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Precompactness in matrix weighted Bourgain-Morrey spaces

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Abstract. In this paper, we introduce matrix weighted Bourgain-Morrey spaces and obtain two sufficient conditions for precompact sets in matrix weighted Bourgain-Morrey spaces. We prove that the dyadic average operator is bounded on some matrix weighted Bourgain-Morrey spaces. With this result, we obtain the necessity for precompact sets in some matrix weighted Bourgain-Morrey spaces. The results are new even for the unweighted Bourgain-Morrey spaces.

1. Introduction

In 1931, Kolmogorov [22] first discovered the characterization of precompact sets in $L^p([0,1])$ for $p \in (1,\infty)$. After that, there are many criteria for compactness of sets in Lebesgue spaces, which are called the Kolmogorov-Riesz compactness theorems on the Lebesgue spaces. More details and the history, we refer the reader to [16].

Inspired by [16], Clop-Cruz [9] obtained a compactness criterion in scalar weighted Lebesgue spaces $L^p(\omega)$ for $1 < p < \infty$ with a scalar wight ω in Muckenhoupt class A_p. In [15], Guo and Zhao improved the result in [9] and obtained a compactness criterion in $L^p(\omega)$ for $p \in (0,\infty)$ with locally integrable weight ω. In [24], Liu, Yang and Zhuo proved the Kolmogorov-Riesz compactness theorem in matrix weighted Lebesgue spaces $L^p(W)$ with $1 < p < \infty$.

The theory of matrix weighted function spaces goes back to [35]. Indeed, in 1958, Wiener and Masani [35, Section 4] studied the matrix weighted $L^2(W)$ for the prediction theory for multivariate stochastic processes. In [32], Treil and Volberg introduced matrix class $\mathcal{A}_2$. Nazarov and Treil [26] and Volberg [33] extended $\mathcal{A}_2$ to $\mathcal{A}_p$ with $p \in (1,\infty)$. In [13], Goldberg showed that the matrix $\mathcal{A}_p$ condition leads to L^p boundedness of a Hardy-Littlewood maximal function and obtained the boundedness of matrix weighted singular integral operators in Lebesgue spaces $L^p, 1 < p < \infty$. In [27–29], Roudenko introduced the matrix-weighted homogeneous Besov spaces $\dot{B}^{s,q}_p(W)$ and matrix-weighted sequence Besov spaces $\dot{b}^{s,q}_p(W)$ and showed their equivalence via φ-transform and wavelets. In [11], Frazier and Roudenko introduced the matrix class

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$\mathcal{A}_p$, $(0 < p \le 1)$ and studied the continuous and discrete matrix-weighted Besov spaces $\dot{B}_p^{s,q}(W)$ and $\dot{b}_p^{s,q}(W)$ with $0 < p \le 1$. In [12], Frazier and Roudenko introduced the homogeneous matrix-weighted Triebel-Lizorkin spaces $\dot{F}_p^{s,q}(W)$ for $s \in \mathbb{R}, 0 < p < \infty, 0 < q \le \infty$ and obtained the Littlewood-Paley characterizations of matrix-weighted Lebesgue spaces $L^p(W)$ and matrix-weighted Sobolev space $L_k^p(W)$ for $k \in \mathbb{N}, 1 < p < \infty$. In [33], Volberg also introduced an analogue condition for the matrix class $\mathcal{A}_\infty$. In [3–5], Bu, Hytönen, Yang, Yuan studied the matrix weighted Besov-type and Triebel-Lizorkin-type spaces. Specifically, they introduced a new concept of the $\mathcal{A}_p$-dimension $\tilde{d}$, which is useful to the proof of main results of this paper. In [7], Bu, Hytönen, Yang, Yuan obtained several new characterizations of $\mathcal{A}_{p,\infty}$-matrix weights. In [8], Bu et al. studied the inhomogeneous Besov-type and Triebel–Lizorkin-type spaces with the result in [7]. In [2], Bu et al. introduced the matrix weighted Hardy spaces and obtained characterizations of these spaces via maximal function, atom. As applications, they established the finite atomic characterization of matrix weighted Hardy spaces and obtained a criterion on the boundedness of sublinear operators from matrix weighted Hardy spaces to any γ-quasi-Banach space ($\gamma \in (0, 1]$). The boundedness of Calderón-Zygmund operators on matrix weighted Hardy spaces was also obtained. In [23], Li, Yang and Yuan introduced the matrix-weighted Besov-Triebel-Lizorkin spaces with logarithmic smoothness and characterize these spaces via Peetre-type maximal functions. In [37, 38], Zhao et al. introduced (generalized grand) Besov-Bourgain-Morrey spaces and explored various real-variable properties of these spaces, which are a bridge connecting Bourgain-Morrey spaces with amalgam-type spaces. Moreover, some real-variable properties and boundedness of classical operators were studied in their article. For many other results on the matrix class $\mathcal{A}_p$ and matrix weighted function spaces, we refer the reader to [6, 10–13, 28, 29, 34].

Bourgain [1] introduced a special case of Bourgain-Morrey spaces to study the Stein-Tomas (Strichartz) estimate. In [18], Hatano, Nogayama, Sawano, and Hakim researched the Bourgain-Morrey spaces from the viewpoints of harmonic analysis and functional analysis. In [21], Hu, Li and Yang introduced the Triebel-Lizorkin-Bourgain-Morrey spaces which connect Bourgain-Morrey spaces and global Morrey spaces.

Motivated by above literature, we will introduce matrix weighted Bourgain-Morrey spaces and research precompact sets in these spaces. The paper is organized as follows. In Section 2, dyadic cubes, the matrix class $\mathcal{A}_p$, $\mathcal{A}_p$-dimension $\tilde{d}$, matrix weighted Bourgain-Morrey spaces are given. The first result lies in Section 3. Specifically, a sufficient condition for totally bounded set in matrix weighted spaces $M_p^{t,r}(W)$ with $1 \le p < r < \infty$ or $1 \le p \le t < r = \infty$ is obtained in Theorem 3.2. As a application, we get a criterion for totally bounded set in degenerate Bourgain-Morrey spaces with matrix weight. The second result (Theorem 4.1) is replacing the translation operator by the average operator. Note that the translation operator is not bounded on $L^p(W)$ and $M_p^{t,r}(W)$ in general. We prove that dyadic average operator is bounded on matrix weighted Bourgain-Morrey spaces with some conditions in Theorem 4.6. Using this result, we obtain the Kolmogorov-Riesz compactness theorem in matrix weighted Bourgain-Morrey spaces. These results are new even for the unweighted Bourgain-Morrey spaces.

Throughout this paper, we let c, C denote constants that are independent of the main parameters involved but whose value may differ from line to line. Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ and $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$. Let $\mathbb{Z}$ be the set of integers. Let χ_E be the characteristic function of the set $E \subset \mathbb{R}^n$. By $A \lesssim B$ we mean that $A \le CB$ with some positive constant C independent of appropriate quantities. By $A \approx B$, we mean that $A \lesssim B$ and $B \lesssim A$.

2. Preliminaries

For $j \in \mathbb{Z}$, $m \in \mathbb{Z}^n$, let $Q_{j,m} := \prod_{i=1}^n [2^{-j} m_i, 2^{-j}(m_i + 1))$. For a cube Q, $\ell(Q)$ stands for the length of cube Q. We denote by $\mathcal{D}$ the the family of all dyadic cubes in $\mathbb{R}^n$, while $\mathcal{D}_j$ is the set of all dyadic cubes with $\ell(Q) = 2^{-j}, j \in \mathbb{Z}$. Let x_Q be the lower left corner of $Q \in \mathcal{D}$. For $\lambda > 0$, let λQ be the cube with the same center of Q and the edge length $\lambda \ell(Q)$. For $k \in \mathbb{N}$, let $k_{\mathrm{pa}}Q$ be the k-th dyadic parent of Q, which is the dyadic cube in $\mathcal{D}$ satisfying $Q \subset k_{\mathrm{pa}}Q$ and $\ell(k_{\mathrm{pa}}Q) = 2^k \ell(Q)$.

2.1. Matrix weights

First, we recall some basic concepts and results from the theory of matrix weights.

For any $d, n \in \mathbb{N}$, denote by $M_{d,n}(\mathbb{C})$ the set of all $d \times n$ complex-valued matrices. $M_{d,d}(\mathbb{C})$ is simply denoted by $M_d(\mathbb{C})$. The zero matrix in $M_{d,n}(\mathbb{C})$ (or $M_d(\mathbb{C})$) is denoted by $O_{d,n}$ (or O_d). Denote by A^* the conjugate transpose of $A \in M_{d,n}(\mathbb{C})$. A matrix $A \in M_d(\mathbb{C})$ is called a *Hermitian matrix* if $A = A^*$ and is called a *unitary matrix* if $A^*A = I_d$ where I_d is the identity matrix. We denote a diagonal matrix by $\operatorname{diag}(\lambda_1, \dots, \lambda_d) = \operatorname{diag}(\lambda_i)$.

For a vector $x \in \mathbb{C}^d$, let $|x| = (\sum_{i=1}^d |x_i|^2)^{1/2}$. For $1 \le p < \infty$, let $|x|_p = (\sum_{i=1}^d |x_i|^p)^{1/p}$. For $p = \infty$, let $|x|_\infty = \max(x_1, \dots, x_d)$. In the finite dimension space, the norms are equivalent. That is, for $1 \le p < q \le \infty$,

$$|x|_q \le |x|_p \le d^{1/p}|x|_\infty \le d^{1/p}|x|_q. \tag{1}$$

For $A \in M_d(\mathbb{C})$, let

$$\|A\| := \sup_{\vec{z} \in \mathbb{C}^d, |\vec{z}|=1} \left|A\vec{z}\right|.$$

We say that a matrix $A \in M_d(\mathbb{C})$ is *positive definite* if, for any $\vec{x} \in \mathbb{C}^d \setminus \{\vec{0}\}$, $\vec{z}^* A\vec{z} > 0$. And a matrix $A \in M_d(\mathbb{C})$ is called *nonnegative definite* if, for any $\vec{x} \in \mathbb{C}^d$, $\vec{z}^* A\vec{z} \ge 0$.

From [20, Theorem 4.1.4], any nonnegative definite matrix is always Hermitian. Hence any nonnegative definite matrix is self-adjoint.

Let $A \in M_d(\mathbb{C})$ be a positive definite matrix and have eigenvalues $\{\lambda_i\}_{i=1}^d$. From [20, Theorem 2.5.6(c)], there exists a unitary matrix $U \in M_d(\mathbb{C})$ such that

$$A = U \operatorname{diag}(\lambda_1, \dots, \lambda_d) U^*. \tag{2}$$

Moreover, by [20, Theorem 4.1.8], we find $\{\lambda_i\}_{i=1}^d \subset (0, \infty)$. The following definition is based on these conclusions.

Definition 2.1. *Let $A \in M_d(\mathbb{C})$ be a positive definite matrix with positive eigenvalues $\{\lambda_i\}_{i=1}^d$. For any $\alpha \in \mathbb{R}$, define*

$$A^\alpha := U \operatorname{diag}\left(\lambda_1^\alpha, \dots, \lambda_m^\alpha\right) U^*, \tag{3}$$

where U is the same as in (2).

Remark 2.2. *From [19, p. 408], we obtain that A^α is independent of the choices of the order of $\{\lambda_i\}_{i=1}^m$ and U, and hence A^α is well defined.*

Now, we recall some concepts of matrix wights.

Definition 2.3. *A matrix-valued function $W : \mathbb{R}^n \to M_d(\mathbb{C})$ is called a matrix weight if W satisfies that*
(i) *for any $x \in \mathbb{R}^n$, $W(x)$ is nonnegative definite;*
(ii) *for almost every $x \in \mathbb{R}^n$, $W(x)$ is invertible;*
(iii) *the entries of W is locally integrable.*

Definition 2.4. *Let $p \in (0, \infty)$, W be a matrix weight. Suppose that $E \subset \mathbb{R}^n$ is a bounded measurable set satisfying $0 < |E| < \infty$. Then the matrix $A_E \in M_d(\mathbb{C})$ is called a reducing operator of order p for W if A_E is positive definite and, for any $\vec{z} \in \mathbb{C}^d$,*

$$\left|A_E\vec{z}\right| \approx \left(\frac{1}{|E|} \int_E \left|W^{\frac{1}{p}}(x)\vec{z}\right|^p \, \mathrm{d}x\right)^{\frac{1}{p}}, \tag{4}$$

where the positive equivalence constants depend only on d and p.

Next we recall the concepts of scalar weight class A_p (see [14, Definitions 7.1.1, 7.1.3]) and matrix weight class $\mathcal{A}_p$ (see [27] for $1 < p < \infty$, [11] for $0 < p \le 1$).

Definition 2.5. *A weight ω is a nonnegative locally integrable function on $\mathbb{R}^n$ such that $0 < \omega(x) < \infty$ for almost all $x \in \mathbb{R}^n$.*

A weight ω is called an A_1 weight if $\mathcal{M}(\omega)(x) \le c\omega(x)$ for almost all $x \in \mathbb{R}^n$ where $\mathcal{M}$ is the Hardy-Littlewood maximal operator.

For $1 < p < \infty$, a weight ω is said to be of class A_p if

$$\sup_{Q \text{ cubes in } \mathbb{R}^n} \left(\frac{1}{|Q|}\int_Q \omega(x)\mathrm{d}x\right)\left(\frac{1}{|Q|}\int_Q \omega(x)^{-1/(p-1)}\mathrm{d}x\right)^{p-1} < \infty.$$

For $1 < p < \infty$, a matrix weight $W \in \mathcal{A}_p(\mathbb{R}^n)$ if and only if

$$\sup_Q \frac{1}{|Q|}\int_Q \left(\frac{1}{|Q|}\int_Q \|W^{1/p}(x)W^{-1/p}(y)\|^{p'}\mathrm{d}y\right)^{p/p'} \mathrm{d}x < \infty,$$

where $p' = p/(p-1)$ is the conjugate index of p, and the supremum is taken over all cubes $Q \subset \mathbb{R}^n$.

For $0 < p \le 1$, a matrix weight $W \in \mathcal{A}_p(\mathbb{R}^n)$ if and only if

$$\sup_Q \operatorname*{ess\,sup}_{y \in Q} \frac{1}{|Q|}\int_Q \|W^{1/p}(x)W^{-1/p}(y)\|^p \mathrm{d}x < \infty.$$

We write $\mathcal{A}_p := \mathcal{A}_p(\mathbb{R}^n)$ for brevity.

Given any matrix weight W and $0 < p < \infty$, there exists (see e.g., [13, Proposition 1.2] for $p > 1$ and [11, p.1237] for $0 < p \le 1$) a sequence $\{A_Q\}_{Q \in \mathcal{D}}$ of positive definite $d \times d$ matrices such that

$$c_1|A_Q\vec{y}| \le \Big(\frac{1}{|Q|}\int_Q |W^{1/p}(x)\vec{y}|^p \mathrm{d}x\Big)^{1/p} \le c_2|A_Q\vec{y}|,$$

with positive constants c_1, c_2 independent of $\vec{y} \in \mathbb{C}^d$ and $Q \in \mathcal{D}$. In this case, we call $\{A_Q\}_{Q \in \mathcal{D}}$ a sequence of reducing operators of order p for W.

Definition 2.6. *A matrix weight W is called a doubling matrix weight of order $p > 0$ if the scalar measures $w_{\vec{y}}(x) = |W^{1/p}(x)\vec{y}|^p$, for $\vec{y} \in \mathbb{C}^d$, are uniformly doubling: there exists $c > 0$ such that for all cubes $Q \subset \mathbb{R}^n$ and all $\vec{y} \in \mathbb{C}^d$,*

$$\int_{2Q} w_{\vec{y}}(x)dx \le c\int_Q w_{\vec{y}}(x)\mathrm{d}x.$$

If $c = 2^\beta$ is the smallest constant for which this inequality holds, we say that β is the doubling exponent of W.

From [17, Proposition 2.10], we know that β is always not less than n.

In [3], Bu, Hytönen, Yang and Yuan introduced the $\mathcal{A}_p$-dimension of matrix weights, which will be used in Theorems 4.6 and 4.7.

Definition 2.7. *Let $0 < p < \infty$, $\tilde{d} \in \mathbb{R}$. A matrix weight W has the $\mathcal{A}_p$-dimension $\tilde{d}$, denoted by $W \in \mathbb{D}_{p,\tilde{d}}(\mathbb{R}^n, \mathbb{C}^n)$, if there exists a positive constant C such that for any cube $Q \subset \mathbb{R}^n$ and any $i \in \mathbb{N}_0$,*

$$\operatorname*{ess\,sup}_{y \in 2^iQ} \frac{1}{|Q|}\int_Q \left\|W^{1/p}(x)W^{-1/p}(y)\right\|^p \mathrm{d}x \le C2^{i\tilde{d}}, \quad \text{for } 0 < p \le 1, \tag{5}$$

or,

$$\frac{1}{|Q|}\int_Q \left(\frac{1}{|2^iQ|}\int_{2^iQ} \left\|W^{1/p}(x)W^{-1/p}(y)\right\|^{p'} \mathrm{d}y\right)^{p/p'} \mathrm{d}x \le C2^{i\tilde{d}}, \quad \text{for } 1 < p < \infty,$$

where $1/p + 1/p' = 1$.

We denote $\mathbb{D}_{p,\tilde{d}}(\mathbb{R}^n,\mathbb{C}^n)$ simply by $\mathbb{D}_{p,\tilde{d}}$.

The following lemma says that if $W \in \mathcal{A}_p$ for $p \in (0,\infty)$, then $W \in \mathbb{D}_{p,\tilde{d}}(\mathbb{R}^n,\mathbb{C}^n)$.

Lemma 2.8 (Proposition 2.27, [3]). *Let $p \in (0,\infty)$ and $W \in \mathcal{A}_p$. Then there exists $\tilde{d} \in [0,n)$ such that W has the $\mathcal{A}_p$-dimension $\tilde{d}$.*

Lemma 2.9 (Corollary 2.32, [3]). *Let $0 < p < \infty$, let $W \in \mathcal{A}_p$ with the $\mathcal{A}_p$-dimension $\tilde{d} \in [0,n)$, and let $\{A_Q\}_{\mathrm{cube}\,Q}$ be the reducing operator of order p for W.* (i) *If $1 < p < \infty$, let $\widetilde{W} := W^{-1/(p-1)}$ (which belongs to $\mathcal{A}_{p'}$) with the $\mathcal{A}_{p'}$-dimension $\tilde{\tilde{d}}$. Then there exists a positive constant C such that, for any cubes Q and R of $\mathbb{R}^n$,*

$$\|A_Q A_R^{-1}\| \le C \max\left(\left[\frac{\ell(R)}{\ell(Q)}\right]^{\tilde{d}/p}, \left[\frac{\ell(Q)}{\ell(R)}\right]^{\tilde{\tilde{d}}/p'}\right)\left[1 + \frac{|x_Q - x_R|}{\max(\ell(Q),\ell(R))}\right]^{\tilde{d}/p+\tilde{\tilde{d}}/p'}.$$

(ii) *If $0 < p \le 1$, then there exists a positive constant C such that, for any cubes Q and R of $\mathbb{R}^n$,*

$$\|A_Q A_R^{-1}\| \le C \max\left(\left[\frac{\ell(R)}{\ell(Q)}\right]^{\tilde{d}/p}, 1\right)\left[1 + \frac{|x_Q - x_R|}{\max(\ell(Q),\ell(R))}\right]^{\tilde{d}/p}.$$

2.2. *Matrix weighted Bourgain-Morrey spaces*

Definition 2.10. *Let $0 < p < \infty$, $d \in \mathbb{N}$ and $W : \mathbb{R}^n \to M_d(\mathbb{C})$ be a matrix weight. The space $L^p_{\mathrm{loc}}(W)$ collects all measurable functions $\vec{f} : \mathbb{R}^n \to \mathbb{C}^d$ such that for each compact set K,*

$$\|\vec{f}\chi_K\|_{L^p(W)} = \left(\int_K |W^{1/p}(x)\vec{f}(x)|^p \mathrm{d}x\right)^{1/p} < \infty.$$

Let $\Omega \subset \mathbb{R}^n$ be an open set. The space $L^p(W,\Omega)$ collects all measurable functions $\vec{f} : \Omega \to \mathbb{C}^d$ such that

$$\|\vec{f}\|_{L^p(W,\Omega)} = \left(\int_\Omega |W^{1/p}(x)\vec{f}(x)|^p \mathrm{d}x\right)^{1/p} < \infty.$$

Definition 2.11. *Let $\mathcal{D} = \{Q_{j,k}\}_{j\in\mathbb{Z},k\in\mathbb{Z}^n}$ be the standard dyadic system. Let $0 < p < t < r < \infty$ or $0 < p \le t < r = \infty$. Let $W : \mathbb{R}^n \to M_d(\mathbb{C})$ be a matrix weight. Define $M^{t,r}_p(W)$ as the set of all $\vec{f} \in L^p_{\mathrm{loc}}(W)$ such that*

$$\|\vec{f}\|_{M^{t,r}_p(W)} := \left\|\left\{W(Q_{j,k})^{1/t-1/p}\left(\int_{Q_{j,k}} |W^{1/p}(y)\vec{f}(y)|^p \mathrm{d}y\right)^{1/p}\right\}_{j\in\mathbb{Z},k\in\mathbb{Z}^n}\right\|_{\ell^r} < \infty,$$

where $W(Q_{j,k}) = \int_{Q_{j,k}} \|W(y)\|\mathrm{d}y$.

Let $\{A_Q\}_{Q\in\mathcal{D}}$ be the reducing operator of order p for W. Define $M^{t,r}_p(\{A_Q\})$ as the set of all $\vec{f} \in L^p_{\mathrm{loc}}(W)$ such that

$$\|\vec{f}\|_{M^{t,r}_p(\{A_Q\})} := \left\|\left\{\left(\|A_{Q_{j,k}}\|^p |Q_{j,k}|\right)^{1/t-1/p}\left(\int_{Q_{j,k}} |W^{1/p}(x)\vec{f}(y)|^p \mathrm{d}y\right)^{1/p}\right\}_{j\in\mathbb{Z},k\in\mathbb{Z}^n}\right\|_{\ell^r} < \infty.$$

Remark 2.12. *By [3, Lemma 2.11] with $M = I_m$, we conclude that, for any cube $Q \subset \mathbb{R}^n$,*

$$\|A_Q\|^p \approx \frac{1}{|Q|}\int_Q \left\|W^{1/p}(x)\right\|^p \mathrm{d}x = \frac{1}{|Q|}\int_Q \|W(x)\|\,\mathrm{d}x = \frac{1}{|Q|}W(Q).$$

Thus $M^{t,r}_p(W)$ is same with $M^{t,r}_p(\{A_Q\})$ in meaning of the equivalent quasi-norms.

If $d = 1, W \equiv 1$, $M_p^{t,r}(W)$ is the classical Bourgain-Morrey space $M_p^{t,r}$ in [18]. We define the scalar weighted Bourgain-Morrey space $M_p^{t,r}(\omega)$ by

$$\|f\|_{M_p^{t,r}(\omega)} := \left\| \left\{ (\omega(Q_{j,k}))^{1/t-1/p} \left(\int_{Q_{j,k}} |f(y)|^p \omega(y) \mathrm{d}y \right)^{1/p} \right\}_{j\in\mathbb{Z},k\in\mathbb{Z}^n} \right\|_{\ell^r} < \infty. \tag{6}$$

That is $M_p^{t,r}(\omega)$ is the case $d = 1$ of matrix weighted Bourgain-Morrey space $M_p^{t,r}(W)$.

The following lemma is proved in [25, 31].

Lemma 2.13. *Let $1 \le p < \infty$ and $W : \mathbb{R}^n \to M_d(\mathbb{C})$ be a matrix weight. Let $\Omega \subset \mathbb{R}^n$ be an open set. Then matrix weighted Lebesgue space $L^p(W, \Omega)$ is a Banach space.*

Theorem 2.14. *Let $1 \le p < \infty$ and $W : \mathbb{R}^n \to M_d(\mathbb{C})$ be a matrix weight. Then the space $L^p_{\mathrm{loc}}(W)$ is complete.*

Proof. Let $\{\vec{f_k}\}_{k=1}^\infty$ be a Cauchy sequence in $L^p_{\mathrm{loc}}(W)$. That is, for any compact set K, any $\epsilon > 0$, there exists $N > 0$ such that if $j, k > N$, $\|(\vec{f_j} - \vec{f_k})\chi_K\|_{L^p(W)} < \epsilon$. Then for any compact set K, $\{\vec{f_k}\chi_K\}_{k=1}^\infty$ is a Cauchy sequence in $L^p(W)$. Since $L^p(W)$ is complete (Lemma 2.13), there exists $\vec{f}\chi_K$ in $L^p(W)$ such that $\vec{f_k}\chi_K \to \vec{f}\chi_K$. By the Fatou lemma, we have

$$\|\vec{f}\chi_K\|_{L^p(W)} \le \liminf_{k\to\infty} \|\vec{f_k}\chi_K\|_{L^p(W)} < \infty.$$

By the dominated convergence theorem, for $j, k > N$, we obtain

$$\|(\vec{f} - \vec{f_k})\chi_K\|_{L^p(W)} = \lim_{j\to\infty} \|(\vec{f_j} - \vec{f_k})\chi_K\|_{L^p(W)} < \epsilon.$$

Hence $\vec{f_k} \to \vec{f}$ in $L^p_{\mathrm{loc}}(W)$ and the space $L^p_{\mathrm{loc}}(W)$ is complete. □

In what follows, the symbol $\hookrightarrow$ always stands for continuous embedding.

Proposition 2.15. *Let $W : \mathbb{R}^n \to M_d(\mathbb{C})$ be a matrix weight.*
(i) *If $0 < p < t < r_1 < r_2 \le \infty$, then*

$$M_p^{t,r_1}(W) \hookrightarrow M_p^{t,r_2}(W).$$

(ii) *If $0 < p_1 < p_2 < t < r < \infty$ or $0 < p_1 < p_2 \le t < r = \infty$, then*

$$M_{p_2}^{t,r}(W) \hookrightarrow M_{p_1}^{t,r}(W).$$

(iii) *If $0 < p \le t < r = \infty$, then*

$$L^t(W) \hookrightarrow M_p^{t,r}(W) \hookrightarrow L^p_{\mathrm{loc}}(W).$$

Proof. (i) It comes from $\ell^{r_1} \hookrightarrow \ell^{r_2}$ since $0 < r_1 < r_2 \le \infty$.

(ii) It comes from the Hölder inequality. Indeed, for each $Q \in \mathcal{D}$,

$$\begin{aligned}
&\left(\frac{1}{W(Q)}\int_Q |W^{1/p_1}(y)\vec{f}(y)|^{p_1}\mathrm{d}y\right)^{1/p_1}\\
&=\left(\frac{1}{W(Q)}\int_Q |W^{1/p_1-1/p_2}(y)W^{1/p_2}(y)\vec{f}(y)|^{p_1}\mathrm{d}y\right)^{1/p_1}\\
&\le\left(\frac{1}{W(Q)}\int_Q \|W^{1/p_1-1/p_2}(y)\|^{p_1}|W^{1/p_2}(y)\vec{f}(y)|^{p_1}\mathrm{d}y\right)^{1/p_1}\\
&\le\left(\frac{1}{W(Q)}\int_Q \|W^{1/p_1-1/p_2}(y)\|^{p_1p_2/(p_2-p_1)}\mathrm{d}y\right)^{1/p_1-1/p_2}\left(\frac{1}{W(Q)}\int_Q |W^{1/p_2}(y)\vec{f}(y)|^{p_2}\mathrm{d}y\right)^{1/p_2}\\
&=\left(\frac{1}{W(Q)}\int_Q \|W(y)\|\mathrm{d}y\right)^{1/p_1-1/p_2}\left(\frac{1}{W(Q)}\int_Q |W^{1/p_2}(y)\vec{f}(y)|^{p_2}\mathrm{d}y\right)^{1/p_2}\\
&=\left(\frac{1}{W(Q)}\int_Q |W^{1/p_2}(y)\vec{f}(y)|^{p_2}\mathrm{d}y\right)^{1/p_2}.
\end{aligned}$$

Hence,

$$\|\vec{f}\|_{M^{t,r}_{p_1}(W)} \le \|\vec{f}\|_{M^{t,r}_{p_2}(W)}.$$

Thus we prove (ii).

(iii) The first embedding comes from the fact that

$$L^t(W) = M^{t,\infty}_t(W) \hookrightarrow M^{t,\infty}_p(W).$$

For any compact set $K \subset \mathbb{R}^n$, there exist at most 2^n dyadic cubes Q_j such that $K \subset \bigcup_{j=1}^{2^n} Q_j$. Hence

$$\|\vec{f}\chi_K\|_{L^p(W)} \le \sum_{j=1}^{2^n} \frac{|Q_j|^{1/t-1/p}}{|Q_j|^{1/t-1/p}}\|\vec{f}\chi_{Q_j}\|_{L^p(W)} \le \|\vec{f}\|_{M^{t,\infty}_p(W)} \sum_{j=1}^{2^n} \frac{1}{|Q_j|^{1/t-1/p}} < \infty.$$

This shows that $M^{t,r}_p(W) \hookrightarrow L^p_{\mathrm{loc}}(W)$. □

Proposition 2.16. *The matrix weighted Bourgain-Morrey space $M^{t,r}_p(W)$ is a Banach space when $1 \le p < t < r < \infty$ or $1 \le p \le t < r = \infty$.*

Proof. The argument are standard, see, for example, [30, Theorem 2.4]. We only show that $M^{t,r}_p(W)$ is complete since others are simple.

Let $\{\vec{f}_j\}_{j=1}^\infty$ be a Cauchy sequence in $M^{t,r}_p(W)$. By Proposition 2.15, we have that $\{\vec{f}_j\}_{j=1}^\infty$ is also a Cauchy sequence in $L^p_{\mathrm{loc}}(W)$. Since $L^p_{\mathrm{loc}}(W)$ is complete, there exists a vector function $\vec{f}$ in $L^p_{\mathrm{loc}}(W)$ such that $\vec{f}_j \to \vec{f}$. By the Fatou lemma,

$$\|\vec{f}\|_{M^{t,r}_p(W)} = \|\liminf_{j\to\infty}\vec{f}_j\|_{M^{t,r}_p(W)} \le \liminf_{j\to\infty}\|\vec{f}_j\|_{M^{t,r}_p(W)} < \infty.$$

Thus $\vec{f} \in M^{t,r}_p(W)$. Therefore,

$$\limsup_{j\to\infty}\|\vec{f}-\vec{f}_j\|_{M^{t,r}_p(W)} \le \limsup_{j\to\infty}\left(\liminf_{k\to\infty}\|\vec{f}_k-\vec{f}_j\|_{M^{t,r}_p(W)}\right) = 0.$$

Thus, the sequence $\{\vec{f}_j\}_{j=1}^\infty$ is convergent to $\vec{f}$ in $M^{t,r}_p(W)$. □

Recall that a matrix weight W is almost everywhere invertible. Hereafter, we define $w(x) := \|W^{-1}(x)\|^{-1}$ when a matrix weight W is invertible at $x \in \mathbb{R}^n$.

Lemma 2.17 (Proposition 3.2,[10]). *Let $1 \le p < \infty$. For a matrix weight $W : \mathbb{R}^n \to M_d(\mathbb{C})$, we have $0 < w(x) \le \|W(x)\| < \infty$ for a.e. $x \in \mathbb{R}^n$. Furthermore, W satisfies a two weight, degenerate ellipticity condition: for $\xi \in \mathbb{R}^d$,*

$$w(x)|\xi|^p \le |W^{1/p}(x)\xi|^p \le \|W(x)\||\xi|^p. \tag{7}$$

Remark 2.18. *In [10, Proposition 3.2], it is assumed that $\xi \in \mathbb{R}^d$ but it also works for $\xi \in \mathbb{C}^d$.*

Proposition 2.19. *Let $1 \le p < t < r < \infty$ or $1 \le p \le t < r = \infty$. Let W be a matrix weight and $\vec{f}, \vec{g} \in M_p^{t,r}(W)$. Then $\|\vec{f} - \vec{g}\|_{M_p^{t,r}(W)} = 0$ if and only if $\vec{f}(x) = \vec{g}(x)$ a.e..*

Proof. Clearly, if $\vec{f}(x) = \vec{g}(x)$ a.e., then $\|\vec{f} - \vec{g}\|_{M_p^{t,r}(W)} = 0$. Then we apply Lemma 2.17 to prove the converse. By the degenerate ellipticity condition (7), we have

$$0 = \|\vec{f} - \vec{g}\|_{M_p^{t,r}(W)} \ge \left\| \left\{ W(Q_{j,k})^{1/t-1/p} \left(\int_{Q_{j,k}} |\vec{f}(y) - \vec{g}(y)|^p w(y) \mathrm{d}y \right)^{1/p} \right\}_{j\in\mathbb{Z},k\in\mathbb{Z}^n} \right\|_{\ell^r}.$$

Since $w(y) > 0$ a.e., it follows that $\vec{f}(y) - \vec{g}(y) = 0$ a.e.. □

3. Sufficient conditions for precompact sets

Definition 3.1. (i) *Suppose that (X, ρ) is a metric space. Let A be a set of X and let $\epsilon > 0$. A set $E \subset X$ is called an ϵ-net for A if every point $a \in A$, there exists a point $e \in E$ such that $\rho(a, e) < \epsilon$.*

(ii) *A set A is called totally bounded if, for each $\epsilon > 0$, it possesses a finite ϵ-net.*

(iii) *A subset in a topological space is precompact if its closure is compact.*

It is well known that in a complete metric space, a set is precompact if and only if it is totally bounded.

Theorem 3.2. *Let $1 \le p < t < r < \infty$ or $1 \le p \le t < r = \infty$. Let $W : \mathbb{R}^n \to M_d(\mathbb{C})$ be a matrix weight. A subset $\mathcal{F} \subset M_p^{t,r}(W)$ is totally bounded if the following conditions are valid:*

(i) *$\mathcal{F}$ uniformly vanishes at infinity, that is,*

$$\lim_{R\to\infty} \sup_{\vec{f}\in\mathcal{F}} \|\vec{f}\chi_{B^c(0,R)}\|_{M_p^{t,r}(W)} = 0;$$

(ii) *$\mathcal{F}$ is equicontinuous, that is,*

$$\lim_{b\to 0} \sup_{\vec{f}\in\mathcal{F}} \sup_{y\in B(0,b)} \|\vec{f} - \tau_y\vec{f}\|_{M_p^{t,r}(W)} = 0. \tag{8}$$

Here and what follows, τ_y denotes the translation operator: $\tau_y\vec{f}(x) := \vec{f}(x - y)$.

Remark 3.3. *In this Remark, we will prove that the conditions (i) and (ii) in Theorem 3.2 together imply that the set $\mathcal{F}$ is bounded. Indeed, choose $b > 0$ such that for all $h \in \mathbb{R}^n$, $|h| \le b$, all $\vec{f} \in \mathcal{F}$,*

$$\|\vec{f} - \tau_y\vec{f}\|_{M_p^{t,r}(W)} \le 1.$$

Choose $R > 0$ such that for all $\vec{f} \in \mathcal{F}$,

$$\|\vec{f}\chi_{B^c(0,R)}\|_{M_p^{t,r}(W)} \le 1.$$

Fix h with $|h| = b$. Then for all $\vec{f} \in \mathcal{F}$, $x \in \mathbb{R}^n$, we have

$$\begin{aligned}\|\vec{f}\chi_{B(x,R)}\|_{M_p^{t,r}(W)} &\le \|(\vec{f} - \tau_h\vec{f})\chi_{B(x,R)}\|_{M_p^{t,r}(W)} + \|\tau_h\vec{f}\chi_{B(x,R)}\|_{M_p^{t,r}(W)} \\ &= \|(\vec{f} - \tau_h\vec{f})\chi_{B(x,R)}\|_{M_p^{t,r}(W)} + \|\vec{f}\chi_{B(x+h,R)}\|_{M_p^{t,r}(W)} \\ &\le 1 + \|\vec{f}\chi_{B(x+h,R)}\|_{M_p^{t,r}(W)}.\end{aligned}$$

Hence, by induction,

$$\|\vec{f}\chi_{B(0,R)}\|_{M_p^{t,r}(W)} \le N + \|\vec{f}\chi_{B(Nh,R)}\|_{M_p^{t,r}(W)}.$$

Now choose $N \ge 1$ such that $Nb = N|h| > 2R$. Then $B(Nh, R) \subset B^c(0, R)$. Hence

$$\|\vec{f}\|_{M_p^{t,r}(W)} \le \|\vec{f}\chi_{B(0,R)}\|_{M_p^{t,r}(W)} + \|\vec{f}\chi_{B^c(0,R)}\|_{M_p^{t,r}(W)} \le N + \|\vec{f}\chi_{B(Nh,R)}\|_{M_p^{t,r}(W)} + \|\vec{f}\chi_{B^c(0,R)}\|_{M_p^{t,r}(W)} \le N + 2.$$

This proves that $\mathcal{F}$ is bounded.

Now we begin to show Theorem 3.2.

Proof. Assume that $\mathcal{F} \subset M_p^{t,r}(W)$ satisfies (i) and (ii). Given $\epsilon > 0$ small enough, to prove the total boundedness of $\mathcal{F}$, it suffices to find a finite ϵ-net of $\mathcal{F}$. Denote by $R_i := [-2^i, 2^i)^n$ for $i \in \mathbb{Z}$. Then from condition (i), there exist a positive integer m large enough such that

$$\sup_{\vec{f}\in\mathcal{F}} \|\vec{f} - \vec{f}\chi_{R_m}\|_{M_p^{t,r}(W)} < \epsilon. \tag{9}$$

Moreover, by condition (ii), there exists a negative integer a such that

$$\sup_{\vec{f}\in\mathcal{F}} \sup_{y\in R_a} \|\vec{f} - \tau_y\vec{f}\|_{M_p^{t,r}(W)} < \epsilon. \tag{10}$$

There exists a sequence $\{Q_j\}_{j=1}^N$ of disjoint cubes in $\mathcal{D}_{-a}$ such that $R_m = \bigcup_{i=1}^N Q_j$, where $N = 2^{(m+1-a)n}$. For any $\vec{f} \in \mathcal{F}$ and $x \in \mathbb{R}^n$, let

$$\Phi(\vec{f})(x) := \begin{cases} \vec{f}_{Q_j} := \frac{1}{|Q_j|}\int_{Q_j} \vec{f}(y)\mathrm{d}y, & x \in Q_j, j = 1, 2, \dots, N, \\ \vec{0}, & \text{otherwise}. \end{cases}$$

Then for each fixed $x \in \mathbb{R}^n$, we have

$$\begin{aligned}\left|W^{1/p}(x)\left(\vec{f}(x) - \vec{f}_{Q_j}\right)\right|\chi_{Q_j}(x) &= \left|\frac{1}{|Q_j|}\int_{Q_j} W^{1/p}(x)\left(\vec{f}(x) - \vec{f}(y)\right)\mathrm{d}y\right|\chi_{Q_j}(x) \\ &\le \frac{1}{|Q_j|}\int_{Q_j}\left|W^{1/p}(x)\left(\vec{f}(x) - \vec{f}(y)\right)\right|\mathrm{d}y\chi_{Q_j}(x).\end{aligned} \tag{11}$$

We split $\|(\vec{f}-\Phi(\vec{f}))\chi_{R_m}\|_{M_p^{t,r}(W)}$ into three parts:

$$\begin{aligned}
&\left\{\sum_{k\in\mathbb{Z}}\sum_{Q\in\mathcal{D}_k} W(Q)^{r/t-r/p}\left(\int_Q\left|W^{1/p}(x)\left(\vec{f}(x)\chi_{R_m}-\Phi(\vec{f})(x)\right)\right|^p \mathrm{d}x\right)^{r/p}\right\}^{1/r}\\
&\le\left\{\sum_{k>-a}\sum_{Q\in\mathcal{D}_k} W(Q)^{r/t-r/p}\left(\int_Q\left|W^{1/p}(x)\left(\vec{f}(x)\chi_{R_m}-\vec{f}_{Q_j}\right)\right|^p \mathrm{d}x\right)^{r/p}\right\}^{1/r}\\
&\quad+\left\{\sum_{k=-m}^{-a}\sum_{Q\in\mathcal{D}_k} W(Q)^{r/t-r/p}\left(\int_Q\left|W^{1/p}(x)\left(\vec{f}(x)\chi_{R_m}-\Phi(\vec{f})(x)\right)\right|^p \mathrm{d}x\right)^{r/p}\right\}^{1/r}\\
&\quad+\left\{\sum_{k<-m}\sum_{Q\in\mathcal{D}_k} W(Q)^{r/t-r/p}\left(\int_Q\left|W^{1/p}(x)\left(\vec{f}(x)\chi_{R_m}-\Phi(\vec{f})(x)\right)\right|^p \mathrm{d}x\right)^{r/p}\right\}^{1/r}\\
&=: S_1+S_2+S_3.
\end{aligned}$$

We first estimate S_1. By (11), Jensen's inequality ($p \ge 1$), the Fubini theorem, we have

$$\begin{aligned}
S_1 &\le\left\{\sum_{k>-a}\sum_{Q\in\mathcal{D}_k,Q\subset R_m} W(Q)^{r/t-r/p}\left(\int_Q\left|\frac{1}{|Q_j|}\int_{Q_j}\left|W^{1/p}(x)\left(\vec{f}(x)-\vec{f}(y)\right)\right|\mathrm{d}y\chi_{Q_j}(x)\right|^p \mathrm{d}x\right)^{r/p}\right\}^{1/r}\\
&\le\left\{\sum_{k>-a}\sum_{Q\in\mathcal{D}_k,Q\subset R_m} W(Q)^{r/t-r/p}\left(\frac{1}{|Q_j|}\int_{Q_j}\int_Q\left|W^{1/p}(x)\left(\vec{f}(x)-\vec{f}(y)\right)\right|^p \mathrm{d}x\mathrm{d}y\right)^{r/p}\right\}^{1/r}\\
&=\left\{\sum_{k>-a}\sum_{Q\in\mathcal{D}_k,Q\subset R_m} W(Q)^{r/t-r/p}\left(\frac{1}{|Q_j|}\int_Q\int_{Q_j}\left|W^{1/p}(x)\left(\vec{f}(x)-\vec{f}(y)\right)\right|^p \mathrm{d}y\mathrm{d}x\right)^{r/p}\right\}^{1/r}\\
&=2^{-an/p}\left\{\sum_{k>-a}\sum_{Q\in\mathcal{D}_k,Q\subset R_m} W(Q)^{r/t-r/p}\left(\int_Q\int_{x-Q_j}\left|W^{1/p}(x)\left(\vec{f}(x)-\vec{f}(x-y)\right)\right|^p \mathrm{d}y\mathrm{d}x\right)^{r/p}\right\}^{1/r} \qquad (12)
\end{aligned}$$

where $x-Q_j := \{x-y : y\in Q_j\}$. Since $x\in Q\subset Q_j$ for some $j\in\{1,2,\dots,N\}$, we have $x-Q_j\subset R_a$. Hence by (10), we obtain

$$(12)\le 2^{-an/p}|Q_j|^{1/p}\sup_{y\in R_a}\left\{\sum_{k>-a}\sum_{Q\in\mathcal{D}_k,Q\subset R_m} W(Q)^{r/t-r/p}\left(\int_Q\left|W^{1/p}(x)(\vec{f}(x)-\vec{f}(x-y))\right|^p \mathrm{d}x\right)^{r/p}\right\}^{1/r}\le\epsilon.$$

As for S_2, for each $Q\in\mathcal{D}_k$ where $k=-m,-m+1,\dots,-a$, there are $2^{(-k-a)n}$ cubes $Q_j\in\mathcal{D}_{-a}$ such that $\cup Q_j = Q$. Denote by Q_{j_ℓ}, $\ell = 1,2,\dots,2^{(-k-a)n}$ these cubes. Then by (11), Jensen's inequality ($p\ge 1$), and the Fubini theorem, we obtain

$$\begin{aligned} S_2 &= \left\{\sum_{k=-m}^{-a}\sum_{Q\in\mathcal{D}_k,Q\subset R_m} W(Q)^{r/t-r/p}\left(\sum_{\ell=1}^{2^{(-k-a)n}}\int_{Q_{j_\ell}}\left|W^{1/p}(x)\left(\vec{f}(x)\chi_{R_m}-\vec{f}_{Q_{j_\ell}}\right)\right|^p dx\right)^{r/p}\right\}^{1/r} \\ &\le \left\{\sum_{k=-m}^{-a}\sum_{Q\in\mathcal{D}_k,Q\subset R_m} W(Q)^{r/t-r/p}\left(\sum_{\ell=1}^{2^{(-k-a)n}}\int_{Q_{j_\ell}}\left|\frac{1}{|Q_{j_\ell}|}\int_{Q_{j_\ell}}\left|W^{1/p}(x)\left(\vec{f}(x)-\vec{f}(y)\right)\right|dy\right|^p dx\right)^{r/p}\right\}^{1/r} \\ &\le \left\{\sum_{k=-m}^{-a}\sum_{Q\in\mathcal{D}_k,Q\subset R_m} W(Q)^{r/t-r/p}\left(\sum_{\ell=1}^{2^{(-k-a)n}}\frac{1}{|Q_{j_\ell}|}\int_{Q_{j_\ell}}\int_{Q_{j_\ell}}\left|W^{1/p}(x)\left(\vec{f}(x)-\vec{f}(y)\right)\right|^p dxdy\right)^{r/p}\right\}^{1/r} \\ &= 2^{-an/p}\left\{\sum_{k=-m}^{-a}\sum_{Q\in\mathcal{D}_k,Q\subset R_m} W(Q)^{r/t-r/p}\left(\sum_{\ell=1}^{2^{(-k-a)n}}\int_{Q_{j_\ell}}\int_{Q_{j_\ell}}\left|W^{1/p}(x)\left(\vec{f}(x)-\vec{f}(y)\right)\right|^p dydx\right)^{r/p}\right\}^{1/r} \\ &= 2^{-an/p}\left\{\sum_{k=-m}^{-a}\sum_{Q\in\mathcal{D}_k,Q\subset R_m} W(Q)^{r/t-r/p}\left(\sum_{\ell=1}^{2^{(-k-a)n}}\int_{Q_{j_\ell}}\int_{x-Q_{j_\ell}}\left|W^{1/p}(x)\left(\vec{f}(x)-\vec{f}(x-y)\right)\right|^p dydx\right)^{r/p}\right\}^{1/r} \qquad (13) \end{aligned}$$

where $x-Q_{j_\ell} := \{x-y : y\in Q_{j_\ell}\}$. Note that $x-Q_{j_\ell}\subset R_a$ when $x\in Q_{j_\ell}$. By (10), we have

$$(13) \le 2^{-an/p}2^{an/p}\sup_{y\in R_a}\left\{\sum_{k=-m}^{-a}\sum_{Q\in\mathcal{D}_k,Q\subset R_m} W(Q)^{r/t-r/p}\left(\sum_{\ell=1}^{2^{(-k-a)n}}\int_{Q_{j_\ell}}\left|W^{1/p}(x)\left(\vec{f}(x)-\vec{f}(x-y)\right)\right|^p dx\right)^{r/p}\right\}^{1/r} \le \epsilon.$$

As for S_3, for each set $\mathcal{D}_k$, there are only 2^n cubes $Q\in\mathcal{D}_k$ such that $Q\cap R_m\neq\emptyset$. We denote by Q_ℓ, $\ell=1,2,\dots,2^n$ these cubes. And for each cube Q_ℓ, there are $2^{(m-a)n}$ cubes $Q_j\in\mathcal{D}_{-a}$ such that $\cup_j Q_j = Q_\ell$. Then

$$\begin{aligned} S_3 &= \left\{\sum_{k<-m}\sum_{\ell=1}^{2^n} W(Q_\ell)^{r/t-r/p}\left(\int_{Q_\ell}\left|W^{1/p}(x)\left(\vec{f}(x)\chi_{R_m}-\Phi(\vec{f})(x)\right)\right|^p dx\right)^{r/p}\right\}^{1/r} \\ &= \left\{\sum_{k<-m}\sum_{\ell=1}^{2^n} W(Q_\ell)^{r/t-r/p}\left(\sum_{j=1}^{2^{(m-a)n}}\int_{Q_j}\left|W^{1/p}(x)\left(\vec{f}(x)\chi_{R_m}-\vec{f}_{Q_j}\right)\right|^p dx\right)^{r/p}\right\}^{1/r}. \end{aligned}$$

Similarly as S_2, from (11), Jensen's inequality ($p\ge 1$), the Fubini theorem, and (10), we get that $S_3\le\epsilon$. Together with the estimates of S_1, S_2, S_3, we obtain

$$\left\{\sum_{k\in\mathbb{Z}}\sum_{Q\in\mathcal{D}_k} W(Q)^{r/t-r/p}\left(\int_Q\left|W^{1/p}(x)(\vec{f}(x)\chi_{R_m}-\Phi(\vec{f})(x))\right|^p dx\right)^{r/p}\right\}^{1/r} \le 3\epsilon. \qquad (14)$$

Note that

$$\|\vec{f}-\Phi(\vec{f})\|_{M_p^{t,r}(W)} \le \|(\vec{f}-\Phi(\vec{f}))\chi_{R_m}\|_{M_p^{t,r}(W)} + \|(\vec{f}-\Phi(\vec{f}))\chi_{R_m^c}\|_{M_p^{t,r}(W)} = \|(\vec{f}-\Phi(\vec{f}))\chi_{R_m}\|_{M_p^{t,r}(W)} + \|\vec{f}\chi_{R_m^c}\|_{M_p^{t,r}(W)}.$$

Thus via (9) and (14), we have

$$\sup_{\vec{f}\in\mathcal{F}}\|\vec{f}-\Phi(\vec{f})\|_{M_p^{t,r}(W)} \le 4\epsilon. \qquad (15)$$

From (15), it suffices to show that $\Phi(\mathcal{F})$ is totally bounded in $M_p^{t,r}(W)$.

From Remark 3.3, we have

$$\sup_{\vec{f}\in\mathcal{F}} \|\Phi(\vec{f})\|_{M_p^{t,r}(W)} \le \sup_{\vec{f}\in\mathcal{F}} \|\Phi(\vec{f}) - \vec{f}\|_{M_p^{t,r}(W)} + \sup_{\vec{f}\in\mathcal{F}} \|\vec{f}\|_{M_p^{t,r}(W)} < \infty.$$

Thus, for any $\vec{f} \in \mathcal{F}$,

$$|W^{1/p}(x)\Phi(\vec{f})(x)| < \infty \quad \text{a.e.} \quad x \in \mathbb{R}^n. \tag{16}$$

Since W is a matrix weight, by (i), (ii) of Definition 2.3, we have for almost everywhere $x \in \mathbb{R}^n$, $W(x)$ is a positive definite matrix. Therefore, we obtain

$$|\Phi(\vec{f})(x)| < \infty \quad \text{a.e.} \quad x \in \mathbb{R}^n. \tag{17}$$

which implies $|\vec{f}_{Q_j}| < \infty$, $j = 1, 2, \dots, N$. From this and the entries of W is locally integrable, we see that Φ is a map from $\mathcal{F}$ to $\mathcal{B}$, a finite dimensional Banach subspace of $M_p^{t,r}(W)$. Note that $\Phi(\mathcal{F}) \subset \mathcal{B}$ is bounded, and hence is totally bounded. The proof of Theorem 3.2 is complete. □

Then we give an application in degenerate Bourgain-Morrey-Sobolev spaces with matrix weights.

Definition 3.4. *Let $1 \le p < t < r < \infty$ or $1 \le p \le t < r = \infty$. Let $W : \mathbb{R}^n \to M_n(\mathbb{C})$ be a matrix weight and set the scalar weight $\omega := \|W\|$. We define the degenerate Bourgain-Morrey-Sobolev spaces $\mathcal{W}^{1,p,t,r}(W)$ by the set of all Lebesgue measurable functions on $\mathbb{R}^n$ such that*

$$\|f\|_{\mathcal{W}^{1,p,t,r}(W)} := \|f\|_{M_p^{t,r}(\omega)} + \|\nabla f\|_{M_p^{t,r}(W)} < \infty$$

where $\nabla f = (\partial_{x_1} f, \dots, \partial_{x_n} f)^T$ is the gradient of f and

$$\|f\|_{M_p^{t,r}(\omega)} := \left\| \left\{ \omega(Q_{j,k})^{1/t-1/p} \left(\int_{Q_{j,k}} |f(y)|^p \omega(y) \mathrm{d}y \right)^{1/p} \right\}_{j\in\mathbb{Z}, k\in\mathbb{Z}^n} \right\|_{\ell^r},$$

is the norm of the scalar weighted Bourgain-Morrey space in (6).

Lemma 3.5 (p. 262, [36]). *A set $\mathcal{F}$ in metric space X is totally bounded if and only if it is Cauchy-precompact, that is, every sequence has a Cauchy subsequence.*

Corollary 3.6. *Let $1 \le p < t < r < \infty$ or $1 \le p \le t < r = \infty$. Let $W : \mathbb{R}^n \to M_n(\mathbb{C})$ be a matrix weight. A subset $\mathcal{F} \subset \mathcal{W}^{1,p,t,r}(W)$ is totally bounded if the following conditions are valid:*

(i) *$\mathcal{F}$ uniformly vanishes at infinity, that is,*

$$\lim_{R\to\infty} \sup_{\vec{f}\in\mathcal{F}} \|\vec{f}\chi_{B^c(0,R)}\|_{\mathcal{W}^{1,p,t,r}(W)} = 0;$$

(ii) *$\mathcal{F}$ is equicontinuous, that is,*

$$\lim_{a\to 0} \sup_{\vec{f}\in\mathcal{F}} \sup_{y\in B(0,a)} \|\vec{f} - \tau_y \vec{f}\|_{\mathcal{W}^{1,p,t,r}(W)} = 0.$$

Proof. Note that $\mathcal{F} \subset \mathcal{W}^{1,p,t,r}(W)$ satisfies conditions (i)-(ii) if and only if $\mathcal{F} \subset M_p^{t,r}(\omega)$ satisfies conditions (i)-(ii) of Theorem 3.2 ($d = 1$) and $\nabla\mathcal{F} := \{\nabla f : f \in \mathcal{F}\} \subset M_p^{t,r}(W)$ satisfies (i)-(ii) of Theorem 3.2 ($d = n$). Hence by Theorem 3.2, we obtain that both $\mathcal{F} \subset M_p^{t,r}(\omega)$ and $\nabla\mathcal{F} \subset M_p^{t,r}(W)$ are totally bounded. Then Corollary 3.6 follows from Lemma 3.5. □

4. Dyadic average operator conditions for precompact sets

In this section, we give a criterion for precompactness by dyadic average operator. Let $\mathcal{F}$ be a subset of $M_p^{t,r}(W)$. $\mathcal{F}$ is equicontinuous by means of the dyadic average operator if

$$\lim_{a\to-\infty, a\in\mathbb{Z}} \sup_{\vec{f}\in\mathcal{F}} \|\vec{f} - E_{d,a}\vec{f}\|_{M_p^{t,r}(W)} = 0, \tag{18}$$

where

$$E_{d,a}\vec{f}(x) := \sum_{Q\in\mathcal{D}_{-a}} \frac{\chi_Q(x)}{|Q|}\int_Q \vec{f}(y)\mathrm{d}y. \tag{19}$$

Recall that x_Q is the lower left corner of $Q \in \mathcal{D}$. Then for $x \in Q \in \mathcal{D}_{-a}$

$$E_{d,a}\vec{f}(x) = \frac{\chi_{Q_{-a,0}}(x - x_Q)}{|Q_{-a,0}|}\int_{Q_{-a,0}} \vec{f}(y - x_Q)\mathrm{d}y.$$

Then we will prove that (8) is stronger than (18). Indeed, suppose that condition (8) holds. For any $\epsilon > 0$, there exists a cube R with center 0 and side length $2^{a+1} > 0$ such that

$$\sup_{\vec{f}\in\mathcal{F}} \sup_{y\in R} \|\vec{f} - \tau_y\vec{f}\|_{M_p^{t,r}(W)} < \epsilon. \tag{20}$$

If $x, y \in Q \in \mathcal{D}_{-a}$, then $x - y \in R$. By Jensen's inequality ($p \geq 1$), Hölder's inequality ($r/p \geq 1$), and (20), we obtain

$$\begin{aligned}
&\|\vec{f} - E_{d,a}\vec{f}\|_{M_p^{t,r}(W)}\\
&= \left(\sum_{Q\in\mathcal{D}} W(Q)^{r/t-r/p}\left(\int_Q \left|\frac{1}{|Q_{-a,0}|}\int_{Q_{-a,0}} W^{1/p}(x)\left(\vec{f}(x) - \chi_{Q_{-a,0}}(x - x_Q)\vec{f}(y - x_Q)\right)\mathrm{d}y\right|^p \mathrm{d}x\right)^{r/p}\right)^{1/r}\\
&\leq \left(\sum_{Q\in\mathcal{D}} W(Q)^{r/t-r/p}\left(\frac{1}{|Q_{-a,0}|}\int_{Q_{-a,0}}\int_Q \left|W^{1/p}(x)\left(\vec{f}(x) - \chi_{Q_{-a,0}}(x - x_Q)\vec{f}(y - x_Q)\right)\right|^p \mathrm{d}x\mathrm{d}y\right)^{r/p}\right)^{1/r}\\
&\leq \left(\sum_{Q\in\mathcal{D}} W(Q)^{r/t-r/p}\frac{1}{|Q_{-a,0}|}\int_{Q_{-a,0}}\left(\int_Q \left|W^{1/p}(x)\left(\vec{f}(x) - \chi_{Q_{-a,0}}(x - x_Q)\vec{f}(y - x_Q)\right)\right|^p \mathrm{d}x\right)^{r/p}\mathrm{d}y\right)^{1/r}\\
&\leq \frac{1}{|Q_{-a,0}|^{1/r}}|Q_{-a,0}|^{1/r}\sup_{y\in R}\left(\sum_{Q\in\mathcal{D}} W(Q)^{r/t-r/p}\left(\int_Q \left|W^{1/p}(x)(\vec{f}(x) - \vec{f}(x - y))\right|^p \mathrm{d}x\right)^{r/p}\right)^{1/r} \leq \epsilon,
\end{aligned}$$

modified when $r = \infty$. Hence we prove that (8) is stronger than (18).

Replacing the translation operator by the average operator, we have the following result.

Theorem 4.1. *Let $1 \leq p < t < r < \infty$ or $1 \leq p \leq t < r = \infty$. Let $W : \mathbb{R}^n \to M_d(\mathbb{C})$ be a matrix weight. A subset $\mathcal{F} \subset M_p^{t,r}(W)$ is totally bounded if the following conditions are valid:*

(i) *$\mathcal{F}$ is bounded, that is,*

$$\sup_{\vec{f}\in\mathcal{F}} \|\vec{f}\|_{M_p^{t,r}(W)} < \infty;$$

(ii) *$\mathcal{F}$ uniformly vanishes at infinity, that is,*

$$\lim_{R\to\infty} \sup_{\vec{f}\in\mathcal{F}} \|\vec{f}\chi_{B^c(0,R)}\|_{M_p^{t,r}(W)} = 0;$$

(iii) $\mathcal{F}$ *is equicontinuous by means of the dyadic average operator, that is, for any*

$$\lim_{a\to-\infty, a\in\mathbb{Z}} \sup_{\vec{f}\in\mathcal{F}} \|\vec{f} - E_{d,a}\vec{f}\|_{M_p^{t,r}(W)} = 0.$$

where $E_{d,a}$ *is same as (19).*

Proof. Assume that $\mathcal{F} \subset M_p^{t,r}(W)$ satisfies (i)-(iii). Given $\epsilon > 0$ small enough, to prove the total boundedness of $\mathcal{F}$, it suffices to find a finite ϵ-net of $\mathcal{F}$. Denote by $R_i := [-2^i, 2^i)^n$ for $i \in \mathbb{Z}$. Then from condition (ii), there exists a positive integer m large enough such that

$$\sup_{\vec{f}\in\mathcal{F}} \|\vec{f} - \vec{f}\chi_{R_m}\|_{M_p^{t,r}(W)} < \epsilon. \tag{21}$$

Moreover, by condition (iii), there exists an integer $a < 0$ such that

$$\sup_{\vec{f}\in\mathcal{F}} \|\vec{f} - E_{d,a}\vec{f}\|_{M_p^{t,r}(W)} < \epsilon. \tag{22}$$

There exists a sequence $\{Q_j\}_{j=1}^N$ of disjoint cubes in $\mathcal{D}_{-a}$ such that $R_m = \bigcup_{i=1}^N Q_j$, where $N = 2^{(m+1-a)n}$. For any $\vec{f} \in \mathcal{F}$ and $x \in \mathbb{R}^n$, let

$$\Phi(\vec{f})(x) := \begin{cases} \vec{f}_{Q_j} := \frac{1}{|Q_j|}\int_{Q_j} \vec{f}(y)\mathrm{d}y, & x \in Q_j, j = 1, 2, \ldots, N, \\ \vec{0}, & \text{otherwise}. \end{cases}$$

Note that for $x \in R_m$,

$$\Phi(\vec{f})(x) = E_{d,a}\vec{f}(x).$$

Hence via (21) and (22), we have

$$\begin{aligned} \|\vec{f} - \Phi(\vec{f})\|_{M_p^{t,r}(W)} &\le \|(\vec{f} - \Phi(\vec{f}))\chi_{R_m}\|_{M_p^{t,r}(W)} + \|(\vec{f} - \Phi(\vec{f}))\chi_{R_m^c}\|_{M_p^{t,r}(W)} = \|(\vec{f} - E_{d,a}\vec{f})\chi_{R_m}\|_{M_p^{t,r}(W)} + \|\vec{f}\chi_{R_m^c}\|_{M_p^{t,r}(W)} \\ &\le \|(\vec{f} - E_{d,a}\vec{f})\|_{M_p^{t,r}(W)} + \epsilon \le 2\epsilon. \end{aligned} \tag{23}$$

From (23), it suffices to show that $\Phi(\mathcal{F})$ is totally bounded in $M_p^{t,r}(W)$. And we have proved that $\Phi(\mathcal{F})$ is totally bounded in $M_p^{t,r}(W)$ in Theorem 3.2. Thus the proof of Theorem 4.1 is complete. □

Lemma 4.2 (Lemma 4.5, [10]). *Let* $1 \le p < \infty$ *and* $W \in \mathcal{A}_p$. *Then* $\|W\|$ *and* $\|W^{-1}\|^{-1}$ *are scalar* A_p *weights.*

The following is a vector-valued extension of the Lebesgue differentiation theorem on matrix weighted spaces $M_p^{t,r}(W)$. For any cube Q, $\vec{f} \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, define

$$E_Q(\vec{f})(x) = \frac{1}{|Q|}\int_Q \vec{f}(y)\mathrm{d}y.$$

Theorem 4.3. *Let* $1 \le p < t < r < \infty$ *or* $1 \le p \le t < r = \infty$. *If* $W \in \mathcal{A}_p$, *then for any* $\vec{f} \in M_p^{t,r}(W)$,

$$\lim_{\ell(Q)\to 0} \left| E_Q\vec{f}(x) - \vec{f}(x) \right| = 0 \quad \text{a.e.} \quad x \in \mathbb{R}^n.$$

Proof. First, for any $\vec{f}=(f_1,f_2,\dots,f_d)^T\in M_p^{t,r}(W)$, it suffices to show that $f_i\in L^1_{\text{loc}}(\mathbb{R}^n)$ for each $i=1,2,\dots,d$. Since $W\in\mathcal{A}_p$, then by (3)

$$|\vec{f}|^p=|W^{-1/p}W^{1/p}\vec{f}|^p\le\|W^{-1/p}\|^p|W^{1/p}\vec{f}|^p=\|W^{-1}\||W^{1/p}\vec{f}|^p.$$

It follows that

$$|\vec{f}|^p\|W^{-1}\|^{-1}\le|W^{1/p}\vec{f}|^p.$$

From Lemma 4.2, we conclude that $\|W^{-1}\|^{-1}$ is a scalar A_p weight, hence $|\vec{f}|\in L^1_{\text{loc}}(\mathbb{R}^n)$. Indeed, since any compact set $K\subset\mathbb{R}^n$ can be contained in a finite set of dyadic cubes, it suffices to show that for any $Q\in\mathcal{D}$, $|\vec{f}|\in L^1(Q)$. By Hölder's inequality, we have

$$\begin{aligned}\int_Q|\vec{f}(x)|\mathrm{d}x&=\int_Q|\vec{f}(x)|\|W^{-1}(x)\|^{-1/p}\|W^{-1}(x)\|^{1/p}\mathrm{d}x\le\left(\int_Q|\vec{f}(x)|^p\|W^{-1}(x)\|^{-1}\mathrm{d}x\right)^{1/p}\left(\int_Q\|W^{-1}(x)\|^{p'/p}\mathrm{d}x\right)^{1/p'}\\&\le C_{Q,W,p}\left(\int_Q|W^{1/p}(x)\vec{f}(x)|^p\mathrm{d}x\right)^{1/p}\lesssim\|\vec{f}\|_{M_p^{t,r}(W)}<\infty.\end{aligned}$$

Hence, $f_i\in L^1_{\text{loc}}(\mathbb{R}^n)$ for each $i=1,2,\dots,d$. By the classical Lebesgue differentiation theorem, for each $1\le i\le d$, we have

$$\lim_{\ell(Q)\to0}\left|E_Qf_i(x)-f_i(x)\right|=0\quad\text{a.e.}\quad x\in\mathbb{R}^n.$$

Theorem 4.3 comes from the fact that for any $x\in\mathbb{R}^n$,

$$\left|E_Q\vec{f}(x)-\vec{f}(x)\right|\le d^{1/2}\max_i\left|E_Qf_i(x)-f_i(x)\right|.$$

Thus the proof is finished. □

Next we need the boundedness of dyadic average operator. Given any collection $\mathcal{Q}$ of pairwise disjoint cubes $Q\subset\mathbb{R}^n$, define the averaging operator $A_{\mathcal{Q}}$ by

$$A_{\mathcal{Q}}\vec{f}(x)=\sum_{Q\in\mathcal{Q}}\frac{\chi_Q(x)}{|Q|}\int_Q\vec{f}(y)\mathrm{d}y.$$

Lemma 4.4 (Proposition 4.7,[10]). *Let $1\le p<\infty$. Let $W:\mathbb{R}^n\to M_d(\mathbb{C})$ be a matrix weight. Then $W\in\mathcal{A}_p$ if and only it satisfies*

$$\|A_{\mathcal{Q}}\vec{f}\|_{L^p(W)}\le C_{n,d,p,W}\|\vec{f}\|_{L^p(W)}.$$

Remark 4.5. *In the sequel, let $p,p',W,\tilde{d},\widetilde{W},\tilde{\tilde{d}}$ have the same meaning as in Lemma 2.9. Let*

$$\tilde{\beta}:=\begin{cases}n, & \text{if } p=1,\\ n+\tilde{\tilde{d}}p/p', & \text{if } 1<p<\infty.\end{cases}\tag{24}$$

Then we claim that for $j+a\in\mathbb{N}_0$,

$$W((j+a)_{\text{pa}}Q)\lesssim\begin{cases}2^{(j+a)n}W(Q), & \text{if } p=1,\\ 2^{(j+a)(n+\tilde{\tilde{d}}p/p')}W(Q), & \text{if } 1<p<\infty.\end{cases}$$

Indeed, when $p = 1$, by Lemma 2.9, we have

$$W((j+a)_{\rm pa}Q) \approx |(j+a)_{\rm pa}Q| \|A_{(j+a)_{\rm pa}Q}\| \lesssim 2^{(j+a)n}|Q|\|A_Q\| \approx 2^{(j+a)n}W(Q).$$

When $p \in (1, \infty)$, by Lemma 2.9, we obtain

$$W((j+a)_{\rm pa}Q) \approx |(j+a)_{\rm pa}Q| \|A_{(j+a)_{\rm pa}Q}\|^p \lesssim 2^{(j+a)n}|Q|2^{(j+a)\tilde{\tilde{d}}p/p'}\|A_Q\|^p \approx 2^{(j+a)(n+\tilde{\tilde{d}}p/p')}W(Q).$$

Thus we prove the claim. Now we show this estimate is better than $W((j+a)_{\rm pa}Q) \lesssim 2^{(j+a)\beta}W(Q)$. Indeed, when $p = 1$, it is obvious since $\beta \geq n$. If $p > 1$, from [3, Lemma 2.11 and Corollary 2.16], for $i \in \mathbb{N}_0$, we deduce that

$$\|A_{i_{\rm pa}Q}A_Q^{-1}\|^p \approx \frac{1}{|i_{\rm pa}Q|}\int_{i_{\rm pa}Q}\left\|W^{1/p}(x)A_Q^{-1}\right\|^p {\rm d}x \approx \frac{1}{|i_{\rm pa}Q|}\int_{i_{\rm pa}Q}\left(\frac{1}{|Q|}\int_Q \left\|W^{1/p}(x)W^{-1/p}(y)\right\|^{p'} {\rm d}y\right)^{p/p'} {\rm d}x.$$

From [12, Lemma 2.2], we have $\|A_{i_{\rm pa}Q}A_Q^{-1}\|^p \lesssim 2^{i(\beta-n)}$. Hence

$$\frac{1}{|i_{\rm pa}Q|}\int_{i_{\rm pa}Q}\left(\frac{1}{|Q|}\int_Q \left\|W^{1/p}(x)W^{-1/p}(y)\right\|^{p'} {\rm d}y\right)^{p/p'} {\rm d}x \lesssim 2^{i(\beta-n)}.$$

This, together with [3, Proposition 2.28(ii)], further implies that $W^{-1/(p-1)} \in \mathcal{A}_{p'}$ has the $\mathcal{A}_{p'}$-dimension $\tilde{\tilde{d}} := (\beta-n)/(p-1)$ and hence

$$n + \tilde{\tilde{d}}p/p' = \beta.$$

Hence we show this estimate is better than $W((j+a)_{\rm pa}Q) \lesssim 2^{(j+a)\beta}W(Q)$.

Theorem 4.6. *Let $W \in \mathcal{A}_p$ has the $\mathcal{A}_p$-dimension $\tilde{d} \in [0, n)$. Let $\tilde{\beta}$ be the same with (24).*

(i) *Let $1 \leq p < t < r < \infty$. If $-nr/p + \tilde{d}r/p + n - \tilde{\beta}r(1/t - 1/p) < 0$, then for each $a \in \mathbb{Z}$, $E_{d,a}$ is a bounded operator on matrix weighted Bourgain-Morrey spaces $M_p^{t,r}(W)$.*

(ii) *Let $1 \leq p \leq t < r = \infty$. If $\tilde{d}/p - n/p - \tilde{\beta}(1/t - 1/p) \leq 0$, then for each $a \in \mathbb{Z}$, $E_{d,a}$ is a bounded operator on matrix weighted Bourgain-Morrey spaces $M_p^{t,r}(W)$.*

Proof. (i) For a fixed $a \in \mathbb{Z}$, we have

$$\begin{aligned}\|E_{d,a}\vec{f}\|_{M_p^{t,r}(W)} \leq &\left(\sum_{j\leq -a}\sum_{Q\in\mathcal{D}_j} W(Q)^{r/t-r/p}\left(\int_Q |W^{1/p}(x)E_{d,a}\vec{f}(x)|^p {\rm d}x\right)^{r/p}\right)^{1/r}\\ &+\left(\sum_{j> -a}\sum_{Q\in\mathcal{D}_j} W(Q)^{r/t-r/p}\left(\int_Q |W^{1/p}(x)E_{d,a}\vec{f}(x)|^p {\rm d}x\right)^{r/p}\right)^{1/r} =: S_1 + S_2.\end{aligned}$$

By Lemma 4.4, we obtain

$$S_1 \leq C_{n,d,p,W}\left(\sum_{j\leq -a}\sum_{Q\in\mathcal{D}_j} W(Q)^{r/t-r/p}\left(\int_Q |W^{1/p}(x)\vec{f}(x)|^p {\rm d}x\right)^{r/p}\right)^{1/r} \lesssim \|\vec{f}\|_{M_p^{t,r}(W)}.$$

When $j > -a$, we denote by $(j+a)_{\rm pa}Q \in \mathcal{D}_{-a}$ the $(j+a)$-th dyadic parent of $Q \in \mathcal{D}_j$. Let $\{A_Q\}_{Q\in\mathcal{D}}$ be the reducing operator of order p for W. By Lemmas 2.9, we have

$$\|A_Q A^{-1}_{(j+a)_{\rm pa}Q}\| \leq c2^{(j+a)\tilde{d}/p}. \tag{25}$$

In S_2, using (5) and (25) , we have

$$
\begin{aligned}
S_2^r &= \sum_{j>-a}\sum_{Q\in\mathcal{D}_j} W(Q)^{r/t-r/p}\left(\int_Q |W^{1/p}(x)E_{d,a}\vec{f}(x)|^p\mathrm{d}x\right)^{r/p}\\
&= \sum_{j>-a}\sum_{Q\in\mathcal{D}_j} W(Q)^{r/t-r/p}\left(\frac{|Q|}{|Q|}\int_Q \left|W^{1/p}(x)\frac{1}{|(j+a)_{\mathrm{pa}}Q|}\int_{(j+a)_{\mathrm{pa}}Q}\vec{f}(y)\mathrm{d}y\right|^p\mathrm{d}x\right)^{r/p}\\
&\approx \sum_{j>-a}\sum_{Q\in\mathcal{D}_j} W(Q)^{r/t-r/p}|Q|^{r/p}\left(\left|A_Q\frac{1}{|(j+a)_{\mathrm{pa}}Q|}\int_{(j+a)_{\mathrm{pa}}Q}\vec{f}(y)\mathrm{d}y\right|\right)^r\\
&\lesssim \sum_{j>-a}\sum_{Q\in\mathcal{D}_j} W(Q)^{r/t-r/p}|Q|^{r/p}\left(2^{(j+a)\tilde{d}/p}\left|A_{(j+a)_{\mathrm{pa}}Q}\frac{1}{|(j+a)_{\mathrm{pa}}Q|}\int_{(j+a)_{\mathrm{pa}}Q}\vec{f}(y)\mathrm{d}y\right|\right)^r\\
&\approx \sum_{j>-a}\sum_{Q\in\mathcal{D}_j} W(Q)^{r/t-r/p}|Q|^{r/p}2^{(j+a)\tilde{d}r/p}\left(\frac{1}{|(j+a)_{\mathrm{pa}}Q|}\int_{(j+a)_{\mathrm{pa}}Q}|W^{1/p}(x)\frac{1}{|(j+a)_{\mathrm{pa}}Q|}\int_{(j+a)_{\mathrm{pa}}Q}\vec{f}(y)\mathrm{d}y|^p\mathrm{d}x\right)^{r/p}\\
&= \sum_{j>-a}\sum_{Q\in\mathcal{D}_j} W(Q)^{r/t-r/p}2^{-jnr/p}2^{(j+a)\tilde{d}r/p}\left(\frac{1}{|(j+a)_{\mathrm{pa}}Q|}\int_{(j+a)_{\mathrm{pa}}Q}|W^{1/p}(x)E_{d,a}\vec{f}(x)|^p\mathrm{d}x\right)^{r/p}\\
&= \sum_{j>-a}\sum_{Q\in\mathcal{D}_j} W(Q)^{r/t-r/p}2^{-jnr/p}2^{(j+a)\tilde{d}r/p}2^{-anr/p}\left(\int_{(j+a)_{\mathrm{pa}}Q}|W^{1/p}(x)E_{d,a}\vec{f}(x)|^p\mathrm{d}x\right)^{r/p}.
\end{aligned}
$$

Remark that given $S\in\mathcal{D}_{-a}$, there are $2^{(j+a)n}$ cubes R such that $(j+a)_{\mathrm{pa}}R = S$. From Remark 4.5 and $1/t-1/p<0$, we have

$$
W(Q)^{1/t-1/p} \lesssim \begin{cases} 2^{-(j+a)n(1/t-1)}W((j+a)_{\mathrm{pa}}Q)^{1/t-1}, & \text{if } p=1,\\ 2^{-(j+a)(n+\tilde{\tilde{d}}p/p')(1/t-1/p)}W((j+a)_{\mathrm{pa}}Q)^{1/t-1/p}, & \text{if } 1<p<\infty. \end{cases}
$$

That is $W(Q)^{1/t-1/p}\lesssim 2^{-(j+a)\tilde{\beta}(1/t-1/p)}W((j+a)_{\mathrm{pa}}Q)^{1/t-1/p}$. Hence, by Lemma 4.4 and $-nr/p+\tilde{d}r/p+n-\tilde{\beta}r(1/t-1/p)<0$, we have

$$
\begin{aligned}
&\sum_{j>-a}\sum_{Q\in\mathcal{D}_j} W(Q)^{r/t-r/p}2^{-jnr/p}2^{(j+a)\tilde{d}r/p}2^{-anr/p}\left(\int_{(j+a)_{\mathrm{pa}}Q}|W^{1/p}(x)E_{d,a}\vec{f}(x)|^p\mathrm{d}x\right)^{r/p}\\
&= \sum_{j>-a}2^{-jnr/p}2^{(j+a)\tilde{d}r/p}2^{-anr/p}\sum_{Q\in\mathcal{D}_j} W(Q)^{r/t-r/p}\left(\int_{(j+a)_{\mathrm{pa}}Q}|W^{1/p}(x)E_{d,a}\vec{f}(x)|^p\mathrm{d}x\right)^{r/p}\\
&\le \sum_{j>-a}2^{-jnr/p}2^{(j+a)\tilde{d}r/p}2^{-anr/p}2^{(j+a)n}2^{-(j+a)\tilde{\beta}(r/t-r/p)}\sum_{S\in\mathcal{D}_{-a}} W(S)^{r/t-r/p}\left(\int_S|W^{1/p}(x)E_{d,a}\vec{f}(x)|^p\mathrm{d}x\right)^{r/p}\\
&\lesssim \|\vec{f}\|^r_{M_p^{t,r}(W)}.
\end{aligned}
$$

(ii) Fix $Q_{j,k}\in\mathcal{D}$. If $j\le -a$, then by Lemma 4.4, we have

$$
W(Q_{j,k})^{1/t-1/p}\left(\int_{Q_{j,k}}|W^{1/p}(x)E_{d,a}\vec{f}(x)|^p\mathrm{d}x\right)^{1/p} \lesssim W(Q_{j,k})^{1/t-1/p}\left(\int_{Q_{j,k}}|W^{1/p}(x)\vec{f}(x)|^p\mathrm{d}x\right)^{1/p}.
$$

If $j > -a$, we denote by $(j+a)_{\rm pa}Q_{j,k} \in \mathcal{D}_{-a}$ the $(j+a)$-th dyadic parent of $Q_{j,k} \in \mathcal{D}_j$.

$$\begin{aligned}
&W(Q_{j,k})^{1/t-1/p}\left(\int_{Q_{j,k}} |W^{1/p}(x)E_{d,a}\vec{f}(x)|^p \mathrm{d}x\right)^{1/p}\\
&= W(Q_{j,k})^{1/t-1/p}\left(\frac{|Q_{j,k}|}{|Q_{j,k}|}\int_{Q_{j,k}} \left|W^{1/p}(x)\frac{1}{|(j+a)_{\rm pa}Q_{j,k}|}\int_{(j+a)_{\rm pa}Q_{j,k}} \vec{f}(y)\mathrm{d}y\right|^p \mathrm{d}x\right)^{1/p}\\
&\approx W(Q_{j,k})^{1/t-1/p}|Q_{j,k}|^{1/p}\left|A_{Q_{j,k}}\frac{1}{|(j+a)_{\rm pa}Q_{j,k}|}\int_{(j+a)_{\rm pa}Q_{j,k}} \vec{f}(y)\mathrm{d}y\right|\\
&\lesssim W(Q_{j,k})^{1/t-1/p}|Q_{j,k}|^{1/p}2^{(j+a)\tilde{d}/p}\left|A_{(j+a)_{\rm pa}Q_{j,k}}\frac{1}{|(j+a)_{\rm pa}Q_{j,k}|}\int_{(j+a)_{\rm pa}Q_{j,k}} \vec{f}(y)\mathrm{d}y\right|\\
&\approx W(Q_{j,k})^{1/t-1/p}2^{-jn/p}2^{(j+a)\tilde{d}/p}\left(\frac{1}{|(j+a)_{\rm pa}Q_{j,k}|}\int_{(j+a)_{\rm pa}Q_{j,k}} \left|W^{1/p}(x)\frac{1}{|(j+a)_{\rm pa}Q_{j,k}|}\int_{(j+a)_{\rm pa}Q_{j,k}} \vec{f}(y)\mathrm{d}y\right|^p \mathrm{d}x\right)^{1/p}.
\end{aligned}$$

Then by Lemma 4.4 and $\tilde{d}/p - n/p - \tilde{\beta}(1/t-1/p) \le 0$, we obtain

$$\begin{aligned}
&W(Q_{j,k})^{1/t-1/p}\left(\int_{Q_{j,k}} |W^{1/p}(x)E_{d,a}\vec{f}(x)|^p \mathrm{d}x\right)^{1/p}\\
&\lesssim W(Q_{j,k})^{1/t-1/p}2^{-jn/p}2^{(j+a)\tilde{d}/p}\left(\frac{1}{|(j+a)_{\rm pa}Q_{j,k}|}\int_{(j+a)_{\rm pa}Q_{j,k}} \left|W^{1/p}(x)\vec{f}(x)\right|^p \mathrm{d}x\right)^{1/p}\\
&= W(Q_{j,k})^{1/t-1/p}2^{-jn/p}2^{(j+a)\tilde{d}/p}2^{-an/p}\left(\int_{(j+a)_{\rm pa}Q_{j,k}} \left|W^{1/p}(x)\vec{f}(x)\right|^p \mathrm{d}x\right)^{1/p}\\
&\lesssim 2^{(j+a)(\tilde{d}/p-n/p)}2^{-(j+a)\tilde{\beta}(1/t-1/p)}W((j+a)_{\rm pa}Q_{j,k})^{1/t-1/p}\left(\int_{(j+a)_{\rm pa}Q_{j,k}} \left|W^{1/p}(x)\vec{f}(x)\right|^p \mathrm{d}x\right)^{1/p}\\
&\lesssim W((j+a)_{\rm pa}Q_{j,k})^{1/t-1/p}\left(\int_{(j+a)_{\rm pa}Q_{j,k}} \left|W^{1/p}(x)\vec{f}(x)\right|^p \mathrm{d}x\right)^{1/p}.
\end{aligned}$$

Finally, take the supremum over the cubes $Q_{j,k} \in D$, we obtain

$$\|E_{d,a}\vec{f}\|_{M_p^{t,\infty}(W)} \lesssim \|\vec{f}\|_{M_p^{t,\infty}(W)}.$$

Hence we finish the proof. □

Finally, we have the following Kolmogorov-Riesz compactness theorem for matrix weighted Bourgain-Morrey spaces.

Theorem 4.7. *Let $1 \le p < t < r < \infty$. Let $W \in \mathcal{A}_p$ with the $\mathcal{A}_p$-dimension $\tilde{d} \in [0, n)$. Let $\tilde{\beta}$ be the same with (24). Let $-nr/p + \tilde{d}r/p + n - \tilde{\beta}r(1/t - 1/p) < 0$. A subset $\mathcal{F}$ of $M_p^{t,r}(W)$ is totally bounded if and only if the following conditions hold:*

(i) *$\mathcal{F}$ is bounded, that is,*

$$\sup_{\vec{f}\in\mathcal{F}} \|\vec{f}\|_{M_p^{t,r}(W)} < \infty;$$

(ii) *$\mathcal{F}$ uniformly vanishes at infinity, that is,*

$$\lim_{R\to\infty}\sup_{\vec{f}\in\mathcal{F}} \|\vec{f}\chi_{B^c(0,R)}\|_{M_p^{t,r}(W)} = 0;$$

(iii) $\mathcal{F}$ *is equicontinuous by means of the dyadic average operator, that is, for any*

$$\lim_{a\to-\infty, a\in\mathbb{Z}} \sup_{\vec{f}\in\mathcal{F}} \|\vec{f}-E_{d,a}\vec{f}\|_{M_p^{t,r}(W)} = 0.$$

where $E_{d,a}$ is same as (19).

Proof. The sufficiency is due to Theorem 4.1. Now we prove the necessity. Assume that $\mathcal{F}$ of $M_p^{t,r}(W)$ is totally bounded. For any given $\epsilon > 0$, there exists $\{\vec{f_k}\}_{k=1}^{N_0} \subset \mathcal{F}$ such that $\{\vec{f_k}\}_{k=1}^{N_0}$ is an ϵ-net of $\mathcal{F}$, that is, for any $\vec{f}\in\mathcal{F}$, there exists $\vec{f_k}, k\in\{1,2,\dots,N_0\}$ such that $\|\vec{f}-\vec{f_k}\|_{M_p^{t,r}(W)} < \epsilon$.

Clearly, (i) is true.

For any $\epsilon > 0$ and $k \in \{1,2,\dots,N_0\}$, since $r < \infty$, there exists $R_k > 0$ such that

$$\|\vec{f_k}\chi_{B^c(0,R_k)}\|_{M_p^{t,r}(W)} < \epsilon.$$

Taking $R = \max_{k\in\{1,\dots,N_0\}}\{R_k\}$, we have $\|\vec{f_k}\chi_{B^c(0,R)}\|_{M_p^{t,r}(W)} < \epsilon$. Then for any $\vec{f}\in\mathcal{F}$,

$$\|\vec{f}\chi_{B^c(0,R)}\|_{M_p^{t,r}(W)} \le \|(\vec{f}-\vec{f_k})\chi_{B^c(0,R)}\|_{M_p^{t,r}(W)} + \|\vec{f_k}\chi_{B^c(0,R)}\|_{M_p^{t,r}(W)} < 2\epsilon.$$

Thus we prove that

$$\lim_{R\to\infty}\sup_{\vec{f}\in\mathcal{F}} \|\vec{f}\chi_{B^c(0,R)}\|_{M_p^{t,r}(W)} = 0.$$

Therefore, we prove that $\mathcal{F}$ satisfies condition (ii).

As for (iii), for each $1 \le k \le N_0$, by Theorem 4.6, we have for each $a \in \mathbb{Z}$,

$$\|\vec{f_k} - E_{d,a}\vec{f_k}\|_{M_p^{t,r}(W)} \le (1+c)\|\vec{f_k}\|_{M_p^{t,r}(W)}.$$

Thus using the dominated convergence theorem to obtain that there exists $a_0 \in \mathbb{Z}$ such that for any $a \le a_0$,

$$\max_{1\le k\le N_0} \|\vec{f_k} - E_{d,a}\vec{f_k}\|_{M_p^{t,r}(W)} < \epsilon.$$

Now if $a \le a_0$, then

$$\begin{aligned}\|\vec{f}-E_{d,a}\vec{f}\|_{M_p^{t,r}(W)} &\le \|E_{d,a}\vec{f}-E_{d,a}\vec{f_k}\|_{M_p^{t,r}(W)} + \|E_{d,a}\vec{f_k}-\vec{f_k}\|_{M_p^{t,r}(W)} + \|\vec{f_k}-\vec{f}\|_{M_p^{t,r}(W)}\\ &\le c\|\vec{f}-\vec{f_k}\|_{M_p^{t,r}(W)} + \epsilon + \epsilon \lesssim \epsilon.\end{aligned}$$

Then the proof is complete. □

Remark 4.8. *Theorem 4.7 is not true for $1 < p < t < r = \infty$. Indeed, let $d = 1, W \equiv 1$, $0 < p < t < r = \infty$. Let $f(x) = |x|^{-n/t}$. Then*

$$\|f\|_{M_p^{t,\infty}} = \sup_{Q\in\mathcal{D}} |Q|^{1/t-1/p}\left(\int_Q |y|^{-np/t}\mathrm{d}y\right)^{1/p} \approx 1.$$

Let $f_j = \chi_{B(0,2^j)}f$ for $j \in \mathbb{N}$. Note that f and f_j are radial and symmetric functions. Let $x = (x_1, 0, \dots, 0)$ where $x_1 = 2^j + 2\times 2^j$. Let $s = x_1/10$. Then $B(x,s) \subset B(0,2^j)^c$. Then

$$\begin{aligned}\|f-f_j\|_{M_p^{t,\infty}} &\ge cs^{-n(1/t-1/p)}\left(\int_{B(x,s)}|y|^{-np/t}\mathrm{d}y\right)^{1/p} \ge cs^{-n(1/t-1/p)}\left(\int_{|x|-|s|}^{|x|+|s|}\eta^{-np/t}\eta^{n-1}\mathrm{d}\eta\right)^{1/p}\\ &= c\left(\frac{|x|}{10}\right)^{-n(1/t-1/p)}\left(\left(1+\frac{1}{10}\right)^{n-np/t} - \left(1-\frac{1}{10}\right)^{n-np/t}\right)^{1/p}|x|^{n(1/t-1/p)}\\ &= c > 0.\end{aligned}$$

This shows that the totally bounded set $\mathcal{F} = \{f\}$ of $M_p^{t,\infty}$ is not uniformly vanishes at infinity.

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