Filomat 39:18 (2025), 6281–6293
https://doi.org/10.2298/FIL2518281B

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Approximation by bivariate Szász–Mirakyan operators preserving $e^{-2(p_1+p_2)}$

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Abstract. The present paper is dedicated to the modification of the bivariate generalized Szász–Mirakyan operators while preserving the exponential functions $\exp(2,2)$ where $\exp(\tau_1,\tau_2) = e^{-\tau_1 p_1 - \tau_2 p_2}, \tau_1, \tau_2 \in \mathbb{R}_0^+$, and $p_1, p_2 \geq 0$. We thoroughly investigate the weighted approximation properties and also obtain the convergence rate for these operators by utilizing a weighted modulus of continuity. Additionally, we delve into the order of approximation by investigating local approximation results through Peetre's $\mathcal{K}$-functional. Furthermore, we present the GBS (Generalized Boolean Sum) operators of Szász–Mirakyan operators and obtain the order of approximation in terms of the Lipschitz class of Bögel continuous functions and the mixed modulus of smoothness. In order to enhance our theoretical findings and effectively showcase the efficiency of our developed operators, we have included a wide range of numerical examples using various values.

1. Introduction

The classical Szász–Mirakyan operators [14, 22, 26] are defined by

$$S_n(c)(p_1) := S_n(c;p_1) = \sum_{\eta_1=0}^{\infty} e^{-np_1} \frac{(np_1)^{\eta_1}}{\eta_1!} c\left(\frac{\eta_1}{n}\right) \tag{1}$$

where $p_1 \geq 0$, $n \in \mathbb{N}$ and c is any real valued function defined on $[0,\infty)$ such that $(S_n|c|)(p_1) < \infty$.

In 1944, the celebrated French mathematician Jean Favard [14] also introduced the classical bivariate Szász–Mirakyan operators as

$$S_{n,m}(c;p_1,p_2) = \sum_{\eta_1=0}^{\infty} \sum_{\eta_2=0}^{\infty} e^{-np_1} e^{-mp_2} \frac{(np_1)^{\eta_1}}{\eta_1!} \frac{(mp_2)^{\eta_2}}{\eta_2!} c\left(\frac{\eta_1}{n}, \frac{\eta_2}{m}\right) \tag{2}$$

where $n, m \in \mathbb{N}$ and for $R_+ := [0,\infty)$, $(p_1,p_2) \in \mathbb{R}_+ \times \mathbb{R}_+$. Several papers have been published on the bivariate Szász–Mirakyan operators such as [13, 19, 23, 24, 27] and references therein.

2020 *Mathematics Subject Classification*. Primary 41A36; Secondary 41A25.
Keywords. Szász–Mirakyan operators, exponential functions, weighted modulus of continuity, Peetre's $\mathcal{K}$-functional, GBS operators, mixed modulus of smoothness.
Received: 22 November 2024; Accepted: 29 June 2025
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In 2003, King [21] introduced a significant class of positive linear operators, known as King type operators, which preserve $e_i(\mu) = \mu^i$, $i = 0, 2$ test functions and achieve a better approximation compared to the classical type operators on certain intervals. Despite the plethora of publications on King's method, we have deliberately narrowed our focus to the works of Aldaz and Render [3], and Birou papers [7] because of the close objectives in this paper.

In 2017, Acar et al. [1] proposed the univariate Szász–Mirakyan type operators which reproduce the functions e_0 and $e^{2kp_1}, k > 0$ fixed as

$$R^*_{n,k}(c;p_1) := R^*_n(c;p_1) := \sum_{\eta_1=0}^{\infty} e^{-n\alpha_n(p_1)} \frac{(n\alpha_n(p_1))^{\eta_1}}{\eta_1!} c\left(\frac{\eta_1}{n}\right) \tag{3}$$

where $p_1 \geq 0$, $n \in \mathbb{N}$ and $c \in C[0,\infty)$. Here,

$$\alpha_{n,k}(p_1) := \alpha_n(p_1) = \frac{2kp_1}{n(e^{2k/n}-1)}.$$

In the same year, Gupta and Malik [18] present the Szász–Mirakyan type operators that reproduce the functions e_0 and e^{-2p_1} as

$$S^*_n(c;p_1) := \sum_{\eta_1=0}^{\infty} e^{-n\gamma_n(p_1)} \frac{(n\gamma_n(p_1))^{\eta_1}}{\eta_1!} c\left(\frac{\eta_1}{n}\right) \tag{4}$$

where $p_1 \geq 0$, $n \in \mathbb{N}$ and $c \in C[0,\infty)$. Here,

$$\gamma_n(p_1) = \frac{2p_1 e^{2/n}}{n(e^{2/n}-1)}.$$

For $\gamma_n(p_1) = p_1$, the operators represented as (4) reduce to the classical Szász–Mirakyan operators (1) and also $\lim_{n\to\infty} \gamma_n(p_1) = p_1$.

The authors have provided better approximation and demonstrated uniform convergence. Additionally, they conducted a comparative analysis.

So far, many papers have been published on preserving exponential functions. Some of these papers, including bivariate ones, provide historical background (see references [2, 8, 25]).

Inspired by the aforementioned articles, we present a novel bivariate modification of the Szász–Mirakyan operators which reproduces exp(2, 2) as

$$\mathcal{S}_{n,m}(c;p_1,p_2) = \sum_{\eta_1=0}^{\infty}\sum_{\eta_2=0}^{\infty} e^{-n\gamma_n(p_1)} e^{-m\gamma_m(p_2)} \frac{(n\gamma_n(p_1))^{\eta_1}}{\eta_1!} \frac{(m\gamma_m(p_2))^{\eta_2}}{\eta_2!} c\left(\frac{\eta_1}{n}, \frac{\eta_2}{m}\right) \tag{5}$$

where

$$\gamma_n(p_1) = \frac{2p_1 e^{2/n}}{n(e^{2/n}-1)} = p_1\beta_n$$

and

$$\gamma_m(p_2) = \frac{2p_2 e^{2/m}}{m(e^{2/m}-1)} = p_2\beta_m.$$

Here, $n, m \in \mathbb{N}$, $T := ((p_1,p_2) \in \mathbb{R}_+ \times \mathbb{R}_+ : 0 \leq p_1, p_2 \leq \infty)$ and $c \in C_B(T)$ where $C_B(T)$ be the space of all continuous and bounded real valued functions on T.

Hence, the explicit form of $\mathcal{S}_{n,m}$ can be written as

$$\mathcal{S}_{n,m}(c;p_1,p_2) = \sum_{\eta_1=0}^{\infty}\sum_{\eta_2=0}^{\infty} e^{\frac{2p_1 e^{2/n}}{(1-e^{2/n})}} e^{\frac{2p_2 e^{2/m}}{(1-e^{2/m})}} \frac{(2p_1 e^{2/n})^{\eta_1}}{\eta_1!(e^{2/n}-1)^{\eta_1}} \frac{(2p_2 e^{2/m})^{\eta_2}}{\eta_2!(e^{2/m}-1)^{\eta_2}} c\left(\frac{\eta_1}{n}, \frac{\eta_2}{m}\right). \tag{6}$$

This article goals to introduce the bivariate Szász–Mirakyan operators that preserve the exponential functions exp(2, 2). Our investigation delves into the weighted approximation properties and examines the degree of approximation using Peetre's $\mathcal{K}$-functional. Furthermore, we bring in the Generalized Boolean Sum (GBS) operators of Szász–Mirakyan operators, emphasizing their ability to preserve the exponential functions exp(2, 2), and we also analyze the order of approximation using some techniques. We conclusively demonstrate the convergence of the classical bivariate Szász–Mirakyan operators and our operators with various parameters through numerical illustrations.

Now, we provide basic lemmas that will be necessary to prove the main results presented in the paper. For the sake of brevity, we will not prove them, but a detailed version can be seen in [19].

Lemma 1.1. *Let $n, m \in \mathbb{N}_0$ $(\mathbb{N} \cup \{0\})$ and $e_{i,j} = \mu^i \nu^j$ with $0 \le i + j \le 4$ be the bivariate test functions. Then, the bivariate Szász–Mirakyan operators $\mathcal{S}_{n,m}$ satisfy*

$$\begin{aligned}
&\mathcal{S}_{n,m}(e_{0,0}; p_1, p_2) = 1,\\
&\mathcal{S}_{n,m}(e_{1,0}; p_1, p_2) = \gamma_n(p_1) = p_1\beta_n,\\
&\mathcal{S}_{n,m}(e_{0,1}; p_1, p_2) = \gamma_m(p_2) = p_2\beta_m,\\
&\mathcal{S}_{n,m}(e_{2,0}; p_1, p_2) = \gamma_n^2(p_1) + \frac{\gamma_n(p_1)}{n},\\
&\mathcal{S}_{n,m}(e_{0,2}; p_1, p_2) = \gamma_m^2(p_2) + \frac{\gamma_m(p_2)}{m},\\
&\mathcal{S}_{n,m}(e_{3,0}; p_1, p_2) = \gamma_n^3(p_1) + \frac{3\gamma_n^2(p_1)}{n} + \frac{\gamma_n(p_1)}{n^2},\\
&\mathcal{S}_{n,m}(e_{0,3}; p_1, p_2) = \gamma_m^3(p_2) + \frac{3\gamma_m^2(p_2)}{m} + \frac{\gamma_m(p_2)}{m^2},\\
&\mathcal{S}_{n,m}(e_{4,0}; p_1, p_2) = \gamma_n^4(p_1) + \frac{6\gamma_n^3(p_1)}{n} + \frac{7\gamma_n^2(p_1)}{n^2} + \frac{\gamma_n(p_1)}{n^3},\\
&\mathcal{S}_{n,m}(e_{0,4}; p_1, p_2) = \gamma_m^4(p_2) + \frac{6\gamma_m^3(p_2)}{m} + \frac{7\gamma_m^2(p_2)}{m^2} + \frac{\gamma_m(p_2)}{m^3}.
\end{aligned}$$

Lemma 1.2. *Let $n, m \in \mathbb{N}_0$, $\mu = e_{1,0}$ and $\nu = e_{0,1}$. From Lemma 1.1, we possess*

$$\mathcal{S}_{n,m}\left((\mu - p_1)^1; p_1, p_2\right) = \gamma_n(p_1) - p_1,$$

$$\mathcal{S}_{n,m}\left((\nu - p_2)^1; p_1, p_2\right) = \gamma_m(p_2) - p_2,$$

$$\mathcal{S}_{n,m}\left((\mu - p_1)^2; p_1, p_2\right) = (\gamma_n(p_1) - p_1)^2 + \frac{\gamma_n(p_1)}{n} = p_1^2(\beta_n - 1)^2 + p_1\frac{\beta_n}{n} := \varrho_n(p_1), \tag{7}$$

$$\mathcal{S}_{n,m}\left((\nu - p_2)^2; p_1, p_2\right) = (\gamma_m(p_2) - p_2)^2 + \frac{\gamma_m(p_2)}{m} = p_2^2(\beta_m - 1)^2 + p_2\frac{\beta_m}{m} := \varrho_m(p_2), \tag{8}$$

$$\mathcal{S}_{n,m}\left((\mu - p_1)^3; p_1, p_2\right) = (\gamma_n(p_1) - p_1)^3 + \frac{3\gamma_n^2(p_1) - 3p_1\gamma_n(p_1)}{n} + \frac{\gamma_n(p_1)}{n^2},$$

$$\mathcal{S}_{n,m}\left((\nu - p_2)^3; p_1, p_2\right) = (\gamma_m(p_2) - p_2)^3 + \frac{3\gamma_m^2(p_2) - 3p_2\gamma_m(p_2)}{m} + \frac{\gamma_m(p_2)}{m^2},$$

$$\begin{aligned}\mathcal{S}_{n,m}\left((\mu - p_1)^4; p_1, p_2\right) = &(\gamma_n(p_1) - p_1)^4 + \frac{6\gamma_n^3(p_1) - 12p_1\gamma_n^2(p_1) + 6p_1^2\gamma_n(p_1)}{n}\\
&+ \frac{7\gamma_n^2(p_1) - 4p_1\gamma_n(p_1)}{n^2} + \frac{\gamma_n(p_1)}{n^3} := \varsigma_n(p_1),\end{aligned} \tag{9}$$

$$\begin{aligned}\mathcal{S}_{n,m}\left((\nu - p_2)^4; p_1, p_2\right) = &(\gamma_m(p_2) - p_2)^4 + \frac{6\gamma_m^3(p_2) - 12p_2\gamma_m^2(p_2) + 6p_2^2\gamma_m(p_2)}{m}\\
&+ \frac{7\gamma_m^2(p_2) - 4p_2\gamma_m(p_2)}{m^2} + \frac{\gamma_m(p_2)}{m^3} := \varsigma_m(p_2).\end{aligned} \tag{10}$$

Lemma 1.3. *Let us take an arbitrary function $c \in C_B(T)$. We have*

$$\|\mathcal{S}_{n,m}(c;\cdot,\cdot)\| \leq \|c\|$$

where $\|\cdot\|$ denotes the uniform norm on $C_B(T)$.

Proof. Considering Lemma 1.1, one can easily prove Lemma 1.3. □

2. Weighted Approximation

In this part, we comprehensively investigate the weighted uniform approximation properties of the sequence of the bivariate Szász–Mirakyan, taking into account a variety of conditions, thereby allowing for a more thorough analysis.
Let $\kappa(p_1,p_2) = 1 + p_1^2 + p_2^2$ be a weight function such that
$B_\kappa\left(\mathbb{R}_+^2\right) = \left\{c : \left|c(p_1,p_2)\right| \leq M_c\kappa(p_1,p_2), M_c > 0\right\}$ and M_c is a constant depending on c. Denote that:

$$C_\kappa\left(\mathbb{R}_+^2\right) = \left\{c \in B_\kappa\left(\mathbb{R}_+^2\right) : c \text{ is continuous}\right\},$$
$$C_\kappa^k\left(\mathbb{R}_+^2\right) = \left\{c \in C_\kappa\left(\mathbb{R}_+^2\right) : \lim_{p_1,p_2\to\infty} \frac{c(p_1,p_2)}{\kappa(p_1,p_2)} = k_c < \infty\right\},$$
$$C_\kappa^0\left(\mathbb{R}_+^2\right)^k = \left\{c \in C_\kappa^k\left(\mathbb{R}_+^2\right) : \lim_{p_1,p_2\to\infty} \frac{c(p_1,p_2)}{\kappa(p_1,p_2)} = 0\right\},$$

respectively, where k_c is a constant depending on c. The norm on B_κ defined as $\|c\|_\kappa = \sup_{p_1,p_2\in\mathbb{R}_+^2} \frac{|c(p_1,p_2)|}{\kappa(p_1,p_2)}$.
For all $c \in C_\kappa^0\left(\mathbb{R}_+^2\right)$ and $\delta_1, \delta_2 > 0$, the weighted modulus of continuity can be given (see [17] and [20]) as

$$\omega_\kappa(c;\delta_1,\delta_2) = \sup_{p_1,p_2\in[0,\infty)} \sup_{0\leq|h_1|\leq\delta_1, 0\leq|h_2|\leq\delta_2} \frac{\left|c(p_1+h_1,p_2+h_2) - h(p_1,p_2)\right|}{\kappa(p_1,p_2)\,\kappa(h_1,h_2)}$$

and for any $j_1, j_2 > 0$, it satisfies the inequality

$$\omega_\kappa(c;j_1\delta_1,j_2\delta_2) \leq 4(1+j_1)(1+j_2)\left(1+\delta_1^2\right)\left(1+\delta_2^2\right)\omega_\kappa(c;\delta_1,\delta_2). \tag{11}$$

It also follows that

$$\begin{aligned}\left|c(\mu,\nu) - c(p_1,p_2)\right| &\leq \kappa(p_1,p_2)\,\kappa\left(\left|\mu-p_1\right|,\left|\nu-p_2\right|\right)\omega_\kappa\left(c;\left|\mu-p_1\right|,\left|\nu-p_2\right|\right)\\ &\leq \left(1+p_1^2+p_2^2\right)\left(1+(\mu-p_1)^2\right)\left(1+(\nu-p_2)^2\right)\omega_\kappa\left(c;\left|\mu-p_1\right|,\left|\nu-p_2\right|\right).\end{aligned} \tag{12}$$

Lemma 2.1 ([15, 16]). *For the sequence of positive linear operators $\{M_{n,m}\}_{n,m\geq1}$ acting from $C_\kappa(\mathbb{R}_+^2)$ to $B_\kappa(\mathbb{R}_+^2)$ then there exists some positive C such that*

$$\left\|M_{n,m}(\kappa)\right\|_\kappa \leq C.$$

Theorem 2.2 ([15, 16]). *If a sequence of positive linear operators $\{M_{n,m}\}_{n,m\geq1}$ acting from $C_\kappa(\mathbb{R}_+^2)$ to $B_\kappa(\mathbb{R}_+^2)$ satisfies the conditions*

$$\lim_{n,m\to\infty}\left\|M_{n,m}(e_{0,0}) - e_{0,0}\right\|_\kappa = 0,$$
$$\lim_{n,m\to\infty}\left\|M_{n,m}(e_{1,0}) - e_{1,0}\right\|_\kappa = 0,$$
$$\lim_{n,m\to\infty}\left\|M_{n,m}(e_{0,1}) - e_{0,1}\right\|_\kappa = 0,$$

$$\lim_{n,m\to\infty}\left\|M_{n,m}\left(\left(e_{2,0}+e_{0,2}\right)\right)-\left(e_{2,0}+e_{0,2}\right)\right\|_{\kappa}=0,$$

then, for any function $c\in C_{\kappa}^{0}\left(\mathbb{R}_{+}^{2}\right)$

$$\lim_{n,m\to\infty}\left\|M_{n,m}c-c\right\|_{\kappa}=0,$$

and there exists a function $c^{*}\in C_{\kappa}(\mathbb{R}_{+}^{2})\setminus C_{\kappa}^{0}(\mathbb{R}_{+}^{2})$, *such that*

$$\lim_{n,m\to\infty}\left\|M_{n,m}c^{*}-c^{*}\right\|_{\kappa}\geq 1.$$

Theorem 2.3. *For all* $c\in C_{\kappa}^{0}\left(\mathbb{R}_{+}^{2}\right)$, *the operators* $\{\mathcal{S}_{n,m}\}_{n,m\in\mathbb{N}}$ *satisfy*

$$\lim_{n,m\to\infty}\left\|\mathcal{S}_{n,m}c-c\right\|_{\kappa}=0.$$

Proof. Firstly, we need to show that $\mathcal{S}_{n,m}$ is acting from C_{κ} to B_{κ}.

$$\begin{aligned}\left\|\mathcal{S}_{n,m}(\kappa)\right\|_{\kappa}&=\sup_{(p_1,p_2)\in\mathbb{R}_{+}^{2}}\frac{\left|\mathcal{S}_{n,m}\left(1+\mu^{2}+\nu^{2};p_1,p_2\right)\right|}{1+p_1^{2}+p_2^{2}}\\&\leq\sup_{(p_1,p_2)\in\mathbb{R}_{+}^{2}}\left[\frac{1}{1+p_1^{2}+p_2^{2}}\left|\left(1+\mathcal{S}_{n,m}\left(\mu^{2};p_1,p_2\right)+\mathcal{S}_{n,m}\left(\nu^{2};p_1,p_2\right)\right)\right|\right]\\&=\sup_{(p_1,p_2)\in\mathbb{R}_{+}^{2}}\left[\frac{\left|1+p_1^{2}\beta_n^{2}+p_1\frac{\beta_n}{n}+p_2^{2}\beta_m^{2}+p_2\frac{\beta_m}{m}\right|}{1+p_1^{2}+p_2^{2}}\right]\\&\leq\left[1+\beta_n^{2}+\beta_m^{2}+\frac{\beta_n}{n}+\frac{\beta_m}{m}\right]=1+\Phi_{n,m}+\Pi_{n,m},\end{aligned}$$

where $\Phi_{n,m}=\beta_n^{2}+\beta_m^{2}$ and $\Pi_{n,m}=\frac{1}{n}+\frac{1}{m}$. Since $\lim_{n,m\to\infty}\Phi_{n,m}=2$ and $\lim_{n,m\to\infty}\Pi_{n,m}=0$, there exits a positive constant C. Indeed $\Phi_{n,m}+\Pi_{n,m}<C$. Therefore, we have $\left\|\mathcal{S}_{n,m}(\kappa)\right\|_{\kappa}\leq 1+C$. It suffices to show that the conditions of Theorem 2.2 are satisfied. After applying Lemma 1.1, we can get desired result, easily. □

Theorem 2.4. *For all* $c\in C_{\kappa}^{0}\left(\mathbb{R}_{+}^{2}\right)$, *we have*

$$\frac{\left|\mathcal{S}_{n,m}\left(c;p_1,p_2\right)-c\left(p_1,p_2\right)\right|}{\kappa\left(p_1,p_2\right)}\leq M\omega_{\kappa}\left(c;\delta_{n,p_1},\delta_{m,p_2}\right),$$

where M *is a positive constant,* $\delta_{n,p_1}=(\varrho_n(p_1))^{1/2}$ *as in* (7), $\delta_{m,p_2}=(\varrho_m(p_2))^{1/2}$ *as in* (8), $(\mathcal{S}_{n,m}((\mu-p_1)^{4};p_1,p_2))^{1/2}=(\varsigma_n(p_1))^{1/2}$ *as in* (9) *and* $(\mathcal{S}_{n,m}((\nu-p_2)^{4};p_1,p_2))^{1/2}=(\varsigma_m(p_2))^{1/2}$ *as in* (10).

Proof. Bearing in mind (11) and (12), we can proceed with

$$\begin{aligned}\left|c(\mu,\nu)-c(p_1,p_2)\right|&\leq\kappa(p_1,p_2)\,\kappa\left(\left|\mu-p_1\right|,\left|\nu-p_2\right|\right)\omega_{\kappa}\left(c;\left|\mu-p_1\right|,\left|\nu-p_2\right|\right)\\&\leq\left(1+p_1^{2}+p_2^{2}\right)\left(1+(\mu-p_1)^{2}\right)\left(1+(\nu-p_2)^{2}\right)\omega_{\kappa}\left(c;\left|\mu-p_1\right|,\left|\nu-p_2\right|\right)\\&\leq 4\left(1+p_1^{2}+p_2^{2}\right)\left(1+\delta_{n,p_1}^{2}\right)\left(1+\delta_{m,p_2}^{2}\right)\left(1+(\mu-p_1)^{2}\right)\left(1+(\nu-p_2)^{2}\right)\\&\quad\times\left(1+\frac{\left|\mu-p_1\right|}{\delta_{n,p_1}}\right)\left(1+\frac{\left|\nu-p_2\right|}{\delta_{m,p_2}}\right)\omega_{\kappa}\left(c;\delta_{n,p_1},\delta_{m,p_2}\right)\end{aligned}$$

$$
\begin{aligned}
&=4\left(1+p_1^2+p_2^2\right)\left(1+\delta_{n,p_1}^2\right)\left(1+\delta_{m,p_2}^2\right)\\
&\times\left(1+\frac{\left|\mu-p_1\right|}{\delta_{n,p_1}}+(\mu-p_1)^2+\frac{\left|\mu-p_1\right|}{\delta_{n,p_1}}(\mu-p_1)^2\right)\\
&\times\left(1+\frac{\left|\nu-p_2\right|}{\delta_{m,p_2}}+(\nu-p_2)^2+\frac{\left|\nu-p_2\right|}{\delta_{m,p_2}}(\nu-p_2)^2\right)\omega_\kappa\left(c;\delta_{n,p_1},\delta_{m,p_2}\right).
\end{aligned}
$$

Applying operators $\mathcal{S}_{n,m}$ within monotonicity and linearity, we find

$$
\begin{aligned}
\left|\mathcal{S}_{n,m}(c;p_1,p_2)-c(p_1,p_2)\right| \leq&\mathcal{S}_{n,m}\left(\left|c(\mu,\nu)-c(p_1,p_2)\right|;p_1,p_2\right)\\
\leq&4\left(1+p_1^2+p_2^2\right)\left(1+\delta_{n,p_1}^2\right)\left(1+\delta_{m,p_2}^2\right)\\
&\times\mathcal{S}_{n,m}\left(1+\frac{\left|\mu-p_1\right|}{\delta_{n,p_1}}+(\mu-p_1)^2+\frac{\left|\mu-p_1\right|}{\delta_{n,p_1}}(\mu-p_1)^2;p_1,p_2\right)\\
&\times\mathcal{S}_{n,m}\left(1+\frac{\left|\nu-p_2\right|}{\delta_{m,p_2}}+(\nu-p_2)^2+\frac{\left|\nu-p_2\right|}{\delta_{m,p_2}}(\nu-p_2)^2;p_1,p_2\right)\omega_\kappa\left(c;\delta_{n,p_1},\delta_{m,p_2}\right)\\
=&4\left(1+p_1^2+p_2^2\right)\left(1+\delta_{n,p_1}^2\right)\left(1+\delta_{m,p_2}^2\right)\\
&\times\left(1+\frac{1}{\delta_{n,p_1}}\mathcal{S}_{n,m}\left(\left|\mu-p_1\right|;p_1,p_2\right)+\mathcal{S}_{n,m}\left((\mu-p_1)^2;p_1,p_2\right)\right.\\
&\left.+\frac{1}{\delta_{n,p_1}}\mathcal{S}_{n,m}\left(\left|\mu-p_1\right|(\mu-p_1)^2;p_1,p_2\right)\right)\\
&\times\left(1+\frac{1}{\delta_{m,p_2}}\mathcal{S}_{n,m}\left(\left|\nu-p_2\right|;p_1,p_2\right)+\left(\mathcal{S}_{n,m}(\nu-p_2)^2;p_1,p_2\right)\right.\\
&\left.+\frac{1}{\delta_{m,p_2}}\mathcal{S}_{n,m}\left(\left|\nu-p_2\right|(\nu-p_2)^2;p_1,p_2\right)\right)\omega_\kappa\left(c;\delta_{n,p_1},\delta_{m,p_2}\right).
\end{aligned}
$$

Using Cauchy-Schwarz inequality, we get

$$
\begin{aligned}
\left|\mathcal{S}_{n,m}(c;p_1,p_2)-c(p_1,p_2)\right| \leq&4\left(1+p_1^2+p_2^2\right)\left(1+\delta_{n,p_1}^2\right)\left(1+\delta_{m,p_2}^2\right)\\
&\times\left[1+\frac{1}{\delta_{n,p_1}}\sqrt{\mathcal{S}_{n,m}\left((\mu-p_1)^2;p_1,p_2\right)}+\mathcal{S}_{n,m}\left((\mu-p_1)^2;p_1,p_2\right)\right.\\
&\left.+\frac{1}{\delta_{n,p_1}}\sqrt{\mathcal{S}_{n,m}\left((\mu-p_1)^2;p_1,p_2\right)}\sqrt{\mathcal{S}_{n,m}\left((\mu-p_1)^4;p_1,p_2\right)}\right]\\
&\times\left[1+\frac{1}{\delta_{m,p_2}}\sqrt{\mathcal{S}_{n,m}\left((\nu-p_2)^2;p_1,p_2\right)}+\mathcal{S}_{n,m}\left((\nu-p_2)^2;p_1,p_2\right)\right.\\
&\left.+\frac{1}{\delta_{m,p_2}}\sqrt{\mathcal{S}_{n,m}\left((\nu-p_2)^2;p_1,p_2\right)}\sqrt{\mathcal{S}_{n,m}\left((\nu-p_2)^4;p_1,p_2\right)}\right]\omega_\kappa\left(c;\delta_{n,p_1},\delta_{m,p_2}\right).
\end{aligned}
$$

Lastly choosing $\delta_{n,p_1}=(\varrho_n(p_1))^{1/2}$, $\delta_{m,p_2}=(\varrho_m(p_2))^{1/2}$ and also $(\mathcal{S}_{n,m}((\mu-p_1)^4;p_1,p_2))^{1/2}=(\varsigma_n(p_1))^{1/2}$ and $(\mathcal{S}_{n,m}((\nu-p_2)^4;p_1,p_2))^{1/2}=(\varsigma_m(p_2))^{1/2}$, we reached the desired result. □

3. Local Approximation Result

Let $C_B^2(T)$ be the space of all functions $c \in C_B(T)$, such that $\frac{\partial^i c}{\partial p_1^i}, \frac{\partial^i c}{\partial p_2^i}$, for $i = 1, 2$ belong to the space $C_B(T)$. The appropriate norm on the space $C_B^2(T)$ is defined as [4]

$$\|c\|_{C_B^2(T)} = \|c\|_{C_B(T)} + \sum_{i=1}^{2}\left(\left\|\frac{\partial^i c}{\partial p_1^i}\right\|_{C_B(T)} + \left\|\frac{\partial^i c}{\partial p_2^i}\right\|_{C_B(T)}\right).$$

The Peetre's $\mathcal{K}$-functional of the function $c \in C_B(T)$ is defined by

$$\mathcal{K}(c;\delta) = \inf_{l\in C_B^2(T)}\{\|c-l\|_{C_B(T)} + \delta\|l\|_{C_B^2(T)},\ \delta > 0\}.$$

The following inequality [12]

$$\mathcal{K}(c;\delta) \le C\left\{\omega_2(c;\sqrt{\delta}) + \min(1,\delta)\|c\|_{C_B^2(T)}\right\} \tag{13}$$

holds for all $\delta > 0$. The constant C is independent of δ and c.

Theorem 3.1. *Let $c \in C_B(T)$. Then for every $(p_1, p_2) \in T$,*

$$\left|\mathcal{S}_{n,m}(c;p_1,p_2) - c(p_1,p_2)\right| \le C\left\{\omega_2(c;\sqrt{\Lambda_{n,m}(p_1,p_2)}) + \min(1;\Lambda_{n,m}(p_1,p_2))\right\} + \omega(c,\sqrt{\theta_{n,p_1}^2 + \theta_{m,p_2}^2}),$$

where $C > 0$, $\theta_{n,p_1}^2 = (\gamma_n(p_1) - p_1)^2$, $\theta_{m,p_2}^2 = (\gamma_m(p_2) - p_2)^2$, $\varrho_n(p_1)$ and $\varrho_m(p_2)$ as in (7), (8) *and $\Lambda_{n,m}(p_1,p_2) := \varrho_n(p_1) + \varrho_m(p_2) + \theta_{n,p_1}^2 + \theta_{m,p_2}^2$.*

Proof. Considering the auxiliary operators

$$R_{n,m}(c;p_1,p_2) = \mathcal{S}_{n,m}(c;p_1,p_2) - c(\gamma_n(p_1),\gamma_m(p_2)) + c(p_1,p_2).$$

Let $l \in C_B^2(T)$ and $(\mu,\nu) \in T$. Applying Taylor's formula

$$l(\mu,\nu) - l(p_1,p_2) = \frac{\partial l(p_1,p_2)}{\partial p_1}(\mu - p_1) + \int_{p_1}^{\mu}(\mu - u)\frac{\partial^2(l(u,p_2))}{\partial u^2}du + \frac{\partial l(p_1,p_2)}{\partial p_2}(\nu - p_2) + \int_{p_2}^{\nu}(\nu - v)\frac{\partial^2(l(p_1,v))}{\partial v^2}dv.$$

Applying the operators $R_{n,m}$, we obtain

$$R_{n,m}(l;p_1,p_2) - l(p_1,p_2) = R_{n,m}\left(\int_{p_1}^{\mu}(\mu - u)\frac{\partial^2(l(u,p_2))}{\partial u^2}du;p_1,p_2\right) + R_{n,m}\left(\int_{p_2}^{\nu}(\nu - v)\frac{\partial^2(l(p_1,v))}{\partial v^2}dv;p_1,p_2\right),$$

then

$$R_{n,m}(l;p_1,p_2) - l(p_1,p_2)$$

$$=\mathcal{S}_{n,m}\left(\int\limits_{p_1}^{\mu}(\mu-u)\frac{\partial^2(l(u,p_2))}{\partial u^2}du;p_1,p_2\right)-\left(\int\limits_{p_1}^{\gamma_n(p_1)}(\gamma_n(p_1)-u)\frac{\partial^2(l(u,p_2))}{\partial u^2}du;p_1,p_2\right)$$
$$+\mathcal{S}_{n,m}\left(\int\limits_{p_2}^{\nu}(\nu-v)\frac{\partial^2(l(p_1,v))}{\partial v^2}dv;p_1,p_2\right)-\left(\int\limits_{p_2}^{\gamma_m(p_2)}(\gamma_m(p_2)-v)\frac{\partial^2(l(p_1,v))}{\partial v^2}dv;p_1,p_2\right).$$

Thus,

$$\left|\int\limits_{p_1}^{\mu}(\mu-u)\frac{\partial^2(l(u,p_2))}{\partial u^2}du\right|\leq\int\limits_{p_1}^{\mu}\left|(\mu-u)\frac{\partial^2(l(u,p_2))}{\partial u^2}du\right|\leq\int\limits_{p_1}^{\mu}\left|\mu-u\right|\left|\frac{\partial^2(l(u,p_2))}{\partial u^2}\right|du$$
$$\leq\left\|\frac{\partial^2(l(u,p_2))}{\partial u^2}\right\|_{C_B(T)}\int\limits_{p_1}^{\mu}\left|\mu-u\right|du\leq\|l\|_{C_B^2(T)}(\mu-p_1)^2,$$

$$\left|\int\limits_{p_2}^{\nu}(\nu-v)\frac{\partial^2(l(p_1,v))}{\partial v^2}dv\right|\leq\int\limits_{p_2}^{\nu}\left|(\nu-v)\frac{\partial^2(l(p_1,v))}{\partial v^2}dv\right|\leq\int\limits_{p_2}^{\nu}|(\nu-v)|\left|\frac{\partial^2(l(p_1,v))}{\partial v^2}\right|dv$$
$$\leq\left\|\frac{\partial^2(l(p_1,v))}{\partial v^2}\right\|_{C_B(T)}\int\limits_{p_2}^{\nu}|(\nu-v)|dv\leq\|l\|_{C_B^2(T)}(\nu-p_2)^2,$$

$$\left|\int\limits_{p_1}^{\gamma_n(p_1)}(\gamma_n(p_1)-u)\frac{\partial^2(l(u,p_2))}{\partial u^2}du\right|\leq\int\limits_{p_1}^{\gamma_n(p_1)}\left|(\gamma_n(p_1)-u)\frac{\partial^2(l(u,p_2))}{\partial u^2}du\right|$$
$$\leq\|l\|_{C_B^2(T)}(\gamma_n(p_1)-p_1)^2$$

and

$$\left|\int\limits_{p_2}^{\gamma_m(p_2)}(\gamma_m(p_2)-v)\frac{\partial^2(l(p_1,v))}{\partial v^2}dv\right|\leq\int\limits_{p_2}^{\gamma_m(p_2)}\left|(\gamma_m(p_2)-v)\frac{\partial^2(l(p_1,v))}{\partial v^2}dv\right|$$
$$\leq\|l\|_{C_B^2(T)}(\gamma_m(p_2)-p_2)^2.$$

If we take $\theta_{n,p_1}^2=(\gamma_n(p_1)-p_1)^2$ and $\theta_{m,p_2}^2=(\gamma_m(p_2)-p_2)^2$, then we reach

$$\left|R_{n,m}(l;p_1,p_2)-l(p_1,p_2)\right|\leq\|l\|_{C_B^2(T)}\left\{\mathcal{S}_{n,m}((\mu-p_1)^2;p_1,p_2)+\mathcal{S}_{n,m}((\nu-p_2)^2;p_1,p_2)+\theta_{n,p_1}^2+\theta_{m,p_2}^2\right\}.$$

Besides, we know that

$$\begin{aligned}\left|R_{n,m}(c;p_1,p_2)\right|&=\left|\mathcal{S}_{n,m}(c;p_1,p_2)-c(\gamma_n(p_1),\gamma_m(p_2))+c(p_1,p_2)\right|\\&\leq\left|\mathcal{S}_{n,m}(c;p_1,p_2)\right|+\left|c(\gamma_n(p_1),\gamma_m(p_2))\right|+\left|c(p_1,p_2)\right|\\&\leq\left|\mathcal{S}_{n,m}(c;p_1,p_2)\right|+\|c\|_{C_B(T)}+\|c\|_{C_B(T)}\end{aligned}$$

and from Lemma 1.3, $\left|\mathcal{S}_{n,m}(c;p_1,p_2)\right|\leq\|c\|_{C_B(T)}$. Therefore,

$$\left|R_{n,m}(c;p_1,p_2)\right|\leq 3\|c\|_{C_B(T)}.$$

Finally,

$$\begin{aligned}
&\left|\mathcal{S}_{n,m}\left(c;p_1,p_2\right)-c\left(p_1,p_2\right)\right| \\
\leq &\left|R_{n,m}\left(c-l\right);p_1,p_2\right|+\left|R_{n,m}\left(l;p_1,p_2\right)-l\left(p_1,p_2\right)\right| \\
&+\left|l\left(p_1,p_2\right)-c\left(p_1,p_2\right)\right|+\left|c\left(\gamma_n(p_1),\gamma_m(p_2)\right)-c\left(p_1,p_2\right)\right| \\
\leq &4\left\|c-l\right\|_{C_B(T)}+\left\|l\right\|_{C_B^2(T)}\left\{\mathcal{S}_{n,m}\left(\left(\mu-p_1\right)^2;p_1,p_2\right)+\mathcal{S}_{n,m}\left(\left(\nu-p_2\right)^2;p_1,p_2\right)+\theta_{n,p_1}^2+\theta_{m,p_2}^2\right\} \\
&+\omega\left(c,\sqrt{\theta_{n,p_1}^2+\theta_{m,p_2}^2}\right).
\end{aligned}$$

If we take the infimum on the right-hand side over all $l\in C_B^2\left(T\right)$ and take into account the inequality (13), so we prove the theorem. □

4. Construction of GBS operator of Szász–Mirakyan operators

In this part of the paper, we will extend the concept of operators to encompass a broader range of functionalities, allowing for more diverse and complex operations for (5) in the Bögel space. The Bögel-continuous (B-continuous) functions which are a significant class of functions in mathematical analysis due to their useful properties. B-continuous and B-differentiable functions were investigated by Karl Bögel in [9–11]. Badea et al. in [5, 6] present the Korovkin type theorem for GBS type operators.

Let $c:\mathcal{X}\times\mathcal{Y}\to\mathbb{R}$ for any real compact intervals of $\mathcal{X}$ and $\mathcal{Y}$. For all $(x_0,y_0),(p_1,p_2)\in\mathcal{X}\times\mathcal{Y}$, the bivariate mixed difference operators $\tilde{\Delta}_{(p_1,p_2)}c\left(x_0,y_0\right)$ are defined as

$$\tilde{\Delta}_{(p_1,p_2)}c\left(x_0,y_0\right)=c\left(p_1,p_2\right)-c\left(p_1,y_0\right)-c\left(x_0,p_2\right)+c\left(x_0,y_0\right).$$

If $\lim_{(p_1,p_2)\to(x_0,y_0)}\tilde{\Delta}_{(p_1,p_2)}c\left(x_0,y_0\right)=0$, then at any point $(x_0,y_0)\in\mathcal{X}\times\mathcal{Y}$ the function $c:\mathcal{X}\times\mathcal{Y}\to\mathbb{R}$ is known as Bögel-continuous on $\mathcal{X}\times\mathcal{Y}$.

The mixed modulus of continuity of c defined as:

$$\omega_{\text{mixed}}\left(c;\delta_1,\delta_2\right)=\sup\left\{\tilde{\Delta}_{(p_1,p_2)}c\left(\mu,\nu\right):\left|p_1-\mu\right|\leq\delta_1,\left|p_2-\nu\right|\leq\delta_2\right\}.$$

Hence, for $\mathcal{X}_{ab}=[0,a]\times[0,b]$, we suppose $C_b\left(\mathcal{X}_{ab}\right)$ denote the space of all B-continuous function on $\mathcal{X}_{ab}$.

Let $(p_1,p_2)\in\mathcal{X}_{ab}$ and $m,n>0$ be a positive integer then for all $c\in C_b\left(\mathcal{X}_{ab}\right)$, the GBS (Generalized Boolean Sum) type operators for our operators as

$$\mathcal{F}_{n,m}\left(c\left(\mu,\nu\right);p_1,p_2\right)=\mathcal{S}_{n,m}\left(c\left(\mu,p_2\right)+c\left(p_1,\nu\right)-c\left(\mu,\nu\right);p_1,p_2\right). \tag{14}$$

Additionally, it is worth noting that [2] (and references therein) provides a comprehensive overview (definitions, lemmas, historical background etc.) of the Bögel space.

Theorem 4.1. *Let $(p_1,p_2)\in\mathcal{X}_{ab}$, then for all $c\in C_b\left(\mathcal{X}_{ab}\right)$, we have*

$$\left|\mathcal{F}_{n,m}\left(c\left(\mu,\nu\right);p_1,p_2\right)-c\left(p_1,p_2\right)\right|\leq 4\omega_{mixed}\left(c;\delta_{n,p_1},\delta_{m,p_2}\right),$$

where $\delta_{n,p_1}=\left(\varrho_n(p_1)\right)^{1/2}$ as in (7) *and $\delta_{m,p_2}=\left(\varrho_m(p_2)\right)^{1/2}$ as in* (8).

Proof. For all $(\mu,\nu),(p_1,p_2)\in\mathcal{X}_{ab}$ and $\delta_{n,p_1},\delta_{m,p_2}>0$, we have

$$\begin{aligned}
\left|\tilde{\Delta}_{(p_1,p_2)}c\left(\mu,\nu\right)\right| &\leq\omega_{\text{mixed}}\left(c;\left|\mu-p_1\right|,\left|\nu-p_2\right|\right) \\
&\leq\left(1+\frac{\left|\mu-p_1\right|}{\delta_{n,p_1}}\right)\left(1+\frac{\left|\nu-p_2\right|}{\delta_{m,p_2}}\right)\omega_{\text{mixed}}\left(c;\delta_{n,p_1},\delta_{m,p_2}\right).
\end{aligned}$$

Applying the operators $\mathcal{F}_{n,m}$ then using Cauchy-Schwarz inequality and choosing $\delta_{n,p_1} = \varrho_n(p_1)$, $\delta_{m,p_2} = \varrho_m(p_2)$

$$\begin{aligned}
&\left|\mathcal{F}_{n,m}\left(c\left(\mu,\nu\right);p_1,p_2\right)-c\left(p_1,p_2\right)\right| \\
=&\left|\mathcal{S}_{n,m}\left(c\left(\mu,p_2\right)+c\left(p_1,\nu\right)-c\left(\mu,\nu\right);p_1,p_2\right)-c\left(p_1,p_2\right)\right| \\
=&\left|\mathcal{S}_{n,m}\left(\tilde{\Delta}_{(p_1,p_2)}c\left(\mu,\nu\right);p_1,p_2\right)\right| \\
\leq&\mathcal{S}_{n,m}\left(\left|\tilde{\Delta}_{(p_1,p_2)}c\left(\mu,\nu\right)\right|;p_1,p_2\right) \\
\leq&\Big(\mathcal{S}_{n,m}\left(e_{0,0};p_1,p_2\right)+\frac{1}{\delta_{n,p_1}}\left(\mathcal{S}_{n,m}\Big(\left(\mu-p_1\right)^2;p_1,p_2\Big)\right)^{1/2}+\frac{1}{\delta_{m,p_2}}\left(\mathcal{S}_{n,m}\Big(\left(\nu-p_2\right)^2;p_1,p_2\Big)\right)^{1/2} \\
&+\frac{1}{\delta_{n,p_1}}\left(\mathcal{S}_{n,m}\Big(\left(\mu-p_1\right)^2;p_1,p_2\Big)\right)^{1/2}\frac{1}{\delta_{m,p_2}}\left(\mathcal{S}_{n,m}\Big(\left(\nu-p_2\right)^2;p_1,p_2\Big)\right)^{1/2}\Big)\,\omega_{\text{mixed}}\left(c;\delta_{n,p_1},\delta_{m,p_2}\right) \\
=&4\omega_{\text{mixed}}\left(c;\delta_{n,p_1},\delta_{m,p_2}\right)
\end{aligned}$$

leads us the desired result. □

Now, we investigate the rate of convergence utilizing the functions of Lipschitz class of B-continuous functions.

For $c \in C_b\left(\mathcal{X}_{ab}\right)$, we define the Lipschitz class of B-continuous functions $Lip_L(\chi_1,\chi_2)$ with $\chi_1,\chi_2 \in (0,1]$ as follows:

$$Lip_L(\chi_1,\chi_2)=\left\{c\in C_b\left(\mathcal{X}_{ab}\right):|\tilde{\Delta}_{(p_1,p_2)}c\left(\mu,\nu;p_1,p_2\right)|\leq L|\mu-p_1|^{\chi_1}|\nu-p_2|^{\chi_2}\right\},$$

for $(\mu,\nu),(p_1,p_2)\in\mathcal{X}_{ab}$ and $L>0$.

Theorem 4.2. *Let* $c \in Lip_L(\chi_1,\chi_2)$ *and* $0<\chi\leq 1$*, then for every* $(p_1,p_2)\in\mathcal{X}_{ab}$*, then*

$$\left|\mathcal{F}_{n,m}\left(c\left(\mu,\nu\right);p_1,p_2\right)-c\left(p_1,p_2\right)\right|\leq L\delta_{n,p_1}^{\chi_1}\delta_{m,p_2}^{\chi_2},$$

where $\delta_{n,p_1}=(\varrho_n(p_1))^{1/2}$ *and* $\delta_{m,p_2}=(\varrho_m(p_2))^{1/2}$ *and L is a positive constant.*

Proof. For the operators $\mathcal{F}_{n,m}$,

$$\begin{aligned}
\mathcal{F}_{n,m}\left(c\left(\mu,\nu\right);p_1,p_2\right)=&\mathcal{S}_{n,m}\left(c\left(p_1,\nu\right)+c\left(\mu,p_2\right)-c\left(\mu,\nu\right);p_1,p_2\right) \\
=&\mathcal{S}_{n,m}\left(c\left(p_1,p_2\right)-\tilde{\Delta}_{(p_1,p_2)}c\left(\mu,\nu\right);p_1,p_2\right) \\
=&c\left(p_1,p_2\right)\mathcal{S}_{n,m}\left(e_{0,0};p_1,p_2\right)-\mathcal{S}_{n,m}\left(\tilde{\Delta}_{(p_1,p_2)}c\left(\mu,\nu\right);p_1,p_2\right).
\end{aligned}$$

Keeping in mind that $c\in Lip_L(\chi_1,\chi_2)$, then

$$\begin{aligned}
\left|\mathcal{F}_{n,m}\left(c\left(\mu,\nu\right);p_1,p_2\right)-f\left(p_1,p_2\right)\right|\leq&\mathcal{S}_{n,m}\left(|\tilde{\Delta}_{(p_1,p_2)}c\left(\mu,\nu\right)|;p_1,p_2\right) \\
\leq&L\mathcal{S}_{n,m}\Big(|\mu-p_1|^{\chi_1}|\nu-p_2|^{\chi_2};p_1,p_2\Big) \\
=&L\mathcal{S}_{n,m}\Big(|\mu-p_1|^{\chi_1};p_1,p_2\Big)\mathcal{S}_{n,m}\Big(|\nu-p_2|^{\chi_2};p_1,p_2\Big).
\end{aligned}$$

Now, using the Hölder inequality with $(r_1,q_1)=(2/\chi_1,2/(2-\chi_1))$ and $(r_2,q_2)=(2/\chi_2,2/(2-\chi_2))$,

$$\left|\mathcal{F}_{n,m}\left(c\left(\mu,\nu\right);p_1,p_2\right)-c\left(p_1,p_2\right)\right|$$

$$\leq L\left(\mathcal{S}_{n,m}\left((\mu-p_1)^2;p_1,p_2\right)\right)^{\chi_1/2}\left(\mathcal{S}_{n,m}\left(e_{0,0};p_1,p_2\right)\right)^{(2-\chi_1)/2}$$
$$\times\left(\mathcal{S}_{n,m}\left((\nu-p_2)^2;p_1,p_2\right)\right)^{\chi_2/2}\left(\mathcal{S}_{n,m}\left(e_{0,0};p_1,p_2\right)\right)^{(2-\chi_2)/2}.$$

Taking δ_{n,p_1} and δ_{m,p_2} as stated above that completes the proof. □

Now, we discuss the order of approximation via mixed modulus of smoothness. D_Bc is called the B-derivative of c and the space of all B-differentiable functions is denoted by $D_b(\mathcal{X}_{ab})$.

Theorem 4.3. *For all $c\in D_b(\mathcal{X}_{ab})$ and $D_Bc\in B(\mathcal{X}_{ab})$,*

$$\left|\mathcal{F}_{n,m}\left(c\left(p_1,p_2\right);p_1,p_2\right)-c\left(p_1,p_2\right)\right|$$
$$\leq M\left(3\left\|D_Bc\right\|_\infty\delta_{n,p_1}\delta_{m,p_2}+\left(2\delta_{n,p_1}\delta_{m,p_2}+\frac{1}{\delta_{n,p_1}}(\varsigma_n(p_1))^{1/2}\delta_{m,p_2}+\frac{1}{\delta_{m,p_2}}(\varsigma_m(p_2))^{1/2}\delta_{n,p_1}\right)\right)\omega_{mixed}\left(D_Bc;\delta_{n,p_1},\delta_{m,p_2}\right),$$

where M is a positive constant and $\delta_{n,p_1}=(\varrho_n(p_1))^{1/2}$ and $\delta_{m,p_2}=(\varrho_m(p_2))^{1/2}$.

Proof. Since $c\in D_b(\mathcal{X}_{ab})$, then $p_1<\zeta_1<\mu$, $p_2<\zeta_2<\nu$ and $\tilde{\Delta}_{(p_1,p_2)}c(\mu,\nu)=(\mu-p_1)(\nu-p_2)D_Bc(\zeta_1,\zeta_2)$, so

$$D_Bc\left(\zeta_1,\zeta_2\right)=\tilde{\Delta}_{(p_1,p_2)}D_Bc\left(\zeta_1,\zeta_2\right)+D_Bc\left(\zeta_1,p_2\right)+D_Bc\left(p_1,\zeta_2\right)-D_Bc\left(p_1,p_2\right).$$

Since $D_Bc\in B(\mathcal{X}_{ab})$,

$$\left|\mathcal{F}_{n,m}\left(\tilde{\Delta}_{(p_1,p_2)}c\left(\mu,\nu\right);p_1,p_2\right)\right|$$
$$=\left|\mathcal{F}_{n,m}\left((\mu-p_1)(\nu-p_2)D_Bc\left(\zeta_1,\zeta_2\right);p_1,p_2\right)\right|$$
$$\leq\mathcal{F}_{n,m}\left(\left|\mu-p_1\right|\left|\nu-p_2\right|\left|\tilde{\Delta}_{(p_1,p_2)}D_Bc\left(\zeta_1,\zeta_2\right)\right|;p_1,p_2\right)$$
$$+\mathcal{F}_{n,m}\left(\left|\mu-p_1\right|\left|\nu-p_2\right|\left(\left|D_Bc\left(\zeta_1,p_2\right)\right|+\left|D_Bc\left(p_1,\zeta_2\right)\right|+\left|D_Bc\left(p_1,p_2\right)\right|\right);p_1,p_2\right)$$
$$\leq\mathcal{F}_{n,m}\left(\left|\mu-p_1\right|\left|\nu-p_2\right|\omega_{\text{mixed}}\left(D_Bc;\left|\eta-p_1\right|,\left|\xi-p_2\right|\right);p_1,p_2\right)+3\left\|D_Bc\right\|_\infty\mathcal{F}_{n,m}\left(\left|\mu-p_1\right|\left|\nu-p_2\right|;p_1,p_2\right)$$

For the mixed modulus of continuity, one can be written as

$$\omega_{\text{mixed}}\left(D_Bc;\left|\zeta_1-p_1\right|,\left|\zeta_2-p_2\right|\right)\leq\omega_{\text{mixed}}\left(D_Bc;\left|\mu-p_1\right|,\left|\nu-p_2\right|\right)$$
$$\leq\left(1+\frac{1}{\delta_{n,p_1}}\left|\mu-p_1\right|\right)\left(1+\frac{1}{\delta_{m,p_2}}\left|\nu-p_2\right|\right)\omega_{mixed}\left(D_Bc;\delta_{n,p_1},\delta_{m,p_2}\right).$$

Therefore, taking into account the above property and using the Cauchy-Schwarz inequality, we obtain

$$\left|\mathcal{F}_{n,m}\left(c;p_1,p_2\right)-c\left(p_1,p_2\right)\right|$$
$$=\left|\mathcal{S}_{n,m}\left(\tilde{\Delta}_{(p_1,p_2)}c\left(\mu,\nu\right);p_1,p_2\right)\right|$$
$$\leq3\left\|D_Bc\right\|_\infty\left(\mathcal{S}_{n,m}\left((\mu-p_1)^2(\nu-p_2)^2;p_1,p_2\right)\right)^{\frac{1}{2}}$$
$$+\left(\begin{array}{c}\mathcal{S}_{n,m}\left(\left|\mu-p_1\right|\left|\nu-p_2\right|;p_1,p_2\right)+\frac{1}{\delta_{n,p_1}}\mathcal{S}_{n,m}\left((\mu-p_1)^2\left|\nu-p_2\right|;p_1,p_2\right)\\+\frac{1}{\delta_{m,p_2}}\mathcal{S}_{n,m}\left(\left|\mu-p_1\right|(\nu-p_2)^2;p_1,p_2\right)+\frac{1}{\delta_{n,p_1}}\frac{1}{\delta_{m,p_2}}\mathcal{S}_{n,m}\left((\mu-p_1)^2(\nu-p_2)^2;p_1,p_2\right)\end{array}\right)$$
$$\times\omega_{\text{mixed}}\left(D_Bc;\delta_{n,p_1},\delta_{m,p_2}\right)$$
$$\leq3\left\|D_Bc\right\|_\infty\left(\mathcal{S}_{n,m}\left((\mu-p_1)^2(\nu-p_2)^2;p_1,p_2\right)\right)^{\frac{1}{2}}$$

$$+\left(\begin{array}{c}\left(\mathcal{S}_{n,m}\left((\mu-p_1)^2(\nu-p_2)^2;p_1,p_2\right)\right)^{\frac{1}{2}}+\frac{1}{\delta_{n,p_1}}\left(\mathcal{S}_{n,m}\left((\mu-p_1)^4(\nu-p_2)^2;p_1,p_2\right)\right)^{\frac{1}{2}}\\ +\frac{1}{\delta_{m,p_2}}\left(\mathcal{S}_{n,m}\left((\mu-p_1)^2(\nu-p_2)^4;p_1,p_2\right)\right)^{\frac{1}{2}}+\frac{1}{\delta_{n,p_1}}\frac{1}{\delta_{m,p_2}}\left(\mathcal{S}_{n,m}\left((\mu-p_1)^2(\nu-p_2)^2;p_1,p_2\right)\right)\end{array}\right)$$
$$\times\,\omega_{\text{mixed}}\left(D_Bc;\delta_{n,p_1},\delta_{m,p_2}\right).$$

Keeping in mind that $\delta_{n,p_1}=(\varrho_n(p_1))^{1/2}$ and $\delta_{m,p_2}=(\varrho_m(p_2))^{1/2}$ also $(\mathcal{S}_{n,m}((\mu-p_1)^4;p_1,p_2))^{1/2}=(\varsigma_n(p_1))^{1/2}$ and $(\mathcal{S}_{n,m}((\nu-p_2)^4;p_1,p_2))^{1/2}=(\varsigma_m(p_2))^{1/2}$, the proof is completed. □

5. Numerical Examples

In the last section, through these examples, we aim to demonstrate the effectiveness of the operators in achieving their intended outcomes. By analyzing their convergence, we can gain valuable insights into their performance and suitability for specific applications.

Example 5.1. *Table 1 reveals that our operators provide a more accurate approximation of the function $e^{-4p_1-1}e^{-4p_2-1}$ for $n,m=50$. Remind that $S_{n,m}(c;p_1,p_2)$ are the bivariate classical Szász–Mirakyan operators and $\mathcal{S}_{n,m}(c;p_1,p_2)$ are our operators.*

Table 1: Error of approximation for $\mathcal{S}_{50,50}(c;p_1,p_2)$ and $S_{50,50}(c;p_1,p_2)$.

p_1	p_2	$\lvert\mathcal{S}_{50,50}(c;p_1,p_2)-c(p_1,p_2)\rvert$	$\lvert S_{50,50}(c;p_1,p_2)-c(p_1,p_2)\rvert$
0.1	0.1	9.61277×10^{-4}	1.92489×10^{-3}
0.5	0.5	2.02212×10^{-4}	4.17949×10^{-4}
0.9	0.9	1.53182×10^{-5}	3.27118×10^{-5}
1.3	1.3	9.31498×10^{-7}	2.05719×10^{-6}
1.7	1.7	5.12982×10^{-8}	1.17276×10^{-7}
2.1	2.1	2.66949×10^{-9}	6.32353×10^{-9}
2.5	2.5	1.33921×10^{-10}	3.29012×10^{-10}
2.9	2.9	6.54855×10^{-12}	1.67012×10^{-11}
3.3	3.3	3.14226×10^{-13}	8.32686×10^{-13}
3.7	3.7	1.48611×10^{-14}	4.09563×10^{-14}

Example 5.2. *It is evident from Table 2 that greater values of n and m lead to more accurate approximations. To gain a deeper understanding of the impact of n and m values on the numerical evaluation of function $50e^{-5p_1+8}e^{-5p_2+8}$ and specific points $p_1=4.0$ and $p_2=4.0$, referring to Table 2 would be beneficial.*

Table 2: Error of approximation by $\mathcal{S}_{n,m}(c;4.0,4.0)$ for n and m values.

n	m	$\lvert\mathcal{S}_{n,m}(c;4.0,4.0)-c(4.0,4.0)\rvert$
10	10	3.65028×10^{-7}
20	20	2.94348×10^{-8}
30	30	1.09082×10^{-8}
40	40	6.16861×10^{-9}
50	50	4.18560×10^{-9}
60	60	3.13233×10^{-9}
70	70	2.48925×10^{-9}
80	80	2.05935×10^{-9}
90	90	1.88757×10^{-9}
100	100	1.75315×10^{-9}

Funding
No funding.
Conflict of interest
No conflict of interest.

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