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On some embedding of the strong annihilating-ideal graph of commutative rings

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Abstract. Let $\mathfrak{S}$ be a commutative ring with unity (CRU) and $W(\mathfrak{S})$ be the set of annihilating-ideals of $\mathfrak{S}$. The strong annihilating-ideal graph of $\mathfrak{S}$, denoted by $SAG(\mathfrak{S})$, is an undirected graph with vertex set $W(\mathfrak{S})^*$. Two vertices $\mathfrak{m}$ and $\mathfrak{n}$ are adjacent if and only if $\mathfrak{m} \cap \mathrm{Ann}(\mathfrak{n}) \neq (0)$ and $\mathfrak{n} \cap \mathrm{Ann}(\mathfrak{m}) \neq (0)$. In this paper, we first characterize the Artinian commutative rings $\mathfrak{S}$ for which $SAG(\mathfrak{S})$ has outerplanarity index 2. Then, we classify Artinian commutative rings $\mathfrak{S}$ for which $SAG(\mathfrak{S})$ is double toroidal or Klein-bottle. Finally, we determine the book thickness of $SAG(\mathfrak{S})$ for genus at most one.

1. Introduction

In this paper, unless stated otherwise, we use $\mathfrak{S}$ to refer to a commutative ring with identity, and we assume that $\mathfrak{S}$ is not a field. We denote all commutative rings with unity as *CRU*. For the commutative ring $\mathfrak{S}$, we define $\mathbb{I}(\mathfrak{S})$ as the set of ideals of $\mathfrak{S}$, and we let $\mathbb{I}(\mathfrak{S})^* = \mathbb{I}(\mathfrak{S}) \setminus \{0\}$, which excludes the zero ideal. An ideal $\mathfrak{m}$ of $\mathfrak{S}$ is called an *annihilator ideal* if there exists a nonzero ideal $\mathfrak{n}$ in $\mathfrak{S}$ such that $\mathfrak{m}\mathfrak{n} = (0)$. For any $\mathfrak{m} \in \mathbb{I}(\mathfrak{S})$, we define the *annihilator* of $\mathfrak{m}$ as $Ann(\mathfrak{m}) = \{r \in \mathfrak{S} : r\mathfrak{m} = (0)\}$. We denote the set of annihilator ideals in $\mathfrak{S}$ by $W(\mathfrak{S})$, and we define $W(\mathfrak{S})^* = W(\mathfrak{S}) \setminus \{0\}$ to exclude the zero ideal. The sets of zero-divisors, nilpotent elements, minimal prime ideals, and unit elements of $\mathfrak{S}$ are represented by $Z(\mathfrak{S})$, $Nil(\mathfrak{S})$, $Min(\mathfrak{S})$, and $U(\mathfrak{S})$, respectively. A ring $(\mathfrak{S}, \mathfrak{A})$ is considered local if $\mathfrak{A}$ is the only maximal ideal in $\mathfrak{S}$. The ideal $E_i \in \mathfrak{S}_1 \times \mathfrak{S}_2 \times \cdots \times \mathfrak{S}_n$ is defined as $E_i = (0) \times (0) \times \cdots \times (0) \times \mathfrak{S}_i \times (0) \times \cdots \times (0)$ for each $1 \leq i \leq n$. For further details, we refer the reader to [5].

A graph, denoted as $G(V, E)$, consists of a collection of vertices V and a collection of edges E. A graph is called *complete* when there is an edge connecting every pair of distinct vertices. A complete graph with n vertices is represented as K_n. A k-partite graph is one where the vertices can be divided into k separate sets, called independent sets, such that no two vertices within the same set are connected by an edge. When $k = 2$, the graph is referred to as a *bipartite* graph. A complete k-partite graph is a k-partite graph in which every pair of vertices from different independent sets is connected by an edge. We denote a complete

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bipartite graph as $K_{m,n}$, where m and n indicate the sizes of the two independent sets. For more information on graph theory, we recommend the work of [24].

A graph G is referred to as *planar* if it can be represented on a plane in such a way that its edges only intersect at their endpoints. A *subdivision* of a graph is created by replacing its edges with non-intersecting paths. In 1930, Kuratowski introduced a straightforward method to determine whether a graph is planar. Kuratowski's Theorem states that a graph G is planar if and only if it does not contain a subdivision of K_5 (the complete graph with five vertices) or $K_{3,3}$ (the complete bipartite graph formed by two groups of three vertices each). An embedding ϕ of a planar graph is called *1-outerplanar* if it shows outerplanarity, which means all vertices are connected to the outer face of the embedding. This concept can be generalized: an embedding is termed k-outerplanar if, after removing all vertices on the outer face and their connecting edges, the resulting graph has a $(k-1)$-outerplanar embedding. A graph is considered k-outerplanar if it supports such a k-outerplanar embedding. The *outerplanarity index* of a graph G is the smallest integer k for which G is k-outerplanar. For a planar graph G, the *inner vertex number* $i(G)$ is defined as the smallest number of vertices that are not on the boundary of the outer region in any planar embedding of G. A graph G is known as minimally non-outerplanar if $i(G) = 1$. For a deeper understanding of k-outerplanarity, refer to the works of [11, 14].

The *genus* of a graph G, denoted as $\gamma(G)$, is the smallest integer k for which the graph can be drawn on a surface with k handles (think of a surface with k "loops"). A graph is *planar* if it can be drawn on a sphere without any lines crossing each other, which means its genus is 0. Graphs with genus 1 are known as *toroidal graphs* (imagine a doughnut shape), while those with genus 2 are called *double-toroidal graphs*. For a non-negative integer k, we can create a surface by taking a sphere and attaching k crosscaps to it. Every connected compact surface can be thought of as being similar to this surface, denoted as N_k. The *crosscap number* (also known as non-orientable genus) $\overline{\gamma}(G)$ is the smallest integer k such that G can be drawn on the surface N_k. Graphs with a crosscap number of 1 are known as *projective plane graphs*, while those with a crosscap number of 2 are referred to as *Klein-bottle graphs*. It's important to note that if H is a subgraph of G, then $\gamma(H)$ is always less than or equal to $\gamma(G)$, and $\overline{\gamma}(H)$ is always less than or equal to $\overline{\gamma}(G)$. For more information on graph theory, we recommend consulting works by [25].

An *n-book embedding* is defined as a collection of n half-planes, referred to as pages, which are all connected along a single line known as the spine. When the vertices of a graph can be arranged along the spine, and its edges can be allocated across r pages such that each edge resides in exactly one page and no two edges intersect within any given page, the arrangement is termed an *r-book embedding*. The book thickness of a graph G, denoted as $bt(G)$, is the minimum integer n such that G can be represented with an n-book embedding. For further information on graph embeddings in surfaces and book embeddings, one may consult [9, 22, 25].

Beck [6] introduced the concept of the zero-divisor graph of a commutative ring in 1988, focusing primarily on colorings. In his work, he proposed that $\chi(\mathfrak{S}) = \omega(\mathfrak{S})$ for any commutative ring $\mathfrak{S}$ [6]. He validated this conjecture for certain types of rings, such as reduced rings and principal ideal rings. However, this assertion does not hold true in general. This was demonstrated in 1993 by Anderson and Naseer, who provided a compelling counterexample (see Theorem 2.1 in [3]), disproving Beck's conjecture for general rings. Anderson and Naseer continued their exploration of colorings in commutative rings, defining a graph where the vertex set consists of the ring elements and an edge is established between vertices a and b if and only if $ab = 0$. In [2], Anderson and Livingston defined the zero-divisor graph of $\mathfrak{S}$, denoted by $\Gamma(\mathfrak{S})$, with the vertex set $Z(\mathfrak{S})^*$. For distinct $a, b \in Z(\mathfrak{S})^*$, the vertices a and b are connected by an edge if and only if $ab = 0$.

In 2011, Behboodi and Rakeei [7, 8] introduced a new graph called the *annihilating-ideal graph* $AG(\mathfrak{S})$ on $\mathfrak{S}$, with the vertex set $W(\mathfrak{S})^*$. Two distinct vertices $\mathfrak{m}$ and $\mathfrak{n}$ are adjacent if and only if $\mathfrak{m}\mathfrak{n} = 0$ (see [16–18, 21] for more details).

Tohidi and collaborators [19, 20] introduced and investigated the *strong annihilating-ideal graph* of a commutative ring $\mathfrak{S}$, denoted by $SAG(\mathfrak{S})$, where the vertex set is $W(\mathfrak{S})^*$. Two vertices $\mathfrak{m}$ and $\mathfrak{n}$ are connected by an edge if and only if $\mathfrak{m} \cap \mathrm{Ann}(\mathfrak{n}) \neq (0)$ and $\mathfrak{n} \cap \mathrm{Ann}(\mathfrak{m}) \neq (0)$. In [16], Rehman and coauthors characterized the Artinian commutative ring $\mathfrak{S}$ for which $SAG(\mathfrak{S})$ is a planar or outerplanar graph. They also classified the rings $\mathfrak{S}$ for which $SAG(\mathfrak{S})$ is a toroidal or projective plane graph. Besides research on various graphs associated to commutative rings, which is vast, as it can be seen from [4], there have also been studies on various graphs associated to not necessarily commutative rings, and the reader may consult [1, 10, 12, 13, 23, 24] and their references for more details.

In this paper, we begin by characterizing the Artinian commutative rings $\mathfrak{S}$ for which the strong annihilating-ideal graph $SAG(\mathfrak{S})$ has an outerplanarity index of 2. Next, we classify the Artinian commutative rings $\mathfrak{S}$ for which $SAG(\mathfrak{S})$ is double toroidal or resembles a Klein bottle. Lastly, we investigate the book thickness of $SAG(\mathfrak{S})$ for graphs with genus at most one.

The results listed below are important for the upcoming sections.

Lemma 1.1. *[19, Lemma 2.1] Let $\mathfrak{S}$ be a CRU and $\mathfrak{m}, \mathfrak{n} \in W(\mathfrak{S})^*$. The following assertions are valid:*

1. *If $\mathfrak{m} - \mathfrak{n}$ is not an edge of $SAG(\mathfrak{S})$, then $Ann(\mathfrak{m}\mathfrak{n}) = Ann(\mathfrak{m})$ or $Ann(\mathfrak{m}\mathfrak{n}) = Ann(\mathfrak{n})$. Moreover, if $\mathfrak{S}$ is a reduced ring, then the converse also holds.*
2. *If $\mathfrak{m} - \mathfrak{n}$ is an edge of the annihilating-ideal graph $AG(\mathfrak{S})$, then $\mathfrak{m} - \mathfrak{n}$ is also an edge of the strong annihilating-ideal graph $SAG(\mathfrak{S})$.*
3. *If $Ann(\mathfrak{m}) \not\subseteq Ann(\mathfrak{n})$ and $Ann(\mathfrak{n}) \not\subseteq Ann(\mathfrak{m})$, then $\mathfrak{m} - \mathfrak{n}$ is an edge of the strong annihilating-ideal graph $SAG(\mathfrak{S})$. Moreover, if $\mathfrak{S}$ is a reduced ring, then the converse is also*
4. *Let $n \geq 1$ be a positive integer. Suppose that $\mathfrak{S} \cong \mathfrak{S}_1 \times \mathfrak{S}_2 \times \cdots \times \mathfrak{S}_n$, where $\mathfrak{S}_i$ is a ring for every $1 \leq i \leq n$, and let $\mathfrak{m} = \mathfrak{m}_1 \times \mathfrak{m}_2 \times \cdots \times \mathfrak{m}_n$ and $\mathfrak{n} = \mathfrak{n}_1 \times \mathfrak{n}_2 \times \cdots \times \mathfrak{n}_n$ be two vertices of $SAG(\mathfrak{S})$. If $\mathfrak{m}_i \cap Ann(\mathfrak{n}_i) \neq (0)$ and $\mathfrak{n}_j \cap Ann(\mathfrak{m}_j) \neq (0)$, for some $1 \leq i, j \leq n$, then $\mathfrak{m} - \mathfrak{n}$ is an edge of $SAG(\mathfrak{S})$. In particular, if $\mathfrak{m}_i - \mathfrak{n}_i$ is an edge of $SAG(\mathfrak{S}_i)$ or $\mathfrak{m}_i = \mathfrak{n}_i$, and $\mathfrak{m}_i \cap Ann(\mathfrak{m}_i) \neq (0)$, for some $1 \leq i \leq n$, then $\mathfrak{m} - \mathfrak{n}$ is an edge of $SAG(\mathfrak{S})$.*
5. *If $\mathfrak{m}$ and $\mathfrak{n}$ belong to the essential ideals of $\mathfrak{S}$, denoted by $Ess(\mathfrak{S})$, or if the annihilators of $\mathfrak{m}$ and $\mathfrak{n}$, that is, $Ann(\mathfrak{m})$ and $Ann(\mathfrak{n})$, belong to $Ess(\mathfrak{S})$, then $\mathfrak{m}$ is adjacent to $\mathfrak{n}$.*
6. *If the distance between $\mathfrak{m}$ and $\mathfrak{n}$ in the annihilating-ideal graph $AG(\mathfrak{S})$, denoted by $d_{AG(\mathfrak{S})}(\mathfrak{m}, \mathfrak{n})$, is 3 for some distinct $\mathfrak{m}, \mathfrak{n} \in W(\mathfrak{S})^*$, then the pair $\mathfrak{m} - \mathfrak{n}$ forms an edge in the strong annihilating-ideal graph $SAG(\mathfrak{S})$.*
7. *If $\mathfrak{m}$ and $\mathfrak{n}$ are distinct elements of $W(\mathfrak{S})^*$ and there is no edge between $\mathfrak{m}$ and $\mathfrak{n}$ in $SAG(\mathfrak{S})$, then the distance between $\mathfrak{m}$ and $\mathfrak{n}$ in $AG(\mathfrak{S})$, denoted by $d_{AG(\mathfrak{S})}(\mathfrak{m}, \mathfrak{n})$, is 2.*

Theorem 1.2. *[16, Theorem 2.1] For the local ring $(\mathfrak{S}, \mathfrak{A})$ the graph $SAG(\mathfrak{S})$ is a complete graph.*

2. Outerplanarity of $SAG(\mathfrak{S})$

In this section, we classify the Artinian *CRU* for which $SAG(\mathfrak{S})$ has outerplanarity index 2. Also, we find the inner vertex number of $SAG(\mathfrak{S})$ for Artinian *CRU*.

In the following results, Rehman et al. [16] characterized the Artinian *CRU* for $SAG(\mathfrak{S})$ is planar or outerplanar.

Theorem 2.1. *[16, Theorems 3.2 and 3.3] Let $\mathfrak{S}$ be an Artinian CRU. Then $SAG(\mathfrak{S})$ is planar $\Longleftrightarrow$ one of the following conditions is satisfied:*

1. *$\mathfrak{S}$ is a local ring that contains no more than four non-trivial ideals.*
2. *$\mathfrak{S} \cong \mathfrak{D}_1 \times \mathfrak{D}_2$, where $\mathfrak{D}_1$ and $\mathfrak{D}_2$ are fields.*
3. *$\mathfrak{S} \cong \mathfrak{D}_1 \times \mathfrak{D}_2 \times \mathfrak{D}_3$, where each $\mathfrak{D}_i$ is a field.*
4. *$\mathfrak{S} \cong \mathfrak{D} \times \mathfrak{S}_1$, where $\mathfrak{D}$ is a field and $(\mathfrak{S}_1, \mathfrak{A})$ is a local ring having a unique non-trivial ideal $\mathfrak{A}$.*

Theorem 2.2. *[16, Theorem 3.5] Let $\mathfrak{S}$ be an Artinian CRU. Then $SAG(\mathfrak{S})$ is outerplanar $\Longleftrightarrow$ one of the following conditions is satisfied:*

1. *$\mathfrak{S}$ is a local ring that contains no more than three non-trivial ideals.*
2. *$\mathfrak{S} \cong \mathfrak{D}_1 \times \mathfrak{D}_2$, where $\mathfrak{D}_1$ and $\mathfrak{D}_2$ are fields.*
3. *$\mathfrak{S} \cong \mathfrak{D} \times \mathfrak{S}_1$, where $\mathfrak{D}$ is a field and $(\mathfrak{S}_1, \mathfrak{A})$ is a local ring having a unique non-trivial ideal $\mathfrak{A}$.*

We are prepared to demonstrate the main result of this section.

Theorem 2.3. *Let $\mathfrak{S}$ be an Artinian CRU. Then $SAG(\mathfrak{S})$ has outerplanarity index 2 $\Longleftrightarrow$ one of the following conditions is satisfied:*

1. *$\mathfrak{S} \cong \mathfrak{D}_1 \times \mathfrak{D}_2$, where $\mathfrak{D}_1$ and $\mathfrak{D}_2$ are fields.*
2. *$\mathfrak{S}$ is a local ring possessing exactly four non-trivial ideals.*

Proof. We understand that all 2-outerplanar graphs are also planar graphs. Thus, we have to consider the rings given in Theorem 2.1. If $\mathfrak{S}$ is one of the rings given in Theorem 2.2, then $SAG(\mathfrak{S})$ is 1-outerplanar. If $\mathfrak{S} \cong \mathfrak{D}_1 \times \mathfrak{D}_2 \times \mathfrak{D}_3$, where each $\mathfrak{D}_i$ is a field, then $SAG(\mathfrak{S})$ is given in Figure 1. If we delete the vertices from the outer faces of the drawing, the resultant graph is K_2, which is 1-outerplanar. Hence $SAG(\mathfrak{S})$ has outerplanarity index 2. If $\mathfrak{S}$ is a local ring with $|\mathbb{I}(\mathfrak{S})^*| = 4$, then $SAG(\mathfrak{S}) \cong K_4$ by Theorem 1.2. Again if we delete the vertices from the outer faces of the drawing, the resultant graph is K_1, which is 1-outerplanar. Hence $SAG(\mathfrak{S})$ is again a 2-outerplanar graph. □

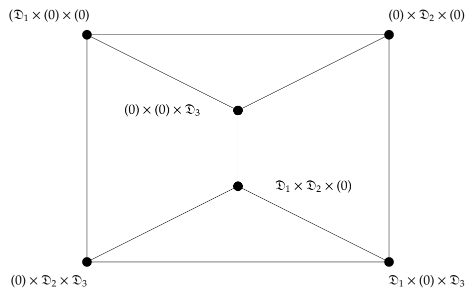

Figure 1. The graph $SAG(\mathfrak{D}_1 \times \mathfrak{D}_2 \times \mathfrak{D}_3)$

Corollary 2.4. *Let $\mathfrak{S}$ be an Artinian CRU. Then $SAG(\mathfrak{S})$ has an outerplanarity index that does not exceed two.*

Lastly, we determine the inner vertex number of $SAG(\mathfrak{S})$ for the class of Artinian rings $\mathfrak{S}$ in the subsequent result.

Theorem 2.5. *Let $\mathfrak{S}$ be an Artinian CRU. Then the inner vertex number of $SAG(\mathfrak{S})$ is determined by:*

$$i(SAG(\mathfrak{S})) = \begin{cases} 2 & \text{if } \mathfrak{S} \cong \mathfrak{D}_1 \times \mathfrak{D}_2 \times \mathfrak{D}_3, \text{ where each } \mathfrak{D}_i \text{ is a field;} \\ 1 & \text{if } \mathfrak{S} \text{ is a local ring with } |\mathbb{I}(\mathfrak{S})^*| = 4; \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The proof is derived directly from Theorems 2.2 and 2.3 as well as Figure 1. □

Corollary 2.6. *Let $\mathfrak{S}$ be an Artinian CRU. Then $SAG(\mathfrak{S})$ is minimally non-outerplanar $\Longleftrightarrow$ either $\mathfrak{S}$ is a local ring featuring exactly four non-trivial ideals.*

3. Embedding of $SAG(\mathfrak{S})$ on double torus

In this section, we classify the Artinian *CRU* for which $SAG(\mathfrak{S})$ is double toroidal, i.e., $\gamma(SAG(\mathfrak{S})) = 2$. The following results provide the genus of K_n and $K_{m,n}$, which will assist us in proving the main result of this section.

Lemma 3.1. *[25] If $r \geq 3$, then*

$$\gamma(K_r) = \left\lceil \frac{(r-3)(r-4)}{12} \right\rceil.$$

Lemma 3.2. *[25] If $r, s \geq 2$, then*

$$\gamma(K_{r,s}) = \left\lceil \frac{(r-2)(s-2)}{4} \right\rceil.$$

Lemma 3.3. *[15, Proposition 4.4.4] Let G be a connected graph having r edges and at least $s \geq 3$ vertices. Then,*

$$\gamma(G) \geq \left\lceil \frac{r}{6} - \frac{s}{2} + 1 \right\rceil.$$

In the following result, Rehman et al. [16] characterized Artinian *CRU* for which $SAG(\mathfrak{S})$ is toroidal graph i.e., $\gamma(SAG(\mathfrak{S})) = 1$.

Theorem 3.4. *[16, Theorems 4.1 and 4.2] Let $\mathfrak{S}$ be a non-local Artinian CRU. Then $\gamma(SAG(\mathfrak{S})) = 1 \iff$ one of the following conditions is satisfied:*

1. $\mathfrak{S} \cong \mathfrak{S}_1 \times \mathfrak{S}_2$*, where each $(\mathfrak{S}_i, \mathfrak{A}_i)$ is local ring with unique non-trivial ideal $\mathfrak{A}_i$.*
2. $\mathfrak{S} \cong \mathfrak{D} \times \mathfrak{S}_1$*, where $\mathfrak{D}$ is a field and $(\mathfrak{S}_1, \mathfrak{A})$ is a local ring featuring the non-trivial ideals $\mathfrak{A}$ and $\mathfrak{A}^2$.*
3. $\mathfrak{S} \cong \mathfrak{D} \times \mathfrak{S}_1$*, where $\mathfrak{D}$ is a field and $(\mathfrak{S}_1, \mathfrak{A})$ is local ring featuring the non-trivial ideals $\mathfrak{A}$, $\mathfrak{A}^2$ and $\mathfrak{A}^3$.*

In the following result, we classify local rings for which $SAG(\mathfrak{S})$ is double-toroidal.

Lemma 3.5. *Let $\mathfrak{S}$ be an Artinian local CRU. Then $\gamma(SAG(\mathfrak{S})) = 2 \iff \mathfrak{S}$ has exactly 8 non-trivial ideals.*

Proof. The conclusion of the proof comes from Theorem 1.2 and Lemma 3.1. □

In the following results, we classify non-local rings for which $SAG(\mathfrak{S})$ is double-toroidal.

Lemma 3.6. *Let $\mathfrak{S} = \mathfrak{D}_1 \times \mathfrak{D}_2 \times \cdots \times \mathfrak{D}_m$ be a CRU, where each $\mathfrak{D}_i$ represents a field and $m \geq 4$. Then $\gamma(SAG(\mathfrak{S})) > 2$.*

Proof. Consider the case when $m = 4$ i.e., $\mathfrak{S} = \mathfrak{D}_1 \times \mathfrak{D}_2 \times \mathfrak{D}_3 \times \mathfrak{D}_4$. The vertices of $SAG(\mathfrak{S})$ are given by $x_1 = E_1$, $x_2 = E_2$, $x_3 = E_3$, $x_4 = E_4$, $x_5 = E_1 + E_2$, $x_6 = E_1 + E_3$, $x_7 = E_1 + E_4$, $x_8 = E_2 + E_3$, $x_8 = E_2 + E_4$, $x_9 = E_3 + E_4$, $x_{10} = E_1 + E_2 + E_3$, $x_{11} = E_1 + E_2 + E_4$, $x_{12} = E_1 + E_3 + E + 4$, $x_{13} = E_1 + E_2 + E_4$ and $x_{14} = E_2 + E_3 + E_4$. The graph generated by the set $\{x_1, \ldots, x_{14}\}$ contains 14 vertices and 55 edges. Thus, by Lemma 3.3, $\gamma(SAG(\mathfrak{S})) > 2$. □

Lemma 3.7. *Let $\mathfrak{S} = \mathfrak{S}_1 \times \mathfrak{S}_2 \times \cdots \times \mathfrak{S}_n$ be a CRU, where $(\mathfrak{S}_i, \mathfrak{A}_i)$ is an Artinian local ring with $\mathfrak{A}_i \neq 0$ for each $1 \leq i \leq n$ and $n \geq 2$. Then $\gamma(SAG(\mathfrak{S})) \neq 2$.*

Proof. Suppose $n \geq 3$. Consider $x_1 = \mathfrak{A}_1 \times (0) \times \cdots \times (0)$, $x_2 = (0) \times \mathfrak{A}_2 \times (0) \times \cdots \times (0)$, $x_3 = (0) \times (0) \times \mathfrak{A}_3 \times (0) \times \cdots \times (0)$, $x_4 = E_1$, $y_1 = E_2$, $y_2 = E_3$, $y_3 = E_1 + E_2$, $y_4 = \mathfrak{A}_1 \times \mathfrak{S}_2 \times (0) \times \cdots \times (0)$, $y_5 = \mathfrak{A}_1 \times (0) \times \mathfrak{S}_3 \times (0) \times \cdots \times (0)$, $y_6 = \mathfrak{A}_1 \times \mathfrak{S}_2 \times \mathfrak{S}_3 \times (0) \times \cdots \times (0)$ and $y_7 = (0) \times \mathfrak{A}_2 \times \mathfrak{A}_3 \times (0) \times \cdots \times (0) \in W(\mathfrak{S})^*$. Since $x_i \cap Ann(y_j) \neq 0$ and $y_j \cap Ann(x_i) \neq 0$, then $K_{4,7}$ is a subset of $SAG(\mathfrak{S})$ generated by the set $\{x_1, \ldots, x_4\} \cup \{y_1, \ldots, y_7\}$. Thus, by using of Lemma 3.2, $\gamma(SAG(\mathfrak{S})) > 2$. Hence $n = 2$.

Define η_i as the nilpotency index of $\mathfrak{A}_i$ for $i = 1, 2$. Suppose $\eta_2 \geq 3$. Consider $a_1 = (0) \times \mathfrak{A}_2$, $a_2 = (0) \times \mathfrak{A}_2^{\eta_2 - 1}$, $a_3 = \mathfrak{A}_1 \times \mathfrak{A}_2^{\eta_2 - 1}$, $a_4 = (0) \times \mathfrak{S}_2$, $a_5 = \mathfrak{A}_1 \times \mathfrak{S}_2$, $b_1 = \mathfrak{S}_1 \times (0)$, $b_2 = \mathfrak{S}_1 \times \mathfrak{A}_2$, $b_3 = \mathfrak{S}_1 \times \mathfrak{A}_2^{\eta_2 - 1}$, $b_4 = \mathfrak{A}_1 \times (0)$ and $b_5 = \mathfrak{A}_1 \times \mathfrak{A}_2 \in W(\mathfrak{S})^*$. Since $a_i \cap Ann(b_j) \neq 0$ and $b_j \cap Ann(a_i) \neq 0$, $K_{5,5}$ is a subgraph of $SAG(\mathfrak{S})$ generated by the set $\{a_1, \ldots, a_5\} \cup \{b_1, \ldots, b_5\}$. Thus, $\gamma(SAG(\mathfrak{S})) > 2$ by the use of Lemma 3.2. Hence $\eta_2 = 2$. Similarly, we can show that $\eta_1 = 2$.

Let $I \in \mathbb{I}(\mathfrak{S}_1)^*$ such that $I \neq \mathfrak{A}_1$. Consider $c_1 = (0) \times \mathfrak{A}_2$, $c_2 = (0) \times I$, $c_3 = \mathfrak{A}_1 \times I$, $c_4 = (0) \times \mathfrak{S}_2$, $c_5 = \mathfrak{A}_1 \times \mathfrak{S}_2$, $d_1 = \mathfrak{S}_1 \times (0)$, $d_2 = \mathfrak{S}_1 \times \mathfrak{A}_2$, $d_3 = \mathfrak{S}_1 \times I$, $d_4 = \mathfrak{A}_1 \times (0)$ and $d_5 = \mathfrak{A}_1 \times \mathfrak{A}_2 \in W(\mathfrak{S})^*$. Since $c_i \cap Ann(d_j) \neq 0$ and $d_j \cap Ann(c_i) \neq 0$, $K_{5,5}$ is a subgraph of $SAG(\mathfrak{S})$ generated by the set $\{c_1, \ldots, c_5\} \cup \{d_1, \ldots, d_5\}$. Thus, $\gamma(SAG(\mathfrak{S})) > 2$ by the use of Lemma 3.2. Hence $\mathfrak{S}_1$ has only one non-trivial ideal $\mathfrak{A}_1$. Analogously, we can prove that $\mathfrak{A}_2$ is the unique non-trivial ideal of $\mathfrak{S}_2$. Thus, again by using of Theorem 3.4, $\gamma(SAG(\mathfrak{S})) = 1$. Hence, considering all the cases, we can conclude that $\gamma(SAG(\mathfrak{S})) \neq 2$. □

Lemma 3.8. *Let $\mathfrak{S} = \mathfrak{D}_1 \times \mathfrak{D}_2 \times \cdots \times \mathfrak{D}_m \times \mathfrak{S}_1 \times \mathfrak{S}_2 \times \cdots \times \mathfrak{S}_n$ be a CRU, where $\mathfrak{D}_i$ is a field, $(\mathfrak{S}_j, \mathfrak{A}_j)$ is an Artinian local ring with $\mathfrak{A}_j \neq 0$ and $m, n \geq 1$. Then $\gamma(SAG(\mathfrak{S})) = 2 \iff m = 2$, $n = 1$ and $\mathfrak{A}_1$ represents the only non-trivial ideal of $\mathfrak{S}_1$.*

Proof. Suppose that $\gamma(SAG(\mathfrak{S})) = 2$. If $m \geq 3$, then by Lemma 3.6, $\gamma(SAG(\mathfrak{S})) > 2$. Thus, $m \leq 2$. Similarly, $n \leq 2$. Take the following cases into consideration:

Case (1): Suppose $m = 2$ and $n = 2$. Again by Lemma 3.6, $\gamma(SAG(\mathfrak{S})) > 2$, a contradiction.

Case (2): Suppose $m = 1$ and $n = 2$. Consider $a_1 = (0) \times \mathfrak{A}_1 \times (0)$, $a_2 = (0) \times \mathfrak{S}_1 \times (0)$, $a_3 = (0) \times \mathfrak{S}_1 \times \mathfrak{S}_2$, $a_4 = (0) \times \mathfrak{S}_1 \times \mathfrak{A}_2$, $a_5 = (0) \times \mathfrak{A}_1 \times \mathfrak{A}_2$, $b_1 = \mathfrak{D}_1 \times (0) \times (0)$, $b_2 = \mathfrak{D}_1 \times (0) \times \mathfrak{S}_2$, $b_3 = \mathfrak{D}_1 \times \mathfrak{A}_1 \times (0)$, $b_4 = \mathfrak{D}_1 \times (0) \times \mathfrak{A}_2$, $b_5 = \mathfrak{D}_1 \times \mathfrak{A}_1 \times \mathfrak{A}_2 \in W(\mathfrak{S})^*$. Since $a_i \cap Ann(b_j) \neq 0)$ and $b_j \cap Ann(a_i)$ for each i, j, $K_{5,5}$ is a subgraph of $SAG(\mathfrak{S})$ generated by $\{a_1, \ldots, a_5\} \cup \{b_1 \ldots, b_5\}$. Thus, by Lemma 3.2, $\gamma(SAG(\mathfrak{S})) > 2$, a contradiction.

Case (3): Suppose $m = 2$ and $n = 1$. Let η_1 be the nilpotency index of $\mathfrak{A}_1$. Suppose $\eta_1 \geq 3$. Consider $x_1 = (0) \times (0) \times \mathfrak{S}_1$, $x_2 = (0) \times \mathfrak{D}_2 \times \mathfrak{A}_1$, $x_3 = (0) \times (0) \times \mathfrak{A}_1$, $x_4 = (0) \times (0) \times \mathfrak{A}_1^{\eta_1 - 1}$, $x_5 = (0) \times \mathfrak{D}_2 \times \mathfrak{S}_1$, $y_1 = \mathfrak{D}_1 \times \mathfrak{D}_2 \times (0)$, $y_2 = \mathfrak{D}_1 \times (0) \times \mathfrak{A}_1$, $y_3 = \mathfrak{D}_1 \times (0) \times \mathfrak{A}_1^{\eta_1 - 1}$, $y_4 = \mathfrak{D}_1 \times \mathfrak{D}_2 \times \mathfrak{A}_1$ and $y_5 = \mathfrak{D}_1 \times \mathfrak{D}_2 \times \mathfrak{A}_1^{\eta_1 - 1} \in W(\mathfrak{S})^*$. Since $x_i \cap Ann(y_j) \neq 0$ and $y_j \cap Ann(x_i) \neq$, the graph $K_{5,5}$ is a subgraph of $SAG(\mathfrak{S})$ generated by the set $\{x_1, \ldots, x_5\} \cup \{y_1, \ldots, y_5\}$. Thus, by using of Lemma 3.2, $\gamma(SAG(\mathfrak{S})) > 2$, a contradiction. Hence $\eta_1 = 2$.

Let $I \in \mathbb{I}(\mathfrak{S}_1)^*$ such that $I \neq \mathfrak{A}_1$. Then the graph $K_{3,3}$ is a subgraph of the graph generated by the set $\{(0) \times (0) \times \mathfrak{S}_1, (0) \times \mathfrak{D}_2 \times \mathfrak{A}_1, (0) \times (0) \times \mathfrak{A}_1, (0) \times (0) \times I, (0) \times \mathfrak{D}_2 \times \mathfrak{S}_1\} \cup \{\mathfrak{D}_1 \times \mathfrak{D}_2 \times (0), \mathfrak{D}_1 \times (0) \times \mathfrak{A}_1, \mathfrak{D}_1 \times (0) \times I, \mathfrak{D}_1 \times \mathfrak{D}_2 \times \mathfrak{A}_1, \mathfrak{D}_1 \times \mathfrak{D}_2 \times I\}$, which leads to a contradiction by employing Lemma 3.2. Hence $\mathfrak{A}_1$ is the only non-trivial ideal of $\mathfrak{S}_1$.

Case (4): Suppose $m = 1 = n$. Suppose $\eta_1 \geq 5$. Consider $c_1 = (0) \times \mathfrak{A}_1^{\eta_1 - 1}$, $c_2 = (0) \times \mathfrak{A}_1^{\eta_1 - 2}$, $c_3 = (0) \times \mathfrak{A}_1^{\eta_1 - 3}$, $c_4 = (0) \times \mathfrak{A}_1^{\eta_1 - 4}$, $c_5 = \mathfrak{D}_1 \times \mathfrak{A}_1^{\eta_1 - 1}$, $c_6 = \mathfrak{D}_1 \times \mathfrak{A}_1^{\eta_1 - 2}$, $c_7 = \mathfrak{D}_1 \times \mathfrak{A}_1^{\eta_1 - 3}$, $c_8 = \mathfrak{D}_1 \times \mathfrak{A}_1^{\eta_1 - 4} \in W(\mathfrak{S})^*$. Since $c_i \cap Ann(c_j) \neq 0$ and $Ann(c_i) \cap c_j \neq 0$ for each i, j, the graph K_8 is a subgraph of $SAG(\mathfrak{S})$ generated by the set $\{c_1, \ldots, c_8\}$. Thus, by using of Lemma 3.1, $\gamma(SAG(\mathfrak{S})) > 2$, a contradiction. Hence $\eta_1 \leq 4$. Then by Theorems 2.1 and 3.4, $\gamma(SAG(\mathfrak{S})) \leq 1$, a contradiction.

The converse is derived from Figure 2. □

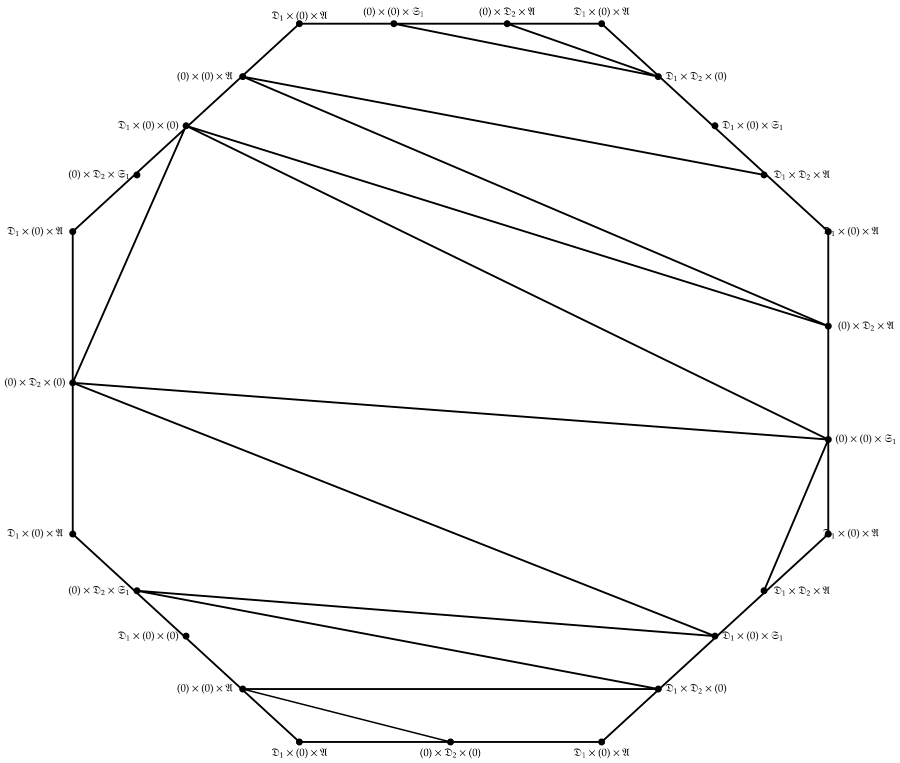

Figure 2. Embedding of $SAG(\mathfrak{D}_1 \times \mathfrak{D}_2 \times \mathfrak{S}_1)$ on double torus,

where $\mathfrak{A}$ being the unique non-trivial ideal of $\mathfrak{S}_1$.

In conclusion, we can state the main result of this section.

Theorem 3.9. *Let $\mathfrak{S}$ be an Artinian CRU. Then $\gamma(SAG(\mathfrak{S})) = 2 \iff$ one of the following conditions is satisfied:*

1. *$\mathfrak{S}$ is a local ring that possesses 7 non-trivial ideals.*
2. *$\mathfrak{S} \cong \mathfrak{D}_1 \times \mathfrak{D}_2 \times \mathfrak{S}_1$, where $\mathfrak{D}_1$ and $\mathfrak{D}_2$ are fields, and $(\mathfrak{S}_1, \mathfrak{A})$ denotes a local ring with $\mathfrak{A}$ being the unique non-trivial ideal of $\mathfrak{S}_1$.*

Proof. The assertion follows from Lemmas 3.5, 3.6, 3.7, and 3.8. □

4. Embedding of $SAG(\mathfrak{S})$ on Klein-bottle

In this section, we classify the Artinian *CRU* for which $SAG(\mathfrak{S})$ is Klein-bottle i.e., $\overline{\gamma}(SAG(\mathfrak{S})) = 2$. The following results give the crosscap of K_r and $K_{r,s}$, which help us to prove the main result of this section.

Theorem 4.1. (1) *Let $r \geq 3$. Then*

$$\overline{\gamma}(K_r) = \begin{cases} \left\lceil \frac{(r-3)(r-4)}{6} \right\rceil & \text{if } r \geq 3 \text{ and } r \neq 7; \\ 3 & \text{if } r = 7. \end{cases}$$

(2) *Let $r, s \geq 2$. Then*

$$\overline{\gamma}(K_{r,s}) = \left\lceil \frac{(r-2)(s-2)}{2} \right\rceil.$$

Lemma 4.2. *[15] Consider a connected graph G that contains $r \geq 3$ vertices and s edges. Then*

$$\overline{\gamma}(G) \geq \left\lceil \frac{s}{3} - r + 2 \right\rceil.$$

In the following result, Rehman et al. [16] characterized Artinian CRU for which $SAG(\mathfrak{S})$ is projective plane i.e., $\overline{\gamma}(SAG(\mathfrak{S})) = 1$.

Theorem 4.3. *[16, Theorems 5.2 and 5.3] Let $\mathfrak{S}$ be a non-local Artinian CRU. Then $\overline{\gamma}(SAG(\mathfrak{S})) = 1 \iff$ one of the following hold:*

1. *$\mathfrak{S} \cong \mathfrak{S}_1 \times \mathfrak{S}_2$, where each $(\mathfrak{S}_i, \mathfrak{A}_i)$ forms a local ring that has exactly one non-trivial ideal $\mathfrak{A}_i$.*
2. *$\mathfrak{S} \cong \mathfrak{D} \times \mathfrak{S}_1$, where $\mathfrak{D}$ is a field and $(\mathfrak{S}_1, \mathfrak{A})$ denotes a local ring that has non-trivial ideals $\mathfrak{A}$ and $\mathfrak{A}^2$.*

In the following result, we characterize local Artinian CRU for which $SAG(\mathfrak{S})$ is a Kline bottle, i.e., $\overline{\gamma}(SAG(\mathfrak{S})) = 2$.

Lemma 4.4. *Let $\mathfrak{S}$ be an Artinian local CRU. Then $\overline{\gamma}(SAG(\mathfrak{S})) \neq 2$.*

Proof. The conclusion is derived from Theorem 1.2 and Lemma 4.1. □

In the subsequent result, we identify the properties of non-local Artinian CRU for which $SAG(\mathfrak{S})$ takes the form of a Kline bottle, indicating that $\overline{\gamma}(SAG(\mathfrak{S})) = 2$.

Lemma 4.5. *Let $\mathfrak{S} = \mathfrak{D}_1 \times \mathfrak{D}_2 \times \cdots \times \mathfrak{D}_m$ be a CRU, where each $\mathfrak{D}_i$ is a field and $m \geq 4$. Then $\overline{\gamma}(SAG(\mathfrak{S})) \neq 2$.*

Proof. The conclusion is obtained from Lemmas 3.6 and 4.2. □

Lemma 4.6. *For the ring $\mathfrak{S} = \mathfrak{S}_1 \times \mathfrak{S}_2 \times \cdots \times \mathfrak{S}_n$, where $(\mathfrak{S}_i, \mathfrak{A}_i)$ is Artinian local ring with $\mathfrak{A}_i \neq 0$ for each $1 \leq i \leq n$ and $n \geq 2$, $\overline{\gamma}(SAG(\mathfrak{S})) \neq 2$.*

Proof. Suppose $n \geq 3$. Consider $a_1 = \mathfrak{A}_1 \times (0) \times \cdots \times (0)$, $a_2 = (0) \times \mathfrak{A}_2 \times (0) \times \cdots \times (0)$, $a_3 = (0) \times (0) \times \mathfrak{A}_3 \times (0) \times \cdots \times (0)$, $b_1 = E_2$, $b_2 = E_3$, $b_3 = E_1 + E_2$, $b_4 = \mathfrak{A}_1 \times \mathfrak{S}_2 \times (0) \times \cdots \times (0)$, $b_5 = \mathfrak{A}_1 \times (0) \times \mathfrak{S}_3 \times (0) \times \cdots \times (0)$, $b_6 = \mathfrak{A}_1 \times \mathfrak{S}_2 \times \mathfrak{S}_3 \times (0) \times \cdots \times (0)$ and $b_7 = (0) \times \mathfrak{A}_2 \times \mathfrak{A}_3 \times (0) \times \cdots \times (0) \in W(\mathfrak{S})^*$. Since $a_i \cap Ann(b_j) \neq 0$ and $b_j \cap Ann(a_i) \neq 0$, then $K_{3,7}$ is a subset of $SAG(\mathfrak{S})$ generated by the set $\{a_1, \ldots, a_3\} \cup \{b_1, \ldots, b_7\}$. Thus, by using of Lemma 4.1, $\overline{\gamma}(SAG(\mathfrak{S})) > 2$. Hence $n = 2$.

Let η_i denote the nilpotency index associated with $\mathfrak{A}_i$ for $i = 1, 2$. Suppose $\eta_2 \geq 3$. Consider $x_1 = \mathfrak{A}_1 \times (0)$, $x_2 = (0) \times \mathfrak{A}_2$, $x_3 = \mathfrak{A}_1 \times \mathfrak{A}_2$, $x_4 = (0) \times \mathfrak{A}_2^{\eta-1}$, $y_1 = \mathfrak{S}_1 \times (0)$, $y_2 = (0) \times \mathfrak{S}_2$, $y_3 = \mathfrak{A}_1 \times \mathfrak{S}_2$, $y_4 = \mathfrak{S}_1 \times \mathfrak{A}_2$ and $y_5 = \mathfrak{S}_1 \times \mathfrak{A}_2^{\eta-1} \in W(\mathfrak{S})^*$. Since $x_i \cap Ann(y_j) \neq 0$ and $y_j \cap Ann(x_i) \neq 0$, $K_{4,5}$ is a subgraph of $SAG(\mathfrak{S})$ generated by the set $\{x_1, \ldots, x_4\} \cup \{y_1, \ldots, y_5\}$. Thus, by using of Lemma 4.1, $\overline{\gamma}(SAG(\mathfrak{S})) > 2$. Hence $\eta_2 = 2$. Similarly, we can show that $\eta_1 = 2$.

Let $J \in \mathbb{I}(\mathfrak{S})^*$ such that $J \neq \mathfrak{A}_1$. $u_1 = \mathfrak{A}_1 \times (0)$, $u_2 = (0) \times \mathfrak{A}_2$, $u_3 = \mathfrak{A}_1 \times \mathfrak{A}_2$, $u_4 = (0) \times J$, $u_5 = \mathfrak{A}_1 \times J$, $v_1 = \mathfrak{S}_1 \times (0)$, $v_2 = (0) \times \mathfrak{S}_2$, $v_3 = \mathfrak{A}_1 \times \mathfrak{S}_2$, $v_4 = \mathfrak{S}_1 \times \mathfrak{A}_2 \in W(\mathfrak{S})^*$. Since $u_i \cap Ann(v_j) \neq 0$ and $v_j \cap Ann(u_i) \neq 0$, $K_{5,4}$ is a subgraph of $SAG(\mathfrak{S})$ generated by the set $\{u_1, \ldots, u_5\} \cup \{v_1, \ldots, v_4\}$. Thus, by using of Lemma 4.1, $\overline{\gamma}(SAG(\mathfrak{S})) > 2$. Hence $\mathfrak{S}_1$ has only one non-trivial ideal $\mathfrak{A}_1$. In a similar manner, it can be demonstrated that $\mathfrak{A}_2$ is the only non-trivial ideal of $\mathfrak{S}_2$. Thus, again by using of Theorem 4.3, $\overline{\gamma}(SAG(\mathfrak{S})) = 1$. Hence, considering all the cases, we can conclude that $\overline{\gamma}(SAG(\mathfrak{S})) \neq 2$. □

Lemma 4.7. *Let $\mathfrak{S} = \mathfrak{D}_1 \times \mathfrak{D}_2 \times \cdots \times \mathfrak{D}_m \times \mathfrak{S}_1 \times \mathfrak{S}_2 \times \cdots \times \mathfrak{S}_n$ be a CRU, where $\mathfrak{D}_i$ is a field, $(\mathfrak{S}_j, \mathfrak{A}_j)$ is Artinian local ring with $\mathfrak{A}_j \neq 0$ and $m, n \geq 1$. Then $\overline{\gamma}(SAG(\mathfrak{S})) = 2 \iff$ one of the following conditions is satisfied:*

1. *$\mathfrak{S} = \mathfrak{D}_1 \times \mathfrak{S}_1$, where $\mathfrak{A}_1$, $\mathfrak{A}_1^2$, and $\mathfrak{A}_1^3$ constitute the only non-trivial ideals of $\mathfrak{S}_1$.*

Proof. Suppose $\overline{\gamma}(SAG(\mathfrak{S})) = 2$. If $n \geq 2$, then by Lemma 4.6, $\overline{\gamma}(SAG(\mathfrak{S})) \neq 2$, a contradiction. Hence $n = 1$. If $m \geq 3$, then by Lemma 4.5, again $\overline{\gamma}(SAG(\mathfrak{S})) > 2$. Hence $m \leq 2$. Let us examine the following cases:

Case(1): Suppose $m = 2$ and $n = 1$. Consider $a_1 = \mathfrak{D}_1 \times (0) \times \mathfrak{A}_1$, $a_2 = (0) \times (0) \times \mathfrak{A}_1$, $a_3 = (0) \times (0) \times \mathfrak{S}_1$, $a_4 = \mathfrak{D}_1 \times (0) \times \mathfrak{S}_1$, $b_1 = \mathfrak{D}_1 \times \mathfrak{D}_2 \times (0)$, $b_2 = \mathfrak{D}_1 \times \mathfrak{D}_2 \times \mathfrak{A}_1$, $b_3 = (0) \times \mathfrak{D}_2 \times (0)$, $b_4 = (0) \times \mathfrak{D}_2 \times \mathfrak{A}_1$,

$b_5 = (0) \times \mathfrak{D}_2 \times \mathfrak{S}_1 \in W(\mathfrak{S})^*$. Since $a_i \cap Ann(b_j) \neq (0)$ and $Ann(a_i) \cap v_j \neq (0)$, the graph $K_{5,4}$ is a subgraph of $SAG(\mathfrak{S})$ generated by the set $\{a_1, \ldots, a_5\} \cup \{b_1, \ldots, b_4\}$. Thus, by using of Lemma 4.1, $\overline{\gamma}(SAG(\mathfrak{S})) > 2$, a contradiction.

Case(2): Suppose $n = m = 1$. Set η_1 to be the nilpotency index of $\mathfrak{S}_1$. Suppose $\eta_1 \geq 5$. Consider $x_1 = (0) \times \mathfrak{A}_1^{\eta_1 - 1}$, $x_2 = (0) \times \mathfrak{A}_1^{\eta_1 - 2}$, $x_3 = (0) \times \mathfrak{A}_1^{\eta_1 - 3}$, $x_4 = (0) \times \mathfrak{A}_1^{\eta_1 - 4}$, $y_1 = \mathfrak{D}_1 \times (0)$, $y_2 = (0) \times \mathfrak{S}_1$, $y_3 = \mathfrak{D}_1 \times \mathfrak{A}_1^{\eta_1 - 1}$, $y_4 = \mathfrak{D}_1 \times \mathfrak{A}_1^{\eta_1 - 2}$, $y_5 = \mathfrak{D}_1 \times \mathfrak{A}_1^{\eta_1 - 3} \in W(\mathfrak{S})^*$. Since $x_i \cap Ann(y_j) \neq 0$ and $y_j \cap Ann(x_i) \neq 0$, the graph $K_{4,5}$ is a subgraph of $SAG(\mathfrak{S})$ generated by the set $\{x_1, \ldots, x_4\} \cup \{y_1, \ldots, y_5\}$. Thus, by using of Lemma 4.1, $\overline{\gamma}(SAG(\mathfrak{S})) > 2$, a contradiction. Hence $\eta_1 \leq 4$. If $\eta_1 \leq 3$, then by Theorems 2.1 and 4.3, $\overline{\gamma}(SAG(\mathfrak{S})) \leq 1$, a contradiction. Hence $\eta_1 = 4$.

Let $I \in \mathbb{I}(\mathfrak{S})^*$ such that $I \neq \mathfrak{A}_1, \mathfrak{A}_1^2, \mathfrak{A}_1^3$. Consider $c_1 = (0) \times \mathfrak{A}_1$, $c_2 = (0) \times \mathfrak{A}_1^2$, $c_3 = (0) \times \mathfrak{A}_1^3$, $c_4 = (0) \times I$, $d_1 = \mathfrak{D}_1 \times (0)$, $d_2 = (0) \times \mathfrak{S}_1$, $d_3 = \mathfrak{D}_1 \times \mathfrak{A}_1$, $d_4 = \mathfrak{D}_1 \times \mathfrak{A}_1^2$, $d_5 = \mathfrak{D}_1 \times \mathfrak{A}_1^3 \in W(\mathfrak{S})^*$. Since $c_i \cap Ann(d_j) \neq 0$ and $d_j \cap Ann(c_i) \neq 0$, the graph $K_{4,5}$ is a subgraph of $SAG(\mathfrak{S})$ generated by the set $\{c_1, \ldots, c_4\} \cup \{d_1, \ldots, d_5\}$. Thus, by using of Lemma 4.1, $\overline{\gamma}(SAG(\mathfrak{S})) > 2$, a contradiction. Consequently, $\mathfrak{A}_1$, $\mathfrak{A}_1^2$, and $\mathfrak{A}_1^3$ represent the only non-trivial ideals of $\mathfrak{S}_1$.
The converse follows from Figure 3.
□

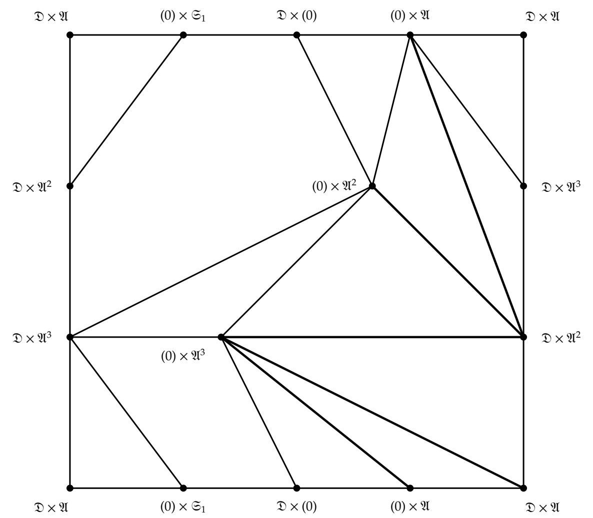

Figure 3. Embedding of $SAG(\mathfrak{D} \times \mathfrak{S}_1)$ on Klein-bottle,
where the non-trivial ideals of $\mathfrak{S}_1$ are exclusively $\mathfrak{A}$, $\mathfrak{A}^2$, and $\mathfrak{A}^3$.

In conclusion, we can summarize the main result of this section.

Theorem 4.8. *Let $\mathfrak{S}$ be an Artinian CRU. Then $\overline{\gamma}(SAG(\mathfrak{S})) = 2 \Longleftrightarrow$ one of the following conditions is satisfied:*

1. *$\mathfrak{S} \cong \mathfrak{D} \times \mathfrak{S}_1$, where $\mathfrak{D}$ is a field and $(\mathfrak{S}_1, \mathfrak{A})$ represents a local ring with $\mathfrak{A}$, $\mathfrak{A}^2$, and $\mathfrak{A}^3$ being the only non-trivial ideals of $\mathfrak{S}_1$.*

Proof. The proof follows from Lemmas 4.4, 4.5, 4.6 and 4.7. □

5. Book thickness of $SAG(\mathfrak{S})$

In this section, we investigate the book thickness of the graph $SAG(\mathfrak{S})$ with a genus of at most one. Firstly, we determine the book thickness of planar $SAG(\mathfrak{S})$ derived from the rings presented in Theorems 2.1, and we demonstrate that all planar $SAG(\mathfrak{S})$ have a book thickness of at most two.

The results presented in [9] will assist us in proving the main results of this section.

Lemma 5.1. *[9, Theorem 2.5] Let G be a connected graph. The following equivalences are valid:*

1. *G has book thickness zero if and only if it is a path.*
2. *G has book thickness less than or equal to* 1 *if and only if it is outerplanar.*

Lemma 5.2. *[9, Theorems 3.4, 3.5, 3.6]*

1. $bt(K_p)$ *is given by* $\left\lceil \frac{p}{2} \right\rceil$, *where* $p \geq 3$.
2. $bt(K_{p,q}) = p$, *where* $p \leq q$ *with* $q \geq p^2 - p + 1$.
3. $bt(K_{3,3}) = 3$ *and* $bt(K_{p,p}) = p - 1$, *where* $p \geq 4$.

In the subsequent result, we assess the book thickness of $SAG(\mathfrak{S})$ in the context of planar graphs.

Theorem 5.3. *Let* $\mathfrak{S}$ *be an Artinian CRU such that* $SAG(\mathfrak{S})$ *is planar. Then the following hold:*

1. $bt(SAG(\mathfrak{S})) = 0 \Longleftrightarrow$ *either* $\mathfrak{S}$ *is a local ring that has no more than two non-trivial ideals or* $\mathfrak{S} \cong \mathfrak{D}_1 \times \mathfrak{D}_2$, *where* $\mathfrak{D}_1$ *and* $\mathfrak{D}_2$ *are fields.*
2. $bt(SAG(\mathfrak{S})) = 1 \Longleftrightarrow$ *either* $\mathfrak{S}$ *is either a local ring with three non-trivial ideals or isomorphic to* $\mathfrak{D} \times \mathfrak{S}_1$, *where* $\mathfrak{D}$ *is a field and* $(\mathfrak{S}_1, \mathfrak{A})$ *is a local ring with a unique non-trivial ideal* $\mathfrak{A}$.
3. $bt(SAG(\mathfrak{S})) = 2 \Longleftrightarrow$ *either* $\mathfrak{S}$ *is a local ring that has precisely four non-trivial ideals or* $\mathfrak{S} \cong \mathfrak{D}_1 \times \mathfrak{D}_2 \times \mathfrak{D}_3$, *where each* $\mathfrak{D}_i$ *is a field.*

Proof. We will consider the rings identified in Theorem 2.1. If $\mathfrak{S}$ is a local ring with $|\mathbb{I}(\mathfrak{S})^*| \leq 4$. If $|\mathbb{I}(\mathfrak{S})^*| \leq 2$, then $SAG(\mathfrak{S}) \cong K_1$ or $K_2(= P_2)$. Thus, by Lemma 5.1, $bt(SAG(\mathfrak{S})) = 0$. If $|\mathbb{I}(\mathfrak{S})^*| = 3$, then $SAG(\mathfrak{S}) \cong K_3$. Thus, by Lemma 5.2, $bt(SAG(\mathfrak{S})) = 1$. If $|\mathbb{I}(\mathfrak{S})^*| = 4$, then $SAG(\mathfrak{S}) \cong K_4$. Thus, by Lemma 5.2, $bt(SAG(\mathfrak{S})) = 2$. If $\mathfrak{S} \cong \mathfrak{D}_1 \times \mathfrak{D}_2$, where $\mathfrak{D}_1$ and $\mathfrak{D}_2$ are fields, then $SAG(\mathfrak{S}) \cong K_2(= P_2)$. Thus, by Lemma 5.1, $bt(SAG(\mathfrak{S})) = 0$. If $\mathfrak{S} \cong \mathfrak{D} \times \mathfrak{S}_1$, where $\mathfrak{D}$ denotes a field and $(\mathfrak{S}_1, \mathfrak{A})$ represents a local ring with its only non-trivial ideal being $\mathfrak{A}$, then $SAG(\mathfrak{S}) \cong C_4$ and hence 1-book embedding of $SAG(\mathfrak{S})$ is given in Figure 4. Finally, if $\mathfrak{S} \cong \mathfrak{D}_1 \times \mathfrak{D}_2 \times \mathfrak{D}_3$, where each $\mathfrak{D}_i$ is a field, then 2-book embedding of $SAG(\mathfrak{S})$ is given in Figure 5. □

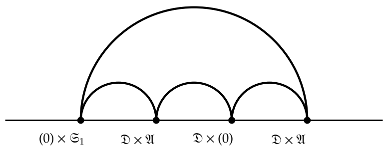

Figure 4. 1-book embedding of $SAG(\mathfrak{D} \times \mathfrak{S}_1)$,
where $\mathfrak{A}$ is the only non-trivial ideal associated with $\mathfrak{S}_1$.

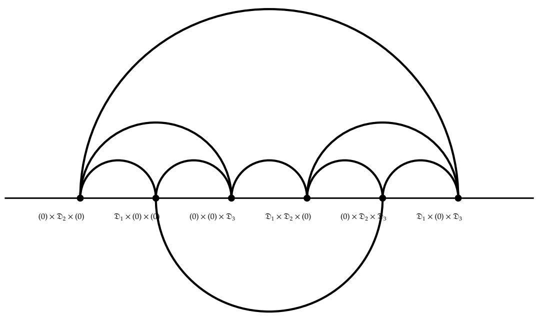

Figure 5. 2-book embedding of $SAG(\mathfrak{D}_1 \times \mathfrak{D}_2 \times \mathfrak{D}_3)$.

In the upcoming result, we establish the book thickness of $SAG(\mathfrak{S})$ with respect to the toroidal graph.

Theorem 5.4. *Let* $\mathfrak{S}$ *be an Artinian CRU with* $\gamma(SAG(\mathfrak{S})) = 1$. *Then the following conditions is satisfied:*

1. $bt(SAG(\mathfrak{S})) = 3 \Longleftrightarrow$ *one of the following conditions is satisfied:*

(a) $\mathfrak{S}$ *is a local ring with* $5 \leq |\mathbb{I}(\mathfrak{S})^*| \leq 6$.

(b) $\mathfrak{S} \cong \mathfrak{S}_1 \times \mathfrak{S}_2$, *where each* $(\mathfrak{S}_i, \mathfrak{A}_i)$ *represents a local ring that has exactly one non-trivial ideal, denoted by* $\mathfrak{A}_i$.

(c) $\mathfrak{S} \cong \mathfrak{D} \times \mathfrak{S}_1$, *where* $\mathfrak{D}$ *denotes a field and* $(\mathfrak{S}_1, \mathfrak{A})$ *represents a local ring that has non-trivial ideals* $\mathfrak{A}$ *and* $\mathfrak{A}^2$.

2. $bt(SAG(\mathfrak{S})) = 4 \iff \mathfrak{S}$ *is either a local ring with 7 non-trivial ideals or isomorphic to* $\mathfrak{D} \times \mathfrak{S}_1$, *where* $\mathfrak{D}$ *represents a field and* $(\mathfrak{S}_1, \mathfrak{A})$ *is a local ring having non-trivial ideals* $\mathfrak{A}$, $\mathfrak{A}^2$, *and* $\mathfrak{A}^3$.

Proof. We need to examine the rings specified in Theorem 3.4. If $\mathfrak{S}$ is a local ring with $5 \leq |\mathbb{I}(\mathfrak{S})^*| \leq 6$, then by Theorem 1.2, $SAG(\mathfrak{S}) \cong K_5$ or K_6. Thus, by Lemma 5.2, $bt(SAG(\mathfrak{S})) = 3$. If $\mathfrak{S}$ is a local ring with $|\mathbb{I}(\mathfrak{S})^*| = 7$, then by Theorem 1.2, $SAG(\mathfrak{S}) \cong K_7$. Thus, by Lemma 5.2, $bt(SAG(\mathfrak{S})) = 4$.If $\mathfrak{S} \cong \mathfrak{S}_1 \times \mathfrak{S}_2$, where each $(\mathfrak{S}_i, \mathfrak{A}_i)$ is local ring with unique non-trivial ideal $\mathfrak{A}_i$, then 3-book embedding of $SAG(\mathfrak{S})$ is given in Figure 6. $\mathfrak{S} \cong \mathfrak{D} \times \mathfrak{S}_1$, where $\mathfrak{D}$ denotes a field, and $(\mathfrak{S}_1, \mathfrak{A})$ represents a local ring characterized by its non-trivial ideals $\mathfrak{A}$ and $\mathfrak{A}^2$, then 3-book embedding of $SAG(\mathfrak{S})$ is given in Figure 7. If $\mathfrak{S} \cong \mathfrak{D} \times \mathfrak{S}_1$, where $\mathfrak{D}$ denotes a field and $(\mathfrak{S}_1, \mathfrak{A})$ is a local ring characterized by its non-trivial ideals $\mathfrak{A}$, $\mathfrak{A}^2$, and $\mathfrak{A}^3$, then 4-book embedding of $SAG(\mathfrak{S})$ is shown in Figure 8. □

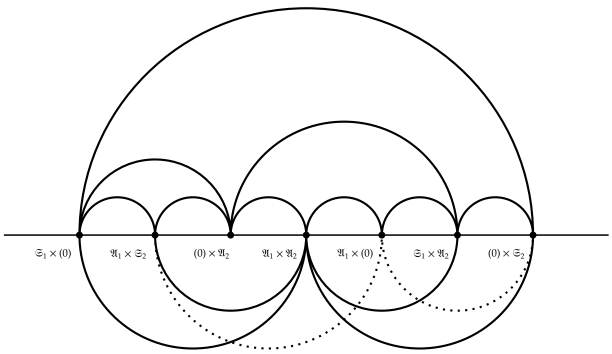

Figure 6. 3-book embedding of $SAG(\mathfrak{S}_1 \times \mathfrak{S}_2)$,

where $\mathfrak{A}_i$ is the only non-trivial ideal associated with $\mathfrak{S}_i$.

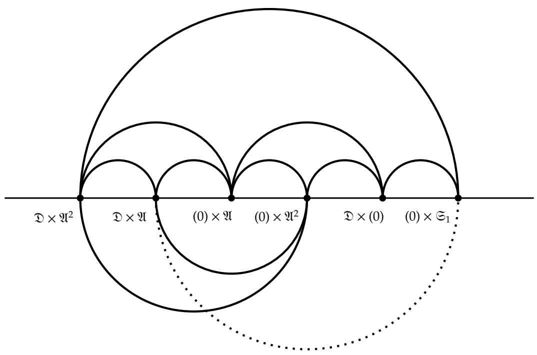

Figure 7. 3-book embedding of $SAG(\mathfrak{D} \times \mathfrak{S}_1)$,

where the only non-trivial ideals present in $\mathfrak{S}_1$ are $\mathfrak{A}$ and $\mathfrak{A}^2$.

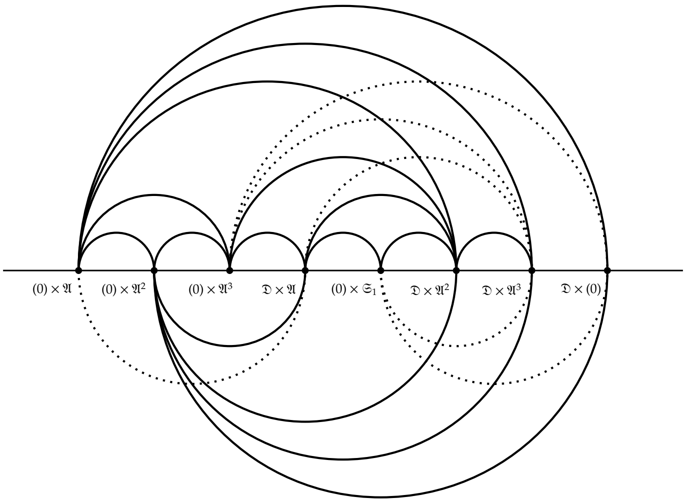

Figure 8. 4-book embedding of $SAG(\mathfrak{D}\times\mathfrak{S}_1)$,

where the non-trivial ideals of $\mathfrak{S}_1$ are exclusively $\mathfrak{A}$, $\mathfrak{A}^2$, and $\mathfrak{A}^3$.

Conclusion

In this article, we have explored various aspects of graph theory as applied to the study of commutative Artinian rings with identity. In the first section, we established some foundational definitions related to rings and graph theory, introducing key graph concepts such as the zero-divisor graph $\Gamma(\mathfrak{S})$, the annihilating-ideal graph $AG(\mathfrak{S})$, and the strong annihilating-ideal graph $SAG(\mathfrak{S})$. Subsequently, we classified the Artinian CRU for which $SAG(\mathfrak{S})$ has an outerplanarity index of 2, while also determining the inner vertex number of $SAG(\mathfrak{S})$ for these rings. In another section, we focused on identifying the Artinian CRU for which $SAG(\mathfrak{S})$ is double toroidal, establishing that $\gamma(SAG(\mathfrak{S}))=2$. Further, we classified the Artinian CRU for which $SAG(\mathfrak{S})$ corresponds to a Klein bottle, showing that $\overline{\gamma}(SAG(\mathfrak{S}))=2$. Finally, we investigated the book thickness of the graph $SAG(\mathfrak{S})$ with a genus of at most one, providing a comprehensive analysis of its structural properties. The findings in this paper contribute to the understanding of the relationships between algebraic structures and their corresponding graph representations, paving the way for future research in this area.

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