Filomat 39:18 (2025), 6295–6310
https://doi.org/10.2298/FIL2518295S

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Toeplitz operators and spectrogram associated with the Sturm-Liouville-Stockwell transform

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Abstract. The Sturm-Liouville-Stockwell transform (SLST) is a novel addition to the class of Stockwell transforms, which has gained a respectable status in the realm of time-frequency signal analysis within a short span of time. Knowing the fact that the study of the time-frequency analysis is both theoretically interesting and practically useful, this article aims to explore two other aspects of time-frequency analysis associated with SLST, namely spectral analysis associated with concentration operators and spectrogram analysis.

1. Introduction

The Fourier transform stands out as a significant discovery in mathematical sciences, that plays a crucial role in modern scientific and technological advancements. In signal processing, extensive research has utilized the Fourier transform to analyze stationary signals or processes with statistically invariant properties over time. However, many signals exhibit non-stationary characteristics, requiring a time-frequency analysis for a comprehensive representation.

Although, Fourier transforms have many successful applications that fascinated the mathematical, physical and engineering communities over decades, they still have numerous shortcomings. One of the significant disadvantages of the Fourier transforms is that they do not give any information about the occurrence of the frequency component at a particular time. They only enable us to analyse the signals either in time domain or frequency domain, but not simultaneously in both domains [12, 33]. A suitable redress of these limitations was given by Gabor [20] in the form of windowed Fourier transform using a Gaussian distribution function as a window function in order to construct efficient time-frequency localized expansions of finite energy signals $f \in L^2(\mathbb{R})$ as

$$V_g(f)(\xi, b) := \int_{\mathbb{R}} f(x)\overline{g(x-b)}e^{-i\xi x}\mathrm{d}x, \quad \xi, b \in \mathbb{R}.$$

2020 *Mathematics Subject Classification*. Primary 42B10; Secondary 44A05, 44A20.
Keywords. Sturm-Liouville-Stockwell transform, Toeplitz operators, spectrogram analysis.
Received: 15 October 2024; Revised: 23 March 2025; Accepted: 08 April 2025
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The spectral contents of non-transient signals in localized neighbourhoods of time can be analyzed. This astonishing feature of the Stockwell transform provides the local characteristics of the Fourier transform with a time resolution equal to the size of the window. The Stockwell transform (ST), also known as the short-time Fourier transform (STFT), see [5, 13], marked a breakthrough in time-frequency analysis. This method involves decomposing non-transient signals using time and frequency-shifted basis functions, termed Stockwell window functions. The ST, with its clear resemblance to the classical Fourier transform, has gained considerable attention in the past few decades. Soon after its inception in quantum mechanics, the Stockwell transform profound influenced diverse branches of science and engineering including harmonic analysis, signal and image processing, pseudo-differential operators, sampling theory, wave propagation, quantum optics, geophysics, astrophysics, medicine [7, 20, 21, 41], and others. Besides its applications, the theoretical skeleton of Stockwell transform has likewise been extensively studied and investigated in other groups including the locally compact Abelian and non-Abelian groups [15, 16, 18], hypergroups [10], Gelfand pairs [40] and so on. For more about Stockwell transforms and their applications, we allude to [17, 22, 23]. Many extensions of the Stockwell transform have been proposed in recent years, see for example [2, 3, 21, 37, 38] and others. Recently, the study of integral transforms in harmonic analysis has known remarkable development (see [25–27, 29, 30]). Another fundamental tool in time-frequency analysis is the Sturm-Liouville-Stockwell transform (SLST), which is the focus of this paper.

We consider the Sturm-Liouville operator defined on $\mathbb{R}_+^*$ by

$$\Delta := \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{A'(x)}{A(x)}\frac{\mathrm{d}}{\mathrm{d}x},$$

where A is a nonnegative function satisfying certain conditions. This operator is the goal of many works in harmonic analysis [4, 8, 28, 39, 44]. Specifically, we consider the Sturm-Liouville transform (SLT)

$$\mathscr{F}(f)(\lambda) := \int_{\mathbb{R}_+} \varphi_\lambda(x) f(x) A(x)\mathrm{d}x, \quad \lambda \in \mathbb{R}_+,$$

where φ_λ is the Sturm-Liouville function given in Section 2 below. The SLT can be considered as a generalization of certain generalized Fourier transforms [2, 19, 21, 31]. Many results have already been demonstrated for the Sturm-Liouville transform $\mathscr{F}$ (see [6, 24, 34–38]).

The Sturm-Liouville function φ_λ satisfies the product formula (see [8, 39])

$$\varphi_\lambda(x)\varphi_\lambda(y) = \int_{\mathbb{R}_+} \varphi_\lambda(z) w(x, y, z) A(z)\mathrm{d}z, \quad \text{for } x, y \in \mathbb{R}_+,$$

where $w(x, y, .)$ is a positive measurable function on $\mathbb{R}_+$, with support in $[|x - y|, x + y]$.

We introduce the Sturm-Liouville translation operators for $f \in L^2(\mathbb{R}_+, A(z)\mathrm{d}z)$ by

$$\tau_y f(x) := \int_{\mathbb{R}_+} f(z) w(x, y, z) A(z)\mathrm{d}z, \quad x, y \in \mathbb{R}_+.$$

Let $g \in L^2(\mathbb{R}_+, \frac{\mathrm{d}t}{2\pi|c(t)|^2})$, where $c(t)$ is the Harish-Chandra function defined in Section 2. The modulation g_y of g by $y \in \mathbb{R}_+$ is defined by

$$g_y := \mathscr{F}\Big(\sqrt{\tau_y |\mathscr{F}^{-1}(g)|^2}\Big),$$

where $\mathscr{F}^{-1}$ is the inverse of the transform $\mathscr{F}$.

Let $g \in L^2(\mathbb{R}_+, \frac{\mathrm{d}t}{2\pi|c(t)|^2})$. The Sturm-Liouville-Stockwell transform (see [37, 38]) is the mapping $\mathscr{S}_g$ defined for $f \in L^2(\mathbb{R}_+, \frac{\mathrm{d}t}{2\pi|c(t)|^2})$ by

$$\mathscr{S}_g(f)(\lambda, y) = \int_{\mathbb{R}_+} f(t)\sigma_\lambda g_y(t)\frac{\mathrm{d}t}{2\pi|c(t)|^2}, \quad \lambda, y \in \mathbb{R}_+,$$

where σ_λ is the operator defined by

$$\mathscr{F}^{-1}(\sigma_\lambda f)(x) = \varphi_\lambda(x)\mathscr{F}^{-1}(f)(x).$$

In this paper, we continue the study of some harmonic analysis problems associated with the Sturm-Liouville-Stockwell transform started in [37, 38]. The aim of this paper is to explore some topics of time-frequency analysis associated with SLST, viz, the spectral analysis for the concentration operators and the spectrogram analysis.

Motivated by Wong's approaches, the aim of the first part of this paper is to study the boundedness and compactness of Toeplitz operators associated with SLST. Our second endeavour is to study the spectral analysis associated with the generalized concentration operator. In particular, we introduce and we study the spectrogram analysis associated with SLST.

The theory of Toeplitz operators was initiated by Daubechies in [11], developed by Wong [42, 43]. Nowadays, Toeplitz operators have found many applications in time-frequency analysis, differential equation theory, quantum mechanics. Arguing from this point of view, many works have been done on them, we refer in particular to the article of Balazs [1], (see also [9, 14, 23, 43]).

The paper is organized as follows. In Section 2, we recall some results about the Sturm-Liouville-Stockwell transform SLST. In Section 3, we recall some boundedness and compactness results for the localization and concentration operators associated with SLST. Section 4 is devoted to introduce and study the boundedness and compactness of the Toeplitz operators associated with SLST. Next, in Section 5, we study the spectrogram analysis associated with SLST. In the last section, we summarize the obtained results and we describe the future work.

2. Sturm-Liouville-Stockwell transform (SLST)

We consider the second-order differential operator Δ defined on $\mathbb{R}_+^*$ by

$$\Delta := \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{A'(x)}{A(x)}\frac{\mathrm{d}}{\mathrm{d}x},$$

where

$$A(x) = x^{2\alpha+1}B(x), \quad \alpha > -1/2,$$

for B a positive, even, infinitely differentiable function on $\mathbb{R}$ such that $B(0) = 1$. Moreover, we assume that A satisfies the following conditions:

(i) A is increasing and $\lim\limits_{x\to\infty} A(x) = \infty$.

(ii) $\dfrac{A'}{A}$ is decreasing and $\lim\limits_{x\to\infty} \dfrac{A'(x)}{A(x)} = 2\rho \geq 0$.

(iii) There exists a constant $\delta > 0$, such that

$$\frac{A'(x)}{A(x)} = 2\rho + e^{-\delta x}D(x), \quad \text{if } \rho > 0,$$

$$\frac{A'(x)}{A(x)} = \frac{2\alpha+1}{x} + e^{-\delta x}D(x), \quad \text{if } \rho = 0,$$

where D is an infinitely differentiable function on $\mathbb{R}_+^*$, bounded and with bounded derivatives on all intervals $[x_0, \infty)$, for $x_0 > 0$.

This operator was studied in [8, 39], and the following results have been established:

(I) For all $\lambda \in \mathbb{C}$, the equation

$$\Delta(u) = -(\lambda^2 + \rho^2)u, \quad u(0) = 1, \quad u'(0) = 0,$$

admits a unique solution, denoted by φ_λ, with the following properties:

• for $x \in \mathbb{R}_+$, the function $\lambda \to \varphi_\lambda(x)$ is analytic on $\mathbb{C}$;
• for $\lambda \in \mathbb{C}$, the function $x \to \varphi_\lambda(x)$ is even and infinitely differentiable on $\mathbb{R}$.
(II) For nonzero $\lambda \in \mathbb{C}$, the equation

$$\Delta(u) = -(\lambda^2 + \rho^2)u,$$

has a solution Φ_λ satisfying

$$\Phi_\lambda(x) = \frac{e^{i\lambda x}}{\sqrt{A(x)}} V(x, \lambda),$$

with

$$\lim_{x\to\infty} V(x, \lambda) = 1.$$

Consequently there exists a function (spectral function) $\lambda \longrightarrow c(\lambda)$, such that

$$\varphi_\lambda(x) = c(\lambda)\Phi_\lambda(x) + c(-\lambda)\Phi_{-\lambda}(x), \quad x \in \mathbb{R}_+,$$

for nonzero $\lambda \in \mathbb{C}$.
Moreover, there exist positive constants k_1, k_2, k, such that

$$k_1|\lambda|^{2\alpha+1} \leq |c(\lambda)|^{-2} \leq k_2|\lambda|^{2\alpha+1},$$

for all λ such that $\mathrm{Im}\lambda \leq 0$ and $|\lambda| \geq k$.
(III) The Sturm-Liouville kernel $\varphi_\lambda(x)$ possesses the following property (see [6, 24]):

$$|\varphi_\lambda(x)| \leq 1, \quad \lambda, x \in \mathbb{R}_+.$$

Examples. 1) (The Bessel case, see [2, 21]). In this case $A(x) = x^{2\alpha+1}$, $\alpha > -1/2$ and $\rho = 0$. The SL-operator Δ is the Bessel operator denoted by Δ_α:

$$\Delta_\alpha = \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{2\alpha+1}{x}\frac{\mathrm{d}}{\mathrm{d}x}.$$

The SL-function $\varphi_\lambda(x)$ is the spherical Bessel function $j_\alpha(\lambda x)$.
2) (The Jacobi case, see [19, 31]). In this case $A(x) = \sinh^{2\alpha+1}(x)\cosh^{2\beta+1}(x)$, $\alpha > \beta \geq -1/2$ and $\rho = \alpha + \beta + 1 > 0$. The SL-operator Δ is the Jacobi operator denoted by $\Delta_{\alpha,\beta}$:

$$\Delta_{\alpha,\beta} = \frac{\mathrm{d}^2}{\mathrm{d}x^2} + [(2\alpha+1)\coth(x) + (2\beta+1)\tanh(x)]\frac{\mathrm{d}}{\mathrm{d}x}.$$

The SL-function $\varphi_\lambda(x)$ is the Jacobi function denoted by $\phi_\lambda^{(\alpha,\beta)}(x)$:

$$\phi_\lambda^{(\alpha,\beta)}(x) = {}_2F_1(\frac{1}{2}(\rho - i\lambda), \frac{1}{2}(\rho + i\lambda), \alpha+1, -\sinh^2(x)),$$

where ${}_2F_1(a, b, c, z)$ is the hypergeometric function.
We denote by
• μ the measure defined on $\mathbb{R}_+$ by $\mathrm{d}\mu(x) := A(x)\mathrm{d}x$; and by $L^p(\mu)$, $p \in [1, \infty]$, the space of measurable functions f on $\mathbb{R}_+$, such that

$$\|f\|_{L^p(\mu)} := \left[\int_{\mathbb{R}_+} |f(x)|^p \mathrm{d}\mu(x)\right]^{1/p} < \infty, \quad p \in [1, \infty),$$
$$\|f\|_{L^\infty(\mu)} := \operatorname*{ess\,sup}_{x\in\mathbb{R}_+} |f(x)| < \infty;$$

• ν the measure defined on $\mathbb{R}_+$ by $\mathrm{d}\nu(\lambda) := \dfrac{\mathrm{d}\lambda}{2\pi|c(\lambda)|^2}$; and by $L^p(\nu)$, $p \in [1, \infty]$, the space of measurable functions f on $\mathbb{R}_+$, such that $\|f\|_{L^p(\nu)} < \infty$.

The Sturm-Liouville transform is the Fourier transform associated with the operator Δ and is defined for $f \in L^1(\mu)$ by

$$\mathscr{F}(f)(\lambda) := \int_{\mathbb{R}_+} \varphi_\lambda(x) f(x) \mathrm{d}\mu(x), \quad \lambda \in \mathbb{R}_+.$$

Some of the properties of the Sturm-Liouville transform $\mathscr{F}$ are collected bellow (see [4, 8, 39]).

Theorem 2.1. (i) $L^1 - L^\infty$-boundedness for $\mathscr{F}$. For all $f \in L^1(\mu)$, $\mathscr{F}(f) \in L^\infty(\nu)$ and

$$\|\mathscr{F}(f)\|_{L^\infty(\nu)} \leq \|f\|_{L^1(\mu)}.$$

(ii) Plancherel formula for $\mathscr{F}$. The Sturm-Liouville transform $\mathscr{F}$ extends uniquely to an isometric isomorphism of $L^2(\mu)$ onto $L^2(\nu)$. In particular,

$$\|f\|_{L^2(\mu)} = \|\mathscr{F}(f)\|_{L^2(\nu)}.$$

(iii) Inversion formula for $\mathscr{F}$. Let $f \in L^1(\mu)$, such that $\mathscr{F}(f) \in L^1(\nu)$. Then

$$f(x) = \int_{\mathbb{R}_+} \varphi_\lambda(x) \mathscr{F}(f)(\lambda) \mathrm{d}\nu(\lambda), \quad \text{a.e. } x \in \mathbb{R}_+.$$

The Sturm-Liouville kernel φ_λ satisfies the product formula [8, 39]

$$\varphi_\lambda(x)\varphi_\lambda(y) = \int_{\mathbb{R}_+} \varphi_\lambda(z) w(x,y,z) \mathrm{d}\mu(z) \quad \text{for } x, y \in \mathbb{R}_+; \tag{2.1}$$

where $w(x, y, .)$ is a measurable positive function on $\mathbb{R}_+$, with support in $[|x-y|, x+y]$, satisfying

$$\int_{\mathbb{R}_+} w(x,y,z) \mathrm{d}\mu(z) = 1, \tag{2.2}$$

$$w(x,y,z) = w(y,x,z) \quad \text{for } z \geq 0, \tag{2.3}$$

$$w(x,y,z) = w(x,z,y) \quad \text{for } z > 0. \tag{2.4}$$

We now define the generalized translation operator induced by (2.1). For $f \in L^1(\mu)$, the linear operator

$$\tau_y f(x) := \int_{\mathbb{R}_+} f(z) w(x,y,z) \mathrm{d}\mu(z), \quad x, y \in \mathbb{R}_+,$$

will be called Sturm-Liouville translation.

As a first remark, we note that the relations (2.2), (2.3) and (2.4) mean that

$$\tau_y f(x) = \tau_x f(y), \quad x, y \in \mathbb{R}_+,$$

and

$$\int_{\mathbb{R}_+} \tau_y f(x) \mathrm{d}\mu(x) = \int_{\mathbb{R}_+} f(x) \mathrm{d}\mu(x), \quad f \in L^1(\mu). \tag{2.5}$$

Theorem 2.2. (See [28, 35, 37]).

(i) For all $y \geq 0$ and $f \in L^p(\mu)$, $p \in [1, \infty]$, we have

$$\|\tau_y f\|_{L^p(\mu)} \leq \|f\|_{L^p(\mu)}.$$

(ii) For $f \in L^2(\mu)$ and $y \in \mathbb{R}_+$, we have

$$\mathscr{F}(\tau_y f)(\lambda) = \varphi_\lambda(y) \mathscr{F}(f)(\lambda), \quad \lambda \in \mathbb{R}_+.$$

Let $f, g \in L^2(\nu)$. We define the convolution product $f\sharp g$ of f and g by

$$f\sharp g(\lambda) := \mathscr{F}(\mathscr{F}^{-1}(f)\mathscr{F}^{-1}(g))(\lambda), \tag{2.6}$$

where $\mathscr{F}^{-1}$ is the inverse of the transform $\mathscr{F}$.

Let $f, g \in L^2(\nu)$. Then

$$\int_{\mathbb{R}_+} |f\sharp g(\lambda)|^2 d\nu(\lambda) = \int_{\mathbb{R}_+} |\mathscr{F}^{-1}(f)(x)|^2 |\mathscr{F}^{-1}(g)(x)|^2 d\mu(x),$$

where both sides are finite or infinite.

We assume that $g \in L^2(\nu)$ and $y \in \mathbb{R}_+$. The modulation of g by y is the function

$$g_y := \mathscr{F}\Big(\sqrt{\tau_y |\mathscr{F}^{-1}(g)|^2}\Big).$$

From (2.5) and Theorem 2.1 (ii) we have

$$\|g_y\|_{L^2(\nu)} = \|g\|_{L^2(\nu)}. \tag{2.7}$$

Let $g \in L^2(\nu)$. The Sturm-Liouville-Stockwell transform (SLST) is the mapping $\mathscr{S}_g$ defined for $f \in L^2(\nu)$ by

$$\mathscr{S}_g(f)(\lambda, y) := f\sharp g_y(\lambda), \quad \lambda, y \in \mathbb{R}_+.$$

From (2.6) and (2.7) we have

$$\|\mathscr{S}_g(f)\|_{L^\infty(\nu\otimes\mu)} \le \|g\|_{L^2(\nu)}\|f\|_{L^2(\nu)}.$$

Theorem 2.3. (See [37]). (Plancherel formula for $\mathscr{S}_g$). Let $g \in L^2(\nu)$ be a non-zero function. Then, for all $f \in L^2(\nu)$, we have

$$\|\mathscr{S}_g(f)\|_{L^2(\nu\otimes\mu)} = \|g\|_{L^2(\nu)}\|f\|_{L^2(\nu)}.$$

Let $f \in L^2(\nu)$ and $\lambda \in \mathbb{R}_+$. We define the operator σ_λ by

$$\mathscr{F}^{-1}(\sigma_\lambda f)(x) = \varphi_\lambda(x)\mathscr{F}^{-1}(f)(x).$$

The operator σ_λ satisfies

$$\sigma_\lambda f(y) = \sigma_y f(\lambda), \quad \|\sigma_\lambda f\|_{L^2(\nu)} \le \|f\|_{L^2(\nu)}. \tag{2.8}$$

Theorem 2.4. (See [38]). Let $f, g \in L^2(\nu)$. Then

$$\mathscr{S}_g(f)(\lambda, y) = \int_{\mathbb{R}_+} f(t)\sigma_\lambda g_y(t) d\nu(t), \quad \lambda, y \in \mathbb{R}_+.$$

For $f, g \in L^2(\nu)$ and $F \in L^2(\nu \otimes \mu)$ we define $\mathscr{S}_g^*$ by

$$\langle \mathscr{S}_g(f), F\rangle_{L^2(\nu\otimes\mu)} = \langle f, \mathscr{S}_g^*(F)\rangle_{L^2(\nu)}. \tag{2.9}$$

Then by Theorem 2.4 and Fubini's theorem we obtain

$$\mathscr{S}_g^*(F)(t) = \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} F(\lambda, y)\sigma_\lambda g_y(t) d\nu(\lambda) d\mu(y), \quad t \in \mathbb{R}_+. \tag{2.10}$$

Theorem 2.5. (See [38]). Let $g \in L^2(\nu)$ be a non-zero function. Then $\mathscr{S}_g(L^2(\nu))$ is a reproducing kernel Hilbert space in $L^2(\nu \otimes \mu)$ with kernel function

$$W_g((\lambda, y); (t, x)) := \overline{\mathscr{S}_g(\sigma_\lambda g_y)(t, x)}. \tag{2.11}$$

3. Concentration operators associated with SLST

In this section, we define the concentration operators for SLST and we prove that they are bounded and compact operators in the so-called Schatten-von Neumann classes.

We denote by $B(L^2(\nu))$ the space of all bounded operators Ψ from $L^2(\nu)$ into itself, equipped with the norm

$$\|\Psi\| := \sup_{\|f\|_{L^2(\nu)}=1} \|\Psi(f)\|_{L^2(\nu)}.$$

For a compact operator $\Psi \in B(L^2(\nu))$, the eigenvalues of the positive self-adjoint operator $|\Psi| := \sqrt{\Psi^*\Psi}$ are called the singular values of Ψ and denoted by $\{s_n(\Psi)\}_{n\in\mathbb{N}}$.

The Schatten-von Neumann class S_p, $p \in [1,\infty)$ is the space of all compact operators Ψ whose singular values $s_n(\Psi)$ lie in $l^p(\mathbb{N})$. The class S_p is provided with the norm

$$\|\Psi\|_{S_p} := \left[\sum_{n=1}^{\infty}(s_n(\Psi))^p\right]^{\frac{1}{p}}.$$

The Schatten-von Neumann class S_∞ is the class of all compact operators with the norm

$$\|\Psi\|_{S_\infty} := \|\Psi\|.$$

We note that the space S_1 is the space of trace-class operators. We define the trace of an operator Ψ in S_1 by

$$\mathrm{Tr}(\Psi) := \sum_{n=1}^{\infty}\langle \Psi(v_n), v_n\rangle_{L^2(\nu)}, \tag{3.1}$$

where $\{v_n\}_{n\in\mathbb{N}}$ is any orthonormal basis of $L^2(\nu)$. Moreover, if Ψ is positive, then

$$\mathrm{Tr}(\Psi) = \|\Psi\|_{S_1}. \tag{3.2}$$

We note that the space S_2 is the space of Hilbert-Schmidt operators. A compact operator Ψ on the Hilbert space $L^2(\nu)$ is called the Hilbert-Schmidt operator, if the positive operator $\Psi^*\Psi$ is in the trace-class S_1. Then for any orthonormal basis $\{v_n\}_{n\in\mathbb{N}}$ of $L^2(\nu)$, we have

$$\|\Psi\|_{HS}^2 = \|\Psi\|_{S_2}^2 = \|\Psi^*\Psi\|_{S_1} = \mathrm{Tr}(\Psi^*\Psi) = \sum_{n=1}^{\infty}\|\Psi(v_n)\|_{L^2(\nu)}^2.$$

Now we are in a position to state the definition of the localization operators associated with SLST. In the following, the function g will be in $L^2(\nu)$ such that $\|g\|_{L^2(\nu)} = 1$ and U be a subset of $\mathbb{R}_+^2$ with $\nu\otimes\mu(U) < \infty$. In this sense, we have the following definition.

Let $\xi \in L^1 \cup L^\infty(\nu\otimes\mu)$. We define the localization operators associated with the Sturm-Liouville-Stockwell transform $\mathscr{S}_g$, for $f \in L^2(\nu)$ by

$$L_{g,\xi}(f)(\lambda) := \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \xi(t,y)\mathscr{S}_g(f)(t,y)\sigma_\lambda g_y(t)\mathrm{d}\nu(t)\mathrm{d}\mu(y), \quad \lambda \in \mathbb{R}_+.$$

For all $f, h \in L^2(\nu)$, we have

$$\langle L_{g,\xi}(f), h\rangle_{L^2(\nu)} = \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \xi(t,y)\mathscr{S}_g(f)(t,y)\overline{\mathscr{S}_g(h)(t,y)}\mathrm{d}\nu(t)\mathrm{d}\mu(y).$$

Therefore, the adjoint of $L_{g,\xi}$ is the operator $L_{g,\xi}^*$ given by

$$L_{g,\xi}^* = L_{g,\overline{\xi}} : L^2(\nu) \to L^2(\nu).$$

Theorem 3.1. (See [38]).

(i) Let $\xi \in L^p(\nu \otimes \mu)$, $p \in [1, \infty]$. Then the localization operator $L_{g,\xi}$ is bounded from $L^2(\nu)$ into itself, and

$$\|L_{g,\xi}\|_{S_\infty} \leq \|\xi\|_{L^p(\nu\otimes\mu)}.$$

(ii) Let $\xi \in L^p(\nu \otimes \mu)$, $p \in [1, \infty)$. Then the localization operator $L_{g,\xi} : L^2(\nu) \to L^2(\nu)$ is compact.

(iii) Let $\xi \in L^1(\nu \otimes \mu)$. We have

$$\mathrm{Tr}(L_{g,\xi}) = \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \xi(t, y) \|\sigma_t g_y\|^2_{L^2(\nu)} \mathrm{d}\nu(t) \mathrm{d}\mu(y),$$

and

$$|\mathrm{Tr}(L_{g,\xi})| \leq \|\xi\|_{L^1(\nu\otimes\mu)}.$$

The Sturm-Liouville-Stockwell concentration operator (SLSCO) is defined for $f \in L^2(\nu)$, by

$$L_{g,U}(f)(\lambda) := \int_U \mathscr{S}_g(f)(t, y) \sigma_\lambda g_y(t) \mathrm{d}\nu(t) \mathrm{d}\mu(y), \quad \lambda \in \mathbb{R}_+.$$

By (2.10) we have

$$L_{g,U}(f)(\lambda) = \mathscr{S}_g^* \left(\chi_U \mathscr{S}_g(f) \right)(\lambda), \quad \lambda \in \mathbb{R}_+, \tag{3.3}$$

where χ_U is the characteristic function of the set U.

For all $f, h \in L^2(\nu)$, we have

$$\left\langle L_{g,U}(f), h \right\rangle_{L^2(\nu)} = \int_U \mathscr{S}_g(f)(t, y) \overline{\mathscr{S}_g(h)(t, y)} \mathrm{d}\nu(t) \mathrm{d}\mu(y). \tag{3.4}$$

Theorem 3.2. (i) The concentration operator $L_{g,U}$ is in S_∞ and we have

$$\|L_{g,U}\|_{S_\infty} \leq 1$$

(ii) The concentration operator $L_{g,U}$ is in S_1 with

$$\mathrm{Tr}(L_{g,U}) = \int_U \|\sigma_t g_y\|^2_{L^2(\nu)} \mathrm{d}\nu(t) \mathrm{d}\mu(y),$$

and

$$|\mathrm{Tr}(L_{g,U})| \leq \nu \otimes \mu(U).$$

Proof. (i) For all functions f and h in $L^2(\nu)$, we have from Hölder's inequality

$$\begin{aligned} \left|\langle L_{g,U}(f), h\rangle_{L^2(\nu)}\right| &\leq \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \left| \mathscr{S}_g(f)(t, y) \overline{\mathscr{S}_g(h)(t, y)} \right| \mathrm{d}\nu(t) \mathrm{d}\mu(y) \\ &\leq \|\mathscr{S}_g(f)\|_{L^2(\nu\otimes\mu)} \|\mathscr{S}_g(h)\|_{L^2(\nu\otimes\mu)}. \end{aligned}$$

Using Theorem 2.3, we get

$$\left| \left\langle L_{g,U}(f), h \right\rangle_{L^2(\nu)} \right| \leq \|f\|_{L^2(\nu)} \|h\|_{L^2(\nu)}.$$

Thus,

$$\|L_{g,U}\|_{S_\infty} \leq 1.$$

(ii) Let $\{v_n\}_{n=1}^{\infty}$ be an orthonormal basis of $L^2(\nu)$. Using (3.4), Fubini's theorem and Theorem 2.4, we obtain

$$\begin{aligned}\sum_{n=1}^{\infty}\langle L_{g,U}(v_n),v_n\rangle_{L^2(\nu\otimes\mu)} &= \sum_{n=1}^{\infty}\int_U|\mathscr{S}_g(v_n)(t,y)|^2\mathrm{d}\nu(t)\mathrm{d}\mu(y)\\ &= \int_U\sum_{n=1}^{\infty}|\mathscr{S}_g(v_n)(t,y)|^2\mathrm{d}\nu(t)\mathrm{d}\mu(y)\\ &= \int_U\sum_{n=1}^{\infty}|\langle v_n,\sigma_t g_y\rangle_{L^2(\nu)}|^2\mathrm{d}\nu(t)\mathrm{d}\mu(y)\\ &= \int_U\|\sigma_t g_y\|^2_{L^2(\nu)}\mathrm{d}\nu(t)\mathrm{d}\mu(y).\end{aligned}$$

Thus from (2.7) and (2.8) we get

$$\sum_{n=1}^{\infty}\langle L_{g,U}(v_n),v_n\rangle_{L^2(\nu\otimes\mu)}\le \nu\otimes\mu(U).$$

Then, the operator $L_{g,U}$ is in S_1 and by relation (3.1) we have

$$\mathrm{Tr}(L_{g,U})=\int_U\|\sigma_t g_y\|^2_{L^2(\nu)}\mathrm{d}\nu(t)\mathrm{d}\mu(y),$$

and

$$|\mathrm{Tr}(L_{g,U})|\le \nu\otimes\mu(U).$$

The theorem is proved. □

4. Toeplitz operators associated with SLST

The first application in this paper is the study of the Toeplitz operators associated with SLST. In the following, the function g will be in $L^2(\nu)$ such that $\|g\|_{L^2(\nu)}=1$ and U be a subset of $\mathbb{R}_+^2$ with $\nu\otimes\mu(U)<\infty$.

We define the orthogonal projection $P_g:L^2(\nu\otimes\mu)\longrightarrow L^2(\nu\otimes\mu)$, by

$$P_g(F)(\lambda,y):=\int_{\mathbb{R}_+}\int_{\mathbb{R}_+}F(t,x)W_g((\lambda,y);(t,x))\mathrm{d}\nu(t)\mathrm{d}\mu(x),$$

where W_g is the kernel given by (2.11).

We define the orthogonal projection $P_U:L^2(\nu\otimes\mu)\longrightarrow L^2(\nu\otimes\mu)$, by

$$P_U(F)(\lambda,y):=\chi_U(\lambda,y)F(\lambda,y).$$

We define the Sturm-Liouville-Stockwell Toeplitz operator (SLSTO), $T_{g,U}:\mathscr{S}_g(L^2(\nu))\to\mathscr{S}_g(L^2(\nu))$, by

$$T_{g,U}(F):=P_gP_U(F). \tag{4.1}$$

We have the following theorem.

Theorem 4.1. The operator $T_{g,U}:\mathscr{S}_g(L^2(\nu))\to\mathscr{S}_g(L^2(\nu))$ satisfies

$$0\le T_{g,U}\le P_U\le I \quad\text{and}\quad T_{g,U}=\mathscr{S}_gL_{g,U}\mathscr{S}_g^*.$$

Proof. Let $F\in\mathscr{S}_g\left(L^2(\nu)\right)$. From (4.1), we have

$$\left\langle T_{g,U}(F),F\right\rangle_{L^2(\nu\otimes\mu)}=\left\langle P_g\left(P_U(F)\right),F\right\rangle_{L^2(\nu\otimes\mu)}=\langle P_U(F),F\rangle_{L^2(\nu\otimes\mu)}.$$

Then

$$\left\langle T_{g,U}(F), F\right\rangle_{L^2(\nu\otimes\mu)} = \int_U |F(t,y)|^2 \mathrm{d}\nu(t)\mathrm{d}\mu(y).$$

Thus we deduce that $0 \le T_{g,U} \le P_U \le I$ and $T_{g,U}$ is bounded and positive.
On the other hand, from (3.3) an by using the fact that $\mathscr{S}_g\mathscr{S}_g^* = P_g$, for $F \in \mathscr{S}_g\left(L^2(\nu)\right)$ we have

$$\mathscr{S}_g L_{g,U}\mathscr{S}_g^*(F) = P_g P_U P_g(F) = P_g P_U(F) = T_{g,U}(F).$$

Therefore the concentration operator $L_{g,U}$ and the Sturm-Liouville-Stockwell Toeplitz operator $T_{g,U}$ are related by

$$T_{g,U} = \mathscr{S}_g L_{g,U}\mathscr{S}_g^*.$$

The theorem is proved. □

We denote by $M_{g,U}$ the quantity

$$M_{g,U} := \int_U \left\|\sigma_t g_y\right\|^2_{L^2(\nu)} \mathrm{d}\nu(t)\mathrm{d}\mu(y). \tag{4.2}$$

We also have the following theorem.

Theorem 4.2. The Sturm-Liouville-Stockwell Toeplitz operator $T_{g,U}$ is compact and of trace-class with

$$\mathrm{Tr}\left(T_{g,U}\right) = \mathrm{Tr}\left(L_{g,U}\right) = M_{g,U}.$$

Proof. From Theorem 4.1, the operator $T_{g,U} : \mathscr{S}_g(L^2(\nu)) \to \mathscr{S}_g(L^2(\nu))$ is bounded and positive. Now, let $\{\phi_n\}_{n=1}^\infty$ be an arbitrary orthonormal basis for $\mathscr{S}_g(L^2(\nu))$.
If we denote by $v_n = \mathscr{S}_g^*(\phi_n)$, then $\{v_n\}_{n=1}^\infty$ is an orthonormal basis for $L^2(\nu)$. Thus by Theorem 4.1, (2.9) and (3.1) we have

$$\begin{aligned}\sum_{n=1}^\infty \langle T_{g,U}(\phi_n), \phi_n\rangle_{L^2(\nu\otimes\mu)} &= \sum_{n=1}^\infty \langle L_{g,U}(\mathscr{S}_g^*(\phi_n)), \mathscr{S}_g^*(\phi_n)\rangle_{L^2(\nu)}\\ &= \sum_{n=1}^\infty \langle L_{g,U}(v_n), v_n\rangle_{L^2(\nu)} = \mathrm{Tr}\left(L_{g,U}\right).\end{aligned}$$

Therefore, by (3.1), (3.2) and Theorem 3.2 (ii), the operator $T_{g,U}$ is trace-class with

$$\left\|T_{g,U}\right\|_{S_1} = \mathrm{Tr}\left(T_{g,U}\right) = M_{g,U}.$$

The theorem is proved. □

Remark 4.3. Since the concentration operator $L_{g,U} = \mathscr{S}_g^* P_U \mathscr{S}_g$ is a compact and self-adjoint operator, the spectral theorem gives the following spectral representation

$$L_{g,U}(f) = \sum_{n=1}^\infty s_n(U)\left\langle f, v_n^U\right\rangle_{L^2(\nu)} v_n^U, \quad f \in L^2(\nu),$$

where $\{s_n(U)\}_{n=1}^\infty$ are the positive eigenvalues arranged in a decreasing manner and $\{v_n^U\}_{n=1}^\infty$ is the corresponding orthonormal set of eigenfunctions. According to Theorem 3.2 (i), we have

$$s_n(U) \le s_1(U) \le 1, \quad n \ge 1.$$

Then by Theorem 4.1, we deduce that the Sturm-Liouville-Stockwell Toeplitz operator $T_{g,U} : \mathscr{S}_g(L^2(\nu)) \to \mathscr{S}_g(L^2(\nu))$ can be diagonalized as

$$T_{g,U}(F) = \sum_{n=1}^\infty s_n(U)\left\langle F, \phi_n^U\right\rangle_{L^2(\nu\otimes\mu)} \phi_n^U, \quad F \in \mathscr{S}_g(L^2(\nu)),$$

where $\phi_n^U = \mathscr{S}_g(v_n^U)$.

In the context of this remark, let θ be the function defined by

$$\theta(t,x) := \int_U |W_g((\lambda,y);(t,x))|^2 d\nu(\lambda)d\mu(y), \quad (t,x) \in \mathbb{R}_+^2, \tag{4.3}$$

where W_g is the kernel given by (2.11). Then we obtain the following result.

Theorem 4.4. For all $(t,x) \in \mathbb{R}_+^2$, we have

$$\theta(t,x) = \sum_{n=1}^{\infty} s_n(U) \left|\phi_n^U(t,x)\right|^2.$$

Proof. From Theorem 2.5, we have for all $(t,x) \in \mathbb{R}_+^2$, the function $W_g(.;(t,x))$ is in $\mathscr{S}_g(L^2(\nu))$. Therefore using the properties of the reproducing kernel Hilbert space, we get

$$\begin{aligned}\left\langle T_{g,U}(W_g(.;(t,x))), W_g(.;(t,x))\right\rangle_{L^2(\nu\otimes\mu)} &= \left\langle P_U(W_g(.;(t,x))), W_g(.;(t,x))\right\rangle_{L^2(\nu\otimes\mu)} \\ &= \int_U \left|W_g((\lambda,y);(t,x))\right|^2 d\nu(\lambda)d\mu(y) \\ &= \theta(t,x).\end{aligned}$$

Let $\left\{w_n^U\right\}_{n=1}^{\infty} \subset \mathscr{S}_g(L^2(\nu))$ be an orthonormal basis of $\operatorname{Ker}\left(T_{g,U}\right)$. Hence, $\left\{\phi_n^U\right\}_{n=1}^{\infty} \cup \left\{w_n^U\right\}_{n=1}^{\infty}$ is an orthonormal basis of $\mathscr{S}_g(L^2(\nu))$ and therefore the reproducing kernel W_g can be written as

$$W_g((\lambda,y);(t,x)) = \sum_{n=1}^{\infty} \overline{\phi_n^U(t,x)}\phi_n^U(\lambda,y) + \sum_{n=1}^{\infty} \overline{w_n^U(t,x)} w_n^U(\lambda,y).$$

Using this, we compute again

$$\begin{aligned}\theta(t,x) &= \left\langle T_{g,U}(W_g(.;(t,x))), W_g(.;(t,x))\right\rangle_{L^2(\nu\otimes\mu)} \\ &= \left\langle T_{g,U}\left(\sum_{n=1}^{\infty} \overline{\phi_n^U(t,x)}\phi_n^U\right), \sum_{k=1}^{\infty} \overline{\phi_k^U(t,x)}\phi_k^U\right\rangle_{L^2(\nu\otimes\mu)} \\ &= \sum_{n,k=1}^{\infty} \overline{\phi_n^U(t,x)}\phi_k^U(t,x)\left\langle T_{g,U}(\phi_n^U), \phi_k^U\right\rangle_{L^2(\nu\otimes\mu)} \\ &= \sum_{n=1}^{\infty} s_n(U)\left|\phi_n^U(t,x)\right|^2.\end{aligned}$$

The theorem is proved. □

5. Spectrogram analysis associated with SLST

The second application in this paper is the study of the spectrogram analysis associated with SLST. In the following, the function g will be in $L^2(\nu)$ such that $\|g\|_{L^2(\nu)} = 1$ and U be a subset of $\mathbb{R}_+^2$ with $\nu\otimes\mu(U) < \infty$.

Let $f \in L^2(\nu)$. We define the Sturm-Liouville-Stockwell spectrogram (SLSS) of f as

$$S_g(f)(t,y) := \left|\mathscr{S}_g(f)(t,y)\right|^2, \quad (t,y) \in \mathbb{R}_+^2.$$

Note that Stockwell spectrograms are a powerful tool for the analysis of non-stationary signals. They are used in a wide variety of applications, including speech recognition, music analysis and medical signal analysis and so on.

From Theorem 2.3, we have

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} S_g(f)(t,y)\mathrm{d}\nu(t)\mathrm{d}\mu(y) = \|f\|^2_{L^2(\nu)},$$

which explains the interpretation of a spectrogram as a time-frequency energy density. Note that also by (3.4), we have

$$\left\langle L_{g,U}f, f\right\rangle_{L^2(\nu)} = \int_U S_g(f)(t,y)\mathrm{d}\nu(t)\mathrm{d}\mu(y).$$

Let V be an N-dimensional subspace of $L^2(\nu)$. We define the orthogonal projection P_V onto V with projection kernel K_V, i.e.

$$P_V f(r) := \int_{\mathbb{R}_+} K_V(r,x)f(x)\mathrm{d}\nu(x), \quad r \in \mathbb{R}_+.$$

Recall that if $\{v_n\}_{n=1}^N$ is an orthonormal basis of V, then

$$K_V(r,x) = \sum_{n=1}^N v_n(r)\overline{v_n(x)}, \quad r,x \in \mathbb{R}_+.$$

The kernel K_V is independent of the choice of orthonormal basis for V.

The spectrogram of the space V with respect g is defined as

$$\mathrm{Spec}_g V(t,y) := \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} K_V(r,x)\overline{\sigma_t g_y(r)}\sigma_t g_y(x)\mathrm{d}\nu(r)\mathrm{d}\nu(x).$$

Then we have the following result.

Theorem 5.1. The spectrogram $\mathrm{Spec}_g V$ is given by

$$\mathrm{Spec}_g V = \sum_{n=1}^N S_g(v_n).$$

Proof. We have

$$\begin{aligned}
\mathrm{Spec}_g V(t,y) &= \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} \sum_{n=1}^N v_n(r)\overline{v_n(x)\sigma_t g_y(r)}\sigma_t g_y(x)\mathrm{d}\nu(r)\mathrm{d}\nu(x) \\
&= \sum_{n=1}^N \langle v_n, \sigma_t g_y\rangle_{L^2(\nu)}\overline{\langle v_n, \sigma_t g_y\rangle}_{L^2(\nu)} \\
&= \sum_{n=1}^N \mathscr{S}_g(v_n)(t,y)\overline{\mathscr{S}_g(v_n)(t,y)} \\
&= \sum_{n=1}^N |\mathscr{S}_g(v_n)(t,y)|^2 = \sum_{n=1}^N S_g(v_n)(t,y).
\end{aligned}$$

This allows us to conclude. □

We define the time-frequency concentration of a subspace V in U as

$$Z_{g,U}(V) := \frac{1}{N}\int_U \mathrm{Spec}_g V(t,y)\mathrm{d}\nu(t)\mathrm{d}\mu(y).$$

Then from Theorem 5.1, we have

$$Z_{g,U}(V) = \frac{1}{N}\sum_{n=1}^N \int_U S_g(v_n)(t,y)\mathrm{d}\nu(t)\mathrm{d}\mu(y).$$

Theorem 5.2. The N-dimensional signal space $V_N = \operatorname{span}\left\{v_n^U\right\}_{n=1}^N$ consisting of the first N eigenfunctions of $L_{g,U}$ corresponding to the N largest eigenvalues $\{s_n(U)\}_{n=1}^N$ maximize the regional concentration $Z_{g,U}(V)$ and

$$Z_{g,U}(V_N) = \frac{1}{N}\sum_{n=1}^N s_n(U).$$

Proof. We have

$$Z_{g,U}(V_N) = \frac{1}{N}\sum_{n=1}^N \int_U S_g\left(v_n^U\right)(t,y)\mathrm{d}\nu(t)\mathrm{d}\mu(y).$$

Moreover, the min-max lemma for self-adjoint operators (see [32], Section 95) states that

$$\begin{aligned} s_n(U) &= \int_U S_g\left(v_n^U\right)(t,y)\mathrm{d}\nu(t)\mathrm{d}\mu(y) \\ &= \max\left\{\left\langle L_{g,U}(f), f\right\rangle_{L^2(\nu)} : \|f\|_{L^2(\nu)} = 1, f \perp v_1^U, \dots, v_{n-1}^U\right\}. \end{aligned}$$

The eigenvalues of $L_{g,U}$ therefore determine the number of orthogonal functions that have a well-concentrated spectrogram in U. So,

$$Z_{g,U}(V_N) = \frac{1}{N}\sum_{n=1}^N s_n(U).$$

The min-max characterization of the eigenvalues of compact operators implies that the first N eigenfunctions of the concentration operator $L_{g,U}$ have optimal cumulative time-frequency concentration inside U, in the sense that

$$\sum_{n=1}^N \left\langle L_{g,U}\left(v_n^U\right), v_n^U\right\rangle_{L^2(\nu)} = \max\left\{\sum_{n=1}^N \left\langle L_{g,U}(v_n), v_n\right\rangle_{L^2(\nu)} : \{v_n\}_{n=1}^N \text{ orthonormal}\right\}.$$

Consequently no N-dimensional subset V of $L^2(\nu)$ can be better concentrated in U than V_N, i.e

$$Z_{g,U}(V) \leq Z_{g,U}(V_N).$$

The proof is complete. □

Remark 5.3. The above result, has important implications for signal processing and time-frequency analysis. It means that the first N eigenfunctions of $L_{g,U}$ can be used to efficiently represent signals that are localized in U. This is because the eigenfunctions are concentrated in U, so they can be used to reconstruct the signal with high accuracy.

The time-frequency concentration of a subspace V_N in U satisfies,

$$s_N(U) \leq Z_{g,U}(V_N) \leq s_1(U) \leq 1.$$

Let $A_{g,U} := \operatorname{Spec}_g V_{N_{g,U}}$, called the accumulated spectrogram, where we assume that $N_{g,U} = \left\lceil M_{g,U}\right\rceil$ is the smallest integer greater than or equal to $M_{g,U}$ (the quantity given by (4.2)) and

$$V_{N_{g,U}} = \operatorname{span}\left\{v_n^U\right\}_{n=1}^{N_{g,U}}.$$

Then

$$A_{g,U}(t,y) = \sum_{n=1}^{N_{g,U}} \left|\mathscr{S}_g\left(v_n^U\right)(t,y)\right|^2 = \sum_{n=1}^{N_{g,U}} \left|\phi_n^U(t,y)\right|^2.$$

Note that

$$\left\|A_{g,U}\right\|_{L^1(\nu\otimes\mu)} = N_{g,U} = M_{g,U} + O(1).$$

Moreover, since

$$\sum_{n=1}^{N_{g,U}} s_n(U) \leq \mathrm{Tr}\left(L_{g,U}\right) = M_{g,U},$$

then we can define the quantity

$$E_{g,U} := 1 - \sum_{n=1}^{N_{g,U}} \frac{s_n(U)}{M_{g,U}}$$

which satisfies,

$$0 \leq E_{g,U} \leq 1.$$

Theorem 5.4. We have

$$\left\|A_{g,U} - \theta\right\|_{L^1(\nu\otimes\mu)} \leq 1 + 2M_{g,U}E_{g,U},$$

where θ is the function given by (4.3).

Proof. From Theorem 4.4, for all $(t, y) \in U$, we have,

$$A_{g,U}(t, y) - \theta(t, y) = \sum_{n=1}^{\infty} (t_n - s_n(U)) \left|\phi_n^U(t, y)\right|^2,$$

where $t_n = 1$ if $n \leq N_{g,U}$ and 0 otherwise. Now since

$$\left\||\phi_n^U|^2\right\|_{L^1(\nu\otimes\mu} = 1 \quad \text{and} \quad \sum_{n=1}^{\infty} s_n(U) = M_{g,U},$$

we obtain

$$\begin{aligned}
\|A_{g,U} - \theta\|_{L^1(\nu\otimes\mu)} &\leq \sum_{n=1}^{\infty} |t_n - s_n(U)| \\
&= \sum_{n=1}^{N_{g,U}} (1 - s_n(U)) + \sum_{n>N_{g,U}} s_n(U) \\
&= N_{g,U} + \sum_{n=1}^{\infty} s_n(U) - 2\sum_{n=1}^{N_{g,U}} s_n(U) \\
&= N_{g,U} + M_{g,U} - 2\sum_{n=1}^{N_{g,U}} s_n(U) \\
&= \left(N_{g,U} - M_{g,U}\right) + 2M_{g,U}E_{g,U} \\
&\leq 1 + 2M_{g,U}E_{g,U}.
\end{aligned}$$

The theorem is proved. □

6. Conclusion and perspective

In this paper, we have examined the Toeplitz operators related to SLST and studied their trace-class and Schatten-von Neumann class properties. The spectrogram analysis associated with SLST is also studied in detail. Finally, we indicate that in our future work, we will study the Benedicks-type uncertainty principles for SLST.

Acknowledgements. The authors are deeply grateful to the referee for his constructive comments and valuable suggestions.

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