \setminus 2B(x,r)$. Then from Minkowski inequality, we have

$$\begin{aligned}\|T(f_1,f_2)\|_{L^{\vec{q},\eta,\psi}(\mathbb{R}^n)} &\leq \|T(f_1\chi_{2B(x,r)}, f_2\chi_{2B(x,r)})\|_{L^{\vec{q},\eta,\psi}(\mathbb{R}^n)} \\ &\quad + \|T(f_1\chi_{2B(x,r)}, f_2\chi_{(2B(x,r))^c})\|_{L^{\vec{q},\eta,\psi}(\mathbb{R}^n)} \\ &\quad + \|T(f_1\chi_{(2B(x,r))^c}, f_2\chi_{2B(x,r)})\|_{L^{\vec{q},\eta,\psi}(\mathbb{R}^n)} \\ &\quad + \|T(f_1\chi_{(2B(x,r))^c}, f_2\chi_{(2B(x,r))^c})\|_{L^{\vec{q},\eta,\psi}(\mathbb{R}^n)} \\ &=: I + II + III + IV.\end{aligned}$$

Noticing that $0 \leq \gamma \leq 1$, then $\frac{\gamma}{1+\gamma} \leq \frac{1}{2}$. And because $n\beta \leq \eta_i < \frac{n}{2} + n\beta (i = 1,2)$ and $\eta = \sum_{i=1}^{2} \eta_i - 2n\beta$, then $n\beta \leq \eta_i < n$ and $0 \leq \eta < n$. Thus, it gets from $\sum_{j=1}^{n} \frac{1}{q_j} = \sum_{j=1}^{n} \frac{1}{p_{1j}} + \sum_{j=1}^{n} \frac{1}{p_{2j}} - 2n\beta$ and Lemma 3.2 that

$$\begin{aligned} I &= \sup_{x\in\mathbb{R}^n, r>0} \frac{\|T(f_1\chi_{2B(x,r)}, f_2\chi_{2B(x,r)})\chi_{B(x,r)}\|_{L^{\vec{q}}(\mathbb{R}^n)}}{[\psi(r)]^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{q_j} - \frac{\eta}{n}}} \\ &\lesssim \sup_{x\in\mathbb{R}^n, r>0} \frac{\|f_1\chi_{2B(x,r)}\|_{L^{\vec{p_1}}(\mathbb{R}^n)} \|f_2\chi_{2B(x,r)}\|_{L^{\vec{p_2}}(\mathbb{R}^n)}}{[\psi(r)]^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{q_j} - \frac{\eta}{n}}} \\ &\leq \prod_{i=1}^{2} \|f_i\|_{L^{\vec{p_i},\eta_i,\psi}(\mathbb{R}^n)} \sup_{x\in\mathbb{R}^n, r>0} \frac{[\psi(2r)]^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{1j}} - \frac{\eta_1}{n}} [\psi(2r)]^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{2j}} - \frac{\eta_2}{n}}}{[\psi(r)]^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{q_j} - \frac{\eta}{n}}} \\ &\leq \prod_{i=1}^{2} \|f_i\|_{L^{\vec{p_i},\eta_i,\psi}(\mathbb{R}^n)} \sup_{x\in\mathbb{R}^n, r>0} \frac{[C_\psi \psi(r)]^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{1j}} + \frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{2j}} - \frac{\eta_1+\eta_2}{n}}}{[\psi(r)]^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{q_j} - \frac{\eta}{n}}}\end{aligned}$$

$$\lesssim \prod_{i=1}^{2} \|f_i\|_{L^{\vec{p_i},\eta_i,\psi}(\mathbb{R}^n)}.$$

Next, we prove II. For $y_1 \in 2B(x,r)$, $y_2 \in 2^{k+1}B(x,r) \setminus 2^kB(x,r)$ and $z \in B(x,r)$, it holds that $|z-y_2| \gtrsim |2^kB(x,r)|^{\frac{1}{n}}$. Thus, by (1.1), we have

$$\begin{aligned}|K_\beta(z,y_1,y_2)| &\lesssim \frac{1}{(\sum_{i=1}^2 |z-y_i|)^{2n(1-\beta)}(1+(\sum_{i=1}^2 |z-y_i|)^n)^N}\\ &\lesssim \frac{1}{|2^kB(x,r)|^{2(1-\beta)}}. \end{aligned} \tag{3.2}$$

Hence, according to (3.2) and Hölder's inequality, we have

$$\begin{aligned}
&|T(f_1\chi_{2B(x,r)}, f_2\chi_{(2B(x,r))^c})(z)|\\
&\le \int_{(\mathbb{R}^n)^2} |K_\beta(z,y_1,y_2)||f_1\chi_{2B(x,r)}(y_1)||f_2\chi_{(2B(x,r))^c}(y_2)|\mathrm{d}y_1\mathrm{d}y_2\\
&\lesssim \left(\int_{2B(x,r)} |f_1(y_1)|\mathrm{d}y_1\right)\left(\sum_{k=1}^{\infty}\int_{2^{k+1}B(x,r)} \frac{|f_2(y_2)|}{|2^kB(x,r)|^{2(1-\beta)}}\mathrm{d}y_2\right)\\
&\le \|f_1\chi_{2B(x,r)}\|_{L^{\vec{p_1}}(\mathbb{R}^n)}\|\chi_{2B(x,r)}\|_{L^{\vec{p_1}'}(\mathbb{R}^n)}\\
&\quad\times\sum_{k=1}^{\infty}\frac{\|f_2\chi_{2^{k+1}B(x,r)}\|_{L^{\vec{p_2}}(\mathbb{R}^n)}\|\chi_{2^{k+1}B(x,r)}\|_{L^{\vec{p_2}'}(\mathbb{R}^n)}}{|2^kB(x,r)|^{2(1-\beta)}}\\
&\lesssim [\psi(2r)]^{\frac{1}{n}\sum_{j=1}^n\frac{1}{p_{1j}}-\frac{\eta_1}{n}}|2B(x,r)|^{1-\frac{1}{n}\sum_{j=1}^n\frac{1}{p_{1j}}}\\
&\quad\times\sum_{k=1}^{\infty}\frac{[\psi(2^{k+1}r)]^{\frac{1}{n}\sum_{j=1}^n\frac{1}{p_{2j}}-\frac{\eta_2}{n}}|2^{k+1}B(x,r)|^{1-\frac{1}{n}\sum_{j=1}^n\frac{1}{p_{2j}}}}{|2^kB(x,r)|^{2(1-\beta)}}\prod_{i=1}^{2}\|f_i\|_{L^{\vec{p_i},\eta_i,\psi}(\mathbb{R}^n)},
\end{aligned}$$

the last inequality follows from the fact $\|\chi_{2B(x,r)}\|_{L^{\vec{p_1}'}(\mathbb{R}^n)} \approx |2B(x,r)|^{1-\frac{1}{n}\sum_{j=1}^n\frac{1}{p_{1j}}}$ in [24]. Therefore, by $0 < C_\psi < 2^n$ and $\frac{\eta_2}{n} \ge \frac{n\beta}{n} = \beta > 2\beta - 1$, we get

$$\begin{aligned}
II &= \sup_{x\in\mathbb{R}^n, r>0}\frac{\|T(f_1\chi_{2B(x,r)}, f_2\chi_{(2B(x,r))^c})\chi_{B(x,r)}\|_{L^{\vec{q}}(\mathbb{R}^n)}}{[\psi(r)]^{\frac{1}{n}\sum_{j=1}^n\frac{1}{q_j}-\frac{\eta}{n}}}\\
&\lesssim \sup_{x\in\mathbb{R}^n, r>0}\sum_{k=1}^{\infty}\frac{[C_\psi\psi(r)]^{\frac{1}{n}\sum_{j=1}^n\frac{1}{p_{1j}}-\frac{\eta_1}{n}}[C_\psi^{k+1}\psi(r)]^{\frac{1}{n}\sum_{j=1}^n\frac{1}{p_{2j}}-\frac{\eta_2}{n}}}{[\psi(r)]^{\frac{1}{n}\sum_{j=1}^n\frac{1}{q_j}-\frac{\eta}{n}}}\\
&\quad\times|B(x,r)|^{1-\frac{1}{n}\sum_{j=1}^n\frac{1}{p_{1j}}+\frac{1}{n}\sum_{j=1}^n\frac{1}{q_j}}|2^{k+1}B(x,r)|^{1-\frac{1}{n}\sum_{j=1}^n\frac{1}{p_{2j}}+2(\beta-1)}\prod_{i=1}^{2}\|f_i\|_{L^{\vec{p_i},\eta_i,\psi}(\mathbb{R}^n)}\\
&\lesssim \prod_{i=1}^{2}\|f_i\|_{L^{\vec{p_i},\eta_i,\psi}(\mathbb{R}^n)}\sum_{k=1}^{\infty}\frac{(C_\psi^{k+1})^{\frac{1}{n}\sum_{j=1}^n\frac{1}{p_{2j}}-\frac{\eta_2}{n}}}{(2^n)^{(k+1)(\frac{1}{n}\sum_{j=1}^n\frac{1}{p_{2j}}+1-2\beta)}}\\
&\le \prod_{i=1}^{2}\|f_i\|_{L^{\vec{p_i},\eta_i,\psi}(\mathbb{R}^n)}\sum_{k=1}^{\infty}\left(\frac{C_\psi}{2^n}\right)^{(k+1)(\frac{1}{n}\sum_{j=1}^n\frac{1}{p_{2j}}-\frac{\eta_2}{n})}\\
&\lesssim \prod_{i=1}^{2}\|f_i\|_{L^{\vec{p_i},\eta_i,\psi}(\mathbb{R}^n)}.
\end{aligned}$$

Similarly, it is not difficult to obtain that

$$III \lesssim \prod_{i=1}^{2} \|f_i\|_{L^{\vec{p_i},\eta_i,\psi}(\mathbb{R}^n)}.$$

Finally, we prove IV. It follows from (3.2) and Hölder's inequality that

$$\begin{aligned}
&|T(f_1\chi_{(2B(x,r))^c}, f_2\chi_{(2B(x,r))^c})(z)| \\
&\leq \int_{(\mathbb{R}^n)^2} |K_\beta(z, y_1, y_2)||f_1\chi_{(2B(x,r))^c}(y_1)||f_2\chi_{(2B(x,r))^c}(y_2)|\mathrm{d}y_1\mathrm{d}y_2 \\
&\lesssim \sum_{k=1}^{\infty} \frac{1}{|2^k B(x,r)|^{2(1-\beta)}} \prod_{i=1}^{2} \int_{2^{k+1}B(x,r)} |f_i(y_i)|\mathrm{d}y_i \\
&\leq \sum_{k=1}^{\infty} \frac{1}{|2^k B(x,r)|^{2(1-\beta)}} \prod_{i=1}^{2} \|f_i\chi_{2^{k+1}B(x,r)}\|_{L^{\vec{p_i}}(\mathbb{R}^n)} \|\chi_{2^{k+1}B(x,r)}\|_{L^{\vec{p_i}'}(\mathbb{R}^n)} \\
&\lesssim \sum_{k=1}^{\infty} \frac{1}{|2^{k+1} B(x,r)|^{2(1-\beta)}} \prod_{i=1}^{2} [\psi(2^{k+1}r)]^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{ij}}-\frac{\eta_i}{n}} \\
&\quad \times |2^{k+1}B(x,r)|^{1-\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{ij}}} \|f_i\|_{L^{\vec{p_i},\eta_i,\psi}(\mathbb{R}^n)}.
\end{aligned}$$

Thus, by $\frac{\eta_1}{n} + \frac{\eta_2}{n} \geq \frac{n\beta+n\beta}{n} = 2\beta$, there holds

$$\begin{aligned}
IV &= \sup_{x\in\mathbb{R}^n, r>0} \frac{\|T(f_1\chi_{(2B(x,r))^c}, f_2\chi_{(2B(x,r))^c})\chi_{B(x,r)}\|_{L^{\vec{q}}(\mathbb{R}^n)}}{[\psi(r)]^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{q_j}-\frac{\eta}{n}}} \\
&\lesssim \sup_{x\in\mathbb{R}^n, r>0} \sum_{k=1}^{\infty} \frac{[C_\psi^{k+1}\psi(r)]^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{1j}}+\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{2j}}-\frac{\eta_1}{n}-\frac{\eta_2}{n}}}{[\psi(r)]^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{q_j}-\frac{\eta}{n}}} |B(x,r)|^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{q_j}} \\
&\quad \times |2^{k+1}B(x,r)|^{1-\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{1j}}+1-\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{2j}}+2(\beta-1)} \prod_{i=1}^{2} \|f_i\|_{L^{\vec{p_i},\eta_i,\psi}(\mathbb{R}^n)} \\
&\leq \prod_{i=1}^{2} \|f_i\|_{L^{\vec{p_i},\eta_i,\psi}(\mathbb{R}^n)} \sum_{k=1}^{\infty} \frac{(C_\psi^{k+1})^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{1j}}+\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{2j}}-\frac{\eta_1}{n}-\frac{\eta_2}{n}}}{(2^n)^{(k+1)(\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{1j}}+\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{2j}}-2\beta)}} \\
&\leq \prod_{i=1}^{2} \|f_i\|_{L^{\vec{p_i},\eta_i,\psi}(\mathbb{R}^n)} \sum_{k=1}^{\infty} \left(\frac{C_\psi}{2^n}\right)^{(k+1)(\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{1j}}+\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{2j}}-\frac{\eta_1}{n}-\frac{\eta_2}{n})} \\
&\lesssim \prod_{i=1}^{2} \|f_i\|_{L^{\vec{p_i},\eta_i,\psi}(\mathbb{R}^n)}.
\end{aligned}$$

The proof is complete. □

4. $T_{\prod \vec{b}}$ and $T_{\sum \vec{b}}$ on space $L^{\vec{q},\eta,\psi}(\mathbb{R}^n)$

The main results of this section are formulated as follows.

Theorem 4.1. *Let $T_{\prod \vec{b}}$ be the m-linear iterative commutator generated by T and $\vec{b}$, $\vec{b} = (b_1, \cdots, b_m)$, $b_i \in \mathrm{BMO}_{\varphi,\beta}$ $(i = 1, \cdots, m)$, $0 \leq \beta < \min\{\frac{\gamma}{1+\gamma}, \frac{1}{m}\}$ with $0 \leq \gamma < \infty$, $h \in \mathrm{Dini}(1)$ and $0 < C_\psi < 2^n$. If $n\beta \leq \eta_i < \frac{n}{m}$ $(i = 1, \cdots, m)$,*

$\eta = \sum_{i=1}^m \eta_i$, $\sum_{j=1}^n \frac{1}{q_j} = \sum_{j=1}^n \sum_{i=1}^m \frac{1}{p_{ij}}$ with $m \geq 2$, $\sum_{j=1}^n \frac{1}{p_{ij}} > \eta_i$ and $1 < \vec{q}, \vec{p_i} < \infty$, then $T_{\prod \vec{b}}$ is bounded from $L^{\vec{p_1},\eta_1,\psi}(\mathbb{R}^n) \times \cdots \times L^{\vec{p_m},\eta_m,\psi}(\mathbb{R}^n)$ to $L^{\vec{q},\eta,\psi}(\mathbb{R}^n)$, i.e.

$$\|T_{\prod \vec{b}}(\vec{f})\|_{L^{\vec{q},\eta,\psi}(\mathbb{R}^n)} \lesssim \prod_{i=1}^m \|b_i\|_{\mathrm{BMO}_{\varphi,\beta}} \|f_i\|_{L^{\vec{p_i},\eta_i,\psi}(\mathbb{R}^n)}.$$

Theorem 4.2. *Suppose that $T_{\sum \vec{b}}$ is the multilinear commutator generated by T and $\vec{b} = (b_1, \cdots, b_m)$, $b_i \in \mathrm{BMO}_{\varphi,\beta}$, $i = 1, \cdots, m$, $0 \leq \beta \leq \frac{\gamma}{1+\gamma}$ with $0 \leq \gamma < \infty$, $h \in \mathrm{Dini}(1)$ and $0 < C_\psi < 2^n$. If $\sum_{j=1}^n \frac{1}{q_j} = \sum_{j=1}^n \sum_{i=1}^m \frac{1}{p_{ij}} - mn\beta + n\beta$ with $m \geq 2$, $\sum_{j=1}^n \frac{1}{p_{ij}} > \eta_i$ $(i = 1, \cdots, m)$, $1 < \vec{q}, \vec{p_i} < \infty$, $n\beta \leq \eta_i < n\beta + \frac{n}{m} - \frac{n\beta}{m}$ and $\eta = \sum_{i=1}^m \eta_i - mn\beta + n\beta$, then $T_{\sum \vec{b}}$ is bounded from $L^{\vec{p_1},\eta_1,\psi}(\mathbb{R}^n) \times \cdots \times L^{\vec{p_m},\eta_m,\psi}(\mathbb{R}^n)$ to $L^{\vec{q},\eta,\psi}(\mathbb{R}^n)$, i.e.*

$$\|T_{\sum \vec{b}}(\vec{f})\|_{L^{\vec{q},\eta,\psi}(\mathbb{R}^n)} \lesssim \|\vec{b}\|_{\mathrm{BMO}_{\varphi,\beta}} \prod_{i=1}^m \|f_i\|_{L^{\vec{p_i},\eta_i,\psi}(\mathbb{R}^n)},$$

where $\|\vec{b}\|_{\mathrm{BMO}_{\varphi,\beta}} = \max\{\|b_1\|_{\mathrm{BMO}_{\varphi,\beta}}, \cdots, \|b_m\|_{\mathrm{BMO}_{\varphi,\beta}}\}$.

Theorem 4.3. *Assume that $0 < \alpha < 1$, $0 \leq \beta < 1 - \frac{\alpha}{n}$, $\alpha + n\beta \leq \eta_i < \min\{\alpha + n\beta + \frac{n}{m}, n\}$ $(i = 1, \cdots, m)$, $\eta = \sum_{i=1}^m \eta_i - mn\beta - m\alpha$ and $0 < C_\psi < 2^n$. If $T_{\prod \vec{b}}$ is the m-linear iterative commutator generated by T and $\vec{b}$, $\vec{b} = (b_1, \cdots, b_m)$, $b_i \in \mathrm{Lip}_\alpha(\mathbb{R}^n)$, $i = 1, \cdots, m$, $\sum_{j=1}^n \frac{1}{q_j} = \sum_{j=1}^n \sum_{i=1}^m \frac{1}{p_{ij}} - mn\beta - m\alpha$, $\sum_{j=1}^n \frac{1}{p_{ij}} > \eta_i$ and $1 < \vec{q}, \vec{p_i} < \infty$, then*

$$\|T_{\prod \vec{b}}(\vec{f})\|_{L^{\vec{q},\eta,\psi}(\mathbb{R}^n)} \lesssim \prod_{i=1}^m \|b_i\|_{\mathrm{Lip}_\alpha(\mathbb{R}^n)} \|f_i\|_{L^{\vec{p_i},\eta_i,\psi}(\mathbb{R}^n)}.$$

Theorem 4.4. *Let $T_{\sum \vec{b}}$ be the multilinear commutator generated by T and $\vec{b} = (b_1, \cdots, b_m)$, $b_i \in \mathrm{Lip}_\alpha(\mathbb{R}^n)$, $i = 1, \cdots, m$, $0 < \alpha < \min\{1, \frac{n}{m-1}\}$, $0 \leq \beta < \min\{\frac{\gamma}{1+\gamma}, 1 - \frac{\alpha}{n}\}$ with $0 \leq \gamma < \infty$, $h \in \mathrm{Dini}(1)$ and $0 < C_\psi < 2^n$. If $\alpha + n\beta \leq \eta_i < \frac{\alpha}{m} + n\beta + \frac{n}{m}$ $(i = 1, \cdots, m)$, $\eta = \sum_{i=1}^m \eta_i - mn\beta - \alpha$, $\sum_{j=1}^n \frac{1}{q_j} = \sum_{j=1}^n \sum_{i=1}^m \frac{1}{p_{ij}} - mn\beta - \alpha$, $m \geq 2$, $\sum_{j=1}^n \frac{1}{p_{ij}} > \eta_i$ and $1 < \vec{q}, \vec{p_i} < \infty$, then*

$$\|T_{\sum \vec{b}}(\vec{f})\|_{L^{\vec{q},\eta,\psi}(\mathbb{R}^n)} \lesssim \|\vec{b}\|_{\mathrm{Lip}_\alpha(\mathbb{R}^n)} \prod_{i=1}^m \|f_i\|_{L^{\vec{p_i},\eta_i,\psi}(\mathbb{R}^n)},$$

where $\|\vec{b}\|_{\mathrm{Lip}_\alpha(\mathbb{R}^n)} = \max\{\|b_1\|_{\mathrm{Lip}_\alpha(\mathbb{R}^n)}, \cdots, \|b_m\|_{\mathrm{Lip}_\alpha(\mathbb{R}^n)}\}$.

In order to prove Theorems 4.1-4.4, we need recall the following lemmas. We first recall the characterizations of bounded mean oscillation BMO on space $L^{\vec{q}}(\mathbb{R}^n)$ in [8].

Lemma 4.5. *Let $b \in \mathrm{BMO}(\mathbb{R}^n)$ and $\vec{q} = (q_1, \cdots, q_n) \in (1, \infty)^n$. Then for any ball $B(x,r) \subset \mathbb{R}^n$ and any $j \in \mathbb{N}$, there holds*

$$\|(b - b_{B(x,r)})\chi_{B(x,r)}\|_{L^{\vec{q}}(\mathbb{R}^n)} \lesssim \|b\|_{\mathrm{BMO}(\mathbb{R}^n)} \|\chi_{B(x,r)}\|_{L^{\vec{q}}(\mathbb{R}^n)},$$

and

$$|b_{2^{j+1}B(x,r)} - b_{B(x,r)}| \lesssim (j+1)\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}.$$

The following characterizations of Lipschitz space have been established in [3].

Lemma 4.6. *Suppose that $0 < \alpha < 1$ and $b \in \mathrm{Lip}_\alpha(\mathbb{R}^n)$. Then for any ball $B(x,r) \subset \mathbb{R}^n$ and $B(x,r) \subset B^*(x,r)$, we have*

$$\|b\|_{\mathrm{Lip}_\alpha(\mathbb{R}^n)} \approx \sup_{x \in \mathbb{R}^n, r>0} \frac{1}{|B(x,r)|^{1+\frac{\alpha}{n}}} \int_{B(x,r)} |b(y) - b_{B(x,r)}| \mathrm{d}y,$$

and

$$|b_{B(x,r)} - b_{B^*(x,r)}| \lesssim \|b\|_{\mathrm{Lip}_\alpha(\mathbb{R}^n)} |B^*(x,r)|^{\frac{\alpha}{n}}.$$

Lemma 4.7. *Assume that $b \in \mathrm{Lip}_\alpha(\mathbb{R}^n)$ with $0 < \alpha < 1$ and $\vec{q} = (q_1, \cdots, q_n) \in (1, \infty)^n$. Then for any ball $B(x, r) \subset \mathbb{R}^n$, there holds*

$$\|(b - b_{B(x,r)})\chi_{B(x,r)}\|_{L^{\vec{q}}(\mathbb{R}^n)} \lesssim \|b\|_{\mathrm{Lip}_\alpha(\mathbb{R}^n)} |B(x, r)|^{\frac{\alpha}{n} + \frac{1}{n}\sum_{j=1}^{n} \frac{1}{q_j}}.$$

Proof. Since $\|\chi_{B(x,r)}\|_{L^{\vec{q}}(\mathbb{R}^n)} \approx |B(x, r)|^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{q_j}}$ in [24], then Lemma 4.7 directly follows from the first part of Lemma 4.5 and Lemma 4.6. The details are omitted here. □

Now, we give the proofs of Theorems 4.1-4.4.

Proof. [The proof of Theorem 4.1] Without loss of generality, we only consider the case $m = 2$, the same method is applied to Theorem 4.2-Theorem 4.4. Let $\vec{f} = (f_1, f_2) \in L^{\vec{p_1},\eta_1,\psi}(\mathbb{R}^n) \times L^{\vec{p_2},\eta_2,\psi}(\mathbb{R}^n)$, $B =: B(x, r)$ for $r > 0$, $2B(x, r) =: B(x, 2r)$ and $(2B(x, r))^c =: \mathbb{R}^n \setminus 2B(x, r)$. Then

$$\begin{aligned}
T_{\prod \vec{b}}(\vec{f})(z) = &(b_1(z) - (b_1)_{B(x,r)})(b_2(z) - (b_2)_{B(x,r)})T(f_1, f_2)(z)\\
&- (b_1(z) - (b_1)_{B(x,r)})T(f_1, (b_2 - (b_2)_{B(x,r)})f_2)(z)\\
&- (b_2(z) - (b_2)_{B(x,r)})T((b_1 - (b_1)_{B(x,r)})f_1, f_2)(z)\\
&+ T((b_1 - (b_1)_{B(x,r)})f_1, (b_2 - (b_2)_{B(x,r)})f_2)(z).
\end{aligned}$$

Then by using Minkowski inequality, we have

$$\begin{aligned}
\|T_{\prod \vec{b}}(\vec{f})\|_{L^{\vec{q},\eta,\psi}(\mathbb{R}^n)} \le &\|(b_1 - (b_1)_{B(x,r)})(b_2 - (b_2)_{B(x,r)})T(f_1, f_2)\|_{L^{\vec{q},\eta,\psi}(\mathbb{R}^n)}\\
&+ \|(b_1 - (b_1)_{B(x,r)})T(f_1, (b_2 - (b_2)_{B(x,r)})f_2)\|_{L^{\vec{q},\eta,\psi}(\mathbb{R}^n)}\\
&+ \|(b_2 - (b_2)_{B(x,r)})T((b_1 - (b_1)_{B(x,r)})f_1, f_2)\|_{L^{\vec{q},\eta,\psi}(\mathbb{R}^n)}\\
&+ \|T((b_1 - (b_1)_{B(x,r)})f_1, (b_2 - (b_2)_{B(x,r)})f_2)\|_{L^{\vec{q},\eta,\psi}(\mathbb{R}^n)}\\
=: &E_1 + E_2 + E_3 + E_4.
\end{aligned}$$

We first estimate E_1. Let $f_i = f_i\chi_{2B(x,r)} + f_i\chi_{(2B(x,r))^c} (i = 1, 2)$. Using Minkowski inequality again, there holds

$$\begin{aligned}
E_1 \le &\|(b_1 - (b_1)_{B(x,r)})(b_2 - (b_2)_{B(x,r)})T(f_1\chi_{2B(x,r)}, f_2\chi_{2B(x,r)})\|_{L^{\vec{q},\eta,\psi}(\mathbb{R}^n)}\\
&+ \|(b_1 - (b_1)_{B(x,r)})(b_2 - (b_2)_{B(x,r)})T(f_1\chi_{2B(x,r)}, f_2\chi_{(2B(x,r))^c})\|_{L^{\vec{q},\eta,\psi}(\mathbb{R}^n)}\\
&+ \|(b_1 - (b_1)_{B(x,r)})(b_2 - (b_2)_{B(x,r)})T(f_1\chi_{(2B(x,r))^c}, f_2\chi_{2B(x,r)})\|_{L^{\vec{q},\eta,\psi}(\mathbb{R}^n)}\\
&+ \|(b_1 - (b_1)_{B(x,r)})(b_2 - (b_2)_{B(x,r)})T(f_1\chi_{(2B(x,r))^c}, f_2\chi_{(2B(x,r))^c})\|_{L^{\vec{q},\eta,\psi}(\mathbb{R}^n)}\\
=: &E_{11} + E_{12} + E_{13} + E_{14}.
\end{aligned}$$

For E_{11}, as $y_1, y_2 \in 2B(x, r)$ and $z \in B(x, r)$, we may choose $\vec{u}$ such that $\frac{1}{\vec{q}} = \frac{1}{\vec{h_1}} + \frac{1}{\vec{h_2}} + \frac{1}{\vec{u}}$ and $\sum_{j=1}^{n}\sum_{i=1}^{2}\frac{1}{h_{ij}} = 2n\beta$, then $\sum_{j=1}^{n}\frac{1}{u_j} = \sum_{j=1}^{n}\sum_{i=1}^{2}\frac{1}{p_{ij}} - 2n\beta$. Let $\frac{1}{\vec{u}} = \sum_{i=1}^{2}\frac{1}{\vec{u_i}}$ and $\sum_{j=1}^{n}\frac{1}{u_{ij}} = \sum_{j=1}^{n}\frac{1}{p_{ij}} - \beta_i$. Then $\beta = \frac{1}{2n}\sum_{i=1}^{2}\beta_i$. By (3.1), we have

$$\begin{aligned}
\mathcal{M}_{\varphi,\beta}(\vec{f})(x) &= \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{(\varphi(|B(x, r)|)|B(x, r)|)^{2(1-\beta)}} \prod_{i=1}^{2} \int_{B(x,r)} |f_i(y_i)|\mathrm{d}y_i\\
&= \sup_{x\in\mathbb{R}^n, r>0} \prod_{i=1}^{2} \frac{1}{[\varphi(|B(x, r)|)]^{1-\beta}|B(x, r)|^{1-\frac{\beta_i}{n}}} \int_{B(x,r)} |f_i(y_i)|\mathrm{d}y_i\\
&\le \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{[\varphi(|B(x, r)|)]^{2(1-\beta)}} \prod_{i=1}^{2} M_{\beta_i}(f_i)(x). \qquad (4.1)
\end{aligned}$$

Thus, by using (4.1) and the similar arguments as in the proof of Lemma 3.2, we have

$$\begin{aligned}&\|T(f_1\chi_{2B(x,r)}, f_2\chi_{2B(x,r)})\chi_{B(x,r)}\|_{L^{\vec{u}}(\mathbb{R}^n)}\\&\lesssim \sup_{x\in\mathbb{R}^n, r>0}\frac{1}{[\varphi(|B(x,r)|)]^{2(1-\beta)}}\prod_{i=1}^{2}\|f_i\chi_{2B(x,r)}\|_{L^{\vec{p_i}}(\mathbb{R}^n)}.\end{aligned}\tag{4.2}$$

Since $n\beta \le \eta_i < \frac{n}{2}$ and $\eta = \sum_{i=1}^{2}\eta_i$, then $n\beta \le \eta_i < n$ and $2n\beta \le \eta < n$. Thus, it gets from (4.2), Hölder's inequality and Lemma 4.5 that

$$\begin{aligned}E_{11} &= \sup_{x\in\mathbb{R}^n, r>0}\frac{\|(b_1-(b_1)_{B(x,r)})(b_2-(b_2)_{B(x,r)})T(f_1\chi_{2B(x,r)}, f_2\chi_{2B(x,r)})\chi_{B(x,r)}\|_{L^{\vec{q}}(\mathbb{R}^n)}}{[\psi(r)]^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{q_j}-\frac{\eta}{n}}}\\
&\le \sup_{x\in\mathbb{R}^n, r>0}\|(b_1-(b_1)_{B(x,r)})\chi_{B(x,r)}\|_{L^{\vec{h_1}}(\mathbb{R}^n)}\|(b_2-(b_2)_{B(x,r)})\chi_{B(x,r)}\|_{L^{\vec{h_2}}(\mathbb{R}^n)}\\
&\quad\times\|T(f_1\chi_{2B(x,r)}, f_2\chi_{2B(x,r)})\chi_{B(x,r)}\|_{L^{\vec{u}}(\mathbb{R}^n)}\frac{1}{[\psi(r)]^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{q_j}-\frac{\eta}{n}}}\\
&\lesssim \sup_{x\in\mathbb{R}^n, r>0}\prod_{i=1}^{2}\|f_i\chi_{2B(x,r)}\|_{L^{\vec{p_i}}(\mathbb{R}^n)}\frac{1}{|B(x,r)|}\int_{B(x,r)}|b_1(y_1)-(b_1)_{B(x,r)}|\mathrm{d}y_1\|\chi_{B(x,r)}\|_{L^{\vec{h_1}}(\mathbb{R}^n)}\\
&\quad\times\frac{1}{|B(x,r)|}\int_{B(x,r)}|b_2(y_2)-(b_2)_{B(x,r)}|\mathrm{d}y_2\|\chi_{B(x,r)}\|_{L^{\vec{h_2}}(\mathbb{R}^n)}\\
&\quad\times\frac{1}{[\varphi(|B(x,r)|)]^{2(1-\beta)}}\frac{1}{[\psi(r)]^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{q_j}-\frac{\eta}{n}}}\\
&\lesssim \prod_{i=1}^{2}\|b_i\|_{\mathrm{BMO}_{\varphi,\beta}}\|f_i\|_{L^{\vec{p_i},\eta_i,\psi}(\mathbb{R}^n)}\sup_{x\in\mathbb{R}^n, r>0}\frac{(\varphi(|B(x,r)|)|B(x,r)|)^{2(1-\beta)}}{|B(x,r)|^2}\\
&\quad\times\frac{|B(x,r)|^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{h_{1j}}+\frac{1}{n}\sum_{j=1}^{n}\frac{1}{h_{2j}}}}{[\varphi(|B(x,r)|)]^{2(1-\beta)}}\frac{[\psi(2r)]^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{1j}}-\frac{\eta_1}{n}+\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{2j}}-\frac{\eta_2}{n}}}{[\psi(r)]^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{q_j}-\frac{\eta}{n}}}\\
&\lesssim \prod_{i=1}^{2}\|b_i\|_{\mathrm{BMO}_{\varphi,\beta}}\|f_i\|_{L^{\vec{p_i},\eta_i,\psi}(\mathbb{R}^n)}.\end{aligned}$$

For E_{12}, as $y_1 \in 2B(x,r)$, $y_2 \in 2^{k+1}B(x,r)\setminus 2^kB(x,r)$ and $z \in B(x,r)$, we have that $|z-y_2| \gtrsim |2^kB(x,r)|^{\frac{1}{n}}$. Thus, by (1.1) and $N = 2\gamma(1-\beta)$, there holds

$$\begin{aligned}|K_\beta(z,y_1,y_2)| &\lesssim \frac{1}{|2^kB(x,r)|^{2(1-\beta)}(1+|2^kB(x,r)|)^N}\\&= \frac{1}{(\varphi(|2^kB(x,r)|)|2^kB(x,r)|)^{2(1-\beta)}}.\end{aligned}\tag{4.3}$$

Thus, from Hölder's inequality and (4.3), we obtain

$$\begin{aligned}&|T(f_1\chi_{2B(x,r)}, f_2\chi_{(2B(x,r))^c})(z)|\\
&\lesssim \left(\int_{2B(x,r)}|f_1(y_1)|\mathrm{d}y_1\right)\left(\sum_{k=1}^{\infty}\int_{2^{k+1}B(x,r)}\frac{|f_2(y_2)|}{(\varphi(|2^kB(x,r)|)|2^kB(x,r)|)^{2(1-\beta)}}\mathrm{d}y_2\right)\\
&\le \|f_1\chi_{2B(x,r)}\|_{L^{\vec{p_1}}(\mathbb{R}^n)}\|\chi_{2B(x,r)}\|_{L^{\vec{p_1}'}(\mathbb{R}^n)}\sum_{k=1}^{\infty}\frac{\|f_2\chi_{2^{k+1}B(x,r)}\|_{L^{\vec{p_2}}(\mathbb{R}^n)}\|\chi_{2^{k+1}B(x,r)}\|_{L^{\vec{p_2}'}(\mathbb{R}^n)}}{(\varphi(|2^{k+1}B(x,r)|)|2^{k+1}B(x,r)|)^{2(1-\beta)}}\end{aligned}$$

$$\times \frac{(\varphi(|2^{k+1}B(x,r)|)|2^{k+1}B(x,r)|)^{2(1-\beta)}}{(\varphi(|2^{k}B(x,r)|)|2^{k}B(x,r)|)^{2(1-\beta)}}$$
$$\lesssim \prod_{i=1}^{2}\|f_i\|_{L^{\vec{p_i},\eta_i,\psi}(\mathbb{R}^n)}[\psi(2r)]^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{1j}}-\frac{\eta_1}{n}}|2B(x,r)|^{1-\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{1j}}}$$
$$\times\sum_{k=1}^{\infty}\frac{[\psi(2^{k+1}r)]^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{2j}}-\frac{\eta_2}{n}}|2^{k+1}B(x,r)|^{1-\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{2j}}}}{(\varphi(|2^{k+1}B(x,r)|)|2^{k+1}B(x,r)|)^{2(1-\beta)}}, \tag{4.4}$$

where

$$\begin{aligned}\frac{(\varphi(|2^{k+1}B(x,r)|)|2^{k+1}B(x,r)|)^{2(1-\beta)}}{(\varphi(|2^{k}B(x,r)|)|2^{k}B(x,r)|)^{2(1-\beta)}} &= \left(\frac{1+|2^{k+1}B(x,r)|}{1+|2^{k}B(x,r)|}\right)^{2\gamma(1-\beta)}\left(\frac{|2^{k+1}B(x,r)|}{|2^{k}B(x,r)|}\right)^{2(1-\beta)}\\
&\le \left(\frac{|2^{k+1}B(x,r)|}{|2^{k}B(x,r)|}\right)^{2\gamma(1-\beta)}\left(\frac{|2^{k+1}B(x,r)|}{|2^{k}B(x,r)|}\right)^{2(1-\beta)}\\
&\le C.\end{aligned}$$

Applying Hölder's inequality and Lemma 4.5 again, there holds for $0 < C_\psi < 2^n$ and $\frac{\eta_2}{n} \ge \frac{n\beta}{n} = \beta > 2\beta - 1$,

$$\begin{aligned}E_{12} &= \sup_{x\in\mathbb{R}^n, r>0}\frac{\|(b_1-(b_1)_{B(x,r)})(b_2-(b_2)_{B(x,r)})T(f_1\chi_{2B(x,r)}, f_2\chi_{(2B(x,r))^c})\chi_{B(x,r)}\|_{L^{\vec{q}}(\mathbb{R}^n)}}{[\psi(r)]^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{q_j}-\frac{\eta}{n}}}\\
&\lesssim \prod_{i=1}^{2}\|f_i\|_{L^{\vec{p_i},\eta_i,\psi}(\mathbb{R}^n)}\sup_{x\in\mathbb{R}^n,r>0}\|(b_1-(b_1)_{B(x,r)})\chi_{B(x,r)}\|_{L^{\vec{h_1}}(\mathbb{R}^n)}\\
&\quad\times\|(b_2-(b_2)_{B(x,r)})\chi_{B(x,r)}\|_{L^{\vec{h_2}}(\mathbb{R}^n)}\|\chi_{B(x,r)}\|_{L^{\vec{u}}(\mathbb{R}^n)}\\
&\quad\times\sum_{k=1}^{\infty}\frac{|B(x,r)|^{1-\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{1j}}}|2^{k+1}B(x,r)|^{1-\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{2j}}}}{(\varphi(|2^{k+1}B(x,r)|)|2^{k+1}B(x,r)|)^{2(1-\beta)}}\\
&\quad\times\frac{[C_\psi\psi(r)]^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{1j}}-\frac{\eta_1}{n}}[C_\psi^{k+1}\psi(r)]^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{2j}}-\frac{\eta_2}{n}}}{[\psi(r)]^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{q_j}-\frac{\eta}{n}}}\\
&\lesssim \prod_{i=1}^{2}\|b_i\|_{\mathrm{BMO}_{\varphi,\beta}}\|f_i\|_{L^{\vec{p_i},\eta_i,\psi}(\mathbb{R}^n)}\sup_{x\in\mathbb{R}^n,r>0}\sum_{k=1}^{\infty}\frac{(\varphi(|2^{k+1}B(x,r)|)|B(x,r)|)^{2(1-\beta)}}{|B(x,r)|^2}\\
&\quad\times\frac{|B(x,r)|^{1-\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{1j}}+\frac{1}{n}\sum_{j=1}^{n}\frac{1}{h_{1j}}+\frac{1}{n}\sum_{j=1}^{n}\frac{1}{h_{2j}}+\frac{1}{n}\sum_{j=1}^{n}\frac{1}{u_j}}}{(\varphi(|2^{k+1}B(x,r)|)|2^{k+1}B(x,r)|)^{2(1-\beta)}}\\
&\quad\times|2^{k+1}B(x,r)|^{1-\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{2j}}}[C_\psi^{k+1}]^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{2j}}-\frac{\eta_2}{n}}\\
&\le \prod_{i=1}^{2}\|b_i\|_{\mathrm{BMO}_{\varphi,\beta}}\|f_i\|_{L^{\vec{p_i},\eta_i,\psi}(\mathbb{R}^n)}\sum_{k=1}^{\infty}\frac{[C_\psi^{k+1}]^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{2j}}-\frac{\eta_2}{n}}}{(2^n)^{(k+1)(\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{2j}}+1-2\beta)}}\\
&\lesssim \prod_{i=1}^{2}\|b_i\|_{\mathrm{BMO}_{\varphi,\beta}}\|f_i\|_{L^{\vec{p_i},\eta_i,\psi}(\mathbb{R}^n)}.\end{aligned}$$

For E_{13}, similar to E_{12}, we have

$$E_{13} \lesssim \prod_{i=1}^{2}\|b_i\|_{\mathrm{BMO}_{\varphi,\beta}}\|f_i\|_{L^{\vec{p_i},\eta_i,\psi}(\mathbb{R}^n)}.$$

For E_{14}, it follows from Hölder's inequality and (4.3) that

$$\begin{aligned}
&|T(f_1\chi_{(2B(x,r))^c}, f_2\chi_{(2B(x,r))^c})(z)| \\
&\lesssim \sum_{k=1}^{\infty} \frac{1}{(\varphi(|2^kB(x,r)|)|2^kB(x,r)|)^{2(1-\beta)}} \prod_{i=1}^{2} \int_{2^{k+1}B(x,r)} |f_i(y_i)|\mathrm{d}y_i \\
&\lesssim \sum_{k=1}^{\infty} \frac{1}{(\varphi(|2^{k+1}B(x,r)|)|2^{k+1}B(x,r)|)^{2(1-\beta)}} \prod_{i=1}^{2} \|f_i\chi_{2^{k+1}B(x,r)}\|_{L^{\vec{p_i}}(\mathbb{R}^n)} \|\chi_{2^{k+1}B(x,r)}\|_{L^{\vec{p_i}'}(\mathbb{R}^n)} \\
&\lesssim \sum_{k=1}^{\infty} \frac{1}{(\varphi(|2^{k+1}B(x,r)|)|2^{k+1}B(x,r)|)^{2(1-\beta)}} \prod_{i=1}^{2} [\psi(2^{k+1}r)]^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{ij}}-\frac{\eta_i}{n}} \\
&\quad \times |2^{k+1}B(x,r)|^{1-\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{ij}}} \|f_i\|_{L^{\vec{p_i},\eta_i,\psi}(\mathbb{R}^n)}.
\end{aligned} \tag{4.5}$$

Since $0 < C_\psi < 2^n$ and $\frac{\eta_1+\eta_2}{n} \geq \frac{2n\beta}{n} = 2\beta$, then using Hölder's inequality and Lemma 4.5, we have

$$\begin{aligned}
E_{14} &= \sup_{x\in\mathbb{R}^n, r>0} \frac{\|(b_1-(b_1)_{B(x,r)})(b_2-(b_2)_{B(x,r)})T(f_1\chi_{(2B(x,r))^c}, f_2\chi_{(2B(x,r))^c})\chi_{B(x,r)}\|_{L^{\vec{q}}(\mathbb{R}^n)}}{[\psi(r)]^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{q_j}-\frac{\eta}{n}}} \\
&\lesssim \prod_{i=1}^{2} \|f_i\|_{L^{\vec{p_i},\eta_i,\psi}(\mathbb{R}^n)} \sup_{x\in\mathbb{R}^n, r>0} \|(b_1-(b_1)_{B(x,r)})\chi_{B(x,r)}\|_{L^{\vec{h_1}}(\mathbb{R}^n)} \\
&\quad \times \|(b_2-(b_2)_{B(x,r)})\chi_{B(x,r)}\|_{L^{\vec{h_2}}(\mathbb{R}^n)} \|\chi_{B(x,r)}\|_{L^{\vec{u}}(\mathbb{R}^n)} \\
&\quad \times \sum_{k=1}^{\infty} \frac{|2^{k+1}B(x,r)|^{1-\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{1j}}+1-\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{2j}}}}{(\varphi(|2^{k+1}B(x,r)|)|2^{k+1}B(x,r)|)^{2(1-\beta)}} \frac{[C_\psi^{k+1}\psi(r)]^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{1j}}-\frac{\eta_1}{n}+\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{2j}}-\frac{\eta_2}{n}}}{[\psi(r)]^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{q_j}-\frac{\eta}{n}}} \\
&\lesssim \prod_{i=1}^{2} \|b_i\|_{\mathrm{BMO}_{\varphi,\beta}} \|f_i\|_{L^{\vec{p_i},\eta_i,\psi}(\mathbb{R}^n)} \sum_{k=1}^{\infty} \frac{[C_\psi^{k+1}]^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{1j}}+\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{2j}}-\frac{\eta_1+\eta_2}{n}}}{(2^n)^{(k+1)(\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{1j}}+\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{2j}}-2\beta)}} \\
&\lesssim \prod_{i=1}^{2} \|b_i\|_{\mathrm{BMO}_{\varphi,\beta}} \|f_i\|_{L^{\vec{p_i},\eta_i,\psi}(\mathbb{R}^n)}.
\end{aligned}$$

Therefore,

$$E_1 \lesssim \prod_{i=1}^{2} \|b_i\|_{\mathrm{BMO}_{\varphi,\beta}} \|f_i\|_{L^{\vec{p_i},\eta_i,\psi}(\mathbb{R}^n)}.$$

Next, we estimate E_2. From Minkowski inequality, we get that

$$\begin{aligned}
E_2 &\leq \|(b_1-(b_1)_{B(x,r)})T(f_1\chi_{2B(x,r)}, (b_2-(b_2)_{B(x,r)})f_2\chi_{2B(x,r)})\|_{L^{\vec{q},\eta,\psi}(\mathbb{R}^n)} \\
&\quad + \|(b_1-(b_1)_{B(x,r)})T(f_1\chi_{2B(x,r)}, (b_2-(b_2)_{B(x,r)})f_2\chi_{(2B(x,r))^c})\|_{L^{\vec{q},\eta,\psi}(\mathbb{R}^n)} \\
&\quad + \|(b_1-(b_1)_{B(x,r)})T(f_1\chi_{(2B(x,r))^c}, (b_2-(b_2)_{B(x,r)})f_2\chi_{2B(x,r)})\|_{L^{\vec{q},\eta,\psi}(\mathbb{R}^n)} \\
&\quad + \|(b_1-(b_1)_{B(x,r)})T(f_1\chi_{(2B(x,r))^c}, (b_2-(b_2)_{B(x,r)})f_2\chi_{(2B(x,r))^c})\|_{L^{\vec{q},\eta,\psi}(\mathbb{R}^n)} \\
&=: E_{21} + E_{22} + E_{23} + E_{24}.
\end{aligned}$$

We only need to consider E_{21}, E_{22} and E_{24} because the estimates of E_{23} and E_{22} are analogous. For E_{21}, let $\frac{1}{\vec{q}} = \frac{1}{\vec{h_1}} + \frac{1}{\vec{s}}$, $\frac{1}{\vec{w_2}} = \frac{1}{\vec{p_2}} + \frac{1}{\vec{h_2}}$ and $\sum_{j=1}^{n}\sum_{i=1}^{2}\frac{1}{h_{ij}} = 2n\beta$, then $\sum_{j=1}^{n}\frac{1}{w_{2j}} = \sum_{j=1}^{n}\frac{1}{p_{2j}} + \sum_{j=1}^{n}\frac{1}{h_{2j}}$ and $\sum_{j=1}^{n}\frac{1}{s_j} = \sum_{j=1}^{n}\frac{1}{p_{1j}} + \sum_{j=1}^{n}\frac{1}{w_{2j}} - 2n\beta$. Assume that $\frac{1}{\vec{s}} = \sum_{i=1}^{2}\frac{1}{\vec{s_i}}$, $\sum_{j=1}^{n}\frac{1}{s_{1j}} = \sum_{j=1}^{n}\frac{1}{p_{1j}} - \beta_1$ and $\sum_{j=1}^{n}\frac{1}{s_{2j}} = \sum_{j=1}^{n}\frac{1}{w_{2j}} - \beta_2$. Then

$\beta = \frac{1}{2n}\sum_{i=1}^{2}\beta_i$. Thus, by using Lemma 4.5, Hölder's inequality, (4.1) and the similar arguments as in the proof of Lemma 3.2, we have

$$\begin{aligned}
&\|T(f_1\chi_{2B(x,r)},(b_2-(b_2)_{B(x,r)})f_2\chi_{2B(x,r)})\chi_{B(x,r)}\|_{L^{\vec{s}}(\mathbb{R}^n)}\\
&\lesssim \sup_{x\in\mathbb{R}^n,r>0}\frac{1}{[\varphi(|B(x,r)|)]^{2(1-\beta)}}\|f_1\chi_{2B(x,r)}\|_{L^{\vec{p}_1}(\mathbb{R}^n)}\|(b_2-(b_2)_{B(x,r)})f_2\chi_{2B(x,r)}\|_{L^{\vec{w}_2}(\mathbb{R}^n)}\\
&\leq \sup_{x\in\mathbb{R}^n,r>0}\frac{1}{[\varphi(|B(x,r)|)]^{2(1-\beta)}}\|(b_2-(b_2)_{B(x,r)})\chi_{2B(x,r)}\|_{L^{\vec{h}_2}(\mathbb{R}^n)}\prod_{i=1}^{2}\|f_i\chi_{2B(x,r)}\|_{L^{\vec{p}_i}(\mathbb{R}^n)}\\
&\leq \sup_{x\in\mathbb{R}^n,r>0}\frac{1}{[\varphi(|B(x,r)|)]^{2(1-\beta)}}(\|(b_2-(b_2)_{2B(x,r)})\chi_{2B(x,r)}\|_{L^{\vec{h}_2}(\mathbb{R}^n)}\\
&\quad+|(b_2)_{2B(x,r)}-(b_2)_{B(x,r)}|\|\chi_{2B(x,r)}\|_{L^{\vec{h}_2}(\mathbb{R}^n)})\prod_{i=1}^{2}\|f_i\chi_{2B(x,r)}\|_{L^{\vec{p}_i}(\mathbb{R}^n)}\\
&\lesssim \sup_{x\in\mathbb{R}^n,r>0}\frac{|2B(x,r)|^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{h_{2j}}}}{[\varphi(|B(x,r)|)]^{2(1-\beta)}}\frac{1}{|B(x,r)|}\int_{B(x,r)}|b_2(y_2)-(b_2)_{B(x,r)}|\mathrm{d}y_2\\
&\quad\times\prod_{i=1}^{2}\|f_i\chi_{2B(x,r)}\|_{L^{\vec{p}_i}(\mathbb{R}^n)}.
\end{aligned}\tag{4.6}$$

Thus, applying (4.6), Hölder's inequality and Lemma 4.5, we have

$$\begin{aligned}
E_{21}&=\sup_{x\in\mathbb{R}^n,r>0}\frac{\|(b_1-(b_1)_{B(x,r)})T(f_1\chi_{2B(x,r)},(b_2-(b_2)_{B(x,r)})f_2\chi_{2B(x,r)})\chi_{B(x,r)}\|_{L^{\vec{q}}(\mathbb{R}^n)}}{[\psi(r)]^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{q_j}-\frac{\eta}{n}}}\\
&\leq \sup_{x\in\mathbb{R}^n,r>0}\|(b_1-(b_1)_{B(x,r)})\chi_{B(x,r)}\|_{L^{\vec{h}_1}(\mathbb{R}^n)}\\
&\quad\times\|T(f_1\chi_{2B(x,r)},(b_2-(b_2)_{B(x,r)})f_2\chi_{2B(x,r)})\chi_{B(x,r)}\|_{L^{\vec{s}}(\mathbb{R}^n)}\frac{1}{[\psi(r)]^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{q_j}-\frac{\eta}{n}}}\\
&\lesssim \prod_{i=1}^{2}\|f_i\|_{L^{\vec{p}_i,\eta_i,\psi}(\mathbb{R}^n)}\sup_{x\in\mathbb{R}^n,r>0}\frac{|2B(x,r)|^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{h_{1j}}+\frac{1}{n}\sum_{j=1}^{n}\frac{1}{h_{2j}}}}{[\varphi(|B(x,r)|)]^{2(1-\beta)}}\\
&\quad\times\frac{1}{|B(x,r)|}\int_{B(x,r)}|b_1(y_1)-(b_1)_{B(x,r)}|\mathrm{d}y_1\frac{1}{|B(x,r)|}\int_{B(x,r)}|b_2(y_2)-(b_2)_{B(x,r)}|\mathrm{d}y_2\\
&\quad\times\frac{[C_\psi\psi(r)]^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{1j}}+\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{2j}}-\frac{\eta_1+\eta_2}{n}}}{[\psi(r)]^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{q_j}-\frac{\eta}{n}}}\\
&\lesssim \prod_{i=1}^{2}\|b_i\|_{\mathrm{BMO}_{\varphi,\beta}}\|f_i\|_{L^{\vec{p}_i,\eta_i,\psi}(\mathbb{R}^n)}\sup_{x\in\mathbb{R}^n,r>0}\frac{(\varphi(|B(x,r)|)|B(x,r)|)^{2(1-\beta)}}{|B(x,r)|^2}\\
&\quad\times\frac{|2B(x,r)|^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{h_{1j}}+\frac{1}{n}\sum_{j=1}^{n}\frac{1}{h_{2j}}}}{[\varphi(|B(x,r)|)]^{2(1-\beta)}}\\
&\leq \prod_{i=1}^{2}\|b_i\|_{\mathrm{BMO}_{\varphi,\beta}}\|f_i\|_{L^{\vec{p}_i,\eta_i,\psi}(\mathbb{R}^n)}.
\end{aligned}$$

For E_{22}, applying Lemma 4.5, (4.3) and Hölder's inequality, there holds

$$|T(f_1\chi_{2B(x,r)},(b_2-(b_2)_{B(x,r)})f_2\chi_{(2B(x,r))^c})(z)|$$

$$\lesssim \left(\int_{2B(x,r)}|f_1(y_1)|\mathrm{d}y_1\right)\left(\sum_{k=1}^{\infty}\int_{2^{k+1}B(x,r)}\frac{|b_2(y_2)-(b_2)_{B(x,r)}||f_2(y_2)|}{(\varphi(|2^kB(x,r)|)|2^kB(x,r)|)^{2(1-\beta)}}\mathrm{d}y_2\right)$$
$$\lesssim \|f_1\chi_{2B(x,r)}\|_{L^{\vec{p_1}}(\mathbb{R}^n)}\|\chi_{2B(x,r)}\|_{L^{\vec{p_1}'}(\mathbb{R}^n)}\sum_{k=1}^{\infty}\frac{\|f_2\chi_{2^{k+1}B(x,r)}\|_{L^{\vec{p_2}}(\mathbb{R}^n)}}{(\varphi(|2^{k+1}B(x,r)|)|2^{k+1}B(x,r)|)^{2(1-\beta)}}$$
$$\times(\|(b_2-(b_2)_{2^{k+1}B(x,r)})\chi_{2^{k+1}B(x,r)}\|_{L^{\vec{p_2}'}(\mathbb{R}^n)}$$
$$+|(b_2)_{2^{k+1}B(x,r)}-(b_2)_{B(x,r)}|\|\chi_{2^{k+1}B(x,r)}\|_{L^{\vec{p_2}'}(\mathbb{R}^n)})$$
$$\lesssim \prod_{i=1}^{2}\|f_i\|_{L^{\vec{p_i},\eta_i,\psi}(\mathbb{R}^n)}\|b_2\|_{\mathrm{BMO}_{\varphi,\beta}}[\psi(2r)]^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{1j}}-\frac{\eta_1}{n}}|2B(x,r)|^{1-\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{1j}}}$$
$$\times\sum_{k=1}^{\infty}\frac{(k+1)[\psi(2^{k+1}r)]^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{2j}}-\frac{\eta_2}{n}}|2^{k+1}B(x,r)|^{-\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{2j}}}}{(\varphi(|2^{k+1}B(x,r)|)|2^{k+1}B(x,r)|)^{1-\beta}}. \tag{4.7}$$

Hence,

$$E_{22}=\sup_{x\in\mathbb{R}^n,r>0}\frac{\|(b_1-(b_1)_{B(x,r)})T(f_1\chi_{2B(x,r)},(b_2-(b_2)_{B(x,r)})f_2\chi_{(2B(x,r))^c})\chi_{B(x,r)}\|_{L^{\vec{q}}(\mathbb{R}^n)}}{[\psi(r)]^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{q_j}-\frac{\eta}{n}}}$$
$$\lesssim\prod_{i=1}^{2}\|f_i\|_{L^{\vec{p_i},\eta_i,\psi}(\mathbb{R}^n)}\|b_2\|_{\mathrm{BMO}_{\varphi,\beta}}\sup_{x\in\mathbb{R}^n,r>0}\|(b_1-(b_1)_{B(x,r)})\chi_{B(x,r)}\|_{L^{\vec{h_1}}(\mathbb{R}^n)}$$
$$\times\|\chi_{B(x,r)}\|_{L^{\vec{s}}(\mathbb{R}^n)}\sum_{k=1}^{\infty}\frac{(k+1)|B(x,r)|^{1-\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{1j}}}|2^{k+1}B(x,r)|^{-\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{2j}}}}{(\varphi(|2^{k+1}B(x,r)|)|2^{k+1}B(x,r)|)^{1-\beta}}$$
$$\times\frac{[C_\psi\psi(r)]^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{1j}}-\frac{\eta_1}{n}}[C_\psi^{k+1}\psi(r)]^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{2j}}-\frac{\eta_2}{n}}}{[\psi(r)]^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{q_j}-\frac{\eta}{n}}}$$
$$\lesssim\prod_{i=1}^{2}\|b_i\|_{\mathrm{BMO}_{\varphi,\beta}}\|f_i\|_{L^{\vec{p_i},\eta_i,\psi}(\mathbb{R}^n)}\sup_{x\in\mathbb{R}^n,r>0}\sum_{k=1}^{\infty}(k+1)[C_\psi^{k+1}]^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{2j}}-\frac{\eta_2}{n}}$$
$$\times\frac{|B(x,r)|^{1-\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{1j}}+\frac{1}{n}\sum_{j=1}^{n}\frac{1}{h_{1j}}+\frac{1}{n}\sum_{j=1}^{n}\frac{1}{s_j}}|2^{k+1}B(x,r)|^{-\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{2j}}}}{(\varphi(|2^{k+1}B(x,r)|)|2^{k+1}B(x,r)|)^{1-\beta}}$$
$$\times\frac{(\varphi(|2^{k+1}B(x,r)|)|B(x,r)|)^{1-\beta}}{|B(x,r)|}$$
$$\le\prod_{i=1}^{2}\|b_i\|_{\mathrm{BMO}_{\varphi,\beta}}\|f_i\|_{L^{\vec{p_i},\eta_i,\psi}(\mathbb{R}^n)}\sum_{k=1}^{\infty}(k+1)\left(\frac{C_\psi}{2^n}\right)^{(k+1)(\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{2j}}-\frac{\eta_2}{n})}$$
$$\lesssim\prod_{i=1}^{2}\|b_i\|_{\mathrm{BMO}_{\varphi,\beta}}\|f_i\|_{L^{\vec{p_i},\eta_i,\psi}(\mathbb{R}^n)}.$$

For E_{24}, from (4.7), we derive

$$|T(f_1\chi_{(2B(x,r))^c},(b_2-(b_2)_{B(x,r)})f_2\chi_{(2B(x,r))^c})(z)|$$
$$\lesssim\prod_{i=1}^{2}\|f_i\|_{L^{\vec{p_i},\eta_i,\psi}(\mathbb{R}^n)}\|b_2\|_{\mathrm{BMO}_{\varphi,\beta}}\sum_{k=1}^{\infty}\frac{|2^{k+1}B(x,r)|^{1-\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{1j}}-\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{2j}}}}{(\varphi(|2^{k+1}B(x,r)|)|2^{k+1}B(x,r)|)^{1-\beta}}$$
$$\times(k+1)[\psi(2^{k+1}r)]^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{1j}}+\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{2j}}-\frac{\eta_1+\eta_2}{n}}. \tag{4.8}$$

Thus, by $\frac{\eta_1+\eta_2}{n} \geq \frac{2n\beta}{n} = 2\beta > \beta$, we have

$$\begin{aligned}
E_{24} &= \sup_{x\in\mathbb{R}^n, r>0} \frac{\|(b_1-(b_1)_{B(x,r)})T(f_1\chi_{(2B(x,r))^c}, (b_2-(b_2)_{B(x,r)})f_2\chi_{(2B(x,r))^c})\chi_{B(x,r)}\|_{L^{\vec{q}}(\mathbb{R}^n)}}{[\psi(r)]^{\frac{1}{n}\sum_{j=1}^n \frac{1}{q_j}-\frac{\eta}{n}}} \\
&\lesssim \prod_{i=1}^2 \|f_i\|_{L^{\vec{p_i},\eta_i,\psi}(\mathbb{R}^n)} \|b_2\|_{\mathrm{BMO}_{\varphi,\beta}} \sup_{x\in\mathbb{R}^n, r>0} \|(b_1-(b_1)_{B(x,r)})\chi_{B(x,r)}\|_{L^{\vec{h_1}}(\mathbb{R}^n)} \\
&\quad \times \|\chi_{B(x,r)}\|_{L^{\vec{s}}(\mathbb{R}^n)} \sum_{k=1}^\infty \frac{(k+1)|2^{k+1}B(x,r)|^{1-\frac{1}{n}\sum_{j=1}^n \frac{1}{p_{1j}}-\frac{1}{n}\sum_{j=1}^n \frac{1}{p_{2j}}}}{(\varphi(|2^{k+1}B(x,r)|)|2^{k+1}B(x,r)|)^{1-\beta}} \\
&\quad \times \frac{[C_\psi^{k+1}\psi(r)]^{\frac{1}{n}\sum_{j=1}^n \frac{1}{p_{1j}}+\frac{1}{n}\sum_{j=1}^n \frac{1}{p_{2j}}-\frac{\eta_1+\eta_2}{n}}}{[\psi(r)]^{\frac{1}{n}\sum_{j=1}^n \frac{1}{q_j}-\frac{\eta}{n}}} \\
&\lesssim \prod_{i=1}^2 \|b_i\|_{\mathrm{BMO}_{\varphi,\beta}} \|f_i\|_{L^{\vec{p_i},\eta_i,\psi}(\mathbb{R}^n)} \sum_{k=1}^\infty (k+1) \frac{[C_\psi^{k+1}]^{\frac{1}{n}\sum_{j=1}^n \frac{1}{p_{1j}}+\frac{1}{n}\sum_{j=1}^n \frac{1}{p_{2j}}-\frac{\eta_1+\eta_2}{n}}}{(2^n)^{(k+1)(\frac{1}{n}\sum_{j=1}^n \frac{1}{p_{1j}}+\frac{1}{n}\sum_{j=1}^n \frac{1}{p_{2j}}-\beta)}} \\
&\lesssim \prod_{i=1}^2 \|b_i\|_{\mathrm{BMO}_{\varphi,\beta}} \|f_i\|_{L^{\vec{p_i},\eta_i,\psi}(\mathbb{R}^n)}.
\end{aligned}$$

Therefore,

$$E_2 \lesssim \prod_{i=1}^2 \|b_i\|_{\mathrm{BMO}_{\varphi,\beta}} \|f_i\|_{L^{\vec{p_i},\eta_i,\psi}(\mathbb{R}^n)}.$$

Now, we estimate E_3. By using the similar arguments as E_2, we can easily check that

$$E_3 \lesssim \prod_{i=1}^2 \|b_i\|_{\mathrm{BMO}_{\varphi,\beta}} \|f_i\|_{L^{\vec{p_i},\eta_i,\psi}(\mathbb{R}^n)}.$$

Finally, we estimate E_4. From Minkowski inequality, it follows that

$$\begin{aligned}
E_4 &\leq \|T((b_1-(b_1)_{B(x,r)})f_1\chi_{2B(x,r)}, (b_2-(b_2)_{B(x,r)})f_2\chi_{2B(x,r)})\|_{L^{\vec{q},\eta,\psi}(\mathbb{R}^n)} \\
&\quad + \|T((b_1-(b_1)_{B(x,r)})f_1\chi_{2B(x,r)}, (b_2-(b_2)_{B(x,r)})f_2\chi_{(2B(x,r))^c})\|_{L^{\vec{q},\eta,\psi}(\mathbb{R}^n)} \\
&\quad + \|T((b_1-(b_1)_{B(x,r)})f_1\chi_{(2B(x,r))^c}, (b_2-(b_2)_{B(x,r)})f_2\chi_{2B(x,r)})\|_{L^{\vec{q},\eta,\psi}(\mathbb{R}^n)} \\
&\quad + \|T((b_1-(b_1)_{B(x,r)})f_1\chi_{(2B(x,r))^c}, (b_2-(b_2)_{B(x,r)})f_2\chi_{(2B(x,r))^c})\|_{L^{\vec{q},\eta,\psi}(\mathbb{R}^n)} \\
&=: E_{41} + E_{42} + E_{43} + E_{44}.
\end{aligned}$$

For E_{41}, let $\frac{1}{\vec{w_1}} = \frac{1}{\vec{p_1}} + \frac{1}{\vec{h_1}}$, $\frac{1}{\vec{w_2}} = \frac{1}{\vec{p_2}} + \frac{1}{\vec{h_2}}$ and $\sum_{j=1}^n \sum_{i=1}^2 \frac{1}{h_{ij}} = 2n\beta$, then $\sum_{j=1}^n \frac{1}{q_j} = \sum_{j=1}^n \frac{1}{w_{1j}} + \sum_{j=1}^n \frac{1}{w_{2j}} - 2n\beta$. Assume that $\frac{1}{\vec{q}} = \sum_{i=1}^2 \frac{1}{\vec{q_i}}$ and $\sum_{j=1}^n \frac{1}{q_{ij}} = \sum_{j=1}^n \frac{1}{w_{ij}} - \beta_i$. Then $\beta = \frac{1}{2n}\sum_{i=1}^2 \beta_i$. Thus, by using Lemma 4.5, Hölder's inequality, (4.1) and the similar arguments as in the proof of Lemma 3.2, we have

$$\begin{aligned}
&\|T((b_1-(b_1)_{B(x,r)})f_1\chi_{2B(x,r)}, (b_2-(b_2)_{B(x,r)})f_2\chi_{2B(x,r)})\chi_{B(x,r)}\|_{L^{\vec{q}}(\mathbb{R}^n)} \\
&\lesssim \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{[\varphi(|B(x,r)|)]^{2(1-\beta)}} \prod_{i=1}^2 \|(b_i-(b_i)_{B(x,r)})f_i\chi_{2B(x,r)}\|_{L^{\vec{w_i}}(\mathbb{R}^n)} \\
&\leq \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{[\varphi(|B(x,r)|)]^{2(1-\beta)}} \prod_{i=1}^2 \|(b_i-(b_i)_{B(x,r)})\chi_{2B(x,r)}\|_{L^{\vec{h_i}}(\mathbb{R}^n)} \|f_i\chi_{2B(x,r)}\|_{L^{\vec{p_i}}(\mathbb{R}^n)}
\end{aligned}$$

$$\leq \sup_{x\in\mathbb{R}^n,r>0} \frac{1}{[\varphi(|B(x,r)|)]^{2(1-\beta)}} \prod_{i=1}^{2} (\|(b_i-(b_i)_{2B(x,r)})\chi_{2B(x,r)}\|_{L^{\vec{h_i}}(\mathbb{R}^n)}$$
$$+ |(b_i)_{2B(x,r)}-(b_i)_{B(x,r)}|\|\chi_{2B(x,r)}\|_{L^{\vec{h_i}}(\mathbb{R}^n)})\|f_i\chi_{2B(x,r)}\|_{L^{\vec{p_i}}(\mathbb{R}^n)}$$
$$\lesssim \sup_{x\in\mathbb{R}^n,r>0} \frac{|2B(x,r)|^{\frac{1}{n}\sum_{j=1}^n \frac{1}{h_{1j}}+\frac{1}{n}\sum_{j=1}^n \frac{1}{h_{2j}}}}{[\varphi(|B(x,r)|)]^{2(1-\beta)}} \prod_{i=1}^{2} \frac{1}{|B(x,r)|}\int_{B(x,r)} |b_i(y_i)-(b_i)_{B(x,r)}|\mathrm{d}y_i$$
$$\times \|f_i\chi_{2B(x,r)}\|_{L^{\vec{p_i}}(\mathbb{R}^n)}.$$

Then

$$E_{41} = \sup_{x\in\mathbb{R}^n,r>0} \frac{\|T((b_1-(b_1)_{B(x,r)})f_1\chi_{2B(x,r)},(b_2-(b_2)_{B(x,r)})f_2\chi_{2B(x,r)})\chi_{B(x,r)}\|_{L^{\vec{q}}(\mathbb{R}^n)}}{[\psi(r)]^{\frac{1}{n}\sum_{j=1}^n \frac{1}{q_j}-\frac{\eta}{n}}}$$
$$\lesssim \prod_{i=1}^{2}\|b_i\|_{\mathrm{BMO}_{\varphi,\beta}}\|f_i\|_{L^{\vec{p_i},\eta_i,\psi}(\mathbb{R}^n)} \sup_{x\in\mathbb{R}^n,r>0} \frac{(\varphi(|B(x,r)|)|B(x,r)|)^{2(1-\beta)}}{|B(x,r)|^2}$$
$$\times \frac{|2B(x,r)|^{\frac{1}{n}\sum_{j=1}^n \frac{1}{h_{1j}}+\frac{1}{n}\sum_{j=1}^n \frac{1}{h_{2j}}}}{[\varphi(|B(x,r)|)]^{2(1-\beta)}} \frac{[C_\psi\psi(r)]^{\frac{1}{n}\sum_{j=1}^n \frac{1}{p_{1j}}+\frac{1}{n}\sum_{j=1}^n \frac{1}{p_{2j}}-\frac{\eta_1+\eta_2}{n}}}{[\psi(r)]^{\frac{1}{n}\sum_{j=1}^n \frac{1}{q_j}-\frac{\eta}{n}}}$$
$$\lesssim \prod_{i=1}^{2}\|b_i\|_{\mathrm{BMO}_{\varphi,\beta}}\|f_i\|_{L^{\vec{p_i},\eta_i,\psi}(\mathbb{R}^n)}.$$

For E_{42}, according to Lemma 4.5, (4.3) and Hölder's inequality, we have

$$|T((b_1-(b_1)_{B(x,r)})f_1\chi_{2B(x,r)},(b_2-(b_2)_{B(x,r)})f_2\chi_{(2B(x,r))^c})(z)|$$
$$\lesssim \int_{2B(x,r)} |b_1(y_1)-(b_1)_{B(x,r)}||f_1(y_1)|\mathrm{d}y_1 \sum_{k=1}^{\infty}\int_{2^{k+1}B(x,r)} \frac{|b_2(y_2)-(b_2)_{B(x,r)}||f_2(y_2)|}{(\varphi(|2^kB(x,r)|)|2^kB(x,r)|)^{2(1-\beta)}}\mathrm{d}y_2$$
$$\lesssim \|f_1\chi_{2B(x,r)}\|_{L^{\vec{p_1}}(\mathbb{R}^n)}(\|(b_1-(b_1)_{2B(x,r)})\chi_{2B(x,r)}\|_{L^{\vec{p_1}'}(\mathbb{R}^n)}$$
$$+ |(b_1)_{2B(x,r)}-(b_1)_{B(x,r)}|\|\chi_{2B(x,r)}\|_{L^{\vec{p_1}'}(\mathbb{R}^n)}) \sum_{k=1}^{\infty} \frac{\|f_2\chi_{2^{k+1}B(x,r)}\|_{L^{\vec{p_2}}(\mathbb{R}^n)}}{(\varphi(|2^{k+1}B(x,r)|)|2^{k+1}B(x,r)|)^{2(1-\beta)}}$$
$$\times (\|(b_2-(b_2)_{2^{k+1}B(x,r)})\chi_{2^{k+1}B(x,r)}\|_{L^{\vec{p_2}'}(\mathbb{R}^n)} + |(b_2)_{2^{k+1}B(x,r)}-(b_2)_{B(x,r)}|\|\chi_{2^{k+1}B(x,r)}\|_{L^{\vec{p_2}'}(\mathbb{R}^n)})$$
$$\lesssim \prod_{i=1}^{2}\|b_i\|_{\mathrm{BMO}_{\varphi,\beta}}\|f_i\|_{L^{\vec{p_i},\eta_i,\psi}(\mathbb{R}^n)}[\psi(2r)]^{\frac{1}{n}\sum_{j=1}^n \frac{1}{p_{1j}}-\frac{\eta_1}{n}}$$
$$\times |B(x,r)|^{-\frac{1}{n}\sum_{j=1}^n \frac{1}{p_{1j}}-\frac{1}{n}\sum_{j=1}^n \frac{1}{p_{2j}}} \sum_{k=1}^{\infty} \frac{(k+1)[\psi(2^{k+1}r)]^{\frac{1}{n}\sum_{j=1}^n \frac{1}{p_{2j}}-\frac{\eta_2}{n}}}{(2^n)^{(k+1)(\frac{1}{n}\sum_{j=1}^n \frac{1}{p_{2j}}+1-\beta)}}.$$

Hence,

$$E_{42} = \sup_{x\in\mathbb{R}^n,r>0} \frac{\|T((b_1-(b_1)_{B(x,r)})f_1\chi_{2B(x,r)},(b_2-(b_2)_{B(x,r)})f_2\chi_{(2B(x,r))^c})\chi_{B(x,r)}\|_{L^{\vec{q}}(\mathbb{R}^n)}}{[\psi(r)]^{\frac{1}{n}\sum_{j=1}^n \frac{1}{q_j}-\frac{\eta}{n}}}$$
$$\lesssim \prod_{i=1}^{2}\|b_i\|_{\mathrm{BMO}_{\varphi,\beta}}\|f_i\|_{L^{\vec{p_i},\eta_i,\psi}(\mathbb{R}^n)} \sup_{x\in\mathbb{R}^n,r>0} \|\chi_{B(x,r)}\|_{L^{\vec{q}}(\mathbb{R}^n)}$$
$$\times |B(x,r)|^{-\frac{1}{n}\sum_{j=1}^n \frac{1}{p_{1j}}-\frac{1}{n}\sum_{j=1}^n \frac{1}{p_{2j}}} \sum_{k=1}^{\infty} \frac{k+1}{(2^n)^{(k+1)(\frac{1}{n}\sum_{j=1}^n \frac{1}{p_{2j}}+1-\beta)}}$$

$$\times \frac{[C_\psi \psi(r)]^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{1j}}-\frac{\eta_1}{n}}[C_\psi^{k+1}\psi(r)]^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{2j}}-\frac{\eta_2}{n}}}{[\psi(r)]^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{q_j}-\frac{\eta}{n}}}$$
$$\lesssim \prod_{i=1}^{2}\|b_i\|_{\mathrm{BMO}_{\varphi,\beta}}\|f_i\|_{L^{\vec{p_i},\eta_i,\psi}(\mathbb{R}^n)}.$$

Similarly,

$$E_{43} \lesssim \prod_{i=1}^{2}\|b_i\|_{\mathrm{BMO}_{\varphi,\beta}}\|f_i\|_{L^{\vec{p_i},\eta_i,\psi}(\mathbb{R}^n)}.$$

For E_{44}, by Lemma 4.5, (4.3) and Hölder's inequality, we have

$$\begin{aligned}
&|T((b_1-(b_1)_{B(x,r)})f_1\chi_{(2B(x,r))^c},(b_2-(b_2)_{B(x,r)})f_2\chi_{(2B(x,r))^c})(z)|\\
&\lesssim \sum_{k=1}^{\infty}\frac{1}{(\varphi(|2^kB(x,r)|)|2^kB(x,r)|)^{2(1-\beta)}}\prod_{i=1}^{2}\int_{2^{k+1}B(x,r)}|b_i(y_i)-(b_i)_{B(x,r)}||f_i(y_i)|\mathrm{d}y_i\\
&\lesssim \sum_{k=1}^{\infty}\frac{1}{(\varphi(|2^{k+1}B(x,r)|)|2^{k+1}B(x,r)|)^{2(1-\beta)}}\prod_{i=1}^{2}\|f_i\chi_{2^{k+1}B(x,r)}\|_{L^{\vec{p_i}}(\mathbb{R}^n)}\\
&\quad\times(\|(b_i-(b_i)_{2^{k+1}B(x,r)})\chi_{2^{k+1}B(x,r)}\|_{L^{\vec{p_i}'}(\mathbb{R}^n)}+|(b_i)_{2^{k+1}B(x,r)}-(b_i)_{B(x,r)}|\|\chi_{2^{k+1}B(x,r)}\|_{L^{\vec{p_i}'}(\mathbb{R}^n)})\\
&\lesssim \prod_{i=1}^{2}\|b_i\|_{\mathrm{BMO}_{\varphi,\beta}}\|f_i\|_{L^{\vec{p_i},\eta_i,\psi}(\mathbb{R}^n)}\sum_{k=1}^{\infty}(k+1)|2^{k+1}B(x,r)|^{-\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{1j}}-\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{2j}}}\\
&\quad\times[\psi(2^{k+1}r)]^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{1j}}-\frac{\eta_1}{n}+\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{2j}}-\frac{\eta_2}{n}}.
\end{aligned}$$

Then

$$\begin{aligned}
E_{44}&=\sup_{x\in\mathbb{R}^n,r>0}\frac{\|T((b_1-(b_1)_{B(x,r)})f_1\chi_{(2B(x,r))^c},(b_2-(b_2)_{B(x,r)})f_2\chi_{(2B(x,r))^c})\chi_{B(x,r)}\|_{L^{\vec{q}}(\mathbb{R}^n)}}{[\psi(r)]^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{q_j}-\frac{\eta}{n}}}\\
&\lesssim \prod_{i=1}^{2}\|b_i\|_{\mathrm{BMO}_{\varphi,\beta}}\|f_i\|_{L^{\vec{p_i},\eta_i,\psi}(\mathbb{R}^n)}.
\end{aligned}$$

Thus,

$$E_4 \lesssim \prod_{i=1}^{2}\|b_i\|_{\mathrm{BMO}_{\varphi,\beta}}\|f_i\|_{L^{\vec{p_i},\eta_i,\psi}(\mathbb{R}^n)}.$$

Consequently,

$$\|T_{\prod\vec{b}}(\vec{f})\|_{L^{\vec{q},\eta,\psi}(\mathbb{R}^n)} \lesssim \prod_{i=1}^{m}\|b_i\|_{\mathrm{BMO}_{\varphi,\beta}}\|f_i\|_{L^{\vec{p_i},\eta_i,\psi}(\mathbb{R}^n)}.$$

This completes the proof of Theorem 4.1. □

Proof. [The proof of Theorem 4.2] Let $\vec{f}=(f_1,f_2)\in L^{\vec{p_1},\eta_1,\psi}(\mathbb{R}^n)\times L^{\vec{p_2},\eta_2,\psi}(\mathbb{R}^n)$, $B=:B(x,r)$ for $r>0$, $2B(x,r)=:B(x,2r)$ and $(2B(x,r))^c=:\mathbb{R}^n\setminus 2B(x,r)$. Then

$$\begin{aligned}
T_{\sum\vec{b}}(\vec{f})(z)&=(b_1(z)-(b_1)_{B(x,r)})T(f_1,f_2)(z)-T((b_1-(b_1)_{B(x,r)})f_1,f_2)(z)\\
&\quad+(b_2(z)-(b_2)_{B(x,r)})T(f_1,f_2)(z)-T(f_1,(b_2-(b_2)_{B(x,r)})f_2)(z).
\end{aligned}$$

Then by using Minkowski inequality, we have

$$\begin{aligned}\|T_{\sum\vec{b}}(\vec{f})\|_{L^{\vec{q},\eta,\psi}(\mathbb{R}^n)} \leq& \|(b_1-(b_1)_{B(x,r)})T(f_1,f_2)\|_{L^{\vec{q},\eta,\psi}(\mathbb{R}^n)}\\ &+\|T((b_1-(b_1)_{B(x,r)})f_1,f_2)\|_{L^{\vec{q},\eta,\psi}(\mathbb{R}^n)}\\ &+\|(b_2-(b_2)_{B(x,r)})T(f_1,f_2)\|_{L^{\vec{q},\eta,\psi}(\mathbb{R}^n)}\\ &+\|T(f_1,(b_2-(b_2)_{B(x,r)})f_2)\|_{L^{\vec{q},\eta,\psi}(\mathbb{R}^n)}\\ =:& F_1+F_2+F_3+F_4.\end{aligned}$$

First, we estimate F_1. Let $f_i=f_i\chi_{2B(x,r)}+f_i\chi_{(2B(x,r))^c}(i=1,2)$. Using Minkowski inequality again, there holds

$$\begin{aligned}F_1\leq& \|(b_1-(b_1)_{B(x,r)})T(f_1\chi_{2B(x,r)},f_2\chi_{2B(x,r)})\|_{L^{\vec{q},\eta,\psi}(\mathbb{R}^n)}\\ &+\|(b_1-(b_1)_{B(x,r)})T(f_1\chi_{2B(x,r)},f_2\chi_{(2B(x,r))^c})\|_{L^{\vec{q},\eta,\psi}(\mathbb{R}^n)}\\ &+\|(b_1-(b_1)_{B(x,r)})T(f_1\chi_{(2B(x,r))^c},f_2\chi_{2B(x,r)})\|_{L^{\vec{q},\eta,\psi}(\mathbb{R}^n)}\\ &+\|(b_1-(b_1)_{B(x,r)})T(f_1\chi_{(2B(x,r))^c},f_2\chi_{(2B(x,r))^c})\|_{L^{\vec{q},\eta,\psi}(\mathbb{R}^n)}\\ =:& F_{11}+F_{12}+F_{13}+F_{14}.\end{aligned}$$

For F_{11}, we may choose $\vec{u}$ such that $\frac{1}{\vec{q}}=\frac{1}{\vec{t_1}}+\frac{1}{\vec{u}}$ and $\sum_{j=1}^n\frac{1}{t_{ij}}=n\beta(i=1,2)$, then $\sum_{j=1}^n\frac{1}{u_j}=\sum_{j=1}^n\sum_{i=1}^2\frac{1}{p_{ij}}-2n\beta$. Noting that $n\beta\leq\eta_i<n\beta+\frac{n}{2}-\frac{n\beta}{2}$ and $\eta=\sum_{i=1}^2\eta_i-2n\beta+n\beta$, then $n\beta\leq\eta_i<(n-\frac{n}{2})\beta+\frac{n}{2}<n$ and $n\beta\leq\eta<n$. Thus, it gets from Lemma 4.5, Hölder's inequality and (4.2) that

$$\begin{aligned}F_{11}=&\sup_{x\in\mathbb{R}^n,r>0}\frac{\|(b_1-(b_1)_{B(x,r)})T(f_1\chi_{2B(x,r)},f_2\chi_{2B(x,r)})\chi_{B(x,r)}\|_{L^{\vec{q}}(\mathbb{R}^n)}}{[\psi(r)]^{\frac{1}{n}\sum_{j=1}^n\frac{1}{q_j}-\frac{\eta}{n}}}\\ \leq&\sup_{x\in\mathbb{R}^n,r>0}\|(b_1-(b_1)_{B(x,r)})\chi_{B(x,r)}\|_{L^{\vec{t_1}}(\mathbb{R}^n)}\\ &\times\|T(f_1\chi_{2B(x,r)},f_2\chi_{2B(x,r)})\chi_{B(x,r)}\|_{L^{\vec{u}}(\mathbb{R}^n)}\frac{1}{[\psi(r)]^{\frac{1}{n}\sum_{j=1}^n\frac{1}{q_j}-\frac{\eta}{n}}}\\ \lesssim&\prod_{i=1}^2\|f_i\|_{L^{\vec{p_i},\eta_i,\psi}(\mathbb{R}^n)}\sup_{x\in\mathbb{R}^n,r>0}\frac{1}{|B(x,r)|}\int_{B(x,r)}|b_1(y_1)-(b_1)_{B(x,r)}|\mathrm{d}y_1\\ &\times\frac{|B(x,r)|^{\frac{1}{n}\sum_{j=1}^n\frac{1}{t_{1j}}}}{[\varphi(|B(x,r)|)]^{2(1-\beta)}}\frac{[C_\psi\psi(r)]^{\frac{1}{n}\sum_{j=1}^n\frac{1}{p_{1j}}+\frac{1}{n}\sum_{j=1}^n\frac{1}{p_{2j}}-\frac{\eta_1+\eta_2}{n}}}{[\psi(r)]^{\frac{1}{n}\sum_{j=1}^n\frac{1}{q_j}-\frac{\eta}{n}}}\\ \lesssim&\|b_1\|_{\mathrm{BMO}_{\varphi,\beta}}\prod_{i=1}^2\|f_i\|_{L^{\vec{p_i},\eta_i,\psi}(\mathbb{R}^n)}\sup_{x\in\mathbb{R}^n,r>0}\frac{[\varphi(|B(x,r)|)]^{2(1-\beta)}|B(x,r)|^{1-\beta}}{|B(x,r)|}\\ &\times\frac{|B(x,r)|^{\frac{1}{n}\sum_{j=1}^n\frac{1}{t_{1j}}}}{[\varphi(|B(x,r)|)]^{2(1-\beta)}}\\ \leq&\|b_1\|_{\mathrm{BMO}_{\varphi,\beta}}\prod_{i=1}^2\|f_i\|_{L^{\vec{p_i},\eta_i,\psi}(\mathbb{R}^n)}.\end{aligned}$$

For F_{12}, applying (4.4), Lemma 4.5, Hölder's inequality and $\frac{\eta_2}{n}\geq\frac{n\beta}{n}=\beta>2\beta-1$, we have

$$F_{12}=\sup_{x\in\mathbb{R}^n,r>0}\frac{\|(b_1-(b_1)_{B(x,r)})T(f_1\chi_{2B(x,r)},f_2\chi_{(2B(x,r))^c})\chi_{B(x,r)}\|_{L^{\vec{q}}(\mathbb{R}^n)}}{[\psi(r)]^{\frac{1}{n}\sum_{j=1}^n\frac{1}{q_j}-\frac{\eta}{n}}}$$

$$\begin{aligned}
&\lesssim \prod_{i=1}^{2}\|f_i\|_{L^{\vec{p_i},\eta_i,\psi}(\mathbb{R}^n)} \sup_{x\in\mathbb{R}^n,r>0}\|(b_1-(b_1)_{B(x,r)})\chi_{B(x,r)}\|_{L^{\vec{t_1}}(\mathbb{R}^n)}\\
&\quad\times\|\chi_{B(x,r)}\|_{L^{\vec{u}}(\mathbb{R}^n)}\sum_{k=1}^{\infty}\frac{|B(x,r)|^{1-\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{1j}}}|2^{k+1}B(x,r)|^{1-\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{2j}}}}{(\varphi(|2^{k+1}B(x,r)|)|2^{k+1}B(x,r)|)^{2(1-\beta)}}\\
&\quad\times\frac{[C_\psi\psi(r)]^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{1j}}-\frac{\eta_1}{n}}[C_\psi^{k+1}\psi(r)]^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{2j}}-\frac{\eta_2}{n}}}{[\psi(r)]^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{q_j}-\frac{\eta}{n}}}\\
&\lesssim\|b_1\|_{\mathrm{BMO}_{\varphi,\beta}}\prod_{i=1}^{2}\|f_i\|_{L^{\vec{p_i},\eta_i,\psi}(\mathbb{R}^n)}\sum_{k=1}^{\infty}\frac{[C_\psi^{k+1}]^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{2j}}-\frac{\eta_2}{n}}}{(2^n)^{(k+1)(\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{2j}}+1-2\beta)}}\\
&\lesssim\|b_1\|_{\mathrm{BMO}_{\varphi,\beta}}\prod_{i=1}^{2}\|f_i\|_{L^{\vec{p_i},\eta_i,\psi}(\mathbb{R}^n)}.
\end{aligned}$$

For F_{13}, similar to F_{12}, we obtain

$$F_{13}\lesssim\|b_1\|_{\mathrm{BMO}_{\varphi,\beta}}\prod_{i=1}^{2}\|f_i\|_{L^{\vec{p_i},\eta_i,\psi}(\mathbb{R}^n)}.$$

For F_{14}, by $\frac{\eta_1+\eta_2}{n}\geq\frac{2n\beta}{n}=2\beta$, (4.5), Hölder's inequality and Lemma 4.5, there holds

$$\begin{aligned}
F_{14}&=\sup_{x\in\mathbb{R}^n,r>0}\frac{\|(b_1-(b_1)_{B(x,r)})T(f_1\chi_{(2B(x,r))^c},f_2\chi_{(2B(x,r))^c})\chi_{B(x,r)}\|_{L^{\vec{q}}(\mathbb{R}^n)}}{[\psi(r)]^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{q_j}-\frac{\eta}{n}}}\\
&\lesssim \prod_{i=1}^{2}\|f_i\|_{L^{\vec{p_i},\eta_i,\psi}(\mathbb{R}^n)} \sup_{x\in\mathbb{R}^n,r>0}\|(b_1-(b_1)_{B(x,r)})\chi_{B(x,r)}\|_{L^{\vec{t_1}}(\mathbb{R}^n)}\\
&\quad\times\|\chi_{B(x,r)}\|_{L^{\vec{u}}(\mathbb{R}^n)}\sum_{k=1}^{\infty}\frac{|2^{k+1}B(x,r)|^{1-\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{1j}}+1-\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{2j}}}}{(\varphi(|2^{k+1}B(x,r)|)|2^{k+1}B(x,r)|)^{2(1-\beta)}}\\
&\quad\times\frac{[C_\psi^{k+1}\psi(r)]^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{1j}}-\frac{\eta_1}{n}+\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{2j}}-\frac{\eta_2}{n}}}{[\psi(r)]^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{q_j}-\frac{\eta}{n}}}\\
&\lesssim\|b_1\|_{\mathrm{BMO}_{\varphi,\beta}}\prod_{i=1}^{2}\|f_i\|_{L^{\vec{p_i},\eta_i,\psi}(\mathbb{R}^n)}\sum_{k=1}^{\infty}\frac{[C_\psi^{k+1}]^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{1j}}+\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{2j}}-\frac{\eta_1+\eta_2}{n}}}{(2^n)^{(k+1)(\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{1j}}+\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{2j}}-2\beta)}}\\
&\lesssim\|b_1\|_{\mathrm{BMO}_{\varphi,\beta}}\prod_{i=1}^{2}\|f_i\|_{L^{\vec{p_i},\eta_i,\psi}(\mathbb{R}^n)}.
\end{aligned}$$

Therefore,

$$F_1\lesssim\|b_1\|_{\mathrm{BMO}_{\varphi,\beta}}\prod_{i=1}^{2}\|f_i\|_{L^{\vec{p_i},\eta_i,\psi}(\mathbb{R}^n)}.$$

Next, we estimate F_2. From Minkowski inequality, we get that

$$\begin{aligned}
F_2&\leq\|T((b_1-(b_1)_{B(x,r)})f_1\chi_{2B(x,r)},f_2\chi_{2B(x,r)})\|_{L^{\vec{q},\eta,\psi}(\mathbb{R}^n)}\\
&\quad+\|T((b_1-(b_1)_{B(x,r)})f_1\chi_{2B(x,r)},f_2\chi_{(2B(x,r))^c})\|_{L^{\vec{q},\eta,\psi}(\mathbb{R}^n)}\\
&\quad+\|T((b_1-(b_1)_{B(x,r)})f_1\chi_{(2B(x,r))^c},f_2\chi_{2B(x,r)})\|_{L^{\vec{q},\eta,\psi}(\mathbb{R}^n)}
\end{aligned}$$

$$+\|T((b_1-(b_1)_{B(x,r)})f_1\chi_{(2B(x,r))^c}, f_2\chi_{(2B(x,r))^c})\|_{L^{\vec{q},\eta,\psi}(\mathbb{R}^n)}$$
$$=: F_{21}+F_{22}+F_{23}+F_{24}.$$

For F_{21}, assume that $\frac{1}{\vec{\varrho}}=\frac{1}{\vec{p_1}}+\frac{1}{\vec{t_1}}$ and $\sum_{j=1}^n\frac{1}{t_{ij}}=n\beta(i=1,2)$, then $\sum_{j=1}^n\frac{1}{q_j}=\sum_{j=1}^n\frac{1}{\varrho_j}+\sum_{j=1}^n\frac{1}{p_{2j}}-2n\beta$. Let $\frac{1}{\vec{q}}=\sum_{i=1}^2\frac{1}{\vec{q_i}}$, $\sum_{j=1}^n\frac{1}{q_{1j}}=\sum_{j=1}^n\frac{1}{\varrho_j}-\beta_1$ and $\sum_{j=1}^n\frac{1}{q_{2j}}=\sum_{j=1}^n\frac{1}{p_{2j}}-\beta_2$. Then $\beta=\frac{1}{2n}\sum_{i=1}^2\beta_i$. Thus, Lemma 4.5 together with Hölder's inequality, (4.1) and the similar arguments as in the proof of Lemma 3.2 implies that

$$\begin{aligned}
F_{21}&=\sup_{x\in\mathbb{R}^n,r>0}\frac{\|T((b_1-(b_1)_{B(x,r)})f_1\chi_{2B(x,r)}, f_2\chi_{2B(x,r)})\chi_{B(x,r)}\|_{L^{\vec{q}}(\mathbb{R}^n)}}{[\psi(r)]^{\frac{1}{n}\sum_{j=1}^n\frac{1}{q_j}-\frac{\eta}{n}}}\\
&\lesssim\sup_{x\in\mathbb{R}^n,r>0}\frac{1}{[\varphi(|B(x,r)|)]^{2(1-\beta)}}\|(b_1-(b_1)_{B(x,r)})f_1\chi_{2B(x,r)}\|_{L^{\vec{\varrho}}(\mathbb{R}^n)}\\
&\quad\times\|f_2\chi_{2B(x,r)}\|_{L^{\vec{p_2}}(\mathbb{R}^n)}\frac{1}{[\psi(r)]^{\frac{1}{n}\sum_{j=1}^n\frac{1}{q_j}-\frac{\eta}{n}}}\\
&\le\sup_{x\in\mathbb{R}^n,r>0}\frac{1}{[\varphi(|B(x,r)|)]^{2(1-\beta)}}\|(b_1-(b_1)_{B(x,r)})\chi_{2B(x,r)}\|_{L^{\vec{t_1}}(\mathbb{R}^n)}\\
&\quad\times\prod_{i=1}^2\|f_i\chi_{2B(x,r)}\|_{L^{\vec{p_i}}(\mathbb{R}^n)}\frac{1}{[\psi(r)]^{\frac{1}{n}\sum_{j=1}^n\frac{1}{q_j}-\frac{\eta}{n}}}\\
&\lesssim\|b_1\|_{\mathrm{BMO}_{\varphi,\beta}}\prod_{i=1}^2\|f_i\|_{L^{\vec{p_i},\eta_i,\psi}(\mathbb{R}^n)}\sup_{x\in\mathbb{R}^n,r>0}\frac{|2B(x,r)|^{\frac{1}{n}\sum_{j=1}^n\frac{1}{t_{1j}}}}{[\varphi(|B(x,r)|)]^{2(1-\beta)}}\\
&\quad\times\frac{[\varphi(|B(x,r)|)]^{2(1-\beta)}|B(x,r)|^{1-\beta}}{|B(x,r)|}\frac{[C_\psi\psi(r)]^{\frac{1}{n}\sum_{j=1}^n\frac{1}{p_{1j}}+\frac{1}{n}\sum_{j=1}^n\frac{1}{p_{2j}}-\frac{\eta_1+\eta_2}{n}}}{[\psi(r)]^{\frac{1}{n}\sum_{j=1}^n\frac{1}{q_j}-\frac{\eta}{n}}}\\
&\lesssim\|b_1\|_{\mathrm{BMO}_{\varphi,\beta}}\prod_{i=1}^2\|f_i\|_{L^{\vec{p_i},\eta_i,\psi}(\mathbb{R}^n)}.
\end{aligned}$$

For F_{22}, by (4.7), there holds

$$\begin{aligned}
F_{22}&=\sup_{x\in\mathbb{R}^n,r>0}\frac{\|T((b_1-(b_1)_{B(x,r)})f_1\chi_{2B(x,r)}, f_2\chi_{(2B(x,r))^c})\chi_{B(x,r)}\|_{L^{\vec{q}}(\mathbb{R}^n)}}{[\psi(r)]^{\frac{1}{n}\sum_{j=1}^n\frac{1}{q_j}-\frac{\eta}{n}}}\\
&\lesssim\|b_1\|_{\mathrm{BMO}_{\varphi,\beta}}\prod_{i=1}^2\|f_i\|_{L^{\vec{p_i},\eta_i,\psi}(\mathbb{R}^n)}\sup_{x\in\mathbb{R}^n,r>0}\|\chi_{B(x,r)}\|_{L^{\vec{q}}(\mathbb{R}^n)}\\
&\quad\times\sum_{k=1}^\infty\frac{|B(x,r)|^{-\frac{1}{n}\sum_{j=1}^n\frac{1}{p_{1j}}}|2^{k+1}B(x,r)|^{1-\frac{1}{n}\sum_{j=1}^n\frac{1}{p_{2j}}}}{|2^{k+1}B(x,r)|^{1-\beta}}\\
&\quad\times\frac{[C_\psi\psi(r)]^{\frac{1}{n}\sum_{j=1}^n\frac{1}{p_{1j}}-\frac{\eta_1}{n}}[C_\psi^{k+1}\psi(r)]^{\frac{1}{n}\sum_{j=1}^n\frac{1}{p_{2j}}-\frac{\eta_2}{n}}}{[\psi(r)]^{\frac{1}{n}\sum_{j=1}^n\frac{1}{q_j}-\frac{\eta}{n}}}\\
&\lesssim\|b_1\|_{\mathrm{BMO}_{\varphi,\beta}}\prod_{i=1}^2\|f_i\|_{L^{\vec{p_i},\eta_i,\psi}(\mathbb{R}^n)}.
\end{aligned}$$

For F_{23}, similar to F_{22}, we have

$$F_{23}\lesssim\|b_1\|_{\mathrm{BMO}_{\varphi,\beta}}\prod_{i=1}^2\|f_i\|_{L^{\vec{p_i},\eta_i,\psi}(\mathbb{R}^n)}.$$

For F_{24}, applying $\frac{\eta_1+\eta_2}{n} \geq \frac{2n\beta}{n} = 2\beta > \beta$ and (4.8), we get

$$\begin{aligned} F_{24} &= \sup_{x\in\mathbb{R}^n, r>0} \frac{\|T((b_1-(b_1)_{B(x,r)})f_1\chi_{(2B(x,r))^c}, f_2\chi_{(2B(x,r))^c})\chi_{B(x,r)}\|_{L^{\vec{q}}(\mathbb{R}^n)}}{[\psi(r)]^{\frac{1}{n}\sum_{j=1}^n \frac{1}{q_j}-\frac{\eta}{n}}} \\ &\lesssim \|b_1\|_{\mathrm{BMO}_{\varphi,\beta}} \prod_{i=1}^2 \|f_i\|_{L^{\vec{p_i},\eta_i,\psi}(\mathbb{R}^n)} \sup_{x\in\mathbb{R}^n, r>0} \|\chi_{B(x,r)}\|_{L^{\vec{q}}(\mathbb{R}^n)} \sum_{k=1}^\infty (k+1) \\ &\quad \times \frac{|2^{k+1}B(x,r)|^{1-\frac{1}{n}\sum_{j=1}^n \frac{1}{p_{1j}}-\frac{1}{n}\sum_{j=1}^n \frac{1}{p_{2j}}}}{|2^{k+1}B(x,r)|^{1-\beta}} \frac{[C_\psi^{k+1}\psi(r)]^{\frac{1}{n}\sum_{j=1}^n \frac{1}{p_{1j}}+\frac{1}{n}\sum_{j=1}^n \frac{1}{p_{2j}}-\frac{\eta_1+\eta_2}{n}}}{[\psi(r)]^{\frac{1}{n}\sum_{j=1}^n \frac{1}{q_j}-\frac{\eta}{n}}} \\ &\lesssim \|b_1\|_{\mathrm{BMO}_{\varphi,\beta}} \prod_{i=1}^2 \|f_i\|_{L^{\vec{p_i},\eta_i,\psi}(\mathbb{R}^n)}. \end{aligned}$$

Therefore,

$$F_2 \lesssim \|b_1\|_{\mathrm{BMO}_{\varphi,\beta}} \prod_{i=1}^2 \|f_i\|_{L^{\vec{p_i},\eta_i,\psi}(\mathbb{R}^n)}.$$

Finally, we estimate F_3 and F_4, it follows from F_1 and F_2 that

$$F_3 \lesssim \|b_2\|_{\mathrm{BMO}_{\varphi,\beta}} \prod_{i=1}^2 \|f_i\|_{L^{\vec{p_i},\eta_i,\psi}(\mathbb{R}^n)},$$

and

$$F_4 \lesssim \|b_2\|_{\mathrm{BMO}_{\varphi,\beta}} \prod_{i=1}^2 \|f_i\|_{L^{\vec{p_i},\eta_i,\psi}(\mathbb{R}^n)}.$$

Thus,

$$\|T_{\sum \vec{b}}(\vec{f})\|_{L^{\vec{q},\eta,\psi}(\mathbb{R}^n)} \lesssim \|\vec{b}\|_{\mathrm{BMO}_{\varphi,\beta}} \prod_{i=1}^m \|f_i\|_{L^{\vec{p_i},\eta_i,\psi}(\mathbb{R}^n)}.$$

The proof is finished. □

Proof. [The proof of Theorem 4.3] Let $\vec{f} = (f_1, f_2) \in L^{\vec{p_1},\eta_1,\psi}(\mathbb{R}^n) \times L^{\vec{p_2},\eta_2,\psi}(\mathbb{R}^n)$, $f_i = f_i\chi_{2B(x,r)} + f_i\chi_{(2B(x,r))^c} (i = 1, 2)$, $B =: B(x,r)$ for $r > 0$, $2B(x,r) =: B(x, 2r)$ and $(2B(x,r))^c =: \mathbb{R}^n \setminus 2B(x,r)$. Then it follows from Minkowski inequality that

$$\begin{aligned} \|T_{\prod \vec{b}}(f_1, f_2)\|_{L^{\vec{q},\eta,\psi}(\mathbb{R}^n)} &\leq \|T_{\prod \vec{b}}(f_1\chi_{2B(x,r)}, f_2\chi_{2B(x,r)})\|_{L^{\vec{q},\eta,\psi}(\mathbb{R}^n)} \\ &\quad + \|T_{\prod \vec{b}}(f_1\chi_{(2B(x,r))^c}, f_2\chi_{2B(x,r)})\|_{L^{\vec{q},\eta,\psi}(\mathbb{R}^n)} \\ &\quad + \|T_{\prod \vec{b}}(f_1\chi_{2B(x,r)}, f_2\chi_{(2B(x,r))^c})\|_{L^{\vec{q},\eta,\psi}(\mathbb{R}^n)} \\ &\quad + \|T_{\prod \vec{b}}(f_1\chi_{(2B(x,r))^c}, f_2\chi_{(2B(x,r))^c})\|_{L^{\vec{q},\eta,\psi}(\mathbb{R}^n)} \\ &=: H_1 + H_2 + H_3 + H_4. \end{aligned}$$

Noticing that $0 < \alpha < 1$, $0 \leq \beta < 1 - \frac{\alpha}{n}$, $\alpha + n\beta \leq \eta_i < \min\{\alpha + n\beta + \frac{n}{2}, n\}$ and $\eta = \sum_{i=1}^2 \eta_i - 2n\beta - 2\alpha$. Then

(a) If $\alpha + n\beta \leq \eta_i < \alpha + n\beta + \frac{n}{2}$, then $\alpha + n\beta \leq \eta_i < n$ and $0 \leq \eta < 2\alpha + 2n\beta + n - 2n\beta - 2\alpha = n$.

(b) If $\alpha + n\beta \leq \eta_i < n$, then $\alpha + n\beta \leq \eta_i < n < \alpha + n\beta + \frac{n}{2}$ and $0 \leq \eta < 2n - 2n\beta - 2\alpha < 2(\alpha + n\beta + \frac{n}{2}) - 2n\beta - 2\alpha = n$.

For H_1, as $y_1, y_2 \in 2B(x,r)$ and $z \in B(x,r)$, then we get from (1.1) that

$$|T_{\prod \vec{b}}(f_1\chi_{2B(x,r)}, f_2\chi_{2B(x,r)})(z)|$$

$$
\begin{aligned}
&\leq \prod_{i=1}^{2} \int_{2B(x,r)} |b_i(z) - b_i(y_i)||K_\beta(z, y_1, y_2)||f_i(y_i)|\mathrm{d}y_i \\
&\lesssim \prod_{i=1}^{2} \|b_i\|_{\mathrm{Lip}_\alpha(\mathbb{R}^n)} \int_{(2B(x,r))^2} \frac{|z-y_1|^\alpha |z-y_2|^\alpha}{(|z-y_1| + |z-y_2|)^{2n(1-\beta)}} |f_1(y_1)||f_2(y_2)|\mathrm{d}y_1\mathrm{d}y_2 \\
&\leq \prod_{i=1}^{2} \|b_i\|_{\mathrm{Lip}_\alpha(\mathbb{R}^n)} \int_{2B(x,r)} \frac{|z-y_1|^\alpha}{|z-y_1|^{n(1-\beta)}} |f_1(y_1)|\mathrm{d}y_1 \int_{2B(x,r)} \frac{|z-y_2|^\alpha}{|z-y_2|^{n(1-\beta)}} |f_2(y_2)|\mathrm{d}y_2 \\
&\leq \prod_{i=1}^{2} \|b_i\|_{\mathrm{Lip}_\alpha(\mathbb{R}^n)} I_{n\beta+\alpha}(f_i \chi_{2B(x,r)})(z). \qquad (4.9)
\end{aligned}
$$

Assume that $\frac{1}{\vec{q}} = \sum_{i=1}^{2} \frac{1}{\vec{v_i}}$. Then $\sum_{j=1}^{n} \frac{1}{v_{ij}} = \sum_{j=1}^{n} \frac{1}{p_{ij}} - n\beta - \alpha$. Since $0 < n\beta + \alpha < n$, then by applying Lemma 2.2, (4.9) and Hölder's inequality, we have

$$
\begin{aligned}
H_1 &= \sup_{x\in\mathbb{R}^n, r>0} \frac{\|T_{\prod \vec{b}}(f_1\chi_{2B(x,r)}, f_2\chi_{2B(x,r)})\chi_{B(x,r)}\|_{L^{\vec{q}}(\mathbb{R}^n)}}{[\psi(r)]^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{q_j} - \frac{\eta}{n}}} \\
&\lesssim \prod_{i=1}^{2} \|b_i\|_{\mathrm{Lip}_\alpha(\mathbb{R}^n)} \sup_{x\in\mathbb{R}^n, r>0} \|I_{n\beta+\alpha}(f_1\chi_{2B(x,r)})\|_{L^{\vec{v_1}}(\mathbb{R}^n)} \\
&\quad \times \|I_{n\beta+\alpha}(f_2\chi_{2B(x,r)})\|_{L^{\vec{v_2}}(\mathbb{R}^n)} \frac{1}{[\psi(r)]^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{q_j} - \frac{\eta}{n}}} \\
&\lesssim \prod_{i=1}^{2} \|b_i\|_{\mathrm{Lip}_\alpha(\mathbb{R}^n)} \sup_{x\in\mathbb{R}^n, r>0} \|f_1\chi_{2B(x,r)}\|_{L^{\vec{p_1}}(\mathbb{R}^n)} \|f_2\chi_{2B(x,r)}\|_{L^{\vec{p_2}}(\mathbb{R}^n)} \frac{1}{[\psi(r)]^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{q_j} - \frac{\eta}{n}}} \\
&\leq \prod_{i=1}^{2} \|b_i\|_{\mathrm{Lip}_\alpha(\mathbb{R}^n)} \|f_i\|_{L^{\vec{p_i},\eta_i,\psi}(\mathbb{R}^n)} \sup_{x\in\mathbb{R}^n, r>0} \frac{[\psi(2r)]^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{1j}} - \frac{\eta_1}{n} + \frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{2j}} - \frac{\eta_2}{n}}}{[\psi(r)]^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{q_j} - \frac{\eta}{n}}} \\
&\lesssim \prod_{i=1}^{2} \|b_i\|_{\mathrm{Lip}_\alpha(\mathbb{R}^n)} \|f_i\|_{L^{\vec{p_i},\eta_i,\psi}(\mathbb{R}^n)}.
\end{aligned}
$$

For H_2, by Lemmas 4.6-4.7, (4.3) and Hölder's inequality, there holds

$$
\begin{aligned}
&|T_{\prod \vec{b}}(f_1\chi_{(2B(x,r))^c}, f_2\chi_{2B(x,r)})(z)| \lesssim \left(\int_{2B(x,r)} |b_2(z) - b_2(y_2)||f_2(y_2)|\mathrm{d}y_2 \right) \\
&\quad \times \left(\sum_{k=1}^{\infty} \int_{2^{k+1}B(x,r)} \frac{|b_1(z) - b_1(y_1)||f_1(y_1)|}{(\varphi(|2^kB(x,r)|)|2^kB(x,r)|)^{2(1-\beta)}} \mathrm{d}y_1 \right) \\
&\leq \left(\|b_2\|_{\mathrm{Lip}_\alpha(\mathbb{R}^n)} \int_{2B(x,r)} |z-y_2|^\alpha |f_2(y_2)|\mathrm{d}y_2 \right) \\
&\quad \times \left(\sum_{k=1}^{\infty} \int_{2^{k+1}B(x,r)} \frac{|b_1(y_1) - (b_1)_{2^{k+1}B(x,r)}||f_1(y_1)|}{(\varphi(|2^kB(x,r)|)|2^kB(x,r)|)^{2(1-\beta)}} \mathrm{d}y_1 \right. \\
&\quad \left. + \sum_{k=1}^{\infty} \frac{|b_1(z) - (b_1)_{B(x,r)}| + |(b_1)_{B(x,r)} - (b_1)_{2^{k+1}B(x,r)}|}{(\varphi(|2^kB(x,r)|)|2^kB(x,r)|)^{2(1-\beta)}} \int_{2^{k+1}B(x,r)} |f_1(y_1)|\mathrm{d}y_1 \right) \\
&\lesssim \|b_2\|_{\mathrm{Lip}_\alpha(\mathbb{R}^n)} |2B(x,r)|^{\frac{\alpha}{n}} \|f_2\chi_{2B(x,r)}\|_{L^{\vec{p_2}}(\mathbb{R}^n)} \|\chi_{2B(x,r)}\|_{L^{\vec{p_2}'}(\mathbb{R}^n)}
\end{aligned}
$$

$$
\begin{aligned}
&\times\Bigg(\sum_{k=1}^{\infty}\frac{\|f_1\chi_{2^{k+1}B(x,r)}\|_{L^{\vec{p_1}}(\mathbb{R}^n)}\|(b_1-(b_1)_{2^{k+1}B(x,r)})\chi_{2^{k+1}B(x,r)}\|_{L^{\vec{p_1}'}(\mathbb{R}^n)}}{(\varphi(|2^{k+1}B(x,r)|)|2^{k+1}B(x,r)|)^{2(1-\beta)}}\\
&+\sum_{k=1}^{\infty}\frac{|b_1(z)-(b_1)_{B(x,r)}|+|(b_1)_{B(x,r)}-(b_1)_{2^{k+1}B(x,r)}|}{(\varphi(|2^kB(x,r)|)|2^kB(x,r)|)^{2(1-\beta)}}\\
&\times\|f_1\chi_{2^{k+1}B(x,r)}\|_{L^{\vec{p_1}}(\mathbb{R}^n)}\|\chi_{2^{k+1}B(x,r)}\|_{L^{\vec{p_1}'}(\mathbb{R}^n)}\Bigg)\\
&\lesssim\|b_2\|_{\mathrm{Lip}_\alpha(\mathbb{R}^n)}\|f_2\|_{L^{\vec{p_2},\eta_2,\psi}(\mathbb{R}^n)}|2B(x,r)|^{\frac{\alpha}{n}+1-\frac{1}{n}\sum_{j=1}^n\frac{1}{p_{2j}}}[\psi(2r)]^{\frac{1}{n}\sum_{j=1}^n\frac{1}{p_{2j}}-\frac{\eta_2}{n}}\\
&\times\Bigg(\|b_1\|_{\mathrm{Lip}_\alpha(\mathbb{R}^n)}\|f_1\|_{L^{\vec{p_1},\eta_1,\psi}(\mathbb{R}^n)}\sum_{k=1}^{\infty}\frac{|2^{k+1}B(x,r)|^{\frac{\alpha}{n}+1-\frac{1}{n}\sum_{j=1}^n\frac{1}{p_{1j}}}[\psi(2^{k+1}r)]^{\frac{1}{n}\sum_{j=1}^n\frac{1}{p_{1j}}-\frac{\eta_1}{n}}}{(\varphi(|2^{k+1}B(x,r)|)|2^{k+1}B(x,r)|)^{2(1-\beta)}}\\
&+\|f_1\|_{L^{\vec{p_1},\eta_1,\psi}(\mathbb{R}^n)}\sum_{k=1}^{\infty}\frac{|b_1(z)-(b_1)_{B(x,r)}|+\|b_1\|_{\mathrm{Lip}_\alpha(\mathbb{R}^n)}|2^{k+1}B(x,r)|^{\frac{\alpha}{n}}}{(\varphi(|2^{k+1}B(x,r)|)|2^{k+1}B(x,r)|)^{2(1-\beta)}}\\
&\times|2^{k+1}B(x,r)|^{1-\frac{1}{n}\sum_{j=1}^n\frac{1}{p_{1j}}}[\psi(2^{k+1}r)]^{\frac{1}{n}\sum_{j=1}^n\frac{1}{p_{1j}}-\frac{\eta_1}{n}}\Bigg)\\
&\lesssim\|b_2\|_{\mathrm{Lip}_\alpha(\mathbb{R}^n)}\|f_2\|_{L^{\vec{p_2},\eta_2,\psi}(\mathbb{R}^n)}|2B(x,r)|^{\frac{\alpha}{n}+1-\frac{1}{n}\sum_{j=1}^n\frac{1}{p_{2j}}}[\psi(2r)]^{\frac{1}{n}\sum_{j=1}^n\frac{1}{p_{2j}}-\frac{\eta_2}{n}}\\
&\times\Bigg(\|b_1\|_{\mathrm{Lip}_\alpha(\mathbb{R}^n)}\|f_1\|_{L^{\vec{p_1},\eta_1,\psi}(\mathbb{R}^n)}\sum_{k=1}^{\infty}\frac{|2^{k+1}B(x,r)|^{\frac{\alpha}{n}+1-\frac{1}{n}\sum_{j=1}^n\frac{1}{p_{1j}}}[\psi(2^{k+1}r)]^{\frac{1}{n}\sum_{j=1}^n\frac{1}{p_{1j}}-\frac{\eta_1}{n}}}{(\varphi(|2^{k+1}B(x,r)|)|2^{k+1}B(x,r)|)^{2(1-\beta)}}\\
&+|b_1(z)-(b_1)_{B(x,r)}|\|f_1\|_{L^{\vec{p_1},\eta_1,\psi}(\mathbb{R}^n)}\\
&\times\sum_{k=1}^{\infty}\frac{|2^{k+1}B(x,r)|^{1-\frac{1}{n}\sum_{j=1}^n\frac{1}{p_{1j}}}[\psi(2^{k+1}r)]^{\frac{1}{n}\sum_{j=1}^n\frac{1}{p_{1j}}-\frac{\eta_1}{n}}}{(\varphi(|2^{k+1}B(x,r)|)|2^{k+1}B(x,r)|)^{2(1-\beta)}}\Bigg). \qquad (4.10)
\end{aligned}
$$

Thus, Lemma 4.7 together with (4.10), $\frac{\eta_1}{n}\geq\frac{n\beta+\alpha}{n}>\frac{\alpha}{n}+2\beta-1>2\beta-1$ and $0<C_\psi<2^n$, we have

$$
\begin{aligned}
H_2&=\sup_{x\in\mathbb{R}^n,r>0}\frac{\|T_{\prod\vec{b}}(f_1\chi_{(2B(x,r))^c},f_2\chi_{2B(x,r)})\chi_{B(x,r)}\|_{L^{\vec{q}}(\mathbb{R}^n)}}{[\psi(r)]^{\frac{1}{n}\sum_{j=1}^n\frac{1}{q_j}-\frac{\eta}{n}}}\\
&\lesssim\prod_{i=1}^{2}\|b_i\|_{\mathrm{Lip}_\alpha(\mathbb{R}^n)}\|f_i\|_{L^{\vec{p_i},\eta_i,\psi}(\mathbb{R}^n)}\sup_{x\in\mathbb{R}^n,r>0}\|\chi_{B(x,r)}\|_{L^{\vec{q}}(\mathbb{R}^n)}\\
&\quad\times|2B(x,r)|^{\frac{\alpha}{n}+1-\frac{1}{n}\sum_{j=1}^n\frac{1}{p_{2j}}}\sum_{k=1}^{\infty}[\varphi(|2^{k+1}B(x,r)|)]^{2(1-\beta)}\\
&\quad\times\frac{|2^{k+1}B(x,r)|^{\frac{\alpha}{n}+1-\frac{1}{n}\sum_{j=1}^n\frac{1}{p_{1j}}}}{(\varphi(|2^{k+1}B(x,r)|)|2^{k+1}B(x,r)|)^{2(1-\beta)}}\frac{[C_\psi\psi(r)]^{\frac{1}{n}\sum_{j=1}^n\frac{1}{p_{2j}}-\frac{\eta_2}{n}}[C_\psi^{k+1}\psi(r)]^{\frac{1}{n}\sum_{j=1}^n\frac{1}{p_{1j}}-\frac{\eta_1}{n}}}{[\psi(r)]^{\frac{1}{n}\sum_{j=1}^n\frac{1}{q_j}-\frac{\eta}{n}}}\\
&\quad+\|b_2\|_{\mathrm{Lip}_\alpha(\mathbb{R}^n)}\prod_{i=1}^{2}\|f_i\|_{L^{\vec{p_i},\eta_i,\psi}(\mathbb{R}^n)}\sup_{x\in\mathbb{R}^n,r>0}\|(b_1-(b_1)_{B(x,r)})\chi_{B(x,r)}\|_{L^{\vec{q}}(\mathbb{R}^n)}\\
&\quad\times|2B(x,r)|^{\frac{\alpha}{n}+1-\frac{1}{n}\sum_{j=1}^n\frac{1}{p_{2j}}}\sum_{k=1}^{\infty}\frac{|2^{k+1}B(x,r)|^{1-\frac{1}{n}\sum_{j=1}^n\frac{1}{p_{1j}}}[\varphi(|2^{k+1}B(x,r)|)]^{2(1-\beta)}}{(\varphi(|2^{k+1}B(x,r)|)|2^{k+1}B(x,r)|)^{2(1-\beta)}}\\
&\quad\times\frac{[C_\psi\psi(r)]^{\frac{1}{n}\sum_{j=1}^n\frac{1}{p_{2j}}-\frac{\eta_2}{n}}[C_\psi^{k+1}\psi(r)]^{\frac{1}{n}\sum_{j=1}^n\frac{1}{p_{1j}}-\frac{\eta_1}{n}}}{[\psi(r)]^{\frac{1}{n}\sum_{j=1}^n\frac{1}{q_j}-\frac{\eta}{n}}}
\end{aligned}
$$

$$\lesssim \prod_{i=1}^{2}\|b_i\|_{\mathrm{Lip}_\alpha(\mathbb{R}^n)}\|f_i\|_{L^{\vec{p_i},\eta_i,\psi}(\mathbb{R}^n)}\sup_{x\in\mathbb{R}^n,r>0}|B(x,r)|^{\frac{\alpha}{n}+1+\frac{1}{n}\sum_{j=1}^{n}\frac{1}{q_j}-\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{2j}}}$$
$$\times\sum_{k=1}^{\infty}\frac{|2^{k+1}B(x,r)|^{\frac{\alpha}{n}+1-\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{1j}}}}{|2^{k+1}B(x,r)|^{2(1-\beta)}}[C_\psi^{k+1}]^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{1j}}-\frac{\eta_1}{n}}$$
$$+\prod_{i=1}^{2}\|b_i\|_{\mathrm{Lip}_\alpha(\mathbb{R}^n)}\|f_i\|_{L^{\vec{p_i},\eta_i,\psi}(\mathbb{R}^n)}\sup_{x\in\mathbb{R}^n,r>0}|B(x,r)|^{\frac{2\alpha}{n}+1+\frac{1}{n}\sum_{j=1}^{n}\frac{1}{q_j}-\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{2j}}}$$
$$\times\sum_{k=1}^{\infty}\frac{|2^{k+1}B(x,r)|^{1-\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{1j}}}}{|2^{k+1}B(x,r)|^{2(1-\beta)}}[C_\psi^{k+1}]^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{1j}}-\frac{\eta_1}{n}}$$
$$\leq\left(\sum_{k=1}^{\infty}\frac{[C_\psi^{k+1}]^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{1j}}-\frac{\eta_1}{n}}}{(2^n)^{(k+1)(\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{1j}}+1-\frac{\alpha}{n}-2\beta)}}+\sum_{k=1}^{\infty}\frac{[C_\psi^{k+1}]^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{1j}}-\frac{\eta_1}{n}}}{(2^n)^{(k+1)(\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{1j}}+1-2\beta)}}\right)$$
$$\times\prod_{i=1}^{2}\|b_i\|_{\mathrm{Lip}_\alpha(\mathbb{R}^n)}\|f_i\|_{L^{\vec{p_i},\eta_i,\psi}(\mathbb{R}^n)}$$
$$\lesssim\prod_{i=1}^{2}\|b_i\|_{\mathrm{Lip}_\alpha(\mathbb{R}^n)}\|f_i\|_{L^{\vec{p_i},\eta_i,\psi}(\mathbb{R}^n)}.$$

For H_3, similar to H_2, we obtain

$$H_3\lesssim\prod_{i=1}^{2}\|b_i\|_{\mathrm{Lip}_\alpha(\mathbb{R}^n)}\|f_i\|_{L^{\vec{p_i},\eta_i,\psi}(\mathbb{R}^n)}.$$

For H_4, according to (4.3), Lemmas 4.6-4.7 and Hölder's inequality, we have

$$|T_{\prod\vec{b}}(f_1\chi_{(2B(x,r))^c},f_2\chi_{(2B(x,r))^c})(z)|$$
$$\lesssim\sum_{k=1}^{\infty}\frac{1}{(\varphi(|2^kB(x,r)|)|2^kB(x,r)|)^{2(1-\beta)}}\prod_{i=1}^{2}\int_{2^{k+1}B(x,r)}|b_i(z)-b_i(y_i)||f_i(y_i)|\mathrm{d}y_i$$
$$\leq\sum_{k=1}^{\infty}\frac{1}{(\varphi(|2^kB(x,r)|)|2^kB(x,r)|)^{2(1-\beta)}}$$
$$\times\prod_{i=1}^{2}\bigg(\int_{2^{k+1}B(x,r)}|b_i(y_i)-(b_i)_{2^{k+1}B(x,r)}||f_i(y_i)|\mathrm{d}y_i$$
$$+\Big[|b_i(z)-(b_i)_{B(x,r)}|+|(b_i)_{B(x,r)}-(b_i)_{2^{k+1}B(x,r)}|\Big]\int_{2^{k+1}B(x,r)}|f_i(y_i)|\mathrm{d}y_i\bigg)$$
$$\leq\sum_{k=1}^{\infty}\frac{1}{(\varphi(|2^kB(x,r)|)|2^kB(x,r)|)^{2(1-\beta)}}$$
$$\times\prod_{i=1}^{2}\Big(\|(b_i-(b_i)_{2^{k+1}B(x,r)})\chi_{2^{k+1}B(x,r)}\|_{L^{\vec{p_i}'}(\mathbb{R}^n)}\|f_i\chi_{2^{k+1}B(x,r)}\|_{L^{\vec{p_i}}(\mathbb{R}^n)}$$
$$+\Big[|b_i(z)-(b_i)_{B(x,r)}|+|(b_i)_{B(x,r)}-(b_i)_{2^{k+1}B(x,r)}|\Big]$$
$$\times\|f_i\chi_{2^{k+1}B(x,r)}\|_{L^{\vec{p_i}}(\mathbb{R}^n)}\|\chi_{2^{k+1}B(x,r)}\|_{L^{\vec{p_i}'}(\mathbb{R}^n)}\Big)$$
$$\lesssim\prod_{i=1}^{2}\|b_i\|_{\mathrm{Lip}_\alpha(\mathbb{R}^n)}\|f_i\|_{L^{\vec{p_i},\eta_i,\psi}(\mathbb{R}^n)}\sum_{k=1}^{\infty}\frac{|2^{k+1}B(x,r)|^{\frac{2\alpha}{n}+2-\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{1j}}-\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{2j}}}}{(\varphi(|2^{k+1}B(x,r)|)|2^{k+1}B(x,r)|)^{2(1-\beta)}}$$

$$
\begin{aligned}
&\times [\psi(2^{k+1}r)]^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{1j}}+\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{2j}}-\frac{\eta_1+\eta_2}{n}} \\
&+\sum_{k=1}^{\infty}\frac{|2^{k+1}B(x,r)|^{2-\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{1j}}-\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{2j}}}}{(\varphi(|2^{k+1}B(x,r)|)|2^{k+1}B(x,r)|)^{2(1-\beta)}} \\
&\times\prod_{i=1}^{2}\|f_i\|_{L^{\vec{p_i},\eta_i,\psi}(\mathbb{R}^n)}\left[|b_i(z)-(b_i)_{B(x,r)}|+\|b_i\|_{\mathrm{Lip}_\alpha(\mathbb{R}^n)}|2^{k+1}B(x,r)|^{\frac{\alpha}{n}}\right] \\
&\times [\psi(2^{k+1}r)]^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{1j}}+\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{2j}}-\frac{\eta_1+\eta_2}{n}}.
\end{aligned} \tag{4.11}
$$

We may choose $\vec{v_i}$ such that $\frac{1}{\vec{q}}=\sum_{i=1}^{2}\frac{1}{\vec{v_i}}$, then $\sum_{j=1}^{n}\sum_{i=1}^{2}\frac{1}{v_{ij}}=\sum_{j=1}^{n}\sum_{i=1}^{2}\frac{1}{p_{ij}}-2n\beta-2\alpha$. Thus, using Lemma 4.7, Hölder's inequality, (4.11), $\frac{\eta_1+\eta_2}{n}\geq\frac{2n\beta+2\alpha}{n}=2\beta+\frac{2\alpha}{n}>2\beta$ and $0<C_\psi<2^n$, we have

$$
\begin{aligned}
H_4&=\sup_{x\in\mathbb{R}^n,r>0}\frac{\|T_{\prod\vec{b}}(f_1\chi_{(2B(x,r))^c},f_2\chi_{(2B(x,r))^c})\chi_{B(x,r)}\|_{L^{\vec{q}}(\mathbb{R}^n)}}{[\psi(r)]^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{q_j}-\frac{\eta}{n}}} \\
&\lesssim\prod_{i=1}^{2}\|b_i\|_{\mathrm{Lip}_\alpha(\mathbb{R}^n)}\|f_i\|_{L^{\vec{p_i},\eta_i,\psi}(\mathbb{R}^n)}\sup_{x\in\mathbb{R}^n,r>0}\|\chi_{B(x,r)}\|_{L^{\vec{q}}(\mathbb{R}^n)} \\
&\quad\times\sum_{k=1}^{\infty}[\varphi(|2^{k+1}B(x,r)|)]^{2(1-\beta)}\frac{|2^{k+1}B(x,r)|^{\frac{2\alpha}{n}+2-\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{1j}}-\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{2j}}}}{(\varphi(|2^{k+1}B(x,r)|)|2^{k+1}B(x,r)|)^{2(1-\beta)}} \\
&\quad\times\frac{[C_\psi^{k+1}\psi(r)]^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{1j}}+\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{2j}}-\frac{\eta_1+\eta_2}{n}}}{[\psi(r)]^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{q_j}-\frac{\eta}{n}}} \\
&\quad+\prod_{i=1}^{2}\|f_i\|_{L^{\vec{p_i},\eta_i,\psi}(\mathbb{R}^n)}\sup_{x\in\mathbb{R}^n,r>0}\|(b_1-(b_1)_{B(x,r)})\chi_{B(x,r)}\|_{L^{\vec{v_1}}(\mathbb{R}^n)} \\
&\quad\times\|(b_2-(b_2)_{B(x,r)})\chi_{B(x,r)}\|_{L^{\vec{v_2}}(\mathbb{R}^n)}\sum_{k=1}^{\infty}\frac{[C_\psi^{k+1}\psi(r)]^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{1j}}+\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{2j}}-\frac{\eta_1+\eta_2}{n}}}{[\psi(r)]^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{q_j}-\frac{\eta}{n}}} \\
&\quad\times\frac{|2^{k+1}B(x,r)|^{2-\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{1j}}-\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{2j}}}[\varphi(|2^{k+1}B(x,r)|)]^{2(1-\beta)}}{(\varphi(|2^{k+1}B(x,r)|)|2^{k+1}B(x,r)|)^{2(1-\beta)}} \\
&\lesssim\prod_{i=1}^{2}\|b_i\|_{\mathrm{Lip}_\alpha(\mathbb{R}^n)}\|f_i\|_{L^{\vec{p_i},\eta_i,\psi}(\mathbb{R}^n)}\sup_{x\in\mathbb{R}^n,r>0}|B(x,r)|^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{q_j}} \\
&\quad\times\sum_{k=1}^{\infty}\frac{|2^{k+1}B(x,r)|^{\frac{2\alpha}{n}+2-\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{1j}}-\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{2j}}}}{|2^{k+1}B(x,r)|^{2(1-\beta)}}[C_\psi^{k+1}]^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{1j}}+\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{2j}}-\frac{\eta_1+\eta_2}{n}} \\
&\quad+\prod_{i=1}^{2}\|b_i\|_{\mathrm{Lip}_\alpha(\mathbb{R}^n)}\|f_i\|_{L^{\vec{p_i},\eta_i,\psi}(\mathbb{R}^n)}\sup_{x\in\mathbb{R}^n,r>0}|B(x,r)|^{\frac{2\alpha}{n}+\frac{1}{n}\sum_{j=1}^{n}\frac{1}{v_{1j}}+\frac{1}{n}\sum_{j=1}^{n}\frac{1}{v_{2j}}} \\
&\quad\times\sum_{k=1}^{\infty}\frac{|2^{k+1}B(x,r)|^{2-\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{1j}}-\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{2j}}}}{|2^{k+1}B(x,r)|^{2(1-\beta)}}[C_\psi^{k+1}]^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{1j}}+\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{2j}}-\frac{\eta_1+\eta_2}{n}} \\
&\leq\left(\sum_{k=1}^{\infty}\frac{[C_\psi^{k+1}]^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{1j}}+\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{2j}}-\frac{\eta_1+\eta_2}{n}}}{(2^n)^{(k+1)(\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{1j}}+\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{2j}}-\frac{2\alpha}{n}-2\beta)}}+\sum_{k=1}^{\infty}\frac{[C_\psi^{k+1}]^{\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{1j}}+\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{2j}}-\frac{\eta_1+\eta_2}{n}}}{(2^n)^{(k+1)(\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{1j}}+\frac{1}{n}\sum_{j=1}^{n}\frac{1}{p_{2j}}-2\beta)}}\right)
\end{aligned}
$$

$$\times \prod_{i=1}^{2} \|b_i\|_{\mathrm{Lip}_\alpha(\mathbb{R}^n)} \|f_i\|_{L^{\vec{p_i},\eta_i,\psi}(\mathbb{R}^n)}$$
$$\lesssim \prod_{i=1}^{2} \|b_i\|_{\mathrm{Lip}_\alpha(\mathbb{R}^n)} \|f_i\|_{L^{\vec{p_i},\eta_i,\psi}(\mathbb{R}^n)}.$$

Therefore,

$$\|T_{\prod \vec{b}}(\vec{f})\|_{L^{\vec{q},\eta,\psi}(\mathbb{R}^n)} \lesssim \prod_{i=1}^{m} \|b_i\|_{\mathrm{Lip}_\alpha(\mathbb{R}^n)} \|f_i\|_{L^{\vec{p_i},\eta_i,\psi}(\mathbb{R}^n)}.$$

This finishes the proof of Theorem 4.3. □

Proof. [The proof of Theorem 4.4] Since $0 < \alpha < \min\{1, \frac{n}{m-1}\}$ and $0 \leq \beta < \min\{\frac{\gamma}{1+\gamma}, 1 - \frac{\alpha}{n}\}$, then $\frac{\alpha}{m} + n\beta + \frac{n}{m} - n = \frac{\alpha}{m} + \frac{n}{m} + n(\beta - 1) < \frac{\alpha}{m} + \frac{n}{m} + n \times (-\frac{\alpha}{n}) = \frac{\alpha}{m} + \frac{n}{m} - \alpha = (\frac{1}{m} - 1)\alpha + \frac{n}{m} < \frac{1-m}{m} \times \frac{n}{m-1} + \frac{n}{m} = 0$. Thus, $\frac{\alpha}{m} + n\beta + \frac{n}{m} < n$. And because $\alpha + n\beta \leq \eta_i < \frac{\alpha}{m} + n\beta + \frac{n}{m}$ and $\eta = \sum_{i=1}^{m} \eta_i - mn\beta - \alpha$, then $\alpha + n\beta \leq \eta_i < n$ and $m\alpha - \alpha \leq \eta < \alpha + mn\beta + n - mn\beta - \alpha = n$. Thus, it is not hard to get the desired result by applying the similar arguments as F_1 and F_3 in the proof of Theorem 4.2 and from H_1 to H_4 in the proof of Theorem 4.3. The details are omitted here. □

5. Some applications

In this section, we will show that the multilinear fractional new maximal operator $\mathcal{M}_{\varphi,\beta}$ and its commutators $\mathcal{M}_{\vec{b},\varphi,\beta}$ and $[\vec{b}, \mathcal{M}_{\varphi,\beta}]$ are bounded from the product of spaces $L^{\vec{p_1},\eta_1,\psi}(\mathbb{R}^n) \times \cdots \times L^{\vec{p_m},\eta_m,\psi}(\mathbb{R}^n)$ to space $L^{\vec{q},\eta,\psi}(\mathbb{R}^n)$, which can be seen as the applications of Theorem 3.1 and Theorems 4.1-4.4. First, the definition of the commutators of $\mathcal{M}_{\varphi,\beta}$ is given as follows.

Definition 5.1. *[15] Let $0 \leq \beta < 1$, $\vec{f} = (f_1, \cdots, f_m)$ and $\vec{b} = (b_1, \cdots, b_m)$. Then the fractional new maximal commutator of $\mathcal{M}_{\varphi,\beta}$ with $\vec{b}$ is defined by*

$$\mathcal{M}_{\vec{b},\varphi,\beta}(\vec{f})(x) = \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{(\varphi(|B(x,r)|)|B(x,r)|)^{m(1-\beta)}} \int_{B(x,r)^m} \prod_{i=1}^{m} |b_i(x) - b_i(y_i)||f_i(y_i)| d\vec{y},$$

and the multilinear fractional commutator of $\mathcal{M}_{\varphi,\beta}$ with $\vec{b}$ is defined by

$$[\vec{b}, \mathcal{M}_{\varphi,\beta}](\vec{f})(x) = \sum_{i=1}^{m} [\vec{b}, \mathcal{M}_{\varphi,\beta}]_i(\vec{f})(x),$$

where

$$[\vec{b}, \mathcal{M}_{\varphi,\beta}]_i(\vec{f})(x) = b_i(x)\mathcal{M}_{\varphi,\beta}(\vec{f})(x) - \mathcal{M}_{\varphi,\beta}(f_1, \cdots, b_i f_i, \cdots, f_m)(x), \quad \text{for } i = 1, \cdots, m.$$

Let $0 \leq \beta < 1$ and $\varpi \in C^\infty(\mathbb{R}^+)$ such that $|\varpi'(t)| \lesssim \frac{1}{t}$ and $\chi_{[0,1]}(t) \leq \varpi(t) \leq \chi_{[0,2]}(t)$. For any $r > 0$, set

$$K_\beta(x, y_1, \cdots, y_m) = \frac{1}{(\varphi(|B(x,r)|)|B(x,r)|)^{m(1-\beta)}} \varpi\left(\frac{|x - y_1| + \cdots + |x - y_m|}{r}\right).$$

From [15], we know that (1.1)-(1.3) hold for $K_\beta(x, y_1, \cdots, y_m)$, where $N = m\gamma(1-\beta)$ and $h\left(\frac{|x-x'|}{\sum_{i=1}^m |x-y_i|}\right) = \frac{|x-x'|}{\sum_{i=1}^m |x-y_i|}$. Then the related maximal operators of T, $T_{\prod \vec{b}}$ and $T_{\sum \vec{b}}$ are defined by

$$T^*(\vec{f})(x) = \sup_{x\in\mathbb{R}^n, r>0} \int_{(\mathbb{R}^n)^m} K_\beta(x, y_1, \cdots, y_m) \prod_{i=1}^{m} |f_i(y_i)| d\vec{y},$$

$$T^*_{\prod\vec{b}}(\vec{f})(x)=\sup_{x\in\mathbb{R}^n,r>0}\int_{(\mathbb{R}^n)^m}\prod_{i=1}^m|b_i(x)-b_i(y_i)|K_\beta(x,y_1,\cdots,y_m)|f_i(y_i)|\mathrm{d}\vec{y},$$

and

$$T^*_{\sum\vec{b}}(\vec{f})(x)=\sum_{i=1}^m T^{*,i}_{b_i}(\vec{f})(x),\quad T^{*,i}_{b_i}(\vec{f})(x)=\sup_{x\in\mathbb{R}^n,r>0}T^i_{b_i}(\vec{f})(x).$$

It is not hard to verify that the conclusions of Theorems 3.1 and 4.1-4.4 are still valid for T^*, $T^*_{\prod\vec{b}}$ and $T^*_{\sum\vec{b}}$. In addition, from [15], we also know that $T^*(\vec{f})(x)\approx\mathcal{M}_{\varphi,\beta}(\vec{f})(x)$, $T^*_{\prod\vec{b}}(\vec{f})(x)\approx\mathcal{M}_{\vec{b},\varphi,\beta}(\vec{f})(x)$, $T^{*,i}_{b_i}(\vec{f})(x)\approx[\vec{b},\mathcal{M}_{\varphi,\beta}]_i(\vec{f})(x)$ and $T^*_{\sum\vec{b}}(\vec{f})(x)\approx[\vec{b},\mathcal{M}_{\varphi,\beta}](\vec{f})(x)$. Thus, it is easy to check that the following conclusions are correct.

Theorem 5.2. *Suppose that $0\le\gamma\le m-1$, $m\ge 2$, $0\le\beta\le\frac{\gamma}{1+\gamma}$ and $0<C_\psi<2^n$. If $h\in\mathrm{Dini}(1)$, $n\beta\le\eta_i<\frac{n}{m}+n\beta$ $(i=1,\cdots,m)$, $\eta=\sum_{i=1}^m\eta_i-mn\beta$, $\sum_{j=1}^n\frac{1}{q_j}=\sum_{j=1}^n\sum_{i=1}^m\frac{1}{p_{ij}}-mn\beta$ with $\sum_{j=1}^n\frac{1}{p_{ij}}>\eta_i$ and $1<\vec{q},\vec{p_i}<\infty$, then $\mathcal{M}_{\varphi,\beta}$ is bounded from $L^{\vec{p_1},\eta_1,\psi}(\mathbb{R}^n)\times\cdots\times L^{\vec{p_m},\eta_m,\psi}(\mathbb{R}^n)$ to $L^{\vec{q},\eta,\psi}(\mathbb{R}^n)$, i.e.*

$$\|\mathcal{M}_{\varphi,\beta}(\vec{f})\|_{L^{\vec{q},\eta,\psi}(\mathbb{R}^n)}\lesssim\prod_{i=1}^m\|f_i\|_{L^{\vec{p_i},\eta_i,\psi}(\mathbb{R}^n)}.$$

The estimates for $\mathcal{M}_{\vec{b},\varphi,\beta}$ and $[\vec{b},\mathcal{M}_{\varphi,\beta}]$ associated with $\mathrm{BMO}_{\varphi,\beta}$ functions on space $L^{\vec{q},\eta,\psi}(\mathbb{R}^n)$ are established as follows.

Theorem 5.3. *Let $\vec{b}=(b_1,\cdots,b_m)$, $b_i\in\mathrm{BMO}_{\varphi,\beta}$, $i=1,\cdots,m$, $0\le\beta<\min\{\frac{\gamma}{1+\gamma},\frac{1}{m}\}$ with $0\le\gamma<\infty$, $h\in\mathrm{Dini}(1)$, $n\beta\le\eta_i<\frac{n}{m}$ $(i=1,\cdots,m)$, $\eta=\sum_{i=1}^m\eta_i$, $\sum_{j=1}^n\frac{1}{q_j}=\sum_{j=1}^n\sum_{i=1}^m\frac{1}{p_{ij}}$ with $m\ge 2$, $\sum_{j=1}^n\frac{1}{p_{ij}}>\eta_i$, $1<\vec{q},\vec{p_i}<\infty$ and $0<C_\psi<2^n$. Then*

$$\|\mathcal{M}_{\vec{b},\varphi,\beta}(\vec{f})\|_{L^{\vec{q},\eta,\psi}(\mathbb{R}^n)}\lesssim\prod_{i=1}^m\|b_i\|_{\mathrm{BMO}_{\varphi,\beta}}\|f_i\|_{L^{\vec{p_i},\eta_i,\psi}(\mathbb{R}^n)}.$$

Theorem 5.4. *Assume that $\vec{b}=(b_1,\cdots,b_m)$, $b_i\in\mathrm{BMO}_{\varphi,\beta}$, $i=1,\cdots,m$, $0\le\beta\le\frac{\gamma}{1+\gamma}$ with $0\le\gamma<\infty$, $h\in\mathrm{Dini}(1)$, $\sum_{j=1}^n\frac{1}{q_j}=\sum_{j=1}^n\sum_{i=1}^m\frac{1}{p_{ij}}-mn\beta+n\beta$ with $m\ge 2$, $\sum_{j=1}^n\frac{1}{p_{ij}}>\eta_i$ $(i=1,\cdots,m)$, $1<\vec{q},\vec{p_i}<\infty$, $n\beta\le\eta_i<n\beta+\frac{n}{m}-\frac{n\beta}{m}$, $\eta=\sum_{i=1}^m\eta_i-mn\beta+n\beta$, $0<C_\psi<2^n$ and $\|\vec{b}\|_{\mathrm{BMO}_{\varphi,\beta}}=\max\{\|b_1\|_{\mathrm{BMO}_{\varphi,\beta}},\cdots,\|b_m\|_{\mathrm{BMO}_{\varphi,\beta}}\}$. Then*

$$\|[\vec{b},\mathcal{M}_{\varphi,\beta}](\vec{f})\|_{L^{\vec{q},\eta,\psi}(\mathbb{R}^n)}\lesssim\|\vec{b}\|_{\mathrm{BMO}_{\varphi,\beta}}\prod_{i=1}^m\|f_i\|_{L^{\vec{p_i},\eta_i,\psi}(\mathbb{R}^n)}.$$

Next, we give the boundedness for $\mathcal{M}_{\vec{b},\varphi,\beta}$ and $[\vec{b},\mathcal{M}_{\varphi,\beta}]$ associated with Lipschitz functions on space $L^{\vec{q},\eta,\psi}(\mathbb{R}^n)$.

Theorem 5.5. *Let $\vec{b}=(b_1,\cdots,b_m)$, $b_i\in\mathrm{Lip}_\alpha(\mathbb{R}^n)$, $i=1,\cdots,m$, $0<\alpha<1$, $0\le\beta<1-\frac{\alpha}{n}$, $\alpha+n\beta\le\eta_i<\min\{\alpha+n\beta+\frac{n}{m},n\}$ $(i=1,\cdots,m)$, $\eta=\sum_{i=1}^m\eta_i-mn\beta-m\alpha$ and $0<C_\psi<2^n$. If $\sum_{j=1}^n\frac{1}{q_j}=\sum_{j=1}^n\sum_{i=1}^m\frac{1}{p_{ij}}-mn\beta-m\alpha$, $\sum_{j=1}^n\frac{1}{p_{ij}}>\eta_i$ and $1<\vec{q},\vec{p_i}<\infty$, then*

$$\|\mathcal{M}_{\vec{b},\varphi,\beta}(\vec{f})\|_{L^{\vec{q},\eta,\psi}(\mathbb{R}^n)}\lesssim\prod_{i=1}^m\|b_i\|_{\mathrm{Lip}_\alpha(\mathbb{R}^n)}\|f_i\|_{L^{\vec{p_i},\eta_i,\psi}(\mathbb{R}^n)}.$$

Theorem 5.6. *Suppose that* $\vec{b} = (b_1, \cdots, b_m)$, $b_i \in \mathrm{Lip}_\alpha(\mathbb{R}^n)$, $i = 1, \cdots, m$, $0 < \alpha < \min\{1, \frac{n}{m-1}\}$, $0 \le \beta < \min\{\frac{\gamma}{1+\gamma}, 1 - \frac{\alpha}{n}\}$ *with* $0 \le \gamma < \infty$, $h \in \mathrm{Dini}(1)$ *and* $0 < C_\psi < 2^n$. *If* $\alpha + n\beta \le \eta_i < \frac{\alpha}{m} + n\beta + \frac{n}{m}$ $(i = 1, \cdots, m)$, $\eta = \sum_{i=1}^m \eta_i - mn\beta - \alpha$, $\sum_{j=1}^n \frac{1}{q_j} = \sum_{j=1}^n \sum_{i=1}^m \frac{1}{p_{ij}} - mn\beta - \alpha$, $m \ge 2$, $\|\vec{b}\|_{\mathrm{Lip}_\alpha(\mathbb{R}^n)} = \max\{\|b_1\|_{\mathrm{Lip}_\alpha(\mathbb{R}^n)}, \cdots, \|b_m\|_{\mathrm{Lip}_\alpha(\mathbb{R}^n)}\}$, $\sum_{j=1}^n \frac{1}{p_{ij}} > \eta_i$ *and* $1 < \vec{q}, \vec{p_i} < \infty$, *then*

$$\|[\vec{b}, \mathcal{M}_{\varphi,\beta}](\vec{f})\|_{L^{\vec{q},\eta,\psi}(\mathbb{R}^n)} \lesssim \|\vec{b}\|_{\mathrm{Lip}_\alpha(\mathbb{R}^n)} \prod_{i=1}^m \|f_i\|_{L^{\vec{p_i},\eta_i,\psi}(\mathbb{R}^n)}.$$

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