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Some properties of remainders of uniform spaces and uniformly continuous mappings

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Abstract. In this paper we study remainders and completions of uniform spaces and uniformly continuous mappings. Necessary and sufficient conditions have been found for remainders to be precompact, complete, compact, uniformly connected, uniformly chained.

1. Introduction and preliminaries

When non-metrizable spaces came to mathematics, demand on creation of a natural structure implementing idea of uniformity appeared, firstly, the concepts of completeness and of uniformly continuous function. This led the Serbian mathematician D. Kurepa [10, 11] and French mathematician A. Weil [12] to define the concept of uniform space and to create the foundations of the theory of uniform spaces and uniformly continuous mappings.

Nowadays the theory of uniform spaces and uniformly continuous mappings is historically and logically substantiated.

It is of interest to compare the properties of a uniform space (X, U) and the space $(\tilde{X}\backslash X, \tilde{U}_{\tilde{X}\backslash X})$, called the remainder of the space (X, U), in completion $(\tilde{X}, \tilde{U})$. The problem of characterizing the properties of the "dual" space $(\tilde{X}\backslash X, \tilde{U}_{\tilde{X}\backslash X})$ by the properties of the space (X, U) naturally arises. This problem does not always have a solution: it is not clear, for example, what properties of $(\tilde{X}\backslash X, \tilde{U}_{\tilde{X}\backslash X})$ are related to uniform paracompactness or metrizability of (X, U). An example of a simple positive result from this area is the statement: the remainder $(\tilde{X}\backslash X, \tilde{U}_{\tilde{X}\backslash X})$ of a uniform space (X, U) is complete if and only if (X, U) is locally complete [4]. Therefore, the study of remainders and completions of uniform spaces is relevant. Some results in this direction are obtained in the theses [7, 8]. In works [1–3, 6] the remainders of topological spaces and topological groups were investigated.

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For a cover α of a set X and $x \in X$, $H \subset X$, we have: $St(\alpha, x) = \{A \in \alpha : A \ni x\}$, $\alpha(x) = \bigcup St(\alpha, x)$, $St(\alpha, H) = \{A \in \alpha : A \bigcap H \neq \emptyset\}$, $\alpha(H) = \bigcup St(\alpha, H)$. For covers α and β of the set X, the symbol $\alpha \succ \beta$ means that the cover α is a refinement of the cover β, i.e. for any $A \in \alpha$ there exists $B \in \beta$ such that $A \subset B$. The symbol $\alpha * \succ \beta$ means that the cover α is a strongly star refinement of the cover β, i.e. for any $A \in \alpha$ there exists $B \in \beta$ such that $\alpha(A) \subset B$, and, for covers α and β of a set X, we have: $\alpha \wedge \beta = \{A \bigcap B : A \in \alpha, B \in \beta\}$.

A uniformity on a nonempty set X is a family U of covers of X which satisfies the following conditions:

(U1) if $\alpha \in U$ and β is a cover of X such that $\alpha \succ \beta$, then $\beta \in U$;

(U2) if $\alpha_1, \alpha_2 \in U$, then there exists $\alpha \in U$ such that $\alpha \succ \alpha_1$ and $\alpha \succ \alpha_2$;

(U3) if $\alpha \in U$, then there exists $\beta \in U$ such that $\beta * \succ \alpha$;

(U4) for any two distinct points x and y in X there exists an $\alpha \in U$ such that no member of α contains both x and y [4, 9].

The covers from U are called uniform covers, and the pair (X, U) a uniform space [4]. If U_1 and U_2 are two uniformities on a set X and $U_1 \supset U_2$, then we say that the uniformity U_1 is finer than the uniformity U_2. The least upper bound of all uniformities on a Tychonoff space X is called the universal uniformity and is denoted by U_X.

For a uniformity U by τ_U we denote the topology generated by the uniformity.

A uniform space (X, U) is called:

(i) precompact if it has a base consisting of finite covers [4, 9];

(ii) pre-Lindelöf if it has a base consisting of countable covers [9];

(iii) complete if every Cauchy filter in it converges [4];

(iv) compact if it is complete and precompact [4];

(v) uniformly connected if for any cover $\alpha \in U$ and any points $x, y \in X$ there is a finite chained sequence $\{A_1, A_2, ..., A_n\} \subset \alpha$ such that $x \in A_1$ and $y \in A_n$ [4];

(vi) uniformly chained if for any cover $\alpha \in U$ there is a positive integer n such that for every pair of points $x, y \in X$ one can choose a chained sequence $\{A_1, A_2, ..., A_k\} \subset \alpha$ such that $k \leq n$, $x \in A_1$ and $y \in A_k$ [4].

A finite sequence $\{A_1, A_2, ..., A_n\}$ of subsets of a uniform space (X, U) is called chained if $A_i \bigcap A_{i+1} \neq \emptyset$ for each $i = 1, 2, ..., n-1$ [4].

Let (X, τ) be a topological space. A filter F_x in X is said to be a filter of neighborhoods of a point x in (X, τ) if each neighborhood of the point x is an element of the filter F_x. It is said that a filter F converges to a point $x \in X$, if F is finer than the filter F_x of neighborhoods of the point x. A filter F in a uniform space (X, U) is called convergent to a point $x \in X$ and x is a limit of the filter F, if it converges to the point x in (X, τ_U). A point $x \in X$ is called a cluster point of a filter F in (X, U) if x belongs to the closure of every member of F in (X, τ_U). Every cluster point of a filter F in (X, U) is a limit point [4]. The filter F is called a Cauchy filter if $\alpha \bigcap F \neq \emptyset$ for any $\alpha \in U$ [4]; a Cauchy filter F is called a free Cauchy filter if $\bigcap\{[M] : M \in F\} = \emptyset$, where $[M]$ is the closure of the set M [4];

For every uniform space (X, U) there is a precompact uniformity U^P on X, satisfying to the following conditions:

a) $U^P \subset U$;

b) The topologies generated by U^P and U coincide;

c) U^P is the largest precompact uniformity contained in U.

A uniformity U^P is called the precompact uniform reflection of the uniformity U and the completion $(\tilde{X}, \tilde{U}^P)$ of the uniform space (X, U^P) is called the compact extension, or the Samuel compactification of the uniform space (X, U). The Samuel compactification of a uniform space (X, U) is denoted by (sX, sU). If U_X is a universal uniformity of a Tychonoff space X, then (sX, sU) is the Stone-Čech compactification of the space X [4, 5].

Let $f : X \to Y$ be a mapping. If α and β are e covers of X and Y, respectively, then $f\alpha = \{fA : A \in \alpha\}$ and $f^{-1}\beta = \{f^{-1}B : B \in \beta\}$ is a covers of Y and X, respectively. A mapping $f : (X, U) \to (Y, V)$ of a uniform space (X, U) to a uniform space (Y, V) is called uniformly continuous if for any $\beta \in V$ there exists $\alpha \in U$ such that $f\alpha \succ \beta$. It is easy to see that a mapping f between uniform spaces (X, U) and (Y, V) is uniformly continuous if and only if $f^{-1}\beta \in U$ for any $\beta \in V$. A bijective mapping $f : (X, U) \to (Y, V)$ of a uniform space (X, U) onto a uniform space (Y, V) is called uniform homeomorphism or a uniform isomorphism if both mappings $f : (X, U) \to (Y, V)$ and $f^{-1} : (Y, V) \to (X, U)$ are uniformly continuous. Uniform spaces (X, U) and (Y, V) are called uniformly homeomorphic, or uniformly isomorphic if there is a uniform homeomorphism $f : (X, U) \to (Y, V)$ [4].

Let $f : (X, U) \to (Y, V)$ be a uniformly continuous mapping of a uniform space (X, U) to a uniform space (Y, V). A uniformly continuous mapping $\hat{f} : (\hat{X}, \hat{U}) \to (Y, V)$ of the uniform space $(\hat{X}, \hat{U})$ to the uniform space (Y, V) is called the completion of the mapping f if the following conditions hold:

1. The uniform space (X, U) is a dense uniform subspace of the uniform space $(\hat{X}, \hat{U})$;
2. $f = \hat{f}|_X$;
3. The mapping $\hat{f}$ is complete [4].

A uniformly continuous mapping $f : (X, U) \to (Y, V)$ of a uniform space (X, U) to a uniform space (Y, V) is called complete if every Cauchy filter F in (X, U), for which fF converges in (Y, V), converges in (X, U) [4].

Throughout this paper all uniform spaces are assumed to be Hausdorff, topological space are Tychonoff and mappings are uniformly continuous.

2. Some properties of remainders of uniform spaces and uniformly continuous mappings

Let $(\tilde{X}, \tilde{U})$ be a completion and $(\tilde{X}\backslash X, \tilde{U}_{\tilde{X}\backslash X})$ be a remainder of a uniform space (X, U). Let P and Q be properties of uniform spaces. The problem naturally arises: If a uniform space (X, U) has the property P, then under what necessary and sufficient conditions its remainder $(\tilde{X}\backslash X, \tilde{U}_{\tilde{X}\backslash X})$ has the property Q?

In this paper necessary and sufficient conditions are found for the remainder to precompact, complete, compact, uniformly connected, uniformly chained. We also study remainders of uniformly continuous mappings.

Proposition 2.1. *The remainder $(\tilde{X}\backslash X, \tilde{U}_{\tilde{X}\backslash X})$ of a uniform space (X, U) is precompact if and only if every cover $\alpha \in U$ contains a finite family $\alpha_0 \subset \alpha$ such that $\alpha_0 \bigcap F \neq \varnothing$ for any free Cauchy filter F in (X, U).*

Proof. Necessity. Let $(\tilde{X}\backslash X, \tilde{U}_{\tilde{X}\backslash X})$ be precompact and $\alpha \in U$ be an arbitrary cover. Then $\tilde{\alpha} \wedge \{\tilde{X}\backslash X\} = \hat{\alpha}$, where $\tilde{\alpha} = \{\tilde{A} : A \in \alpha\}$, $\tilde{A} = \tilde{X}\backslash [X\backslash A]_{\tilde{X}}$. Since $\tilde{\alpha} \in \tilde{U}$, then $\hat{\alpha} \in \tilde{U}_{\tilde{X}\backslash X}$. The space $(\tilde{X}\backslash X, \tilde{U}_{\tilde{X}\backslash X})$ is precompact, so $\hat{\alpha}$ contains a finite subcover $\hat{\alpha}_0 = \{\tilde{A}_1 \bigcap(\tilde{X}\backslash X), \tilde{A}_2 \bigcap(\tilde{X}\backslash X), ..., \tilde{A}_n \bigcap(\tilde{X}\backslash X)\}$. Denote $\alpha_0 = \{A_1, A_2, ..., A_n\}$. We show that $\alpha_0 \bigcap F \neq \varnothing$ for any free Cauchy filter F in (X, U). Let F be an arbitrary free Cauchy filter in (X, U). Due to the freeness of the Cauchy filter F, it converges to some point $\hat{x} \in \tilde{X}\backslash X$. Then there is a number $i_0 \leq n$ such that $\tilde{A}_{i_0} \bigcap(\tilde{X}\backslash X) \ni \hat{x}$. Let $\tilde{B}(\hat{x})$ denote the filter of the neighborhood of the point $\hat{x}$. Put $B = \tilde{B}(\hat{x}) \bigcap X$. It is clear that $F \supset B$ and $A_{i_0} \in F$, i.e. $\alpha_0 \bigcap F \neq \varnothing$.

Sufficiency. Let $\hat{\alpha} \in \tilde{U}_{\tilde{X}\backslash X}$ be an arbitrary cover, $\hat{\alpha} = \tilde{\alpha} \wedge \{\tilde{X}\backslash X\}$, $\tilde{\alpha} = \{\tilde{A} : A \in \alpha\}$, $\tilde{A} = \tilde{X}\backslash [X\backslash A]_{\tilde{X}}$. Then $\alpha \in U$. According to the condition of the proposition, the cover α contains a finite subfamily $\alpha_0 \subset \alpha$, $\alpha_0 = \{A_1, A_2, ..., A_n\}$ such that $\alpha_0 \bigcap F \neq \varnothing$ for each free Cauchy filter F in (X, U). It is easy to see that $\hat{\alpha}_0 = \{\tilde{A}_1 \bigcap(\tilde{X}\backslash X), \tilde{A}_2 \bigcap(\tilde{X}\backslash X), ..., \tilde{A}_n \bigcap(\tilde{X}\backslash X)\}$ is subcover of the cover $\hat{\alpha}$. Thus, $(\tilde{X}\backslash X, \tilde{U}_{\tilde{X}\backslash X})$ is precompact. □

Proposition 2.2. *The remainder $(\tilde{X}\backslash X, \tilde{U}_{\tilde{X}\backslash X})$ of a uniform space (X, U) is pre-Lindelöf if and only if every cover $\alpha \in U$ contains countable family $\alpha_0 \subset \alpha$ such that $\alpha_0 \bigcap F \neq \varnothing$ for every Cauchy filter F in (X, U).*

The proof, with minor modifications, is similar to the proof of Proposition 2.1.

Proposition 2.3. *The remainder $(\tilde{X}\backslash X, \tilde{U}_{\tilde{X}\backslash X})$ of a uniform space (X, U) is complete if and only if the space (X, U) is open in the completion $(\tilde{X}, \tilde{U})$.*

Proof. Necessity. Let $(\tilde{X}\backslash X, \tilde{U}_{\tilde{X}\backslash X})$ be complete. Then the space $(\tilde{X}\backslash X, \tilde{U}_{\tilde{X}\backslash X})$ is closed in the completion $(\tilde{X}, \tilde{U})$. Therefore, the uniform space (X, U) is open.

Sufficiency. Let the uniform space (X, U) be open in the completion $(\tilde{X}, \tilde{U})$. Then the remainder $(\tilde{X}\backslash X, \tilde{U}_{\tilde{X}\backslash X})$ is closed in the completion $(\tilde{X}, \tilde{U})$. □

Theorem 2.4. *The remainder* $(\tilde{X}\backslash X, \tilde{U}_{\tilde{X}\backslash X})$ *of a uniform space* (X, U) *is compact if and only if the uniform space* (X, U) *is open in* $(\tilde{X}, \tilde{U})$ *and every cover* $\alpha \in U$ *contains a finite family* $\alpha_0 \subset \alpha$ *such that* $\alpha_0 \bigcap F \neq \varnothing$ *for any free Cauchy filter* F *in* (X, U).

Proof. Necessity. If the remainder $(\tilde{X}\backslash X, \tilde{U}_{\tilde{X}\backslash X})$ of the space (X, U) is compact, i.e. closed, then by Proposition 2.3 (X, U) is open in the completion $(\tilde{X}, \tilde{U})$, and if $(\tilde{X}\backslash X, \tilde{U}_{\tilde{X}\backslash X})$ is compact, i.e. precompact, then by Proposition 2.1 every uniform cover $\alpha \in U$ contains a finite family $\alpha_0 \subset \alpha$ such that $\alpha_0 \bigcap F \neq \varnothing$ for any free Cauchy filter F in (X, U).

Sufficiency. If (X, U) is open in the completion $(\tilde{X}, \tilde{U})$, then by Proposition 2.3 $(\tilde{X}\backslash X, \tilde{U}_{\tilde{X}\backslash X})$ is closed, i.e. complete in the completion $(\tilde{X}, \tilde{U})$. If for every uniform cover $\alpha \in U$ the condition of the theorem is satisfied, then by Proposition 2.1 the remainder $(\tilde{X}\backslash X, \tilde{U}_{\tilde{X}\backslash X})$ of the space (X, U) is precompact. Since a complete and precompact space is compact, then $(\tilde{X}\backslash X, \tilde{U}_{\tilde{X}\backslash X})$ is compact. □

Theorem 2.5. *The remainder* $(sX\backslash X, sU_{sX\backslash X})$ *of a uniform space* (X, U) *is compact if and only if the uniform space* (X, U) *is open in the Samuel comapctification* (sX, sU).

Proof. Necessity. If the remainder $(sX\backslash X, sU_{sX\backslash X})$ of the space (X, U) is compact, then it is closed, therefore,(X, U) is open in the Samuel compactification (sX, sU).

Sufficiency. Let (X, U) be open in the Samuel compactification (sX, sU). Then the remainder $(sX\backslash X, sU_{sX\backslash X})$ of the space (X, U) is closed in the Samuel compactification (sX, sU). Since every closed subspace of the compact space is compact, then the remainder $(sX\backslash X, sU_{sX\backslash X})$ is compact. □

It is known that if U_X is the universal uniformity of the Tychonoff space X, then (sX, sU_X) is the Stone-Čech compactification of the space X. Then from Theorem 2.4 it follows:

Corollary 2.6. *The remainder* βX *of a Tychonoff space* X *is compact if and only if the space* X *is open in the Stone-Čech compactification* βX.

A finite sequence $\{A_1, A_2, ..., A_n\}$ of subsets of a uniform space (X, U) is called u-chained if $A_i \bigcap A_{i+1}$ is contained in some free Cauchy filter F in (X, U) for each $i = 1, 2, ..., n-1$.

Theorem 2.7. *The remainder* $(\tilde{X}\backslash X, \tilde{U}_{\tilde{X}\backslash X})$ *of a uniform space* (X, U) *is uniformly connected if and only if for each cover* $\alpha \in U$ *and for any two free Cauchy filters* F' *and* F'' *in* (X, U) *there exists a finite u-chained sequence* $\{A_1, A_2, ..., A_n\} \subset \alpha$ *such that* $A_1 \in F'$ *and* $A_n \in F''$.

Proof. Necessity. Let $(\tilde{X}\backslash X, \tilde{U}_{\tilde{X}\backslash X})$ be a uniformly connected space, $\alpha \in U$ be an arbitrary cover, F' and F'' be arbitrary free Cauchy filters in (X, U). Then $\hat{\alpha} \in \tilde{U}_{\tilde{X}\backslash X}$, $\hat{\alpha} = \tilde{\alpha} \wedge \{\tilde{X}\backslash X\}$, $\tilde{\alpha} = \{\tilde{A} : A \in \alpha\}$, $\tilde{A} = \tilde{X}\backslash [X\backslash A]_{\tilde{X}}$. The Cauchy filters F' and F'' converge to some points $\hat{x}' \in \tilde{X}\backslash X$ and $\hat{x}'' \in F''$, respectively. Since the remainder $(\tilde{X}\backslash X, \tilde{U}_{\tilde{X}\backslash X})$ is uniformly connected, there exists a finite chained sequence $\{\hat{A}_1, \hat{A}_2, ..., \hat{A}_n\} \subset \hat{\alpha}$ such that $\hat{x}' \in \hat{A}_1$ and $\hat{x}'' \in \hat{A}_n$, $\hat{A}_n = \tilde{A}_n \bigcap (\tilde{X}\backslash X)$. By $\tilde{B}(\hat{x}')$ and $\tilde{B}(\hat{x}'')$ we denote the filter of neighborhoods of the points $\hat{x}'$ and $\hat{x}''$, respectively. It is easy to see that $\tilde{B}(\hat{x}') \wedge \{X\} \subset F'$ and $\tilde{B}(\hat{x}'') \wedge \{X\} \subset F''$. Therefore, $A_1 \in F'$ and $A_n \in F''$. We show that $\{A_1, A_2, ..., A_n\} \subset \alpha$ is u-chained. Since $\{\hat{A}_1, \hat{A}_2, ..., \hat{A}_n\} \subset \hat{\alpha}$ is chained sequence, then $\hat{A}_i \bigcap \hat{A}_{i+1} \neq \varnothing$ for each $i = 1, 2, ..., n-1$. We select element $\hat{x}_i$ from $\hat{A}_i \bigcap \hat{A}_{i+1}$. By $\tilde{B}(\hat{x}_i)$ we denote the filter of neighborhoods of the point $\hat{x}_i$ in $(\tilde{X}\backslash X, \tilde{U}_{\tilde{X}\backslash X})$. The trace filter F_i of neighborhoods $\tilde{B}(\hat{x}_i)$ on X is a free Cauchy filter in (X, U). Therefore, $A_i \bigcap A_{i+1} \in F_i$. So $\{A_1, A_2, ..., A_n\} \subset \alpha$ is a u-chained sequence.

Sufficiency. Let $\hat{\alpha} \in \tilde{U}_{\tilde{X}\backslash X}$ be an arbitrary cover and $\hat{x}, \hat{y} \in \tilde{X}\backslash X$ be arbitrary points, where $\hat{\alpha} = \tilde{\alpha} \wedge \{\tilde{X}\backslash X\}$, $\tilde{\alpha} = \{\tilde{A} : A \in \alpha\}$, $\tilde{A} = \tilde{X}\backslash [X\backslash A]_{\tilde{X}}$. By $\tilde{B}(\hat{x})$ and $\tilde{B}(\hat{y})$ we denote the filter of neighborhoods of the points $\hat{x}$ and $\hat{y}$ in $(\tilde{X}\backslash X, \tilde{U}_{\tilde{X}\backslash X})$ respectively. Put $\tilde{B}(\hat{x}) \wedge \{X\} = F_{\hat{x}}$ and $\tilde{B}(\hat{y}) \wedge \{X\} = F_{\hat{y}}$. It is easy to see that $F_{\hat{x}}$ and $F_{\hat{y}}$ are free Cauchy filters in (X, U). Then there exists a u-checked sequence $\{A_1, A_2, ..., A_n\} \subset \alpha$, such that $A_1 \in F_{\hat{x}}$ and $A_n \in F_{\hat{y}}$. Since $\{A_1, A_2, ..., A_n\}$ is a u-chained sequence, then $\{\hat{A}_1, \hat{A}_2, ..., \hat{A}_n\}$ is chained. From $A_1 \in F_{\hat{x}}$ and $A_n \in F_{\hat{y}}$ follows that $\hat{x} \in \hat{A}_1$ and $\hat{y} \in \hat{A}_n$. □

Corollary 2.8. *The remainder* $(sX\backslash X, sU_{sX\backslash X})$ *of a uniform space* (X, U) *is uniformly connected if and only if for each cover* $\alpha \in U$ *and for any two free Cauchy filters* F' *and* F'' *in* (X, U) *there exists a finite u-chained sequence* $\{A_1, A_2, ..., A_n\} \subset \alpha$ *such that* $A_1 \in F'$ *and* $A_n \in F''$*.*

Theorem 2.9. *The remainder* $(\tilde{X}\backslash X, \tilde{U}_{\tilde{X}\backslash X})$ *of a uniform space* (X, U) *is uniformly chained if and only if for each cover* $\alpha \in U$ *there exists a number* n *such that for any pair of free Cauchy filters* F'*and* F'' *in* (X, U) *one can select a u-chained sequence* $\{A_1, A_2, ..., A_k\} \subset \alpha$ *such that for* $k \leq n$, $A_1 \in F'$ *and* $A_k \in F''$*.*

Proof. Necessity. Let the remainder $(\tilde{X}\backslash X, \tilde{U}_{\tilde{X}\backslash X})$ of a uniform space (X, U) be uniformly chained and $\alpha \in U$ be an arbitrary cover. Put $\hat{\alpha} = \tilde{\alpha} \wedge \{\tilde{X}\backslash X\}$, $\tilde{\alpha} = \{\tilde{A} : A \in \alpha\}$, $\tilde{A} = \tilde{X}\backslash [X\backslash A]_{\tilde{X}}$. Then $\hat{\alpha} \in \tilde{U}_{\tilde{X}\backslash X}$. For a cover $\hat{\alpha}$ there is a number n that for every $\hat{x} \in \tilde{X}\backslash X$ and $\hat{y} \in \tilde{X}\backslash X$ there exists a chained sequence $\{\hat{A}_1, \hat{A}_2, ..., \hat{A}_k\}$ such that $\hat{x} \in \hat{A}_1$ and $\hat{y} \in \hat{A}_k$, where $\hat{A}_i = \tilde{A}_i \bigcap (\tilde{X}\backslash X)$, $i = 1, 2, ..., k$. We show that for a cover $\alpha \in U$ the number n satisfies the conditions of the theorem. Denote by $\tilde{B}_{\hat{x}}$ and $\tilde{B}_{\hat{y}}$ the filter of neighborhoods of the points $\hat{x}$ and $\hat{y}$ in $(\tilde{X}\backslash X, \tilde{U}_{\tilde{X}\backslash X})$. Put $\tilde{B}_{\hat{x}} \bigcap X = F'$ and $\tilde{B}_{\hat{y}} \bigcap X = F''$. It is easy to see that F' and F'' are free Cauchy filters in (X, U). It is clear that $A_1 = \tilde{A}_1 \bigcap X \in F'$ and $A_k = \tilde{A}_k \bigcap X \in F''$. Put $\{A_1 = \tilde{A}_1 \bigcap X, A_2 = \tilde{A}_2 \bigcap X, ..., A_k = \tilde{A}_k \bigcap X\}$, $k \leq n$. Then $\{A_1, A_2, ..., A_k\} \subset \alpha$ is a u-chained sequence.

Sufficiency. Let $\hat{\alpha} \in \tilde{U}_{\tilde{X}\backslash X}$ be an arbitrary cover. Then there exists $\alpha \in U$ such that $\tilde{\alpha} \wedge \{\tilde{X}\backslash X\} = \hat{\alpha}$. For α there is a number n that satisfies the conditions of theorem, i.e. for any pair of free Cauchy filters $F'$and F'' in (X, U) one can select a u-chained sequence $\{A_1, A_2, ..., A_k\} \subset \alpha$ such that for $k \leq n$, $A_1 \in F'$ and $A_k \in F''$. We show that for a cover $\hat{\alpha}$ the number $n + 2$ satisfies the conditions of the definition of uniformly chained spacesaa. Let the free Cauchy filters F' and F'' converge to points $\hat{x} \in \tilde{X}\backslash X$ and $\hat{y} \in \tilde{X}\backslash X$ respectively. Then there exist $\hat{A}_{\hat{x}} \in \hat{\alpha}$ and $\hat{A}_{\hat{y}} \in \hat{\alpha}$ such that $\hat{x} \in \hat{A}_{\hat{x}}$, $\hat{y} \in \hat{A}_{\hat{y}}$, $\hat{A}_{\hat{x}} = \tilde{A}_{\hat{x}} \bigcap (\tilde{X}\backslash X)$, $\hat{A}_{\hat{y}} = \tilde{A}_{\hat{y}} \bigcap (\tilde{X}\backslash X)$, $A_{\hat{x}} \in \alpha$, $A_{\hat{y}} \in \alpha$, $A_{\hat{x}} = \tilde{A}_{\hat{x}} \bigcap X$, $A_{\hat{y}} = \tilde{A}_{\hat{y}} \bigcap X$. Put $\{\hat{A}_{\hat{x}}, \hat{A}_1, \hat{A}_2, ..., \hat{A}_k, \hat{A}_{\hat{y}}\}$. It is easy to see that $A_{\hat{x}} \in F'$ and $A_{\hat{y}} \in F''$. Then $A_{\hat{x}} \bigcap A_1 \neq \emptyset$ and $A_k \bigcap A_{\hat{y}} \neq \emptyset$, i.e. $\hat{A}_{\hat{x}} \bigcap \hat{A}_1 \neq \emptyset$ and $\hat{A}_k \bigcap \hat{A}_{\hat{y}} \neq \emptyset$. Thus, $\{\hat{A}_{\hat{x}}, \hat{A}_1, \hat{A}_2, ..., \hat{A}_k, \hat{A}_{\hat{y}}\}$ is a chained sequence. □

Theorem 2.10. *The Tychonoff space* X *is connected if and only if the universal uniformity* U_X *of the space* X *is connected.*

Proof. Let X be a connected Tychonoff space and U_X be the universal uniformity on X. Suppose that there exists a disjunctive cover $\alpha \in U_X$ containing more than one element. Let $A \in \alpha$ an arbitrary element. Put $B = \bigcup\{A' \in \alpha : A' \neq A\}$. Then $X = A \bigcup B$, $A \bigcap B = \emptyset$. Hence, X is not connected. A contradiction. Thus, the universal uniformity U_X is connected.

Conversely, let U_X be the connected universal uniformity of the Tychonoff space X. Suppose that X is not a connected Tychonoff space. Then there is disjunctive open two-element cover $\alpha = \{A, B\}$ of the space X. It is easy to see that $\alpha \in U_X$. Consequently, U_X is not a connected uniformity. A contradiction. Thus, the Tychonoff space X is connected. □

Proposition 2.11. *Completion of a uniformly connected space is uniformly connected.*

Proof. Let $(\tilde{X}, \tilde{U})$, be a completion of a uniformly connected space (X, U), $\tilde{\alpha} \in \tilde{U}$, $\tilde{\alpha} = \{\tilde{A} : A \in \alpha\}$, $\tilde{A} = \tilde{X}\backslash[X\backslash A]_{\tilde{X}}$ be an arbitrary cover and $\tilde{x}, \tilde{y} \in \tilde{X}$ be arbitrary points. Then $\alpha \in U$, $\alpha = \{\tilde{A} \bigcap X : \tilde{A} \in \tilde{\alpha}\}$ and there exist $\tilde{A}_{\tilde{x}}, \tilde{A}_{\tilde{y}} \in \tilde{\alpha}$, such that $\tilde{x} \in \tilde{A}_{\tilde{x}}$, $\tilde{y} \in \tilde{A}_{\tilde{y}}$. Let $x \in \tilde{A}_{\tilde{x}} \bigcap X$ and $y \in \tilde{A}_{\tilde{y}} \bigcap X$. Then there is a finite chain $\{A_1, A_2, ..., A_n\} \subset \alpha$ such that $x \in A_1$ and $y \in A_n$. Let $\tilde{A}_i \in \tilde{\alpha}$ be the elements such that $A_i = \tilde{A}_i \bigcap X$, $i = 1, 2, ..., n$. Since $(\tilde{A}_{\tilde{x}} \bigcap X) \bigcap A_1 \neq \emptyset$ and $(\tilde{A}_{\tilde{y}} \bigcap X) \bigcap A_n \neq \emptyset$, then $\tilde{A}_{\tilde{x}} \bigcap \tilde{A}_1 \neq \emptyset$ and $\tilde{A}_{\tilde{y}} \bigcap \tilde{A}_n \neq \emptyset$. Then the sequence $\{\tilde{A}_{\tilde{x}}, \tilde{A}_1, \tilde{A}_2, ..., \tilde{A}_n, \tilde{A}_{\tilde{y}}\} \subset \tilde{\alpha}$ will be chained. □

Theorem 2.12. *A Tychonoff space* X *has connected Stone-Čech compactification* βX *if and only if its universal uniformity is connected.*

Proof. Necessity. Let X be a connected Tychonoff space with connected Stone-Čech compactification βX. Then by Theorem 2.10, the universal uniformity U_X is connected.

Sufficiency. Let the universal uniformity U_X be connected. We denote by U_X^p the maximal precompact uniformity contained in U_X. It is easy to see that the uniformity U_X^p is also connected. Then the completion $(\tilde{X}, \tilde{U}_X^p)$ of the precompact uniform space (X, U_X^p) is compact and by Proposition 2.11, the uniform space $(\tilde{X}, \tilde{U}_X^p)$ is uniformly connected. As it is known, the space $(\tilde{X}, \tilde{U}_X^p)$ is a Stone-Čech compactification of the space X. Therefore, by Theorem 2.10, βX is connected. □

Let $f : (X, U) \to (Y, V)$ be a uniformly continuous mapping of a uniform space (X, U) onto a uniform space (Y, V). Let $\hat{f} : (\hat{X}, \hat{U}) \to (Y, V)$ be a completion of the mapping f. We denote by $\hat{f}|_{\hat{X}\setminus X} : (\hat{X}\setminus X, \hat{U}_{\hat{X}\setminus X}) \to (Y, V)$ the remainder of the mapping f.

Theorem 2.13. *The remainder $\hat{f}|_{\hat{X}\setminus X} : (\hat{X}\setminus X, \hat{U}_{\hat{X}\setminus X}) \to (Y, V)$ of a uniformly continuous mapping $f : (X, U) \to (Y, V)$ is complete if and only if the uniform space (X, U) is open subspace of the space $(\hat{X}, \hat{U})$.*

Proof. Let the remainder $\hat{f}|_{\hat{X}\setminus X} : (\hat{X}\setminus X, \hat{U}_{\hat{X}\setminus X}) \to (Y, V)$ of a uniform space f be complete. Then $\hat{X}\setminus X$ is closed in $(\hat{X}, \hat{U})$. Thus, (X, U) is open in $(\hat{X}, \hat{U})$.

Conversely, let X be open in $(\hat{X}, \hat{U})$. Then $(\hat{X}\setminus X, \hat{U}_{\hat{X}\setminus X})$ is closed in $(\hat{X}, \hat{U})$. Thus, $\hat{f}|_{\hat{X}\setminus X} : (\hat{X}\setminus X, \hat{U}_{\hat{X}\setminus X}) \to (Y, V)$ is complete. □

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