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On k-circulant matrices with the Mersenne numbers having arithmetic indices

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Abstract. Let k be a non-zero complex number. In this paper, we consider a k-circulant matrix whose first row is $(M_s, M_{s+t}, M_{s+2t}, \ldots, M_{s+(n-2)t}, M_{s+(n-1)t})$, where M_n is the n^{th} Mersenne number, s is a non-negative integer and t is a positive integer. The formulae for the eigenvalues of such matrix are obtained. That formulae improve the result of Theorem 2.3. [20] (because the result of Theorem 2.3. [20] can not be applied in some cases) and show that there are cases when the result of Theorem 2.9. [20] can also not be applied. Then, we consider the norms of such matrix. Namely, the obtained formulae for the 1−norm, the ∞−norm, the Euclidean norm and the spectral norm of such matrix extend (and correct) the results of, respectively, Theorem 3.3. [20], Theorem 3.4. [20] and Theorem 3.6. [20]. At the end of the paper, we also obtain the bounds for the spectral norm of a k-circulant matrix whose first row is $(M_s^{-1}, M_{s+t}^{-1}, M_{s+2t}^{-1}, \ldots, M_{s+(n-2)t}^{-1}, M_{s+(n-1)t}^{-1})$ provided that s is a positive integer.

1. Introduction

Let $C \in \mathbb{C}^{m\times n}$, where $\mathbb{C}^{m\times n}$ is the set of all $m \times n$ complex matrices i.e. $C = \left[c_{i,j}\right]_{m\times n}$, where $c_{i,j} \in \mathbb{C}$, $i = \overline{1,m}$, $j = \overline{1,n}$. By $C_{i\rightarrow}$ and $C_{\downarrow j}$ we denote, respectively, the i^{th} row and the j^{th} column of C $\left(i = \overline{1,m}, j = \overline{1,n}\right)$ i.e. $C_{i\rightarrow} = (c_{i,1}, c_{i,2}, \ldots, c_{i,n})$ and $C_{\downarrow j} = \left(c_{1,j}, c_{2,j}, \ldots, c_{m,j}\right)^T$, where $\left(c_{1,j}, c_{2,j}, \ldots, c_{m,j}\right)^T$ is the transpose of $\left(c_{1,j}, c_{2,j}, \ldots, c_{m,j}\right)$. In this paper we consider the properties (the eigenvalues, the determinant and the norms) of a k-circulant matrix, where k is a non-zero complex number, with the Mersenne numbers having arithmetic indices.

Definition 1.1. *(A k-**circulant matrix**) Let $C \in \mathbb{C}^{n\times n}$ and $C_{1\rightarrow} = (c_0, c_1, c_2, \ldots, c_{n-1})$. A matrix C is called a k-circulant matrix if C satisfy the following conditions:*

$$c_{i,j} = \begin{cases} c_{j-i}, & i \le j \\ kc_{n+j-i}, & otherwise \end{cases}, \; i = \overline{2,n}, \; j = \overline{1,n} \tag{1}$$

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i.e. C has the following form:

$$\begin{bmatrix} c_0 & c_1 & c_2 & \dots & c_{n-2} & c_{n-1} \\ kc_{n-1} & c_0 & c_1 & \dots & c_{n-3} & c_{n-2} \\ kc_{n-2} & kc_{n-1} & c_0 & \dots & c_{n-4} & c_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ kc_2 & kc_3 & kc_4 & \dots & c_0 & c_1 \\ kc_1 & kc_2 & kc_3 & \dots & kc_{n-1} & c_0 \end{bmatrix}. \Diamond \tag{2}$$

Since a k-circulant matrix is completely determined by k and its first row, if C is a k-circulant matrix (of order n) whose first row is $(c_0, c_1, c_2, \dots, c_{n-1})$, we shall write $C = \text{circ}_n\{_k(c_0, c_1, c_2, \dots, c_{n-1})\}$. If the order of a matrix is known, then the designation for the order of a matrix can be omitted. If $k = 1$ ($k = -1$), then k-circulant matrices are called circulant (skew circulant) matrices. Let us note that k-circulant matrices belong to the class of matrices having constant main diagonals known as Toeplitz matrices, named after the German mathematician *Otto Toeplitz* (1881–1940).

Definition 1.2. *(**A Toeplitz matrix**) Let $T \in \mathbb{C}^{n\times n}$, $T_{1\rightarrow} = (t_0, t_1, t_2, \dots, t_{n-1})$ and $T_{\downarrow 1} = (t_0, t_{-1}, t_{-2}, \dots, t_{1-n})^T$. A matrix T is called a Toeplitz matrix if T satisfy the following conditions:*

$$t_{i,j} = t_{j-i},\ i = \overline{2,n},\ j = \overline{2,n} \tag{3}$$

i.e. T has the following form:

$$\begin{bmatrix} t_0 & t_1 & t_2 & \dots & t_{n-2} & t_{n-1} \\ t_{-1} & t_0 & t_1 & \dots & t_{n-3} & t_{n-2} \\ t_{-2} & t_{-1} & t_0 & \dots & t_{n-4} & t_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ t_{2-n} & t_{3-n} & t_{4-n} & \dots & t_0 & t_1 \\ t_{1-n} & t_{2-n} & t_{3-n} & \dots & t_{-1} & t_0 \end{bmatrix}. \Diamond \tag{4}$$

For more information regarding Toeplitz matrices, we recommend [7], [11], [14], [19] and [24].

In the paper [5] R. E. Cline, R. J. Plemmons and G. Worm presented necessary and sufficient conditions for a complex square matrix to be a k-circulant matrix.

Lemma 1.3. *(Lemma 2. [5]) Let A be any complex matrix of order n. Then, A is a k-circulant matrix (of order n) if and only if A commutes with the matrix $K = circ_n\{_k(0, 1, 0, \dots, 0)\}$. In this case A can be expressed as*

$$A = \sum_{i=0}^{n-1} a_i K^i, \tag{5}$$

where $A_{1\rightarrow} = (a_0, a_1, a_2, \dots, a_{n-1})$.▼

Lemma 1.4. *(Lemma 3. [5]) Let A be any complex matrix of order n and*

$$\Psi = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \psi & 0 & \cdots & 0 & 0 \\ 0 & 0 & \psi^2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \psi^{n-2} & 0 \\ 0 & 0 & 0 & \cdots & 0 & \psi^{n-1} \end{bmatrix}, \tag{6}$$

where ψ is any n^{th} root of k. Then, A is a k-circulant matrix if and only if

$$A = \Psi Q \Psi^{-1} \tag{7}$$

for some circulant matrix Q .▼

In recent years k-circulant matrices present one of the most important research fields of applied mathematics and computational mathematics. These matrices have a wide range of applications in many areas such as signal and image processing, coding theory, probability, statistics, partial and ordinary differential equations, economy, engineering model, communications, numerical analysis and vibration analysis (see [2], [4], [8], [9], [15], [22], [23], [40], [42] and [43]). Many authors considered k-circulant matrices with well – known number sequences and obtained some good results in this area. In the paper [10] C. He, J. Ma, K. Zhang and Z. Wang considered k-circulant matrices with the first rows $(F_0, F_1, F_2, \ldots, F_{n-1})$, $(L_0, L_1, L_2, \ldots, L_{n-1})$, $(F_0^2, F_1^2, F_2^2, \ldots, F_{n-1}^2)$, $(L_0^2, L_1^2, L_2^2, \ldots, L_{n-1}^2)$ and $(F_0L_0, F_1L_1, F_2L_2, \ldots, F_{n-1}L_{n-1})$, where F_n is the n^{th} Fibonacci number ($F_0=0, F_1=1, F_n=F_{n-2}+F_{n-1}$, $n \geq 2$) and L_n is the n^{th} Lucas number ($L_0=2, L_1=1, L_n=L_{n-2}+L_{n-1}$, $n \geq 2$), and obtained the upper bounds for the spectral norms of such matrices. The obtained results in relation to the upper bounds for the spectral norms of k-circulant matrices with the Fibonacci and Lucas numbers are more accurate than the corresponding results presented in the paper [33]. The purpose of the paper [1] is to investigate the norms of k-circulant matrices with the hyper - Fibonacci numbers ($F_n^k = \sum_{s=0}^{n} F_s^{k-1}, F_n^0 = F_n, F_0^k = 0, F_1^k = 1$ i.e. $F_n^k = F_{n-1}^k + F_n^{k-1}$) and the hyper - Lucas numbers ($L_n^k = \sum_{s=0}^{n} L_s^{k-1}, L_n^0 = L_n, L_0^k = 2, L_1^k = 2k+1$ i.e. $L_n^k = L_{n-1}^k + L_n^{k-1}$). C. Köme and Y. Yazlik considered k-circulant matrices with the biperiodic Fibonacci numbers $\left(q_0=0, q_1=1, q_n=\begin{cases} q_{n-2}+aq_{n-1}, & n \text{ is even} \\ q_{n-2}+bq_{n-1}, & n \text{ is odd} \end{cases}, a,b \in \mathbb{R}\setminus\{0\}\right)$ and the biperiodic Lucas numbers $\left(l_0=2, l_1=a, l_n=\begin{cases} l_{n-2}+bl_{n-1}, & n \text{ is even} \\ l_{n-2}+al_{n-1}, & n \text{ is odd} \end{cases}, a,b \in \mathbb{R}\setminus\{0\}\right)$, and obtained the bounds for the spectral norms of such matrices (and their Kronecker and Hadamard products) in the paper [17], and the determinants and the inverses of such matrices in the paper [18]. In the paper [16] Z Jiang, J. Li and N. Shen considered k-circulant matrices with the first rows $(P_1, P_2, P_3, \ldots, P_n)$, $(Q_1, Q_2, Q_3, \ldots, Q_n)$, $(J_1, J_2, J_3, \ldots, J_n)$ and $(j_1, j_2, j_3, \ldots, j_n)$, where P_n is the n^{th} Pell number ($P_0=0, P_1=1, P_n=P_{n-2}+2P_{n-1}$, $n \geq 2$), Q_n is the n^{th} Pell - Lucas number ($Q_0=2, Q_1=2, Q_n=Q_{n-2}+2Q_{n-1}$, $n \geq 2$), J_n is the n^{th} Jacobsthal number ($J_0=0, J_1=1, J_n=2J_{n-2}+J_{n-1}, n \geq 2$), j_n is the n^{th} Jacobsthal - Lucas number ($j_0=2, j_1=1, j_n=2j_{n-2}+j_{n-1}, n \geq 2$), and obtained the determinants of such matrices and investigated the singularity of such matrices. In the paper [41] Y. Yazlik and N. Taskara considered a k-circulant matrix with the first row $(H_{s,0}, H_{s,1}, H_{s,2}, \ldots, H_{s,n-1})$, where $\{H_{s,n}\}_{n\in\mathbb{N}}$ is the generalized s-Horadam sequence $\left(H_{s,0} = a, H_{s,1} = b, H_{s,n} = f(s)H_{s,n-1} + g(s)H_{s,n-2}, n \geq 2, a,b \in \mathbb{R}, s \in \mathbb{R}^+, f^2(s) + 4g^2(s) > 0\right)$, and gave the eigenvalues, the determinant and the bounds for the spectral norm of such matrix. In the paper [34] W. Sintunavarat considered k-circulant and symmetric k-circulant matrices with the first rows $(P_1, P_2, P_3, \ldots, P_n)$, $(P_0, P_1, P_2, \ldots, P_{n-1})$, $(P_1^2, P_2^2, P_3^2, \ldots, P_n^2)$ and $(P_0^2, P_1^2, P_2^2, \ldots, P_{n-1}^2)$, where P_n is the n^{th} Padovan number ($P_0=0, P_1=P_2=1, P_n=P_{n-3}+P_{n-2}$, $n \geq 3$), and obtained the upper bounds for the spectral norms of such matrices. In the paper [26] M. Pešović and Z. Pucanović considered a k-circulant matrix with the generalized Narayana sequence $\{u_n\}_{n\in\mathbb{N}}$ $\left(u_0=a, u_1=b, u_3=c, u_n=u_{n-3}+u_{n-1}, n \geq 3, a, b, c \in \mathbb{N}_0 \; (= \mathbb{N}\cup\{0\}), \text{ such that } a^2+b^2+c^2 \neq 0\right)$, obtained the eigenvalues and the bounds for the spectral norm of such matrix and presented necessary and sufficient conditions for the circulant matrix and the skew circulant matrix with the generalized Narayana numbers to be invertible.The following papers are also devoted to k-circulant matrices: [27], [29]-[32], [35]-[39], [41].

For any complex matrix $A = \left[a_{i,j}\right]$ of order n, the symbols $\lambda_j(A), j = \overline{0, n-1}$, $|A|$, $A^{\circ -1}$, $\|A\|_1$, $\|A\|_\infty$, $\|A\|_E$ and $\|A\|_2$ denote the eigenvalues, the determinant, the Hadamard inverse (provided that $a_{i,j} \neq 0$ for all $i, j = \overline{1, n}$), the 1−norm (the column−sum norm), the ∞−norm (the row−sum norm), the Euclidean norm and the spectral norm of A, respectively.

Definition 1.5. *Let $A = \left[a_{i,j}\right]$ be a complex matrix of order n.*

1) $A^{\circ -1} = \left[a_{i,j}^{-1}\right]$, *if* $a_{i,j} \neq 0$ *for all* $i, j = \overline{1, n}$, 2) $\|A\|_1 = \max_{1\leq j\leq n} \sum_{i=1}^{n} |a_{i,j}|$, 3) $\|A\|_\infty = \max_{1\leq i\leq n} \sum_{j=1}^{n} |a_{i,j}|$,

4) $\|A\|_E = \sqrt{\sum_{i,j=1}^{n} |a_{i,j}|^2}$, 5) $\|A\|_2 = \sqrt{\max_{1 \le i \le n} \lambda_i(A^* A)}$, *where* A^* *is the conjugate transpose of* A . ◊

Let us also mention that, based on the following theorem:

Theorem 1.6. *(Equivalence of the matrix norms, Theorem 1. [44]) Let* A *be any* $m \times n$ *complex matrix. For any pair of matrix norms* $\|A\|_p$ *and* $\|A\|_q$ *there exist positive constants* c_1 *and* c_2 *such that:*

$$c_1 \|A\|_p \le \|A\|_q \le c_2 \|A\|_p \,. \blacklozenge, \tag{8}$$

and the following table:

$\|A\|_q$	$\|A\|_p = \|A\|_1$	$\|A\|_2$	$\|A\|_\infty$	$\|A\|_E$	$\|A\|_M$	$\|A\|_G$
			c_1			
$\|A\|_1$	–	$\frac{1}{\sqrt{n}}$	$\frac{1}{n}$	$\frac{1}{\sqrt{n}}$	$\frac{1}{max}$	$\frac{1}{\sqrt{mn}}$
$\|A\|_2$	$\frac{1}{\sqrt{m}}$	–	$\frac{1}{\sqrt{n}}$	$\frac{1}{\sqrt{r}}$	$\frac{1}{max}$	$\frac{1}{\sqrt{mn}}$
$\|A\|_\infty$	$\frac{1}{m}$	$\frac{1}{\sqrt{m}}$	–	$\frac{1}{\sqrt{m}}$	$\frac{1}{max}$	$\frac{1}{\sqrt{mn}}$
$\|A\|_E$	$\frac{1}{\sqrt{m}}$	1	$\frac{1}{\sqrt{n}}$	–	$\frac{1}{max}$	$\frac{1}{\sqrt{mn}}$
$\|A\|_M$	$\frac{max}{m}$	$\frac{max}{\sqrt{mn}}$	$\frac{max}{n}$	$\frac{max}{\sqrt{mn}}$	–	$\frac{\max}{\sqrt{mn}}$
$\|A\|_G$	$\sqrt{\frac{n}{m}}$	1	$\sqrt{\frac{m}{n}}$	1	$\frac{\sqrt{mn}}{max}$	–
			c_2			
$\|A\|_1$	–	$\sqrt{m}$	m	$\sqrt{m}$	$\frac{m}{max}$	$\sqrt{\frac{m}{n}}$
$\|A\|_2$	$\sqrt{n}$	–	$\sqrt{m}$	1	$\frac{\sqrt{mn}}{max}$	1
$\|A\|_\infty$	n	$\sqrt{n}$	–	$\sqrt{n}$	$\frac{n}{max}$	$\sqrt{\frac{n}{m}}$
$\|A\|_E$	$\sqrt{n}$	$\sqrt{r}$	$\sqrt{m}$	–	$\frac{\sqrt{mn}}{max}$	1
$\|A\|_M$	max	max	max	max	–	$\frac{max}{\sqrt{mn}}$
$\|A\|_G$	$\sqrt{mn}$	$\sqrt{mn}$	$\sqrt{mn}$	$\sqrt{mn}$	$\frac{\sqrt{mn}}{max}$	–

,

where $max = \max(m, n)$, $r = \text{rank}(A)$, $\|A\|_M$ is the maximum norm and $\|A\|_G$ is the $G-$norm (geometric-mean norm), it follows that the following inequalities hold for any complex matrix A of order n:

$$\frac{\|A\|_E}{\sqrt{n}} \le \|A\|_2 \le \|A\|_E \,. \tag{9}$$

Before we present our main results, let us recall that *the Mersenne numbers* $\{M_n\}$ satisfy the following:

$$M_n = \begin{cases} 0, & n=0 \\ 1, & n=1 \\ 3M_{n-2} - 2M_{n-1}, & n \geq 2 \end{cases} . \tag{10}$$

These numbers are named after the French theologian, philosopher, mathematician and music theorist *Marin Mersenne* (1588 –1648).

Let α and β be the roots of the equation $x^2 - 3x + 2 = 0$ i.e.

$$\alpha = 2,\ \beta = 1,\ \alpha\beta = 2,\ \alpha + \beta = 3 \quad \text{and} \quad \alpha - \beta = 1 . \tag{11}$$

Binet's formula for the Mersenne numbers is:

$$M_n = \alpha^n - \beta^n = 2^n - 1 . \tag{12}$$

Lemma 1.7. *The Mersenne numbers can be also expressed in general by the following recursive equality:*

$$M_n = 2^{n-1} + M_{n-1} . \tag{13}$$

Proof.

$$M_n = 2^n - 1 = 2 \cdot 2^{n-1} - 1 = 2^{n-1} + 2^{n-1} - 1 = 2^{n-1} + M_{n-1} . \blacktriangledown$$

Let us also recall that *the Fermat numbers* $\{R_n\}$ satisfy the following:

$$R_n = \begin{cases} 2, & n=0 \\ 3, & n=1 \\ 3R_{n-2} - 2R_{n-1}, & n \geq 2 \end{cases} \tag{14}$$

and

$$R_n = \alpha^n + \beta^n = 2^n + 1 \tag{15}$$

is *Binet's formula* for the Fermat numbers. These numbers are named after the French lawyer and mathematician *Pierre de Fermat* (1601 –1665).

The first few terms of the Mersenne sequence (and the Fermat sequence) are given by the following table.

n	0	1	2	3	4	5	6	...
M_n	0	1	3	7	15	31	63	...
R_n	2	3	5	9	17	33	65	...

The following identities hold for the Mersenne numbers:

$$\sum_{i=0}^{n-1} M_i = M_n - n \quad \text{and} \quad \sum_{i=0}^{n-1} M_i^2 = \frac{M_{2n} - 6M_n + 3n}{3} \qquad \textit{(Lemma 3.1. [20])} \tag{16}$$

More information about these numbers can be found in [3], [6], [25] and [28].

The motivation for this paper was found in the paper [20]. In that paper M. Kumari, K. Prasad, J. Tanti and E. Özkan considered k-circulant matrices with the Mersenne numbers having arithmetic indices i.e. matrices having the following forms:

$$\begin{bmatrix} M_s & M_{s+t} & M_{s+2t} & \dots & M_{s+(n-2)t} & M_{s+(n-1)t} \\ kM_{s+(n-1)t} & M_s & M_{s+t} & \dots & M_{s+(n-3)t} & M_{s+(n-2)t} \\ kM_{s+(n-2)t} & kM_{s+(n-1)t} & M_s & \dots & M_{s+(n-4)t} & M_{s+(n-3)t} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ kM_{s+2t} & kM_{s+3t} & kM_{s+4t} & \dots & M_s & M_{s+t} \\ kM_{s+t} & kM_{s+2t} & kM_{s+3t} & \dots & kM_{s+(n-1)t} & M_s \end{bmatrix}, \tag{17}$$

where $s, t \in \mathbb{N}_0$ $(= \mathbb{N} \cup \{0\})$ (and, as we have already mentioned, M_n is the n^{th} Mersenne number), and obtained:

• The eigenvalues of such matrices

Theorem 1.8. *(Theorem 2.3. [20]) Let $\mathbb{M}$ be a matrix of the form (17). The eigenvalues of $\mathbb{M}$ are given by the following formulae:*

$$\lambda_j(\mathbb{M}) = \begin{cases} \frac{M_s - kM_{s+nt} - \psi\omega^{-j}\left(2^t M_{s-t} - kM_{s+(n-1)t}\right)}{(1-2^t\psi\omega^{-j})(1-\psi\omega^{-j})}, & s > t \\ \frac{M_s - kM_{(1+n)s} + \psi\omega^{-j} kM_{ns}}{(1-2^s\psi\omega^{-j})(1-\psi\omega^{-j})}, & s = t \\ \frac{M_s - kM_{s+nt} - \psi\omega^{-j}\left(2^s M_{t-s} - kM_{s+(n-1)t}\right)}{(1-2^t\psi\omega^{-j})(1-\psi\omega^{-j})}, & s < t \end{cases}, \tag{18}$$

where ψ is any n^{th} root of k and ω is any primitive n^{th} root of unity. ♦

Remark 1.9. *(in relation to Theorem 1.8.) There is a mistake in Theorem 1.8. Since*

$$2^s - 2^t = \begin{cases} 2^t M_{s-t}, & s > t \\ 0, & s = t \\ -2^s M_{t-s}, & s < t \end{cases} \tag{19}$$

and 2^t is missing (and it should be written with the expression $kM_{s+(n-1)t}$), the result should be as follows:

$$\lambda_j(\mathbb{M}) = \begin{cases} \frac{M_s - kM_{s+nt} - \psi\omega^{-j}\left(2^t M_{s-t} - 2^t kM_{s+(n-1)t}\right)}{(1-2^t\psi\omega^{-j})(1-\psi\omega^{-j})}, & s > t \\ \frac{M_s - kM_{(1+n)s} + \psi\omega^{-j} 2^s kM_{ns}}{(1-2^s\psi\omega^{-j})(1-\psi\omega^{-j})}, & s = t \\ \frac{M_s - kM_{s+nt} + \psi\omega^{-j}\left(2^s M_{t-s} + 2^t kM_{s+(n-1)t}\right)}{(1-2^t\psi\omega^{-j})(1-\psi\omega^{-j})}, & s < t \end{cases}. \ \diamond \tag{20}$$

• The determinant of such matrices

Theorem 1.10. *(Theorem 2.9. [20]) Let $\mathbb{M}$ be a matrix of the form (17). The determinant of $\mathbb{M}$ is given by the following formula:*

$$|\mathbb{M}| = \frac{(M_s - kM_{s+nt})^n - k(2^s - 2^t - 2^t kM_{s+(n-1)t})^n}{1 - kR_{nt} + 2^{nt}k^2}. \ ♦ \tag{21}$$

• The 1−norm and the ∞−norm of such matrices (for $s=0$ and $t=1$)

Theorem 1.11. *(Theorem 3.3. [20]) Let* $\mathbb{M}$ *be a matrix of the form (17) for* $s=0$ *and* $t=1$*. The* 1−*norm and the* ∞−*norm of* $\mathbb{M}$ *are given by the following formula:*

$$||\mathbb{M}||_1 = ||\mathbb{M}||_\infty = |k|\,(M_n - n)\,.\blacklozenge \tag{22}$$

Remark 1.12. *(in relation to Theorem 1.11.) The result of Theorem 1.11. is valid only if* $|k| \geq 1$ *.* ◇

• The Euclidean norm of such matrices (for $s=0$ and $t=1$)

Theorem 1.13. *(Theorem 3.4.[20]) Let* $\mathbb{M}$ *be a matrix of the form (17) for* $s=0$ *and* $t=1$*. The Euclidean norm of* $\mathbb{M}$ *is given by the following formula:*

$$||\mathbb{M}||_E = \sqrt{\sum_{i=0}^{n-1} (R_{2i} - 4)\left[n - i\left(1 - |k|^2\right)\right]}\,.\blacklozenge \tag{23}$$

Remark 1.14. *(in relation to Theorem 1.13.) There is a mistake in Theorem 1.13. The result should be as follows:*

$$||\mathbb{M}||_E = \sqrt{\sum_{i=0}^{n-1} \left(R_{2i} - 2^{j+1}\right)\left[n - i\left(1 - |k|^2\right)\right]}\,.\diamond \tag{24}$$

• The bounds for the spectral norm of such matrices (for $s=0$ and $t=1$)

Theorem 1.15. *(Theorem 3.6.[20]) Let* $\mathbb{M}$ *be a matrix of the form (17) for* $s=0$ *and* $t=1$*. The bounds for the spectral norm of* $\mathbb{M}$ *are given by the following formulae:*

1) If $|k| \geq 1$*, then*

$$\sqrt{\frac{M_{2n} - 6M_n + 3n}{3}} \leq ||\mathbb{M}||_2 \leq |k| \sqrt{\left(1 + \frac{M_{2n} - 6M_n + 3n}{3}\right)\left(\frac{M_{2n} - 6M_n + 3n}{3}\right)}, \tag{25}$$

2) If $|k| < 1$*, then*

$$|k| \sqrt{\frac{M_{2n} - 6M_n + 3n}{3}} \leq ||\mathbb{M}||_2 \leq \sqrt{n\left(\frac{M_{2n} - 6M_n + 3n}{3}\right)}\,.\blacklozenge \tag{26}$$

In this paper we shall consider the matrices of the form (17) (for $t \neq 0$), improve the formulae for the eigenvalues of such matrices because the result of Theorem 1.8 can not be applied in some cases and show that there are cases when the result of Theorem 1.10 can also not be applied. Namely, the result of Theorem 1.8 (and the result of Theorem 1.10) can not be applied if $\psi\omega^{-j} = \frac{1}{2^t}$ or $\psi\omega^{-j} = 1$ for some $j = \overline{0, n-1}$, where ψ is any n^{th} root of k and ω is any primitive n^{th} root of unity. Also, in this paper we shall consider the norms of such matrices and extend (and correct) the results of, respectively, Theorem 1.11, Theorem 1.13 and Theorem 1.15. At the end, we shall give the upper and lower bounds for the matrices $\text{circ}\{_k(M_s^{-1}, M_{s+t}^{-1}, \ldots, M_{s+(n-1)t}^{-1})\}$ (for $s, t \neq 0$) and the obtained results will be illustrated by the example.

2. Main results

Before starting to present our results, let us recall that the n^{th} Mersenne number, any n^{th} root of k and any primitive n^{th} root of unity are denoted by M_n, ψ and ω, respectively. Throughout this section, unless otherwise state, s is a non-negative integer and t is a positive integer. The following lemma presented by R. E. Cline, R. J. Plemmons and G. Worm in the paper [5] will be used.

Lemma 2.1. *(Lemma 4. [5]) The eigenvalues of* $C = circ\{_k(c_0, c_1, c_2, \ldots, c_{n-1})\}$ *are:*

$$\lambda_j(C) = \sum_{i=0}^{n-1} c_i(\psi\omega^{-j})^i, \ j = \overline{0, n-1}. \tag{27}$$

Moreover, in this case

$$c_i = \frac{1}{n}\sum_{j=0}^{n-1} \lambda_j(C)(\psi\omega^{-j})^{-i}, \ i = \overline{0, n-1}\ . \blacktriangledown \tag{28}$$

First, we obtain the eigenvalues of (17).

Theorem 2.2. *Let* $\mathbb{M}$ *be the matrix as in (17). The eigenvalues of* $\mathbb{M}$ *are given by the following formulae:*

1) If $\psi\omega^{-j} = \frac{1}{2^t}$*, then*

$$\lambda_j(\mathbb{M}) = n2^s - \frac{M_{nt}}{2^{(n-1)t}M_t}, \tag{29}$$

2) If $\psi\omega^{-j} = 1$*, then*

$$\lambda_j(\mathbb{M}) = 2^s\frac{M_{nt}}{M_t} - n, \tag{30}$$

3) If $\psi\omega^{-j} \neq \frac{1}{2^t}$ *and* $\psi\omega^{-j} \neq 1$*, then*

$$\lambda_j(\mathbb{M}) = \begin{cases} \frac{M_s - kM_{s+nt} - \psi\omega^{-j}\left(2^t M_{s-t} - 2^t kM_{s+(n-1)t}\right)}{(1-2^t\psi\omega^{-j})(1-\psi\omega^{-j})}, & s > t \\ \frac{M_s - kM_{(1+n)s} + \psi\omega^{-j}2^t kM_{ns}}{(1-2^t\psi\omega^{-j})(1-\psi\omega^{-j})}, & s = t \\ \frac{M_s - kM_{s+nt} + \psi\omega^{-j}\left(2^s M_{t-s} + 2^t kM_{s+(n-1)t}\right)}{(1-2^t\psi\omega^{-j})(1-\psi\omega^{-j})}, & s < t \end{cases}. \tag{31}$$

Proof. *Based on Lemma 2.1. and (12), it follows:*

1) Suppose that $\psi\omega^{-j} = \frac{1}{2^t}$*. Then,*

$$\begin{aligned} \lambda_j(\mathbb{M}) &= \sum_{i=0}^{n-1} M_{s+it}(\psi\omega^{-j})^i = \sum_{i=0}^{n-1}\left(2^{s+it} - 1\right)\frac{1}{2^{it}} = 2^s\sum_{i=0}^{n-1} 1 - \sum_{i=0}^{n-1}\frac{1}{2^{it}} \\ &= 2^s n - \frac{1 - \frac{1}{2^{nt}}}{1 - \frac{1}{2^t}} = n2^s - \frac{2^{nt} - 1}{2^{(n-1)t}(2^t - 1)} = n2^s - \frac{M_{nt}}{2^{(n-1)t}M_t}, \end{aligned}$$

2) Suppose that $\psi\omega^{-j} = 1$*. Then,*

$$\lambda_j(\mathbb{M}) = \sum_{i=0}^{n-1} M_{s+it}(\psi\omega^{-j})^i = \sum_{i=0}^{n-1}\left(2^{s+it} - 1\right) = 2^s\sum_{i=0}^{n-1} 2^{it} - \sum_{i=0}^{n-1} 1 = 2^s\frac{1 - 2^{nt}}{1 - 2^t} - n = 2^s\frac{M_{nt}}{M_t} - n,$$

3) Suppose that $\psi\omega^{-j} \neq \frac{1}{2^t}$ *and* $\psi\omega^{-j} \neq 1$*. Then,* $\lambda_j(\mathbb{M})$ *follows from (20).* ♦

The result of Theorem 2.2 is illustrated by the following example.

Example 2.3. *Let*

$$M = circ\{_{\frac{1}{16}}(1, 3, 7, 15)\}$$

i.e.

$$M=\begin{bmatrix} 1 & 3 & 7 & 15 \\ \frac{15}{16} & 1 & 3 & 7 \\ \frac{7}{16} & \frac{15}{16} & 1 & 3 \\ \frac{3}{16} & \frac{7}{16} & \frac{15}{16} & 1 \end{bmatrix}.$$

Since $s=t=1$, $n=4$ and $k=\frac{1}{16}$ i.e. $\psi=\frac{1}{2}$ and $\omega=i$, based on Theorem 2.2, it follows that

★*: $\psi\omega^0=\frac{1}{2}$, so $\lambda_0(M)$ is obtained based on 1) of Theorem 2.2: $\lambda_0(M)=\frac{49}{8}$.*

★*: $\psi\omega^{-j}\neq\frac{1}{2}$ and $\psi\omega^{-j}\neq 1$, for $j=\overline{1,3}$, so $\lambda_j(M)$, for $j=\overline{1,3}$, are obtained based on 3) of Theorem 2.2: $\lambda_{1,3}(M)=-\frac{3}{4}\pm\frac{3}{8}i$ and $\lambda_2(M)=-\frac{5}{8}$.*

Since $|M|=\prod_{j=0}^{n-1}\lambda_j(M)$, it follows that

$$|M|=-\frac{11025}{4096}.\ \diamond$$

Remark 2.4. *(in relation to Example 2.3.) The determinant of $M=circ\{\frac{1}{16}(1,3,7,15)\}$ is not possible to obtain using the result of Theorem 1.10.* $\diamond$

Next, we determine the $1-$norm and the $\infty-$norm of (17). The following result is needed.

Lemma 2.5. *The following identity holds for the Mersenne numbers:*

$$\sum_{i=0}^{n-1}M_{s+it}=2^s\frac{M_{nt}}{M_t}-n. \tag{32}$$

Proof.

$$\sum_{i=0}^{n-1}M_{s+it}=\sum_{i=0}^{n-1}\left(2^{s+it}-1\right)=2^s\sum_{i=0}^{n-1}2^{it}-\sum_{i=0}^{n-1}1=2^s\frac{1-2^{nt}}{1-2^t}-n=2^s\frac{M_{nt}}{M_t}-n.\ \blacktriangledown$$

Theorem 2.6. *Let $\mathbb{M}$ be the matrix as in (17). The $1-$norm and the $\infty-$norm of $\mathbb{M}$ are given by the following formulae:*

1) If $|k|\geq 1$, then

$$||\mathbb{M}||_1=||\mathbb{M}||_\infty=M_s+|k|\left(2^s\frac{M_{nt}}{M_t}-n-M_s\right), \tag{33}$$

2) If $|k|<1$, then

$$||\mathbb{M}||_1=||\mathbb{M}||_\infty=2^s\frac{M_{nt}}{M_t}-n. \tag{34}$$

Proof. *From the definition of the $1-$norm and the $\infty-$norm of a matrix, we obtain:*

1) If $|k|\geq 1$, then

$$\begin{aligned}||\mathbb{M}||_1=||\mathbb{M}||_\infty &= \max_{1\leq j\leq n}\sum_{i=1}^{n}|m_{i,j}|=\sum_{i=1}^{n}|m_{i,1}|=M_s+|k|\sum_{i=1}^{n-1}M_{s+it}\\ &= M_s+|k|\left(2^s\frac{M_{nt}}{M_t}-n-M_s\right).\end{aligned}$$

2) If $|k|<1$, then

$$||\mathbb{M}||_1=||\mathbb{M}||_\infty=\max_{1\leq j\leq n}\sum_{i=1}^{n}|m_{i,j}|=\sum_{i=1}^{n}|m_{i,n}|=\sum_{i=0}^{n-1}M_{s+it}=2^s\frac{M_{nt}}{M_t}-n.\ \blacklozenge$$

Next, we determine the Euclidean norm of (17). The following formula will be used.

For all x,

$$\sum_{i=1}^{n-1} ix^i = \frac{x - nx^n + (n-1)x^{n+1}}{(1-x)^2} \ . \tag{35}$$

In order to obtain the Euclidean norm of (17) we also need the following result.

Lemma 2.7. *The following identity holds for the Mersenne numbers:*

$$\sum_{i=0}^{n-1} M_{s+it}^2 = 4^s \frac{M_{2nt}}{M_{2t}} - 2^{s+1} \frac{M_{nt}}{M_t} + n \, . \tag{36}$$

Proof.

$$\begin{aligned}
\sum_{i=0}^{n-1} M_{s+it}^2 &= \sum_{i=0}^{n-1} \left(2^{s+it} - 1\right)^2 = \sum_{i=0}^{n-1} \left(2^{2(s+it)} - 2^{s+it+1} + 1\right) = \sum_{i=0}^{n-1} 2^{2(s+it)} - 2^{s+1} \sum_{i=0}^{n-1} 2^{it} + \sum_{i=0}^{n-1} 1 \\
&= 4^s \frac{2^{2nt} - 1}{2^{2t} - 1} - 2^{s+1} \frac{2^{nt} - 1}{2^t - 1} + n = 4^s \frac{M_{2nt}}{M_{2t}} - 2^{s+1} \frac{M_{nt}}{M_t} + n \, . \blacktriangledown
\end{aligned}$$

Theorem 2.8. *Let* $\mathbb{M}$ *be the matrix as in (17). The Euclidean norm of* $\mathbb{M}$ *is:*

$$\|\mathbb{M}\|_E = \sqrt{n\left[4^s \frac{M_{2nt}}{M_{2t}} - 2^{s+1} \frac{M_{nt}}{M_t} + n\right] + (|k|^2 - 1)W} \, , \tag{37}$$

where

$$W = \frac{4^{s+t} - n4^{s+nt} + (n-1)4^{s+(n+1)t}}{M_{2t}^2} - \frac{2^{s+t+1} - n2^{s+nt+1} + (n-1)2^{s+(n+1)t+1}}{M_t^2} + \frac{n(n-1)}{2} \, .$$

Proof. *From the definition of the Euclidean norm of a matrix, using (12), (35) and the result of Lemma 2.7, we obtain:*

$$\begin{aligned}
(\|\mathbb{M}\|_E)^2 &= nM_s^2 + \left[(n-1) + k^2\right] M_{s+t}^2 + \left[(n-2) + 2|k|^2\right] M_{s+2t}^2 + \cdots + \left[1 + (n-1)|k|^2\right] M_{s+(n-1)t}^2 \\
&= \sum_{i=0}^{n-1} (n-i) M_{s+it}^2 + |k|^2 \sum_{i=1}^{n-1} i M_{s+it}^2 = n \sum_{i=0}^{n-1} M_{s+it}^2 + (|k|^2 - 1) \sum_{i=1}^{n-1} i M_{s+it}^2 \\
&= n\left[4^s \frac{M_{2nt}}{M_{2t}} - 2^{s+1} \frac{M_{nt}}{M_t} + n\right] + (|k|^2 - 1) \sum_{i=1}^{n-1} i \left(2^{s+it} - 1\right)^2 \\
&= n\left[4^s \frac{M_{2nt}}{M_{2t}} - 2^{s+1} \frac{M_{nt}}{M_t} + n\right] + (|k|^2 - 1) \sum_{i=1}^{n-1} i \left(2^{2(s+it)} - 2^{s+it+1} + 1\right) \\
&= n\left[4^s \frac{M_{2nt}}{M_{2t}} - 2^{s+1} \frac{M_{nt}}{M_t} + n\right] + (|k|^2 - 1) \left[4^s \sum_{i=1}^{n-1} i4^{it} - 2^{s+1} \sum_{i=1}^{n-1} i2^{it} + \sum_{i=1}^{n-1} i\right]
\end{aligned}$$

$$= n\left[4^s \frac{M_{2nt}}{M_{2t}} - 2^{s+1}\frac{M_{nt}}{M_t} + n\right] +$$

$$(|k|^2-1)\left[4^s \frac{4^t - n4^{nt} + (n-1)4^{(n+1)t}}{(1-4^t)^2} - 2^{s+1}\frac{2^t - n2^{nt} + (n-1)2^{(n+1)t}}{(1-2^t)^2} + \frac{n(n-1)}{2}\right]$$

$$= n\left[4^s \frac{M_{2nt}}{M_{2t}} - 2^{s+1}\frac{M_{nt}}{M_t} + n\right] + (|k|^2-1)\left[\frac{4^{s+t} - n4^{s+nt} + (n-1)4^{s+(n+1)t}}{M_{2t}^2}\right.$$

$$\left.-\frac{2^{s+t+1} - n2^{s+nt+1} + (n-1)2^{s+(n+1)t+1}}{M_t^2} + \frac{n(n-1)}{2}\right].$$

Therefore,

$$||\mathbb{M}||_E = \sqrt{n\left[4^s \tfrac{M_{2nt}}{M_{2t}} - 2^{s+1}\tfrac{M_{nt}}{M_t} + n\right] + (|k|^2-1)W},$$

where

$$W = \frac{4^{s+t}-n4^{s+nt}+(n-1)4^{s+(n+1)t}}{M_{2t}^2} - \frac{2^{s+t+1}-n2^{s+nt+1}+(n-1)2^{s+(n+1)t+1}}{M_t^2} + \frac{n(n-1)}{2}\,.\ \blacklozenge$$

Next, the upper and lower bounds for the spectral norm of (17) will be obtained. Apart from the result of Lemma 2.7, the following lemma will be used.

Lemma 2.9. *([13]) Let* $G=\left[g_{i,j}\right]$ *and* $H=\left[h_{i,j}\right]$ *be matrices of order* $m \times n$*. Then,*

$$||G \circ H||_2 \leq r_1(G) \cdot c_1(H), \tag{38}$$

where $G \circ H = [g_{i,j}h_{i,j}]$ *is the Hadamard product (or the Schur product) of* G *and* H*,*

$$r_1(G) = \max_{1 \leq i \leq m} \sqrt{\sum_{j=1}^{n} |\, g_{i,j}\,|^2} \quad \text{and} \quad c_1(H) = \max_{1 \leq j \leq n} \sqrt{\sum_{i=1}^{m} |\, h_{i,j}\,|^2}.\ \blacktriangledown$$

We recommend the papers [12] and [21] for more information about the Hadamard product of matrices.

Theorem 2.10. *Let* $\mathbb{M}$ *be the matrix as in (17). The upper and lower bounds for the spectral norm of* $\mathbb{M}$ *are*

1) If $|k| \geq 1$*, then*

$$\sqrt{4^s \frac{M_{2nt}}{M_{2t}} - 2^{s+1}\frac{M_{nt}}{M_t} + n} \leq ||\mathbb{M}||_2 \tag{39}$$

and

$$||\mathbb{M}||_2 \leq \sqrt{\left(M_s^2 + |k|^2\left(4^s \frac{M_{2nt}}{M_{2t}} - 2^{s+1}\frac{M_{nt}}{M_t} + n - M_s^2\right)\right)\left(1 + 4^s \frac{M_{2nt}}{M_{2t}} - 2^{s+1}\frac{M_{nt}}{M_t} + n - M_s^2\right)}\,, \tag{40}$$

2) If $|k| < 1$*, then*

$$|k|\sqrt{\left(4^s \frac{M_{2nt}}{M_{2t}} - 2^{s+1}\frac{M_{nt}}{M_t} + n\right)} \leq ||\mathbb{M}||_2 \leq \sqrt{n\left(4^s \frac{M_{2nt}}{M_{2t}} - 2^{s+1}\frac{M_{nt}}{M_t} + n\right)}\,. \tag{41}$$

Proof. From the definition of the Euclidean norm of a matrix, we obtain:

$$||\mathbb{M}||_E^2 = \sum_{i=0}^{n-1}(n-i)M_{s+it}^2 + |k|^2 \sum_{i=1}^{n-1} iM_{s+it}^2 . \tag{42}$$

1) If $|k| \geq 1$, then

$$\begin{aligned} ||\mathbb{M}||_E^2 \geq & \sum_{i=0}^{n-1}(n-i)M_{s+it}^2 + \sum_{i=1}^{n-1} iM_{s+it}^2 = n\sum_{i=0}^{n-1} M_{s+it}^2 \\ = & \ n\left(4^s \frac{M_{2nt}}{M_{2t}} - 2^{s+1}\frac{M_{nt}}{M_t} + n\right). \end{aligned}$$

Therefore,

$$\frac{||\mathbb{M}||_E}{\sqrt{n}} \geq \sqrt{4^s \frac{M_{2nt}}{M_{2t}} - 2^{s+1}\frac{M_{nt}}{M_t} + n}\,.$$

We conclude from (9) that

$$||\mathbb{M}||_2 \geq \sqrt{4^s \frac{M_{2nt}}{M_{2t}} - 2^{s+1}\frac{M_{nt}}{M_t} + n}\ .$$

Now, we shall obtain the upper bound for the spectral norm of $\mathbb{M}$. Let U and V be the following matrices:

$$U = \begin{bmatrix} M_s & 1 & 1 & \cdots & 1 \\ kM_{s+(n-1)t} & M_s & 1 & \cdots & 1 \\ kM_{s+(n-2)t} & kM_{s+(n-1)t} & M_s & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ kM_{s+t} & kM_{s+2t} & kM_{s+3t} & \cdots & M_s \end{bmatrix} \text{ and } V = \begin{bmatrix} 1 & M_{s+t} & M_{s+2t} & \cdots & M_{s+(n-1)t} \\ 1 & 1 & M_{s+t} & \cdots & M_{s+(n-2)t} \\ 1 & 1 & 1 & \cdots & M_{s+(n-3)t} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix}.$$

Then,

$$\begin{aligned} r_1(U) = & \max_{1\leq i\leq n} \sqrt{\sum_{j=1}^{n} |\ u_{i,j}\ |^2} = \sqrt{\sum_{j=1}^{n} |\ u_{n,j}\ |^2} = \sqrt{M_s^2 + |k|^2 \sum_{i=1}^{n-1} M_{s+it}^2} \\ = & \sqrt{M_s^2 + |k|^2\left(\sum_{i=0}^{n-1} M_{s+it}^2 - M_s^2\right)} = \sqrt{M_s^2 + |k|^2\left(4^s \frac{M_{2nt}}{M_{2t}} - 2^{s+1}\frac{M_{nt}}{M_t} + n - M_s^2\right)} \end{aligned}$$

and

$$\begin{aligned} c_1(V) = & \max_{1\leq j\leq n} \sqrt{\sum_{i=1}^{n} |\ v_{i,j}\ |^2} = \sqrt{\sum_{i=1}^{n} |\ v_{i,n}\ |^2} = \sqrt{1 + \sum_{i=1}^{n-1} M_{s+it}^2} \\ = & \sqrt{1 + \sum_{i=0}^{n-1} M_{s+it}^2 - M_s^2} = \sqrt{1 + 4^s \frac{M_{2nt}}{M_{2t}} - 2^{s+1}\frac{M_{nt}}{M_t} + n - M_s^2}\ . \end{aligned}$$

Since $\mathbb{M} = U \circ V$, based on Lemma 2.9, we can write

$$||\mathbb{M}||_2 \leq r_1(U) \cdot c_1(V) = \sqrt{\left(M_s^2 + |k|^2 \left(\sum_{i=0}^{n-1} M_{s+it}^2 - M_s^2\right)\right)\left(1 + \sum_{i=0}^{n-1} M_{s+it}^2 - M_s^2\right)}$$

$$= \sqrt{\left(M_s^2 + |k|^2 \left(4^s \frac{M_{2nt}}{M_{2t}} - 2^{s+1} \frac{M_{nt}}{M_t} + n - M_s^2\right)\right)\left(1 + 4^s \frac{M_{2nt}}{M_{2t}} - 2^{s+1} \frac{M_{nt}}{M_t} + n - M_s^2\right)} .$$

2) If $|k| < 1$, then

$$||\mathbb{M}||_E^2 \geq \sum_{i=0}^{n-1} (n-i)|k|^2 M_{s+it}^2 + \sum_{i=1}^{n-1} i|k|^2 M_{s+it}^2 = n|k|^2 \sum_{i=0}^{n-1} M_{s+it}^2$$

$$= n|k|^2 \left(4^s \frac{M_{2nt}}{M_{2t}} - 2^{s+1} \frac{M_{nt}}{M_t} + n\right) .$$

Therefore,

$$\frac{||\mathbb{M}||_E}{\sqrt{n}} \geq |k| \sqrt{\left(4^s \frac{M_{2nt}}{M_{2t}} - 2^{s+1} \frac{M_{nt}}{M_t} + n\right)}.$$

We conclude from (9) that

$$||\mathbb{M}||_2 \geq |k| \sqrt{\left(4^s \frac{M_{2nt}}{M_{2t}} - 2^{s+1} \frac{M_{nt}}{M_t} + n\right)}.$$

Now, we shall obtain the upper bound for the spectral norm of $\mathbb{M}$. Let P and Q be the following matrices:

$$P = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ k & 1 & 1 & \cdots & 1 \\ k & k & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ k & k & k & \cdots & 1 \end{bmatrix} \text{ and } Q = \begin{bmatrix} M_s & M_{s+t} & M_{s+2t} & \cdots & M_{s+(n-1)t} \\ M_{s+(n-1)t} & M_s & M_{s+t} & \cdots & M_{s+(n-2)t} \\ M_{s+(n-2)t} & M_{s+(n-1)t} & M_s & \cdots & M_{s+(n-3)t} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ M_{s+t} & M_{s+2t} & M_{s+3t} & \cdots & M_s \end{bmatrix} .$$

Then,

$$r_1(P) = \max_{1 \leq i \leq n} \sqrt{\sum_{j=1}^{n} |\, p_{i,j} \,|^2} = \sqrt{\sum_{j=1}^{n} |\, p_{1,j} \,|^2} = \sqrt{n}$$

and

$$c_1(Q) = \max_{1 \leq j \leq n} \sqrt{\sum_{i=1}^{n} |\, q_{i,j} \,|^2} = \sqrt{\sum_{i=1}^{n} |\, q_{i,1} \,|^2} = \sqrt{\sum_{i=0}^{n-1} M_{s+it}^2} = \sqrt{4^s \frac{M_{2nt}}{M_{2t}} - 2^{s+1} \frac{M_{nt}}{M_t} + n} .$$

Since $\mathbb{M} = P \circ Q$, based on Lemma 2.9, we can write

$$||\mathbb{M}||_2 \leq r_1(P) \cdot c_1(Q) = \sqrt{n\left(4^s \frac{M_{2nt}}{M_{2t}} - 2^{s+1} \frac{M_{nt}}{M_t} + n\right)} . \blacklozenge$$

The following example illustrates the result of Theorem 2.10.

Example 2.11. *Let* $\mathsf{M} = circ\{_k(M_1, M_2, \ldots, M_n)\}$.

The lower bounds for the spectral norm of M

a) for $|k| \geq 1$,

n	$\|\mathsf{M}\|_2 \geq$
2	$\sqrt{10} \approx 3.1623$
3	$\sqrt{59} \approx 7.6811$
4	$2\sqrt{71} \approx 16.8523$
5	$\sqrt{1245} \approx 35.2846$
6	$\sqrt{5214} \approx 72.2080$
7	$\sqrt{21343} \approx 146.0924$

b) for $k = -\frac{1}{3}, \frac{1}{3}$,

n	$\|\mathsf{M}\|_2 \geq$
2	$\frac{\sqrt{10}}{3} \approx 1.0541$
3	$\frac{\sqrt{59}}{3} \approx 2.5604$
4	$\frac{2}{3}\sqrt{71} \approx 5.6174$
5	$\sqrt{\frac{415}{3}} \approx 11.7615$
6	$\sqrt{\frac{1738}{3}} \approx 24.0693$
7	$\frac{\sqrt{21343}}{3} \approx 48.6975$

c) for $k = -\frac{1}{2}, \frac{1}{2}$.

n	$\|\mathsf{M}\|_2 \geq$
2	$\sqrt{\frac{5}{2}} \approx 1.5811$
3	$\frac{\sqrt{59}}{2} \approx 3.8406$
4	$\sqrt{71} \approx 8.4262$
5	$\frac{\sqrt{1245}}{2} \approx 17.6423$
6	$\sqrt{\frac{2607}{2}} \approx 36.1040$
7	$\frac{\sqrt{21343}}{2} \approx 73.0462$

The upper bounds for the spectral norm of M

a) for $k = -3, 3$,

n	$\|\mathsf{M}\|_2 \leq$
2	$2\sqrt{205} \approx 28.6356$
3	$\sqrt{30857} \approx 175.6616$
4	$4\sqrt{45227} \approx 850.6656$
5	$\sqrt{13940265} \approx 3733.6664$
6	$2\sqrt{61157613} \approx 15640.6666$
7	$\sqrt{4099542097} \approx 64027.6667$

b) for $k = -2, 2$,

n	$\|\mathsf{M}\|_2 \leq$
2	$\sqrt{370} \approx 19.2354$
3	$\sqrt{13747} \approx 117.2476$
4	$\sqrt{321772} \approx 567.2495$
5	$\sqrt{6196365} \approx 2489.2499$
6	$\sqrt{108722328} \approx 10427.0000$
7	$\sqrt{1854214680} \approx 43060.5931$

c) for $|k| < 1$.

n	$\|\mathsf{M}\|_2 \leq$
2	$2\sqrt{5} \approx 4.4721$
3	$\sqrt{177} \approx 13.3041$
4	$4\sqrt{71} \approx 33.7046$
5	$5\sqrt{249} \approx 78.8987$
6	$6\sqrt{869} \approx 176.8728$
7	$7\sqrt{3049} \approx 386.5243$

. ◊

At the end of this paper, we determine the upper and lower bounds for the spectral norm of $\mathbb{M}_k^{\circ -1} = \text{circ}\{_k(M_s^{-1}, M_{s+t}^{-1}, \ldots, M_{s+(n-1)t}^{-1})\}$ (for $s \neq 0$).

In order to obtain the upper and lower bounds for the spectral norm of $\mathbb{M}_k^{\circ -1}$ we need the following result.

Lemma 2.12. *The following identity holds for the Mersenne numbers:*

$$M_{s+(n-1)t} M_{s+(n-1)t+1} = R_{2(s+(n-1)t)+1} - 3 \cdot 2^{s+(n-1)t}. \tag{43}$$

Proof.

$$\begin{aligned} M_{s+(n-1)t} M_{s+(n-1)t+1} &= \left(2^{s+(n-1)t} - 1\right)\left(2^{s+(n-1)t+1} - 1\right) \\ &= 2^{2(s+(n-1)t)+1} - 2^{s+(n-1)t} - 2^{s+(n-1)t+1} + 1 \\ &= 2^{2(s+(n-1)t)+1} - 2^{s+(n-1)t} - 2 \cdot 2^{s+(n-1)t} + 1 \\ &= R_{2(s+(n-1)t)+1} - 3 \cdot 2^{s+(n-1)t}. \blacktriangledown \end{aligned}$$

Theorem 2.13. *Let* $\mathbb{M}_k^{\circ -1} = circ\{_k(M_s^{-1}, M_{s+t}^{-1}, \ldots, M_{s+(n-1)t}^{-1})\}$.

1) If $|k| \geq 1$, *then*

$$\sqrt{\frac{n}{R_{2(s+(n-1)t)+1} - 3 \cdot 2^{s+(n-1)t}}} \leq \|\mathbb{M}_k^{\circ -1}\|_2 \leq \sqrt{n\left(\frac{1}{M_s^2} + (n-1)|k|^2\right)}, \tag{44}$$

2) If $|k| < 1$, *then*

$$|k| \sqrt{\frac{n}{R_{2(s+(n-1)t)+1} - 3 \cdot 2^{s+(n-1)t}}} \leq \|\mathbb{M}_k^{\circ -1}\|_2 \leq \sqrt{n\left(\frac{1}{M_s^2} + n - 1\right)}. \tag{45}$$

Proof. From the definition of the Euclidean norm of a matrix, we obtain:

$$\|\mathbb{M}_k^{\circ -1}\|_E^2 = \sum_{i=0}^{n-1}(n-i)\frac{1}{M_{s+it}^2} + |k|^2 \sum_{i=1}^{n-1} i \frac{1}{M_{s+it}^2}. \tag{46}$$

1) If $|k| \geq 1$, then

$$\begin{aligned} \|\mathbb{M}_k^{\circ -1}\|_E^2 \geq & \sum_{i=0}^{n-1}(n-i)\frac{1}{M_{s+it}^2} + \sum_{i=1}^{n-1} i\frac{1}{M_{s+it}^2} = n\sum_{i=0}^{n-1}\frac{1}{M_{s+it}^2} > n\sum_{i=0}^{n-1}\frac{1}{M_{s+(n-1)t}^2} \\ = & \left(\frac{n}{M_{s+(n-1)t}}\right)^2 > \frac{n^2}{M_{s+(n-1)t}M_{s+(n-1)t+1}} = \frac{n^2}{R_{2(s+(n-1)t)+1} - 3 \cdot 2^{s+(n-1)t}}. \end{aligned}$$

Therefore,

$$\frac{\|\mathbb{M}_k^{\circ -1}\|_E}{\sqrt{n}} \geq \sqrt{\frac{n}{R_{2(s+(n-1)t)+1} - 3 \cdot 2^{s+(n-1)t}}}.$$

We conclude from (9) that

$$\|\mathbb{M}_k^{\circ -1}\|_2 \geq \sqrt{\frac{n}{R_{2(s+(n-1)t)+1} - 3 \cdot 2^{s+(n-1)t}}}.$$

Now, we shall obtain the upper bound for the spectral norm of $M_k^{\circ -1}$. Let U and V be the following matrices:

$$U = \begin{bmatrix} \frac{1}{M_s} & \frac{1}{M_{s+t}} & \frac{1}{M_{s+2t}} & \cdots & \frac{1}{M_{s+(n-1)t}} \\ k & \frac{1}{M_s} & \frac{1}{M_{s+t}} & \cdots & \frac{1}{M_{s+(n-2)t}} \\ k & k & \frac{1}{M_s} & \cdots & \frac{1}{M_{s+(n-3)t}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ k & k & k & \cdots & \frac{1}{M_s} \end{bmatrix} \text{ and } V = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ \frac{1}{M_{s+(n-1)t}} & 1 & 1 & \cdots & 1 \\ \frac{1}{M_{s+(n-2)t}} & \frac{1}{M_{s+(n-1)t}} & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{M_{s+t}} & \frac{1}{M_{s+2t}} & \frac{1}{M_{s+3t}} & \cdots & 1 \end{bmatrix}.$$

Then,

$$r_1(U) = \max_{1 \leq i \leq n} \sqrt{\sum_{j=1}^{n} |u_{i,j}|^2} = \sqrt{\sum_{j=1}^{n} |u_{n,j}|^2} = \sqrt{\frac{1}{M_s^2} + (n-1)|k|^2}$$

and

$$c_1(V) = \max_{1 \le j \le n} \sqrt{\sum_{i=1}^{n} |\, v_{i,j} \,|^2} = \sqrt{\sum_{i=1}^{n} |\, v_{i,n} \,|^2} = \sqrt{n}\ .$$

Since $\mathbb{M}_k^{\circ-1} = U \circ V$, based on Lemma 2.9, we can write

$$\|\mathbb{M}_k^{\circ-1}\|_2 \le r_1(U) \cdot c_1(V) = \sqrt{n\left(\frac{1}{M_s^2} + (n-1)|k|^2\right)}\ .$$

2) If $|k| < 1$, then

$$\|\mathbb{M}_k^{\circ-1}\|_E^2 \ge \sum_{i=0}^{n-1}(n-i)|k|^2\frac{1}{M_{s+it}^2} + \sum_{i=1}^{n-1} i|k|^2\frac{1}{M_{s+it}^2} = n|k|^2\sum_{i=0}^{n-1}\frac{1}{M_{s+it}^2} > n|k|^2\sum_{i=0}^{n-1}\frac{1}{M_{s+(n-1)t}^2}$$

$$= |k|^2\left(\frac{n}{M_{s+(n-1)t}}\right)^2 > |k|^2\,\frac{n^2}{M_{s+(n-1)t}M_{s+(n-1)t+1}} = |k|^2\,\frac{n^2}{R_{2(s+(n-1)t)+1} - 3\cdot 2^{s+(n-1)t}}\ .$$

Therefore,

$$\frac{\|\mathbb{M}_k^{\circ-1}\|_E}{\sqrt{n}} \ge |k|\,\sqrt{\frac{n}{R_{2(s+(n-1)t)+1} - 3\cdot 2^{s+(n-1)t}}}\ .$$

We conclude from (9) that

$$\|\mathbb{M}_k^{\circ-1}\|_2 \ge |k|\,\sqrt{\frac{n}{R_{2(s+(n-1)t)+1} - 3\cdot 2^{s+(n-1)t}}}\ .$$

Now, we shall obtain the upper bound for the spectral norm of $\mathbb{M}_k^{\circ-1}$. Let P and Q be the following matrices:

$$P = \begin{bmatrix} \frac{1}{M_s} & 1 & 1 & \cdots & 1 \\ \frac{k}{M_{s+(n-1)t}} & \frac{1}{M_s} & 1 & \cdots & 1 \\ \frac{k}{M_{s+(n-2)t}} & \frac{k}{M_{s+(n-1)t}} & \frac{1}{M_s} & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{k}{M_{s+t}} & \frac{k}{M_{s+2t}} & \frac{k}{M_{s+3t}} & \cdots & \frac{1}{M_s} \end{bmatrix} \text{ and } Q = \begin{bmatrix} 1 & \frac{1}{M_{s+t}} & \frac{1}{M_{s+2t}} & \cdots & \frac{1}{M_{s+(n-1)t}} \\ 1 & 1 & \frac{1}{M_{s+t}} & \cdots & \frac{1}{M_{s+(n-2)t}} \\ 1 & 1 & 1 & \cdots & \frac{1}{M_{s+(n-3)t}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix}.$$

Then,

$$r_1(P) = \max_{1 \le i \le n} \sqrt{\sum_{j=1}^{n} |\, p_{i,j} \,|^2} = \sqrt{\sum_{j=1}^{n} |\, p_{1,j} \,|^2} = \sqrt{\frac{1}{M_s^2} + n - 1}$$

and

$$c_1(Q) = \max_{1 \le j \le n} \sqrt{\sum_{i=1}^{n} |\, q_{i,j} \,|^2} = \sqrt{\sum_{i=1}^{n} |\, q_{i,1} \,|^2} = \sqrt{n}\ .$$

Since $\mathbb{M}_k^{\circ-1} = P \circ Q$, based on Lemma 2.9, we can write

$$\|\mathbb{M}_k^{\circ-1}\|_2 \le r_1(P) \cdot c_1(Q) = \sqrt{n\left(\frac{1}{M_s^2} + n - 1\right)}\ . \blacklozenge$$

The following example illustrates the result of Theorem 2.13.

Example 2.14. *Let* $\mathsf{M}_k^{\circ -1} = circ\{_k(M_1^{-1}, M_2^{-1}, \ldots, M_n^{-1})\}$.

The lower bounds for the spectral norm of $\mathsf{M}_k^{\circ -1}$

a) for $|k| \geq 1$,

n	$\|\mathsf{M}_k^{\circ -1}\|_2 \geq$
2	$\sqrt{\frac{2}{21}} \approx 0.308607$
3	$\frac{1}{\sqrt{35}} \approx 0.169031$
4	$\frac{2}{\sqrt{465}} \approx 0.092748$
5	$\frac{1}{3}\sqrt{\frac{5}{217}} \approx 0.050598$
6	$\sqrt{\frac{2}{2667}} \approx 0.027384$
7	$\sqrt{\frac{7}{32385}} \approx 0.014702$

b) for $k = -\frac{1}{3}, \frac{1}{3}$,

n	$\|\mathsf{M}_k^{\circ -1}\|_2 \geq$
2	$\frac{1}{3}\sqrt{\frac{2}{21}} \approx 0.102869$
3	$\frac{1}{3\sqrt{35}} \approx 0.056344$
4	$\frac{2}{3\sqrt{465}} \approx 0.030915$
5	$\frac{1}{9}\sqrt{\frac{5}{217}} \approx 0.016866$
6	$\frac{1}{3}\sqrt{\frac{2}{2667}} \approx 0.009128$
7	$\frac{1}{3}\sqrt{\frac{7}{32385}} \approx 0.004901$

c) for $k = -\frac{1}{2}, \frac{1}{2}$.

n	$\|\mathsf{M}_k^{\circ -1}\|_2 \geq$
2	$\frac{1}{\sqrt{42}} \approx 0.154303$
3	$\frac{1}{2\sqrt{35}} \approx 0.084515$
4	$\frac{1}{\sqrt{465}} \approx 0.046373$
5	$\frac{1}{6}\sqrt{\frac{5}{217}} \approx 0.025299$
6	$\frac{1}{\sqrt{5334}} \approx 0.013692$
7	$\frac{1}{2}\sqrt{\frac{7}{32385}} \approx 0.007351$

The upper bounds for the spectral norm of $\mathsf{M}_k^{\circ -1}$

a) for $k = -3, 3$,

n	$\|\mathsf{M}_k^{\circ -1}\|_2 \leq$
2	$2\sqrt{5} \approx 4.472136$
3	$\sqrt{57} \approx 7.549834$
4	$4\sqrt{7} \approx 10.583005$
5	$\sqrt{185} \approx 13.601471$
6	$2\sqrt{69} \approx 16.613248$
7	$\sqrt{385} \approx 19.621417$

b) for $k = -2, 2$,

n	$\|\mathsf{M}_k^{\circ -1}\|_2 \leq$
2	$\sqrt{10} \approx 3.162278$
3	$3\sqrt{3} \approx 5.196152$
4	$2\sqrt{13} \approx 7.211103$
5	$\sqrt{85} \approx 9.219544$
6	$3\sqrt{14} \approx 11.224972$
7	$5\sqrt{7} \approx 13.228757$

c) for $|k| < 1$.

n	$\|\mathsf{M}_k^{\circ -1}\|_2 \leq$
2	2
3	3
4	4
5	5
6	6
7	7

. ◊

3. Conclusion

In this paper, we considered the matrices of the form

$$\mathbb{M} = \text{circ}\{_k(M_s, M_{s+t}, \ldots, M_{s+(n-1)t})\},$$

where k is a non-zero complex number, M_n is the n^{th} Mersenne number, s is a non-negative integer and t is a positive integer, obtained the formulae for the eigenvalues of such matrices (improving the result of Theorem 1.8) and showed that there are cases when the result of Theorem 1.10 can also not be applied. We also considered the norms of such matrices and extended (and corrected) the results of, respectively, Theorem 1.11, Theorem 1.13 and Theorem 1.15. At the end of this paper, we gave the bounds for the spectral norm of $\mathbb{M}_k^{\circ -1} = \text{circ}\{_k(M_s^{-1}, M_{s+t}^{-1}, \ldots, M_{s+(n-1)t}^{-1})\}$ (for $s \neq 0$).

Using the well known fact: *the eigenvalues of an upper triangular matrix are the diagonal entries*, we conclude that the eigenvalues of $\mathbb{M}$ for $k = 0$ (i.e. a semicirculant matrix) are: $\lambda_j(\mathbb{M}) = M_s$, $j = \overline{0, n-1}$. The 1−norm, the ∞−norm, the Euclidean norm and the bounds for the spectral norm of such matrix can be obtained, respectively, from (34), (37) and (41) i.e. k can be equal to 0 in (34), (37) and (41). The upper and lower bounds for the spectral norm of $\mathbb{M}_k^{\circ -1}$ for $k = 0$ (and $s \neq 0$) can be obtained from (45).

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