Filomat 39:18 (2025), 6345–6360
https://doi.org/10.2298/FIL2518345K

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Insights into the topological nature of neutrosophic quasi-normed spaces: Exploring open mapping and closed graph theorems

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Abstract. We are going to explore neutrosophic quasi-normed space. We will define what a neutrosophic quasi-norm is and show an example. As we know, for $p \in (0, \infty)$ space (ℓ^p, d) is a paratopological vector space but it does not possess a norm such that topology generated by this norm is compatible with topology generated by metric d. We give an example that topology generated by neutrosophic quasi norm is compatible with topology generated by metric d. In this paper, we will prove the open mapping theorem and the closed graph theorem for neutrosophic quasi-normed space.

1. Introduction

In 1965, L. A. Zadeh unveiled a pioneering theory that revolutionized the understanding of sets, introducing the innovative concept of fuzzy sets which expanded the traditional crisp set theory [38]. The exploration of fuzzy norms within linear spaces began with Katsaras' initial proposition [19], later augmented by Felbin's alternative definition in 1992, which incorporated a metric of the Kaleva and Seikkala type [13, 18]. Cheng and Mordeson [12] further delved into this realm in 1994, while recent studies by Xiao et al. [37] delved into the intricate relationships between the axioms of KM fuzzy normed spaces and KM(Kramosil and Michalek) fuzzy metric spaces.

Building upon this foundation, Bag and Samanta [8], Cheng and Mordeson[12] introduced a nuanced variation of fuzzy norms tailored for specific applications. The evolution of fuzzy functional analysis, as discussed in references [1] to [32], has been heavily influenced by these advancements. Alegre and Romaguera [2, 8] proposed the concept of fuzzy quasi-norms with a general t-norm, diverging from the traditional symmetry of fuzzy norms. Furthermore, they extended the application o f fuzzy quasi-norms to characterize paratopological vector spaces [3].

In their work cited in [4], Alegre and Romaguera presented notable findings, including the uniform boundedness theorem, within fuzzy quasi-normed fields. Gao et al. recently contributed a decomposition t heorem for fuzzy quasi-norms [14]. Hussein and Al-Basri explored the completion of quasi-fuzzy normed algebras over fuzzy fields [17]. As noted in [2], fuzzy quasi-normed spaces provide a fitting framework for

2020 *Mathematics Subject Classification*. Primary 40A05; Secondary 40C05.

Keywords. Fuzzy norm, Fuzzy quasi-norm space, $(\mathcal{N}_f)$, fuzzy quasi-norm $(\mathcal{N}_{f_q})$, intuitionistic fuzzy norm, neutrosophic norm $(\mathcal{N}_\mathcal{N})$, neutrosophic quasi norm $(\mathcal{N})$, Neutrosophic quasi-normed space, paratopological vector space.

Received: 21 August 2024; Revised: 06 May 2025; Accepted: 03 June 2025

Communicated by Ljubiša D. R. Kočinac

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analyzing the complexity of various exponential time algorithms and are pivotal in discussions concerning approximation theory and theoretical computer science.

The pursuit of further exploration into fuzzy quasi-normed spaces remains imperative. The open mapping theorem in functional analysis holds profound significance, a fact underscored by Wu and Li recent discussion of it within the context of fuzzy quasi-normed spaces [36].

In 1984, Atanassov [7] made a significant contribution to the realm of fuzzy sets by introducing intuitionistic fuzzy sets. This groundbreaking concept introduced a novel membership function designed to quantify degrees of non-belongingness . Following this pivotal development, the exploration of intuitionistic fuzzy structures continued to evolve. In 2004, Park delved into the exploration of intuitionistic fuzzy metric spaces, further expanding the scope and applications of these fuzzy structures [30]. Building upon Park's work, Saddati and Park made significant strides in 2006 with their groundbreaking research on intuitionistic fuzzy normed spaces, pushing the boundaries of this domain and opening new avenues for exploration [31].

Neutrosophic set theory, as a paradigm distinct from fuzzy set theory, introduces a revolutionary extension by incorporating the notion of neutrality as a third indeterminent component alongside truth and falsehood. In contrast to the binary nature of traditional set theory and gradation of membership degrees in fuzzy sets, neutrosophic sets provide a richer framework that acknowledges the inherent ambiguity and uncertainty in data and decision making processes. Within, this framework, elements can exhibit varying degrees of inclusion, exclusion, or neutrality, enabling a more nuanced representation complex phenomena. This expansion not only enhances our ability to model analyze real-world system but also opens up new avenues for exploring indeterminacy and vagueness in diverse fields ranging from artificial intelligence and decision sciences to engineering and social sciences. Recent studies by Aral, Kandemir and Et M [5, 6] and Chandan et al. [11] have further advanced the understanding of neutrosophic normed space.

Expanding beyond intuitionistic fuzzy sets, the concept of neutrosophic sets emerged as a generalization of fuzzy structures. Introduced by Smarandache et al., neutrosophic sets provided a broader framework for dealing with uncertainty and imprecision [35], [34]. Furthering the study of neutrosophic sets, Kirişci and Şimşek defined a metric and a norm tailored specifically for neutrosophic sets, alongside an exploration of their topological properties [24], [25]. These developments expanded the toolkit available for handling complex and uncertain data, offering new perspectives on fuzzy systems.

Recent studies by Khan and Faisal [20–23] have advanced the understanding of neutrosophic and intuitionistic fuzzy normed spaces through Tauberian theorems, Zweier Sequences, topological analysis and finding new sequence spaces via Jordan totient operator. In the ongoing pursuit of advancing the understanding and application of fuzzy structures, this article introduces the c oncept of neutrosophic quasi-normed spaces. Within this framework, the open mapping theorem and closed graph theorem are established, representing significant milestones in the field's progress. This contribution marks a notable advancement, offering enhanced capabilities for modeling and analyzing uncertain and imprecise data, thus furthering the practical utility of fuzzy theory in various domains.

2. Preliminaries

Definition 2.1. ([33]) A binary operation $\circ : [0,1] \times [0,1] \to [0,1]$ qualifies as a continuous $t-norm$ if it adheres to the following properties:

(a) $s \circ t = t \circ s$ for all $s, t \in [0,1]$;
(b) $s \circ (t \circ u) = (s \circ t) \circ u$ for all $s, t, u \in [0,1]$;
(c) $s \circ t \leq u \circ d$ whenever $s \leq t$ and $u \leq d$ for all $s, t, u, d \in [0,1]$;
(d) $s \circ 1 = s$ for all $s \in [0,1]$;
(e) $\circ$ is continuous.

Definition 2.2. ([33]) A binary operation $\star : [0,1] \times [0,1] \to [0,1]$ qualifies as a continuous $t-conorm$ if it adheres to the following properties:

(a) $s \star t = t \star s$ for all $s, t \in [0,1]$;

(b) $s \star (t \star u) = (s \star t) \star u$ for all $s, t, u \in [0, 1]$;
(c) $s \star t \le u \star d$ whenever $s \le u$ and $t \le d$ for all $s, t, u, d \in [0, 1]$;
(d) $s \star 0 = s$ for all $s \in [0, 1]$;
(e) $\star$ is continuous.

Example 2.3. Let $\circ$ be a binary operation on $[0, 1]$ and defined as $\circ(s, u) = \min\{s, u\}$ for all $s, u \in [0, 1]$. Then $\circ$ is a continuous $t-norm$. Usually this $t-norm$ is denoted by $\wedge$.

Example 2.4. Let $\star$ be a binary operation on $[0, 1]$ and defined as $\star(s, u) = \max\{s, u\}$ for all $s, u \in [0, 1]$. Then $\star$ is a continuous $t-conorm$. Usually this $t-conorm$ is denoted by $\vee$.

Proposition 2.5. ([16]) *Suppose $\circ$ and $\star$ function as continuous $t-norm$ and $t-conorm$, respectively. Then*

(a) *If $0 < \check{c}_1 < \check{c}_2 < 1$, there exists $\check{c}_3, \check{c}_4 \in (0, 1)$ such that $\check{c}_1 \circ \check{c}_3 \ge \check{c}_2$ and $\check{c}_1 \ge \check{c}_4 \star \check{c}_2$.*
(b) *If $0 < \check{c}_5 < 1$, then there exists $\check{c}_6, \check{c}_7 \in (0, 1)$ such that $\check{c}_6 \circ \check{c}_6 \ge \check{c}_5$ and $\check{c}_7 \star \check{c}_7 \le \check{c}_5$.*

Remark 2.6. ([36]P) Let $p \in (0, \infty)$ and ℓ^p be the collection of all $p-summable$ sequences i.e.;

$$\ell^p = \left\{x = (x_n) \in \omega : \sum_{n=1}^{\infty} |x_n|^p < \infty\right\}. \tag{1}$$

We know that $d(x, y) = \sum_{n=0}^{\infty} |x_n - y_n|$, is a translation invariant metric on ℓ^p for all $p \in (0, \infty)$. Let τ_d is the topology generated by metric d. The space (ℓ_p, τ_d) is a topological vector space. But for $0 < p < 1$, ℓ^p does not possess a norm; such that topology generated by this norm is compatible with τ_d. For $1 \le p < \infty$, ℓ^p is a norm linear space with norm defined as follows

$$\|\xi\|_p = \left(\sum_{n=1}^{\infty} |\xi_n|^p\right)^{1/p}.$$

Each neutrosophic norm $\mathcal{N}_\mathcal{N}$ induces a T_0 topology $\tau_{\mathcal{N}_\mathcal{N}}$ on X, generated by the base of open balls

$$\mathfrak{B}(x) = \{B_{\mathcal{N}_\mathcal{N}}(x, \check{c}, \check{t}) : x \in X,\ \check{t} > 0 \text{ and } \check{c} \in (0, 1)\} \tag{2}$$

where

$$B_{\mathcal{N}_\mathcal{N}}(x, \check{c}, \check{t}) = \{y \in X : \mathcal{P}(x - y, \check{t}) > 1 - \check{c},\ \mathcal{Q}(x - y, \check{t}) < \check{c}, \mathcal{R}(x - y, \check{t}) < \check{c}\}. \tag{3}$$

From equation 2, we can define base of open balls center at Θ(origin)

$$\mathfrak{B}(\Theta) = \{B_{\mathcal{N}_\mathcal{N}}(\Theta, \check{c}, \check{t}) : \Theta \in X,\ \check{t} > 0 \text{ and } \check{c} \in (0, 1)\} \tag{4}$$

where

$$B_{\mathcal{N}_\mathcal{N}}(\Theta, \check{c}, \check{t}) = \{y \in X : \mathcal{P}(y, \check{t}) > 1 - \check{c},\ \mathcal{Q}(y, \check{t}) < \check{c},\ \mathcal{R}(y, \check{t}) < \check{c}\}. \tag{5}$$

Definition 2.7. ([36]) A paratopological vector space is denoted by the 4-tuple $(X, +, ., \tau)$, wherein (X, τ) constitutes a T_0 topology on X and the addition operation $+$ is continuous. For any neighborhood B of $\check{t}\xi$, where $\xi \in X$ and $\check{t} \ge 0$, there exists a neighborhood B' of ξ and a positive $\epsilon > 0$ such that the interval $[\check{t}, \check{t} + \epsilon)B' \subseteq B$.

Theorem 2.8. ([36]P) *In the context of a FQNS $\left(X, \mathcal{N}_{f_q}, \circ\right)$ it follows that $\left(X, \tau_{\mathcal{N}_{f_q}}, \circ\right)$ constitutes a quasi-metrizable paratopological vector space.*

To know more about paratopological vector spaces see [3]. Simply paratopological vector space $\left(X,+,.,\tau\right)$ is represented by $\left(X,\tau\right)$, if no confusion arises.

Remark 2.9. ([36]) If $t-norm$ is chosen as $\circ(a,b)=\min\{a,b\}$, then $B_{\mathcal{N}}(\Theta)$ is convex.

Definition 2.10. ([36]) Let M be a subset of real vector space X. Then

(a) M is semibalanced if $x\in M$, $\check{c}x\in M$ whenever $\check{c}\in[0,1]$.
(b) M is absorbing if for each $x\in X$, there is $\check{c}_0>0$ such that $\check{c}_0x\in M$.

Definition 2.11. ([28]) Mapping $L:X\to Y$ is open map, if the set $F(B)$ is open in Y for every open set B in X.

Theorem 2.12. ([36]) *Let $\left(X,\mathcal{N}_{f_q},\circ\right)$ and $(Y,\mathcal{N}'_{f_q},\circ')$ be FQNS. Assume that $\left(X,\mathcal{N}_{f_q},\circ\right)$ is right $\mathcal{N}_{f_q}$-complete and $(Y,\mathcal{N}'_{f_q},\circ')$ is Hausdorff and of the half second category. If $T:X\to Y$ is a linear, surjective, and continuous mapping, then T is open.*

3. Main results

Definition 3.1. Let X be a real vector space, and $\circ,\star$ be continuous $t-norm$ and $t-conorm$ respectively. Let $\mathcal{P},\mathcal{Q}$ and $\mathcal{R}$ be fuzzy sets on $X\times[0,\infty)$. Then $\left(X,\mathcal{P},\mathcal{Q},\mathcal{R},\circ,\star\right)$ is said to be neutrosophic quasi norm on X, if

(a) $\mathcal{P}(\mathsf{x},0)=0;$ for all $\mathsf{x}\in X$
(b) $\mathcal{P}(\mathsf{x},\check{t})=\mathcal{P}(-\mathsf{x},\check{t})=1$ for all $\check{t}>0\iff \mathsf{x}=0;$
(c) $\mathcal{P}(\alpha x,\check{t})=\mathcal{P}\left(\mathsf{x},\dfrac{\check{t}}{\alpha}\right)$ for all $\alpha>0;$
(d) $\mathcal{P}(\mathsf{x},\check{t})\circ\mathcal{P}(\mathsf{y},\check{s})\leq\mathcal{P}(\mathsf{x}+\mathsf{y},\check{t}+\check{s})$ for all $\mathsf{x},\mathsf{y}\in X$ and $\check{s},\check{t}>0;$
(e) $\mathcal{P}(\mathsf{x},-):[0,\infty)\to[0,1]$ is left continuous;
(f) $\lim_{\check{t}\to\infty}\mathcal{P}(\mathsf{x},\check{t})=1;$
(g) $\mathcal{Q}(\mathsf{x},0)=1;$ for all $\mathsf{x}\in X$
(h) $\mathcal{Q}(\mathsf{x},\check{t})=\mathcal{Q}(-\mathsf{x},\check{t})=0$ for all $\check{t}>0\iff \mathsf{x}=0;$
(i) $\mathcal{Q}(\alpha\mathsf{x},\check{t})=\mathcal{Q}\left(\mathsf{x},\dfrac{\check{t}}{\alpha}\right)$ for all $\alpha>0;$
(j) $\mathcal{Q}(\mathsf{x},\check{t})\star\mathcal{Q}(\mathsf{y},\check{s})\geq\mathcal{Q}(\mathsf{x}+\mathsf{y},\check{t}+\check{s})$ for all $\mathsf{x},\mathsf{y}\in X$ and $\check{s},\check{t}>0;$
(k) $\mathcal{Q}(\mathsf{x},-):[0,\infty)\to[0,1]$ is right continuous;
(l) $\lim_{\check{t}\to\infty}\mathcal{Q}(\mathsf{x},\check{t})=0;$
(m) $\mathcal{R}(\mathsf{x},0)=1;$ for all $\mathsf{x}\in X,$
(n) $\mathcal{R}(\mathsf{x},\check{t})=\mathcal{R}(-\mathsf{x},\check{t})=0$ if and only $\mathsf{x}=\mathsf{y}$, for all $\check{t}>0$
(o) $\mathcal{R}\left(\alpha\mathsf{x},\check{t}\right)=\mathcal{R}\left(\mathsf{x},\dfrac{\check{t}}{\alpha}\right)$ for $\alpha>0;$
(p) $\mathcal{R}(\mathsf{x},\check{t})\star\mathcal{Q}(\mathsf{y},\check{s})\geq\mathcal{Q}(\mathsf{x}+\mathsf{y},\check{t}+\check{s});$
(q) $\mathcal{R}(\mathsf{x},-):[0,\infty)\to[0,1]$ is right continuous.

A neutrosophic quasi norm on X is neutrosophic norm if $\mathcal{P}(\alpha\mathsf{x},\check{t})=\mathcal{P}\left(\mathsf{x},\dfrac{\check{t}}{|\alpha|}\right)$, $\mathcal{Q}(\alpha\mathsf{x},\check{t})=\mathcal{Q}\left(\mathsf{x},\dfrac{\check{t}}{|\alpha|}\right)$ and $\mathcal{R}\left(\alpha\mathsf{x},\check{t}\right)=\mathcal{R}\left(\mathsf{x},\dfrac{\check{t}}{|\alpha|}\right)$ for $\alpha\neq0$. We will denote neutrosophic quasi norm by $\mathcal{N}=(\mathcal{P},\mathcal{Q},\mathcal{R})$. If $\mathcal{N}$ is neutrosophic quasi norm on X, than $\mathcal{N}^{-1}$ is also neutrosophic quasi norm, where $\mathcal{N}^{-1}$ is $(\mathcal{P}^{-1}, \mathcal{Q}^{-1}, \mathcal{R}^{-1})$ and $\mathcal{P}^{-1}(\mathsf{x}, \check{t}) = \mathcal{P}(-\mathsf{x}, \check{t})$, $\mathcal{Q}^{-1}(\mathsf{x}, \check{t}) = \mathcal{Q}(-\mathsf{x}, \check{t})$, $\mathcal{R}^{-1}(\mathsf{x}, \check{t}) = \mathcal{R}(-\mathsf{x}, \check{t})$. Moreover, $\mathcal{N}^s$ defined as

$$\mathcal{N}^s = \Big(\min\{\mathcal{P}(\mathsf{x}, \check{t}), \mathcal{P}(-\mathsf{x}, \check{t})\}, \max\{\mathcal{Q}(\mathsf{x}, \check{t}), \mathcal{Q}(-\mathsf{x}, \check{t})\},$$

$$\max\{\mathcal{R}(\mathsf{x}, \check{t}), \mathcal{R}(-\mathsf{x}, \check{t})\}\Big)$$

is neutrosophic norm on X.

Here, we present an exemplary instance of a FQNS, strategically adapted from the literature, spanning references [16], [33], [3], [2], [4], [14] ,[17], and [15]. In their pursuit of a systematic framework to dissect the complexities inherent in certain exponential time algorithms, García-Raffi et al. [15] introduced what they termed the dual p-complexity (quasi-normed) space, denoted as (C_p^*, σ_p).

In this setup, C_p^* encompasses functions $g : \mathbb{N} \to \mathbb{R}$ that satisfy the condition $\sum_{n=0}^{\infty} \frac{1}{2^n}|g(n)|^p < +\infty$. The weight function $\sigma_p(g)$ is crafted as;

$$\sigma_p(g) = \left(\sum_{n=0}^{\infty} (2^{-n} \max\{g(n), 0\})^p\right)^{1/p}$$

catering to all g within C_p^*. Consider the set ℓ^p, comprising $x = (x_n)_{n\in\mathbb{N}}$ for which $\sum_{n=0}^{\infty} |x_n|^p < \infty$. Notably, García-Raffi et al. [15] demonstrated an isometric isomorphism between (C_p^*, σ_p) and the quasi-normed linear space $(\ell^p, \|\cdot\|_{+p})$, where $1 \le p < \infty$. Here, $\|x\|_{+p} = (\sum_{n=0}^{\infty}(x_n \vee 0)^p)^{1/p}$ for all x in ℓ^p.

In the intriguing scenario of $0 < p < 1$, the resulting space is commonly acknowledged as a quasi-metrizable topological vector space devoid of quasi-normability. However, as elaborated in [2], it emerges that every $(\ell^p, \|\cdot\|_{+p})$ space, $0 < p < \infty$, manifests neutrosophic quasi-normability through the devised neutrosophic quasi-norm $(\mathcal{N}, \circ)$, define as in the following example;

Definition 3.2. A sequence $\{x_n\}$ in $(X, \tau_{\mathcal{N}})$ converges to x if;

$$\lim_{n\to\infty} \mathcal{P}(x_n - x, t) = 1,$$

$$\lim_{n\to\infty} \mathcal{Q}(x_n - x, t) = 0$$

and

$$\lim_{n\to\infty} \mathcal{R}(x_n - x, t) = 0$$

for all $t > 0$. We denote closure and interior of a set A in $(X, \tau_{\mathcal{N}})$ by $cl_{\mathcal{N}}A$ and $int_{\mathcal{N}}A$ respectively.

Remark 3.3. $B_{\mathcal{N}}(\mathsf{x}, \check{c}_2, \check{t}) \subseteq B_{\mathcal{N}}(\mathsf{x}, \check{c}_1, \check{t})$, if $\check{c}_1 > \check{c}_2 > 0$ and $\check{t}_1 > \check{t}_2 > 0$ then $B_{\mathcal{N}}(\mathsf{x}, \check{c}_2, \check{t}_2) \subseteq B_{\mathcal{N}}(\mathsf{x}, \check{c}_1, \check{t}_1)$. Now the set $\{B_{\mathcal{N}}(\mathsf{x}, \check{c}_n, \check{t}_n) : \check{c}_n \in (0,1), \check{t}_n > 0, n \in \mathbb{N}\}$ forms a fundamental set of neighborhoods of x in $(X, \tau_{\mathcal{N}})$, where both sequences $\{\mathsf{x}_n\}$ and $\{\check{t}_n\}$ converges to 0.

Example 3.4. Let $(\ell^p, \|\cdot\|_{+p})$ be a quasi normed linear space, where $1 \le p < \infty$ and

$$\|x\|_{+p} = \Big(\sum_{n=0}^{\infty} max\{x_n, 0\}^p\Big)^{1/p}.$$

In the scenario where $x = (x_n) \in \ell^p$ and for $0 < p < 1$, it constitutes a quasi-metrizable topological vector space but lacks quasi-normability. Nevertheless, each $(\ell^p, \|.\|_{+p})$ can be characterized as neutrosophic quasi-normable through an neutrosophic quasi-norm $(\mathcal{P}, \mathcal{Q})$ defined as follows; for $0 < p < 1$,

$$\mathcal{P}_p(x,t) = \begin{cases} \dfrac{t^p}{t^p + \sum_{n=0}^{\infty}\left(max\{x_n, 0\}\right)^p}, & t > 0 \\ 0, & t = 0, \end{cases} \tag{6}$$

$$\mathcal{Q}_p(x,t) = \begin{cases} \dfrac{\sum_{n=0}^{\infty}\left(max\{x_n, 0\}\right)^p}{t^p + \sum_{n=0}^{\infty}\left(max\{x_n, 0\}\right)^p}, & t > 0 \\ 1, & t = 0 \end{cases} \tag{7}$$

and

$$\mathcal{R}_p(x,t) = \begin{cases} \dfrac{2\sum_{n=0}^{\infty}\left(max\{x_n, 0\}\right)^p}{t^p + 2\sum_{n=0}^{\infty}\left(max\{x_n, 0\}\right)^p}, & t > 0 \\ 1, & t = 0. \end{cases} \tag{8}$$

For $1 \leq p < \infty$,

$$\mathcal{P}_p(x,t) = \begin{cases} \dfrac{t}{t + \left(\sum_{n=0}^{\infty}\left(max\{x_n, 0\}\right)^p\right)^{1/p}}, & t > 0 \\ 0, & t = 0, \end{cases} \tag{9}$$

$$\mathcal{Q}_p(x,t) = \begin{cases} \dfrac{\left(\sum_{n=0}^{\infty}(max\{x_n, 0\})^p\right)^{1/p}}{t + \left(\sum_{n=0}^{\infty}(max\{x_n, 0\})^p\right)^{1/p}}, & t > 0 \\ 1, & t = 0 \end{cases} \tag{10}$$

and

$$\mathcal{R}_p(x,t) = \begin{cases} \dfrac{2\left(\sum_{n=0}^{\infty}(max\{x_n, 0\})^p\right)^{1/p}}{t + 2\left(\sum_{n=0}^{\infty}(max\{x_n, 0\})^p\right)^{1/p}}, & t > 0 \\ 1, & t = 0, \end{cases} \tag{11}$$

where $\left(\sum_{n=0}^{\infty}\left(max\{x_n, 0\}\right)^p\right)^{1/p} = \|x_n\|_{+p}$.

4. Open mapping and closed graph theorems

In this segment, we aim to formulate the close mapping theorem within the context of FQNS. To pave the way for this discussion, we initially present several lemmas.

Lemma 4.1. ([36]) *Let* $\left(X, T, I, F, *, \diamond\right)$ *be neutrosophic quasi normed space and* $\mathfrak{B}(\Theta)$ *be the collection of open balls center at origin. Then for* $t > 0$ *and* $t \in (0, 1)$

(a) $B_{\mathcal{N}}(\Theta, r, t)$ *is absorbing .*
(b) $B_{\mathcal{N}}(\Theta, r, t)$ *is semibalanced.*
(c) $\lambda B_{\mathcal{N}}(\Theta, r, t) = B_{\mathcal{N}}\Big(\Theta, r, \lambda t\Big)$ *for every* $\lambda > 0$,
(d) *if* $B \in \mathfrak{B}(\Theta)$, *there is* $B' \in \mathfrak{B}(\Theta)$ *such that* $B' + B' \subseteq B$.
(e) *if* $B, B' \in \mathfrak{B}(\Theta)$, *there is* $B'' \in \mathfrak{B}(\Theta)$ *such that* $B'' \subseteq B \cap B'$.

Lemma 4.2. ([36]) *Let* M *be a subset of neutrosophic quasi normed space* $\Big(V, T, I, F, *, \diamond\Big)$, $t > 0$. *Then:*

(a) $int_{\mathcal{N}}(tM) = t\ int_{\mathcal{N}}(M)$
(b) $cl_{\mathcal{N}}(tM) = t\ cl_{\mathcal{N}}(M)$.

Lemma 4.3. ([36]) *Let* $\left(X, \tau\right)$ *be a paratopological vector space.*

(a) *If* M *is convex subset of* X *and* $int(M) \neq \mathcal{R}$ *then* $(1 - \alpha)intA + \alpha M \subseteq int(M)$, *where* $\alpha \in (0, 1)$ *and consequently* $intM$ *is convex.*
(b) *If* M *is absorbing, convex subset of* X *and* $intM \neq \mathcal{R}$ *then* $\Theta \in int(AM)$.
(c) *If* $\mathfrak{B}(\Theta)$ *is a base of* Θ*-neighborhoods, then* $cl(M) = \{M - B : B \in \mathfrak{B}\}$.

Lemma 4.4. ([36]) *If* A *is absorbent and convex subset of NQNS* $\left(X, \mathcal{P}, \mathcal{Q}, \mathcal{R}, \circ, \star\right)$ *then* $cl(A)$.

Definition 4.5. Let $\left(X, \mathcal{P}, \mathcal{Q}, \mathcal{R}, \circ, \star\right)$ be NQNS then a sequence $\{x_n\}$ in X is left/right $\mathcal{N}-Cauchy/(\mathcal{N}'-Cauchy)$ with respect to topology $\tau_{\mathcal{N}}/(\tau_{\mathcal{N}'})$ if $x_n - x_m \to 0$ as $m, n \to \infty$, for $m > n$ respectively.

Definition 4.6. A NQNS $\left(X, \mathcal{P}, \mathcal{Q}, \mathcal{R}, \circ, \star\right)$ is said to be left/right complete if every left/right $\mathcal{N}-Cauchy/\mathcal{N}'-Cauchy$ sequence is convergent in X.

Definition 4.7. Let S be a subspace of a NQNS $\left(X, \mathcal{P}, \mathcal{Q}, \mathcal{R}, \circ, \star\right)$, then S said to be of half second category if $S = \bigcup_{n=1}^{\infty} M_n$, there exists positive integer m such that

$$int_{\mathcal{N}'}(cl_{\mathcal{N}'^{-1}} M_m) \neq \mathcal{R}.$$

Definition 4.8. Let $\left(X, \mathcal{P}_X, \mathcal{Q}_X, \mathcal{R}_X, \circ, \star\right)$ and $\left(Y, \mathcal{P}_Y, \mathcal{Q}_Y, \mathcal{R}_Y, \circ, \star\right)$ be two neutrosophic normed spaces then their product is defined as; for all $(x, y) \in \left(X \times Y\right)$ and $t > 0$

$$\left(X, \mathcal{P}_X, \mathcal{Q}_X, \mathcal{R}_X, \circ, \star\right) \times \left(Y, \mathcal{P}_Y, \mathcal{Q}_Y, \mathcal{R}_Y, \circ, \star\right) = \Big(X \times Y, \mathcal{P}_X \circ \mathcal{P}_Y, \mathcal{Q}_X \star \mathcal{Q}_Y, \mathcal{R}_X \star \mathcal{R}_Y, \circ, \star\Big)$$

where

$$\mathcal{P}\Big((x, y), t\Big) = \mathcal{P}_X(x, t) \circ \mathcal{P}_Y(y, t) = \mathcal{P}_X \circ \mathcal{P}_Y$$

$$\mathcal{Q}\Big((x, y), t\Big) = \mathcal{Q}_X(x, t) \star \mathcal{Q}_Y(y, t) = \mathcal{Q}_X \star \mathcal{Q}_Y$$

$$\mathcal{R}\Big((x, y), t\Big) = \mathcal{R}_X(x, t) \star \mathcal{R}_Y(y, t) = \mathcal{R}_X \star \mathcal{R}_Y$$

is again a neutrosophic normed space. We will call it product neutrosophic normed space.

Lemma 4.9. *Let* $\left(X,\mathcal{P}_X,\mathcal{Q}_X,\mathcal{R}_X,\circ,\star\right)$ *and* $\left(Y,\mathcal{P}_Y,\mathcal{Q}_Y,\mathcal{R}_Y,\circ,\star\right)$ *be NQNS. Let* $\mathcal{N}=\mathcal{N}_X\times\mathcal{N}_Y$*, where* $\mathcal{N}_X=(\mathcal{P}_X,\mathcal{Q}_X,\mathcal{R}_X)$ *and* $\mathcal{N}_Y=(\mathcal{P}_Y,\mathcal{Q}_Y,\mathcal{R}_Y)$*, for any* $(x,y)\in X\times Y$*,*

(a) *if* $r_1,r_2\in(0,1)$ *and* $t_1,t_2>0$ *then*

$$B_{\mathcal{N}}\Big((x,y),r,t\Big)\subseteq B_{\mathcal{N}_X}\Big(x,r_1,t_1\Big)\times B_{\mathcal{N}_Y}\Big(y,r_2,t_2\Big) \tag{12}$$

where $r=\min(r_1,r_2)$ *and* $t=\min(t_1,t_2)$.

(b) *if* $r\in(0,1)$ *and* $t>0$ *then there exists* $s\in(0,1)$ *such that*

$$B_{\mathcal{N}}\Big((x,y),r,t\Big)\supseteq B_{\mathcal{N}_X}\Big(x,s,t\Big)\times B_{\mathcal{N}_Y}\Big(y,s,t\Big). \tag{13}$$

Proof. (a) Let $(x_1,y_1)\in B_{\mathcal{N}}\Big((x,y),r,t\Big)$ then we have that

$$1-r<\mathcal{P}\Big((x_1-x,y_1-y),t\Big)=\mathcal{P}_X(x_1-x,t)\circ\mathcal{P}_Y(y_1-y,t). \tag{14}$$

Since $r=\min(r_1,r_2)$ we get

$$\begin{aligned}
r_1 &\geq r\\
1-r_1 &\leq 1-r\\
1-r_1 &\leq \mathcal{P}_X(x_1-x,t)\circ\mathcal{P}_Y(y_1-y,t)\\
1-r_1 &< \mathcal{P}_X(x_1-x,t_1).
\end{aligned}$$

This implies that $x_1\in B_{\mathcal{N}_X}\Big(x,r_1,t_1\Big)$. Similarly,

$$\begin{aligned}
r_2 &\geq r\\
1-r_2 &\leq 1-r\\
1-r_2 &\leq \mathcal{P}_X(x_1-x,t)\circ\mathcal{P}_Y(y_1-y,t)\\
1-r_2 &< \mathcal{P}_Y(y_1-y,t_2).
\end{aligned}$$

Now we will check for $\mathcal{Q}$ and $\mathcal{R}$. Since, we assumed that $(x_1,y_1)\in B_{\mathcal{N}}\Big((x,y),r,t\Big)$, we have

$$r>\mathcal{Q}\Big((x_1-x,y_1-y),t\Big)=\mathcal{Q}_X(x_1-x,t)\star\mathcal{Q}_Y(y_1-y,t). \tag{15}$$

Since, $r\leq r_1$ we have

$$\begin{aligned}
r_1 &\geq r\\
&\geq \mathcal{Q}_X(x_1-x,t)\star\mathcal{Q}_Y(y_1-y,t)\\
r_1 &> \mathcal{Q}_X(x_1-x,t_1).
\end{aligned}$$

Similarly

$$\begin{aligned}
r_2 &\geq r\\
&\geq \mathcal{Q}_X(x_1-x,t)\star\mathcal{Q}_Y(y_1-y,t)\\
r_2 &> \mathcal{Q}_Y(y_1-y,t_2).
\end{aligned}$$

In a similar manner for $\mathcal{R}$, we get

$$r_1 > \mathcal{R}_X(x_1 - x, t_1),$$
$$r_2 > \mathcal{R}_Y(y_1 - y, t_2).$$

Hence finally, we get $x_1 \in B_{\mathcal{N}_X}\Big(x, r_1, t_1\Big)$ and $y_1 \in B_{\mathcal{N}_Y}\Big(y, r_2, t_2\Big)$. This implies that $(x_1, y_1) \in B_{\mathcal{N}_X}\Big(x, r_1, t_1\Big) \times B_{\mathcal{N}_Y}\Big(y, r_2, t_2\Big)$. Hence equation 12 holds.

(b) From Proposition 2.5, there exists $s \in (0, 1)$, such that $(1 - s) \circ (1 - s) > 1 - r$. For any $(x_1, y_1) \in B_{\mathcal{N}_X}\big(x, s, t\big) \times B_{\mathcal{N}_Y}\big(y, r, t\big)$, we get

$$\mathcal{P}_X(x_1 - x, t) > 1 - s,$$
$$\mathcal{Q}_X(x_1 - x, t) < s,$$
$$\mathcal{R}_X(x_1 - x, t) < s$$

and

$$\mathcal{P}_Y(y_1 - y, t) > 1 - s$$
$$\mathcal{Q}_Y(y_1 - y, t) < s$$
$$\mathcal{R}_Y(y_1 - y, t) < s.$$

Therefore

$$\begin{aligned}\mathcal{P}\Big((x_1, y_1) - (x, y), t\Big) &= \mathcal{P}\Big((x_1 - x, y_1 - y), t\Big)\\ &= \mathcal{P}_X(x_1 - x, t) \circ \mathcal{P}_Y(y_1 - y, t)\\ &> (1 - s) \circ (1 - s)\\ &> 1 - r,\end{aligned}$$

$$\begin{aligned}\mathcal{Q}\Big((x_1, y_1) - (x, y), t\Big) &= \mathcal{Q}\Big((x_1 - x, y_1 - y), t\Big)\\ &= \mathcal{Q}_X(x_1 - x, t) \star \mathcal{Q}_Y(y_1 - y, t)\\ &< s \star s\\ &< r\end{aligned}$$

and

$$\begin{aligned}\mathcal{R}\Big((x_1, y_1) - (x, y), t\Big) &= \mathcal{R}\Big((x_1 - x, y_1 - y), t\Big)\\ &= \mathcal{R}_X(x_1 - x, t) \star \mathcal{R}_Y(y_1 - y, t)\\ &< s \star s\\ &< r.\end{aligned}$$

Hence $(x_1, y_1) \in B_{\mathcal{N}}\Big((x, y), r, t\Big)$. □

Theorem 4.10. *Let* $\Big(X, \mathcal{N}_1, \circ, \star\Big)$ *and* $\Big(Y, \mathcal{N}_2, \circ, \star\Big)$ *be NQNS. Let* $(X \times Y, \mathcal{N}, \circ, \star)$ *be product NQNS,* $\mathcal{N} = \mathcal{N}_1 \times \mathcal{N}_2$*. Then the following hold;*

(a) $\tau_{\mathcal{N}} = \tau_{\mathcal{N}_1} \times \tau_{\mathcal{N}_2}$,
(b) *If X is right $\mathcal{N}_1$ − complete and Y is right $\mathcal{N}_2$ − complete then $X \times Y$ is $\mathcal{N}_1 \times \mathcal{N}_2$ − complete.*

Proof. (a) The proof is straight and forward from lemma 4.9. (b) Suppose that (x_n, y_n) be a sequence in $X \times Y$ and left $\mathcal{N}$ − *Cauchy* i.e. $(x_m, y_m) - (x_n, -y_n) = (x_m - x_n, y_n - y_m) \longrightarrow (0_X, 0_Y)$ as $m, n \longrightarrow \infty$, with respect to product neutrosophic quasi norm $\tau_{\mathcal{N}}$. Consequently, from theorem 4.10a, we can see that sequences x_n, y_n are left $\mathcal{N}_1$ − *Cauchy* and left $\mathcal{N}_2$ − *Cauchy* respectively.

Since X and Y are right $\mathcal{N}_1$ complete and right $\mathcal{N}_2$ complete respectively. This guarantees that sequences x_n and y_n are convergent to some $x \in X$ and $y \in Y$, with respect to $\tau_{\mathcal{N}_1^{-1}}$ and $\tau_{\mathcal{N}_2^{-1}}$ respectively. Again from theorem 4.10*a*, sequence (x_n, y_n) is convergent to (x, y) with respect to $\tau_{\mathcal{N}^{-1}}$. Hence $X \times Y$ is right $\mathcal{N}_1 \times \mathcal{N}_2$ − *complete.* □

Open Mapping Theorem

Theorem 4.11. *Let $\left(X, \mathcal{P}, \mathcal{Q}, \mathcal{R}, \circ, \star\right)$ and $\left(Y, \mathcal{P}', \mathcal{Q}', \mathcal{R}', \circ', \star\right)$ be NQNS. Suppose that $\left(X, \mathcal{P}, \mathcal{Q}, \mathcal{R}, \circ, \star\right)$ is right $(\mathcal{P}, \mathcal{Q}, \mathcal{R})$− complete and $\left(Y, \mathcal{P}', \mathcal{Q}', \circ', \star'\right)$ is Hausdorff and of half second category. If $T : X \to Y$ is linear, surjective and continuous, then T is open.*

Proof. Let $\mathfrak{B}(\Theta_X)$ be the family of open balls center at origin Θ. By remark 2.9 and lemma 4.1, for any $U = B_{\mathcal{N}}(\Theta_X, \hat{r}, \hat{t}) \in \mathfrak{B}(\Theta_X)$, U is absorbent, semibalanced and convex, hence $X = \bigcup_{n=1}^{\infty} nU$. Since T is onto and linear $Y = T(X) = \bigcup_{n=1}^{\infty} nT(U)$. Since $\left(Y, \mathcal{P}', \mathcal{Q}', \circ', \star'\right)$ is of half second category, there exists $n \in \mathbb{N}$ such that $int_{\mathcal{N}'} cl_{\mathcal{N}'^{-1}} nT(U) \neq \mathcal{R}$. From lemma 4.2, we have $int_{\mathcal{N}'} cl_{\mathcal{N}'^{-1}} T(U) \neq \mathcal{R}$. Since T is linear and onto, $T(U)$ is absorbing and convex. From lemma 4.4, we have $cl_{\mathcal{N}'^{-1}}(T(U))$ is absorbing and convex. from lemma 4.3, $\Theta_Y \in int_{\mathcal{N}'} cl_{\mathcal{N}'^{-1}} T(U)$. By using the definition of interior of a set, there exists an open ball $B_{\mathcal{N}'}(\Theta_Y, , r', t')$ such that

$$\Theta_Y \in B_{\mathcal{N}'}(\Theta_Y, r', t') \subseteq cl_{\mathcal{N}'^{-1}} T(U). \tag{16}$$

Let $U_n = B_{\mathcal{N}}\left(\Theta_X, \dfrac{\hat{r}}{2^n}, \dfrac{\hat{t}}{2^{n+1}}\right)$. Then U_n is a local base at Θ_X. For any U_n, $n \in \mathbb{N}$ from equation 16, there exists $B_{\mathcal{N}'}^{(n)} = B_{\mathcal{N}'}(\Theta_Y, r'_n, t'_n)$, such that

$$B_{\mathcal{N}'}(\Theta_Y, r'_n, t'_n) \subseteq cl_{\mathcal{N}'^{-1}} T(U_n). \tag{17}$$

Where $r'_n \in (0, 1)$ and $t'_n > 0$. And from remark 3.3, we have that $\lim_{n\to\infty} r'_n = 0$ and $\lim_{n\to\infty} t'_n = 0$. By definition of open map, we have to show that T maps open sets in $\left(X, \mathcal{P}, \mathcal{Q}, \mathcal{R}, \circ, \star\right)$ onto open sets in $(Y, \mathcal{P}', \mathcal{Q}', \circ', \star')$ i.e. we will show that

$$B_{\mathcal{N}'}^{(1)} = B_{\mathcal{N}'}(\Theta_Y, r'_1, t'_1), \subseteq T(U). \tag{18}$$

Here, $\mathcal{N} = (\mathcal{P}, \mathcal{Q})$ and $\mathcal{N}' = (\mathcal{P}', \mathcal{Q}')$ is neutrosophic quasi norms on X and Y respectively. From equation 17, we have for $n = 1$

$$B_{\mathcal{N}'}(\Theta_Y, r'_1, t'_1) \subseteq cl_{\mathcal{N}'^{-1}} T(U_1).$$

Let $y \in B_{\mathcal{N}'}(\Theta_Y, r'_1, t'_1)$, there exists $x_1 \in U_1$ such that

$$\mathcal{P}'^{-1}(Tx_1 - y, t'_2) > 1 - r'_2,$$
$$\mathcal{Q}'^{-1}(Tx_1 - y, t'_2) < r'_2,$$
$$\mathcal{R}'^{-1}(Tx_1 - y, t'_2) < r'_2$$

or

$$\mathcal{P}'(y - Tx_1, t_2') > 1 - r_2',$$
$$\mathcal{Q}'(y - Tx_1, t_2') < r_2',$$
$$\mathcal{R}'(y - Tx_1, t_2') < r_2'.$$

This implies that,

$$y - Tx_1 \in B_{\mathcal{N}'}(\Theta_Y, r_2', t_2') \subseteq cl_{\mathcal{N}'^{-1}} T(U_2).$$

So, there exists $x_2 \in U_2$ such that

$$\mathcal{P}'^{-1}(Tx_2 + Tx_1 - y, t_3') > 1 - r_3'$$
$$\mathcal{Q}'^{-1}(Tx_2 + Tx_1 - y, t_3') < r_3'$$
$$\mathcal{R}'^{-1}(Tx_2 + Tx_1 - y, t_3') < r_3'.$$

or

$$\mathcal{P}'^{-1}(y - Tx_2 - Tx_1, t_3') > 1 - r_3',$$
$$\mathcal{Q}'^{-1}(y - Tx_2 - Tx_1, t_3') < r_3',$$
$$\mathcal{R}'^{-1}(y - Tx_2 - Tx_1, t_3') < r_3'.$$

On continuing this process, we have

$$\mathcal{P}'(y - Tx_n - T_{x_{n-1}} - ...... - Tx_2 - Tx_1, t_n' + 1) > 1 - r_{n+1}'$$
$$\mathcal{Q}'(y - Tx_n - T_{x_{n-1}} - ...... - Tx_2 - Tx_1, t_n' + 1) < r_{n+1}'$$
$$\mathcal{R}'(y - Tx_n - T_{x_{n-1}} - ...... - Tx_2 - Tx_1, t_n' + 1) < r_{n+1}'.$$

This implies that sequence $Tx_n - T_{x_{n-1}} - ...... - Tx_2 - Tx_1 \to y$ as $n \to \infty$, for $r_n' \in (0,1)$ and $t_n' > 0$. Since $x_k \in U_k = B_{\mathcal{N}}\left(\Theta_X, \frac{\hat{r}}{2^k}, \frac{\hat{t}}{2^{k+1}}\right)$ i.e., $\mathcal{P}\left(x_k, \frac{\hat{t}}{2^{k+1}}\right) > 1 - \frac{\hat{r}}{2^k}$, $\mathcal{Q}\left(x_k, \frac{\hat{t}}{2^{k+1}}\right) < \frac{\hat{r}}{2^k}$ and $\mathcal{R}\left(x_k, \frac{\hat{t}}{2^{k+1}}\right) < \frac{\hat{r}}{2^k}$. Let $s_n = \sum_{k=1}^{k=n} x_k$, for $m > n$;

$$\begin{aligned}
\mathcal{P}\left(s_m - s_n, \frac{1}{2^{n+1}}\left(1 - \frac{1}{2^{m-n}}\right)\hat{t}\right) &= \mathcal{P}\left(\sum_{k=1}^{m} x_k - \sum_{k=1}^{n} x_k, \frac{1}{2^{n+1}}\left(1 - \frac{1}{2^{m-n}}\right)\hat{t}\right) \\
&= \mathcal{P}\left(\sum_{k=n+1}^{m} x_k, \frac{1}{2^{n+1}}\left(1 - \frac{1}{2^{m-n}}\right)\hat{t}\right) \\
&= \mathcal{P}\left(\sum_{k=n+1}^{m} x_k, \sum_{k=n+1}^{m} \frac{1}{2^{k+1}}\hat{t}\right) \\
&\geq \min_{n+1 \leq k \leq m} \mathcal{P}\left(x_k, \frac{1}{2^{k+1}}\hat{t}\right) \\
&\geq \min_{n+1 \leq k \leq m} \left(1 - \frac{\hat{r}}{2^k}\right).
\end{aligned}$$

Similarly,

$$
\begin{aligned}
\mathcal{Q}\left(s_m - s_n, \frac{1}{2^{n+1}}\left(1 - \frac{1}{2^{m-n}}\right)\hat{t}\right) &= \mathcal{Q}\left(\sum_{k=1}^{m} x_k - \sum_{k=1}^{n} x_k, \frac{1}{2^{n+1}}\left(1 - \frac{1}{2^{m-n}}\right)\hat{t}\right) \\
&= \mathcal{Q}\left(\sum_{k=n+1}^{m} x_k, \frac{1}{2^{n+1}}\left(1 - \frac{1}{2^{m-n}}\right)\hat{t}\right) \\
&= \mathcal{Q}\left(\sum_{k=n+1}^{m} x_k, \sum_{k=n+1}^{m} \frac{1}{2^{k+1}}\hat{t}\right) \\
&\leq \max_{n+1\leq k\leq m} \mathcal{Q}\left(x_k, \frac{1}{2^{k+1}}\hat{t}\right) \\
&\leq \max_{n+1\leq k\leq m} \left(\frac{\hat{r}}{2^k}\right),
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{R}\left(s_m - s_n, \frac{1}{2^{n+1}}\left(1 - \frac{1}{2^{m-n}}\right)\hat{t}\right) &= \mathcal{R}\left(\sum_{k=1}^{m} x_k - \sum_{k=1}^{n} x_k, \frac{1}{2^{n+1}}\left(1 - \frac{1}{2^{m-n}}\right)\hat{t}\right) \\
&= \mathcal{R}\left(\sum_{k=n+1}^{m} x_k, \frac{1}{2^{n+1}}\left(1 - \frac{1}{2^{m-n}}\right)\hat{t}\right) \\
&= \mathcal{R}\left(\sum_{k=n+1}^{m} x_k, \sum_{k=n+1}^{m} \frac{1}{2^{k+1}}\hat{t}\right) \\
&\leq \max_{n+1\leq k\leq m} \mathcal{R}\left(x_k, \frac{1}{2^{k+1}}\hat{t}\right) \\
&\leq \max_{n+1\leq k\leq m} \left(\frac{\hat{r}}{2^k}\right).
\end{aligned}
$$

We get that if $m, n \to \infty$ then $\mathcal{P} \to 1$, $\mathcal{Q} \to 0$ and $\mathcal{R} \to 0.$ Hence sequence s_n is left $\mathcal{N} - Cauchy$. By right $\mathcal{N}$ - completeness of $\left(X, \mathcal{P}, \mathcal{Q}, \mathcal{R}, \circ, \star\right)$, which guarantees that there exists some $x \in X$ such that $s_n \xrightarrow{\mathcal{N}^{-1}} x$ as $n \to \infty$. Since T is continuous, we have

$$
\sum_{k=1}^{n} T x_k \xrightarrow{\mathcal{N}'^{-1}} Tx.
$$

Since $(Y, \tau_{\mathcal{N}})$ is Hausdorff. So $(Y, \tau_{\mathcal{N}'^{-1}})$ will be Hausdorff. Hence, $y = Tx$.

$$\begin{aligned}
\mathcal{P}(x,\hat{t}) &= \mathcal{P}\left(x - s_n + s_n, \frac{\hat{t}}{2} + \frac{\hat{t}}{2}\right)\\
&\geq \min\left\{\mathcal{P}\left(x - s_n, \frac{\hat{t}}{2}\right), \mathcal{P}\left(s_n, \frac{\hat{t}}{2}\right)\right\}\\
&\geq \min\left\{\mathcal{P}^{-1}\left(s_n - x, \frac{\hat{t}}{2}\right), \mathcal{P}\left(s_n, \frac{\hat{t}}{2}\right)\right\}\\
&\geq \min\left\{\mathcal{P}^{-1}\left(s_n - x, \frac{\hat{t}}{2}\right), \mathcal{P}\left(\sum_{k=1}^{n} x_k, \frac{\hat{t}}{2}\right)\right\}\\
&\geq \min\left\{\mathcal{P}^{-1}\left(s_n - x, \frac{\hat{t}}{2}\right), \mathcal{P}\left(\sum_{k=1}^{n} x_k, \frac{\hat{t}}{2}\right)\right\}\\
&\geq \min\left\{\mathcal{P}^{-1}\left(s_n - x, \frac{\hat{t}}{2}\right), \mathcal{P}\left(\sum_{k=1}^{n} x_k, \sum_{k=1}^{n} \frac{\hat{t}}{2^{k+1}}\right)\right\}\\
&\geq \min\left\{\mathcal{P}^{-1}\left(s_n - x, \frac{\hat{t}}{2}\right), \min_{1\leq k\leq n} \mathcal{P}\left(x_k, \frac{\hat{t}}{2^{k+1}}\right)\right\}\\
&\geq \min\left\{\mathcal{P}^{-1}\left(s_n - x, \frac{\hat{t}}{2}\right), \left(1 - \frac{\hat{r}}{2^k}\right)\right\}.
\end{aligned}$$

Since $s_n \xrightarrow{\mathcal{N}^{-1}} x$ that means $\mathcal{P}^{-1}\left(s_n - x, \frac{\hat{t}}{2}\right) > 1 - \hat{r}$, for all $\hat{t} > 0$ and $\hat{r} \in (0,1)$. We have,

$$\mathcal{P}(x,\hat{t}) > 1 - \hat{r}.$$

Similarly, for $\mathcal{Q}(x,\hat{t})$

$$\begin{aligned}
\mathcal{Q}(x,\hat{t}) &= \mathcal{Q}\left(x - s_n + s_n, \frac{\hat{t}}{2} + \frac{\hat{t}}{2}\right)\\
&\leq \max\left\{\mathcal{Q}\left(x - s_n, \frac{\hat{t}}{2}\right), \mathcal{Q}\left(s_n, \frac{\hat{t}}{2}\right)\right\}\\
&\leq \max\left\{\mathcal{Q}^{-1}\left(s_n - x, \frac{\hat{t}}{2}\right), \mathcal{Q}\left(s_n, \frac{\hat{t}}{2}\right)\right\}\\
&\leq \max\left\{\mathcal{Q}^{-1}\left(s_n - x, \frac{\hat{t}}{2}\right), \mathcal{Q}\left(\sum_{k=1}^{n} x_k, \frac{\hat{t}}{2}\right)\right\}\\
&\leq \max\left\{\mathcal{Q}^{-1}\left(s_n - x, \frac{\hat{t}}{2}\right), \mathcal{Q}\left(\sum_{k=1}^{n} x_k, \frac{\hat{t}}{2}\right)\right\}\\
&\leq \max\left\{\mathcal{Q}^{-1}\left(s_n - x, \frac{\hat{t}}{2}\right), \mathcal{Q}\left(\sum_{k=1}^{n} x_k, \sum_{k=1}^{n} \frac{\hat{t}}{2^{k+1}}\right)\right\}\\
&\leq \max\left\{\mathcal{Q}^{-1}\left(s_n - x, \frac{\hat{t}}{2}\right), \max_{1\leq k\leq n} \mathcal{Q}\left(x_k, \frac{\hat{t}}{2^{k+1}}\right)\right\}\\
&\leq \max\left\{\mathcal{Q}^{-1}\left(s_n - x, \frac{\hat{t}}{2}\right), \left(\frac{\hat{r}}{2^k}\right)\right\}
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{R}(x,\hat{t}) &= \mathcal{R}\left(x - s_n + s_n, \frac{\hat{t}}{2} + \frac{\hat{t}}{2}\right) \\
&\leq \max\left\{\mathcal{R}\left(x - s_n, \frac{\hat{t}}{2}\right), \mathcal{R}\left(s_n, \frac{\hat{t}}{2}\right)\right\} \\
&\leq \max\left\{\mathcal{R}^{-1}\left(s_n - x, \frac{\hat{t}}{2}\right), \mathcal{R}\left(s_n, \frac{\hat{t}}{2}\right)\right\} \\
&\leq \max\left\{\mathcal{R}^{-1}\left(s_n - x, \frac{\hat{t}}{2}\right), \mathcal{R}\left(\sum_{k=1}^{n} x_k, \frac{\hat{t}}{2}\right)\right\} \\
&\leq \max\left\{\mathcal{R}^{-1}\left(s_n - x, \frac{\hat{t}}{2}\right), \mathcal{R}\left(\sum_{k=1}^{n} x_k, \frac{\hat{t}}{2}\right)\right\} \\
&\leq \max\left\{\mathcal{R}^{-1}\left(s_n - x, \frac{\hat{t}}{2}\right), \mathcal{R}\left(\sum_{k=1}^{n} x_k, \sum_{k=1}^{n} \frac{\hat{t}}{2^{k+1}}\right)\right\} \\
&\leq \max\left\{\mathcal{R}^{-1}\left(s_n - x, \frac{\hat{t}}{2}\right), \max_{1\leq k\leq n} \mathcal{R}\left(x_k, \frac{\hat{t}}{2^{k+1}}\right)\right\} \\
&\leq \max\left\{\mathcal{R}^{-1}\left(s_n - x, \frac{\hat{t}}{2}\right), \left(\frac{\hat{r}}{2^k}\right)\right\}.
\end{aligned}$$

Since $s_n \xrightarrow{\mathcal{N}^{-1}} x$ that means $\mathcal{Q}^{-1}\left(s_n - x, \frac{\hat{t}}{2}\right) < \hat{r}$, $\mathcal{R}^{-1}\left(s_n - x, \frac{\hat{t}}{2}\right) < \hat{r}$, for all $\hat{t} > 0$ and $\hat{r} \in (0,1)$. Hence $x \in U = B_{\mathcal{N}}(\Theta_X, \hat{r}, \hat{t})$ and shows that $y = Tx \in T(U)$. So, equation 18 holds. This completes the proof.[12] □

Let $(X, \mathcal{N}_1, \circ, \star)$ and $(Y, \mathcal{N}_2, \circ, \star)$ be NQNS and $L : (X, \mathcal{N}_1, \circ, \star) \longrightarrow (Y, \mathcal{N}_2, \circ, \star)$ be a mapping. The set

$$\Gamma_L = \{(x, y) \in X \times Y : y = L(x)\} \tag{19}$$

is called the graph of L. Then is said to be closed if Γ_L is closed with respect to $\tau_{\mathcal{N}_1} \times \tau_{\mathcal{N}_2}$.

From the open mapping theorem, we get the inverse mapping theorem.

Corollary 4.12. *: Consider $(X, \mathcal{N}_1, \circ, \star)$ and $(Y, \mathcal{N}_2, \circ, \star)$ as NQNS. Let $(X, \mathcal{N}, \circ, \star)$ be right $\mathcal{N}_1$ − complete, and $(Y, \mathcal{N}_2, \circ, \star)$ be Hausdorff and of the half second category. If $L : X \longrightarrow Y$ is a linear, bijective, and continuous mapping, then its inverse, if it exists, is also continuous.*

The inverse mapping theorem, stemming from the open mapping theorem, demonstrates that under certain conditions, a linear, bijective, and continuous mapping between FQNS results in topological isomorphism, implying that these two FQNS are topologically equivalent.

Closed Graph Theorem

Theorem 4.13. *Let $(X, \mathcal{N}_1, \circ, \star)$ and $(Y, \mathcal{N}_2, \circ, \star)$ be NQNS, with*

$$L : (X, \mathcal{N}_1, \circ, \star) \longrightarrow (Y, \mathcal{N}_2, \circ, \star)$$

being a linear mapping. If X is both right $\mathcal{N}_1$ − complete and Hausdorff, and belongs to the half second category, while Y is right $\mathcal{N}_2$ − complete, then L is closed if and only if L is continuous.

Proof. Let us suppose L has a closed graph. Define $\mathcal{N} = \mathcal{N}_1 \times \mathcal{N}_2$. As per Theorem 4.10, we establish that the space $(X \times Y, \mathcal{N}_1, \circ, \star)$ is right $\mathcal{N}_1$-complete. Since the graph Γ_L of L is closed within $(X \times Y, \mathcal{N}_1, \circ, \star)$, it also attains right $\mathcal{N}_1$-completeness. Notably, the projections $q_1 : \Gamma_L \longrightarrow X$ and $q_2 : \Gamma_L \longrightarrow Y$, defined as $q_1(x, y) = x$ and $q_2(x, y) = y$, for any $(x, y) \in \Gamma_L$, emerge as both linear and continuous mappings, with q_1 being bijective as well. Hence, as per Corollary 4.12, q_1^{-1} exhibits continuity. Consequently, $L = q_2 \circ q_1^{-1}$ proves continuous.

Now, let us assume that L is continuous. Suppose we have a sequence $(x_n, y_n) \in \Gamma_L$ converging to some $(x, y) \in X \times Y$ concerning the product topology $\tau_{\mathcal{N}} = \tau_{\mathcal{N}_1} \times \tau_{\mathcal{N}_2}$. This convergence translates to x_n approaching x in X and y_n converging to y in Y. Given that $y_n = L(x_n)$, applying the continuity of L and the uniqueness of the limit, we deduce $L(x) = y$, implying $(x, y) \in \Gamma_L$. Therefore, Γ_L is closed regarding the product topology. □

Acknowledgement

We thank the editor and referees for valuable comments and suggestions for improving the paper.

References

[1] C. Alegre, S. Romaguera, *The Hahn-Banach extension theorem for fuzzy normed spaces revisited*, Abstr. Appl. Anal. **2014** (2014), Art. ID 151472.
[2] C. Alegre, S. Romaguera, *Characterizations of metrizable topological vector spaces and their asymmetric generalizations in terms of fuzzy (quasi-) norms*, Fuzzy Sets Syst. **161** (2010), 2181–2192.
[3] C. Alegre, S. Romaguera, *On paratopological vector spaces*, Acta Math. Hungar. **101** (2003), 237–261.
[4] C. Alegre, S. Romaguera, *On the uniform boundedness theorem in fuzzy quasi-normed spaces*, Fuzzy Sets Syst. **282** (2016), 143–153.
[5] N. D. Aral, H. S. Kandemir, M. Et, *Lacunary harmonic summability in neutrosophic normed spaces*, Filomat **38** (2024), 6763–6771.
[6] N. D. Aral, H. S. Kandemir, M. Et, *Strongly lacunary convergence of order β of difference sequences of fractional order in neutrosophic normed spaces*, Filomat **37** (2023), 6443–6451.
[7] K. T. Atanassov, *Intuitionistic fuzzy sets*, Physica-Verlag HD, 1999.
[8] T. Bag, S. K. Samanta, *Finite dimensional fuzzy normed linear spaces*, Ann. Fuzzy Math. Inform. **6** (2013), 271–283.
[9] T. Bag, S. K. Samanta, *Fuzzy bounded linear operators*, Fuzzy Sets Syst. **151** (2005), 513–547.
[10] T. Bag, S. K. Samanta, *Some fixed point theorems in fuzzy normed linear spaces*, Inform. Sci. **117** (2007), 3271–3289
[11] C. Chaurasiya, A. Ahmad, A. Esi, *Neutrosophic fuzzy I-convergent Fibonacci difference sequence spaces*, Math. Found. Comput. **8** (2025), 725-[-733.
[12] S. C. Cheng, J. N. Mordeson, *Fuzzy linear operators and fuzzy normed linear spaces*, Bull. Calcutta Math. Soc. **86** (1994), 429–436.
[13] C. Felbin, *Finite dimensional fuzzy normed linear space*, Fuzzy Sets Syst. **48** (1992), 239–248.
[14] R. Gao, X. X. Li, J. R. Wu, *The decomposition theorem for a fuzzy quasi-norm*, J. Math. **2020** (2020), Art. ID 8845283.
[15] L. M. García-Raffi, S. Romaguera, E. A. Sánchez-Pérez, *Sequence spaces and asymmetric norms in the theory of computational complexity*, Math. Comput. Modelling **36** (2002), 1–11.
[16] A. George, P. Veeramani, *On some results in fuzzy metric spaces*, Fuzzy Sets and Systems **64** (1994), 395–399.
[17] B. Y. Hussein, F. K. Al-Basri, *On the completion of quasi-fuzzy normed algebra over fuzzy field*, J. Interdisciplinary Math. **25** (2020), 1–13.
[18] O. Kaleva, S. Seikkala, *On fuzzy metric spaces*, Fuzzy Sets and Systems **12** (1984), 215–229.
[19] A. K. Katsaras, *Fuzzy topological vector spaces I*, Fuzzy Sets Syst. **6** (1981), 85–95.
[20] V. A. Khan, M. Faisal, *Tauberian theorems for Cesàro summability in neutrosophic normed spaces*, Filomat **37** (2023), 3411–3426.
[21] V. A. Khan, M. Et, M. Faisal, *Some new neutrosophic normed sequence spaces defined by Jordan totient operator*, Filomat **37** (2023), 8953–8968.
[22] V. A. Khan, M. Faisal, *A study on $(\lambda - \mu)$ Zweier sequences and their behaviour in neutrosophic normed spaces*, Neutrosophic Sets Syst. **73** (2024), 570–583.
[23] V. A. Khan, M. Faisal, M. Kamran, *Some topological properties of intuitionistic fuzzy quasi normed space*, Filomat **38** (2024), 8907–8916.
[24] M. Kirişci, N. Şimşek, *Neutrosophic metric spaces*, Math. Sci. **14** (2020), 241–248.
[25] M. Kirişci, N. Şimşek, *Neutrosophic normed spaces and statistical convergence*, J. Analysis **28** (2020), 1059–1073.
[26] K. Y. Lee, *Approximation properties in fuzzy normed spaces*, Fuzzy Sets Syst. **282** (2016), 115–130.
[27] X. X. Li, J. R. Wu, *Norm structures of a fuzzy normed space*, J. Math. **40** (2020), 511–518.
[28] J. R. Munkres, *Topology* (2nd ed.), Prentice Hall, Upper Saddle River, NJ, 2000.
[29] H. A. Nazarkandi, *A characterization for fuzzy normed linear spaces and its applications*, J. Math. Extension **11** (2017), 19–30.
[30] J. H. Park, *Intuitionistic fuzzy metric spaces*, Chaos Solitons Fractals **22** (2004), 1039–1046.
[31] R. Saadati, J. H. Park, *On the intuitionistic fuzzy topological spaces*, Chaos Solitons Fractals **27** (2006), 331–344.
[32] M. Saheli, *Fuzzy topology generated by fuzzy norm*, Iranian J. Fuzzy Syst. **13** (2016), 113–123.
[33] B. Schweizer, A. Sklar, *Statistical metric spaces*, Pacific J. Math. **10** (1960), 313–334.
[34] F. Smarandache, *A unifying field in logics: neutrosophic logic, neutrosophic set, neutrosophic probability*, Infinite Study, 2005.

[35] H. Wang, F. Smarandache, Y. Zhang, R. Sunderraman, *Single valued neutrosophic sets*, Infinite Study **12** (2010).
[36] J. Wu, R. Li, *Open mapping and closed graph theorems in fuzzy quasi-normed spaces*, IEEE Trans. Fuzzy Syst. **30** (2022), 5291–5296.
[37] J. Z. Xiao, X. H. Zhu, H. Zhou, *On the topological structure of KM fuzzy metric spaces and normed spaces*, IEEE Trans. Fuzzy Syst. **28** (2019), 1575–1584.
[38] L. A. Zadeh, *Fuzzy sets*, Information and Control **8** (1965), 338–353.