Filomat 39:18 (2025), 6367–6381
https://doi.org/10.2298/FIL2518367A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

# Hermite–Hadamard–Mercer's-type inequalities for $\mathfrak{h}$-convex function

**Asia Shehzadi[a], Haibo Chen[a,*], Hüseyin Budak[b,c], Wali Haider[d], Muhammad Aamir Ali[e]**

[a] *School of Mathematics and Statistics, Central South University, Changsha 410083, China*
[b] *Department of Mathematics, Saveetha School of Engineering, SIMATS, Saveetha University, Chennai 602105, Tamil Nadu, India*
[c] *Department of Mathematics, Faculty of Science and Arts, Kocaeli University, Kocaeli 41001, Türkiye*
[d] *School of Mathematics and Information Science, Guangzhou University, Guangzhou, 510006, China*
[e] *School of Mathematics Hohai University, Nanjing 210098, China*

**Abstract.** In this paper, we prove significant results related to the Hermite–Hadamard–Mercer-type inequality for $\mathfrak{h}$-convex functions in the framework of Katugampola fractional integral operators. Also, we establish some novel fractional inequalities related to the right and left sides of Hermite–Hadamard–Mercer-type inequalities for differentiable mappings whose derivatives in absolute value are $\mathfrak{h}$-convex function. The findings in this research generalise a number of inequities identified in earlier studies.

## 1. Introduction

The mathematical idea of convexity is extremely important, particularly in the fields of geometry and optimisation. The concept of convexity is considered to have originated in ancient Egypt and Babylon, but it was in the 18th century that mathematicians formalized and studied convex functions. According to the best of our understanding, Karl Hermann Amandus Schwarz [15] introduced the convex function in the late 19th century. His research on convexity made significant contributions to the advancement of mathematical theory. Convex functions are important in economics, biology, and engineering. Convexity is particularly useful in studying mathematical inequalities, such as Jensen inequality, Hermite–Hadamard (H.H) inequality, and Jensen–Mercer inequality, which provide valuable insights and applications for mathematicians and researchers [17, 36]. In [26, 34] Charles Hermite (1822-1901) and Jacques Salomon Hadamard (1822-1963), two French mathematicians introduced H.H inequality. This inequality can be expressed as follows: Let $\Theta : \mathcal{I} \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function and $\nu, \varpi \in \mathcal{I}$ with $\nu < \varpi$, then

$$\Theta\left(\frac{\nu+\varpi}{2}\right) \leq \frac{1}{\varpi-\nu}\int_{\nu}^{\varpi} \Theta(\zeta)d\zeta \leq \frac{\Theta(\nu)+\Theta(\varpi)}{2}. \tag{1}$$

2020 *Mathematics Subject Classification*. Primary 26B25; Secondary 26D10, 26D15
*Keywords*. Convex function, $\mathfrak{h}$-convex function, Jensen–Mercer inequality, Hermite–Hadamard-Mercer-type inequalities
Received: 28 October 2024; Revised: 27 June 2025; Accepted: 09 July 2025
Communicated by Ljubiša D. R. Kočinac
* Corresponding author: Haibo Chen
*Email addresses:* ashehzadi937@gmail.com (Asia Shehzadi), math_chb@csu.edu.cn (Haibo Chen), hsyn.budak@gmail.com (Hüseyin Budak), haiderwali416@gmail.com (Wali Haider), mahr.muhammad.aamir@gmail.com (Muhammad Aamir Ali)
ORCID iDs: https://orcid.org/0009-0005-1101-5536 (Asia Shehzadi), https://orcid.org/0000-0002-9868-7079 (Haibo Chen), https://orcid.org/0000-0001-8843-955X (Hüseyin Budak), https://orcid.org/0009-0001-7065-2755 (Wali Haider), https://orcid.org/0000-0001-5341-4926 (Muhammad Aamir Ali)

Additionally, the direction of the inequality (1) is reversed when the function $\Theta$ is concave. The H.H inequalities are crucial in convex analysis and widely used in various fields like mathematics, optimisation, economics and other disciplines [7, 13, 14, 16, 32].

Jensen's inequality, known as the *king of inequalities*, is a fundamental mathematical inequality that serves as a foundation for other classical inequalities. In recent years, scholars have focused on the Hermite–Jensen–Mercer (HJM) inequality, a novel version of the H.H inequality. In the framework of $k$-fractional conformable integrals, Butt et al [9] have derived unique results of HJM-type inequalities for convex functions by using a new methodology. A generalisation of Mercer's result, known as Jensen's operator inequality for convex functions, is presented by Matkovic et al [28]. The obtained result is utilised to prove Mercer's power means operator for monotonicity property as well as a comparison theorem for quasi-arithmetic means for operators. Xu et al [39] have introduced a generalised form of the Jensen–Mercer inequality specifically for generalised $\mathfrak{h}$-convex functions defined on fractal sets. They established H.H-Mercer local fractional integral inequalities using integral operators associated with the Mittag-Leffler kernel. In [19] authors have developed novel variants of H.H-Mercer-type inequalities via Atangana-Baleanu fractional integral operators with non-local and non-singular kernels. Matkovic and Pecaric [29] have presented several refinements of Jensen–Mercer's inequality.

Abbasi et al [1] have discussed the characterisations of $\mathfrak{h}$-convex functions defined on a convex set within a linear space. By employing this approach, they expand the Jensen–Mercer inequality for $\mathfrak{h}$-convex functions. Moreover, introduced the notion of operator $\mathfrak{h}$-convex functions and established operator versions of Jensen and Jensen–Mercer-type inequalities for certain classes. Vivas-Cortez et al [38] have used generalised fractional integrals to prove H.H-Mercer inequality for convex functions. They introduced novel inequalities of H.H and H.H-Mercer-type for multiple fractional integrals, including Riemann–Liouville, $k$-Riemann–Liouville, conformable, and exponential kernel fractional integrals. Liu [24] has identified certain Simpson-type inequalities for functions with condition whose third derivatives in absolute value are $\mathfrak{h}$-convex and $(\alpha, m)$-convex, respectively. Long and Du [25] have introduced multiplicative $k$-Atangana-Baleanu fractional integrals and studied their properties, constructing H.H-Mercer inequalities. Additionally, they proposed identities for fractional Mercer-type inequalities with applications in differential equations, quadrature formulas, and special means. By utilising Caputo fractional derivatives, Zhao et al [40] derived certain HJM inequalities. Abdeljawad et al [2] gave various H.H-Mercer inequalities involving Riemann–Liouville fractional operators. Additionally, they proved several fractional integral inequalities related to the left side of H.H-Mercer-type inequality for differentiable convex functions. For interval-valued functions, Kara et al [23] first established certain inclusions of fractional H.H-Mercer-type inequalities. Butt et al [8] have proposed novel fractional H.H-Mercer-type inequalities for a differentiable function, with the condition absolute values of its derivatives are convex.

There are numerous results associated with convex functions; two of these are the H.H inequality and the Mercer-type-inequality, which frequently arise in the mathematical literature. Kashuri et al [22] have established novel H.H-type inequalities, specifically designed for approximately $\mathfrak{h}$-convex functions, using the framework of generalised fractional integrals. Furthermore, they include midpoint and trapezoid-type inequalities, also tailored for approximately $\mathfrak{h}$-convex functions and utilising generalised fractional integrals. Ali et al [4] gave a collection of H.H-type inequalities specially designed for $\mathfrak{h}$-convex functions. These inequalities are derived by employing generalised fractional integrals. Budak and Sarikaya [6] have proposed precise definitions that apply to $\mathfrak{h}$-convex stochastic processes. Following this, they proposed a novel version of the H.H inequality specifically designed for $\mathfrak{h}$-convex stochastic processes. Sanja Varosanec, in [37], came up with the idea of $\mathfrak{h}$-convex functions, which are a significant generalisation of convex functions.

The motivation and inspiration for this work are derived from the current research trends and findings in the field, as revealed by the references [1, 3, 4, 10, 12]. In this paper, by using Jensen–Mercer inequality for $\mathfrak{h}$-convex function, we have formulated generalised H.H-Mercer inequalities for Katugampola fractional integrals operator and established some novel H.H-Mercer-type inequalities for differentiable mappings defined by the $\mathfrak{h}$-convexity of their absolute value of derivatives. Further, these results are satisfied for $s$-convex function mentioned in next section.

## 2. Preliminaries

In this section, we present some basic definitions and mathematical preliminaries which extensively used in forthcoming work.

**Definition 2.1.** ([15]) A set $\mathfrak{U}$ in a vector space is termed convex if any two points $\nu, \varpi \in \mathfrak{U}$. The points inside this segment can be stated in the following ways

$$\ell\nu + (1-\ell)\varpi, \quad \forall\, \ell \in [0,1].$$

**Definition 2.2.** ([18])] A function $\Theta : [0,+\infty) \to \mathbb{R}$ is said to be $s$-convex if

$$\ell^s\Theta(\nu) + (1-\ell)^s\Theta(\varpi) \geq \Theta(\ell\nu + (1-\ell)\varpi)$$

for all $\ell \in [0,1], s \in (0,1]$ and $\nu, \varpi \in [0,+\infty)$.

**Definition 2.3.** ([37])] Suppose $\mathfrak{h} : \mathcal{J} \to \mathbb{R}$ is a non-negative function with $\mathfrak{h} \neq 0$. A non-negative function $\Theta : \mathcal{I} = [\nu, \varpi] \to \mathbb{R}$ is said to be $\mathfrak{h}$-convex function, or $\Theta$ belongs to the class $SX(\mathfrak{h}, \mathcal{I})$, then the subsequent inequalities holds:

$$\mathfrak{h}(\ell)\Theta(\nu) + \mathfrak{h}(1-\ell)\Theta(\varpi) \geq \Theta(\ell\nu + (1-\ell)\varpi)$$

for all $\nu, \varpi \in \mathcal{I}$ and $\ell \in [0,1]$.

**Definition 2.4.** ([30]) Consider $\Theta$ be a convex function defined over the real interval $\Omega \subset \mathbb{R}$. If $\vartheta_1, \vartheta_2, \vartheta_3, \ldots, \vartheta_{\mathrm{m}} \in \Omega$ and $\varrho_1, \varrho_2, \varrho_3, \ldots, \varrho_{\mathrm{m}} \geq 0$, then the well-known Jensen inequality stated as

$$\sum_{\jmath=1}^{m} \varrho_\jmath \Theta\left(\vartheta_\jmath\right) \geq \Theta\left(\sum_{\jmath=1}^{m} \varrho_\jmath \vartheta_\jmath\right).$$

**Definition 2.5.** ([27]) Let $\Theta : [\nu, \varpi] \subset \mathbb{R} \to \mathbb{R}$ be convex function. Then, we have

$$\Theta(\nu) + \Theta(\varpi) - \sum_{\jmath=1}^{m} \varrho_\jmath \Theta\left(\vartheta_\jmath\right) \geq \Theta\left(\nu + \varpi - \sum_{\jmath=1}^{m} \varrho_\jmath \vartheta_\jmath\right)$$

for every $\vartheta_\jmath \in [\nu, \varpi]$ and $\varrho_\jmath \in [0,1], (\jmath = 1,2,3,\ldots,m)$ with $\sum_{\jmath=1}^{m} \varrho_\jmath = 1$. The inequality was established by Mercer in [27], and considered as a variant of Jensen's inequality.

**Definition 2.6.** ([20]) Let $\Theta : [\nu, \varpi] \to \mathbb{R}$ be convex function and for every $\varrho, \gamma \in [\nu, \varpi]$. Then, we have

$$\Theta(\nu) + \Theta(\varpi) - \Theta\left(\frac{\varrho+\gamma}{2}\right) \geq \Theta(\nu) + \Theta(\varpi) - \frac{1}{\gamma-\varrho}\int_{\varrho}^{\gamma} \Theta(\zeta)d\zeta \geq \Theta\left(\nu + \varpi - \frac{\varrho+\gamma}{2}\right),$$

or

$$\Theta(\nu) + \Theta(\varpi) - \frac{\Theta(\varrho)+\Theta(\gamma)}{2} \geq \frac{1}{\gamma-\varrho}\int_{\nu+\varpi-\gamma}^{\nu+\varpi-\varrho} \Theta(\zeta)d\zeta \geq \Theta\left(\nu + \varpi - \frac{\varrho+\gamma}{2}\right).$$

**Definition 2.7.** ([37]) Suppose $\mathfrak{h} : \Omega \to \mathbb{R}$ be a non-negative super-multiplicative function and if $\Theta \in X(\mathfrak{h}, \mathcal{I})$, $\varrho_1, \varrho_2, \varrho_3, \ldots, \varrho_m \in \mathcal{I}$, then the following inequality holds

$$\sum_{\jmath=1}^{m} \mathfrak{h}\left(\frac{w_\jmath}{W_m}\right)\Theta\left(\varrho_\jmath\right) \geq \Theta\left(\frac{1}{W_m}\sum_{\jmath=1}^{m} w_\jmath \varrho_\jmath\right),$$

where $W_m = \sum_{\jmath=1}^{m} w_\jmath$ and $w_1, w_2, w_3 \ldots, w_m \in \mathbb{R}^+, (m \geq 2)$.

**Definition 2.8.** ([5]) If $\Theta : \Omega \subset \mathbb{R} \to \mathbb{R}^+$ be $\mathfrak{h}$-convex function on $\Omega$, and $\Theta$ is super-multiplicative function on positive real numbers, then for any positive increasing sequence $\left(\varrho_\jmath\right)_{\jmath=1}^{m} \in \Omega$, and $w_1, w_2, w_3, \ldots, w_m \in \mathbb{R}^+, (m \geq 2)$ such that $W_m = \sum_{\jmath=1}^{m} w_\jmath$ also $\sum_{\jmath=1}^{m} \mathfrak{h}\left(\frac{w_\jmath}{W_m}\right) \leq 1$, then the following inequality holds

$$\Theta(\varrho_1) + \Theta(\varrho_m) - \sum_{\jmath=1}^{m} \mathfrak{h}\left(\frac{w_\jmath}{W_m}\right)\Theta\left(\varrho_\jmath\right) \geq \Theta\left(\varrho_1 + \varrho_m - \frac{1}{W_m}\sum_{\jmath=1}^{m} w_\jmath \varrho_\jmath\right).$$

When inequality is repudiated then, $\Theta$ is a $\mathfrak{h}$-concave.

**Definition 2.9.** ([33]) The Riemann–Liouville fractional integrals of order $\alpha > 0$ with $\nu \geq 0$ are defined by

$$\mathcal{L}^{\alpha}_{\nu+}\Theta(\eta) = \frac{1}{\Gamma(\alpha)}\int_{\nu}^{\eta}(\eta - \zeta)^{\alpha-1}\Theta(\zeta)d\zeta, \quad \eta > \nu$$

and

$$\mathcal{L}^{\alpha}_{\varpi-}\Theta(\eta) = \frac{1}{\Gamma(\alpha)}\int_{\eta}^{\varpi}(\zeta - \eta)^{\alpha-1}\Theta(\zeta)d\zeta, \quad \eta < \varpi,$$

respectively. Here, $\Gamma(\alpha)$ is the Gamma function and $\mathcal{L}^{0}_{\nu+}\Theta(\eta) = \mathcal{L}^{0}_{\varpi-}\Theta(\eta) = \Theta(\eta)$.

**Definition 2.10.** ([35]) The left-sided and right-sided Hadamard fractional integrals of order $\alpha > 0$ of the function $\Theta$ are given by

$$\mathcal{H}^{\alpha}_{\nu^+}\Theta(\eta) = \frac{1}{\Gamma(\alpha)}\int_{\nu}^{\chi}\ln\left(\frac{\eta}{\zeta}\right)^{\alpha-1}\frac{\Theta(\zeta)}{\zeta}d\zeta, \quad (\eta > \nu)$$

and

$$\mathcal{H}^{\alpha}_{\varpi^-}\Theta(\eta) = \frac{1}{\Gamma(\alpha)}\int_{\eta}^{\varpi}\ln\left(\frac{\zeta}{\chi}\right)^{\alpha-1}\frac{\Theta(\zeta)}{\zeta}d\zeta, \quad (\eta < \varpi),$$

respectively.

**Definition 2.11.** ([21]) Let $\nu, \varpi \in \mathbb{R}$ with $\nu < \varpi, \varkappa \in [\nu, \varpi], \rho > 0, r \in \mathbb{R}, p \geq 1$ and $\alpha > 0$. Then the left and right side Katugampola fractional integrals of order $\alpha$ of the function $\Theta \in \Upsilon^p_r(\nu, \varpi)$ are defined by

$$^{\rho}\mathcal{I}^{\alpha}_{\nu^+}\Theta(\chi) = \frac{1}{\Gamma(\alpha)}\int_{\nu}^{\chi}\left(\frac{\chi^{\rho} - \zeta^{\rho}}{\rho}\right)^{\alpha-1}\Theta(\zeta)\zeta^{\rho-1}d\zeta \quad (\chi > \nu)$$

and

$$^{\rho}\mathcal{I}^{\alpha}_{\varpi^-}\Theta(\chi) = \frac{1}{\Gamma(\alpha)}\int_{\chi}^{\varpi}\left(\frac{\zeta^{\rho} - \chi^{\rho}}{\rho}\right)^{\alpha-1}\Theta(\zeta)\zeta^{\rho-1}d\zeta \quad (\chi < \varpi),$$

respectively. Here $\Upsilon^p_r(\nu, \varpi)$, is the space of all the complex-valued Lebesgue measurable function $\Theta$ on $[\nu, \varpi]$ such that $||\Theta||_{\Upsilon^p_r} = \int_{\nu}^{\varpi}|\zeta^r\Theta(\zeta)|^p\frac{d\zeta}{\zeta} < \infty$ for $1 \leq p < \infty$ and $||\Theta||_{\Upsilon^{\infty}_r} = ess \sup_{\nu \leq \zeta \leq \varpi}[\zeta^r|\Theta(\zeta)|] < \infty$.

Here, we recall the following lemmas that are required to prove trapezoidal type inequalities for $\mathfrak{h}$-convex function, and these results are established in [11].

**Lemma 2.12.** *Assume that $\Theta : [\nu^p, \varpi^p] \to \mathbb{R}$ is a differentiable mapping on $(\nu^p, \varpi^p)$, $\alpha, p > 0$ and $\nu, \varpi \in [0, \infty)$ with $\varpi > \nu$. Then, we have*

$$\frac{\Theta(\nu^p + \varpi^p - \vartheta^p) + \Theta(\nu^p + \varpi^p - \varrho^p)}{2} - \frac{\Gamma(\alpha+1)p^{\alpha}}{2\left(\varrho^p - \vartheta^p\right)^{\alpha}}\left[{}^{p}\mathcal{I}^{\alpha}_{(\nu^p+\varpi^p-\varrho^p)^+}\Theta(\nu^p + \varpi^p - \vartheta^p) + {}^{p}\mathcal{I}^{\alpha}_{(\nu^p+\varpi^p-\vartheta^p)^-}\Theta(\nu^p + \varpi^p - \varrho^p)\right]$$

$$= \frac{p(\varpi^p - \nu^p)}{2}\int_0^1[(\ell^p)^{\alpha} - (1 - \ell^p)^{\alpha}]\ell^{p-1}\Theta'(\nu^p + \varpi^p - (\ell^p x^p + (1 - \ell^p)\varrho^p))d\ell$$

*holds for all* $\vartheta, \varrho \in [\nu, \varpi]$. *In order to facilitate, where*

$$\Omega_{\Theta}(\alpha, p, \nu, \varpi) = \frac{\Theta(\nu^p + \varpi^p - \vartheta^p) + \Theta(\nu^p + \varpi^p - \varrho^p)}{2} - \frac{\Gamma(\alpha+1)p^{\alpha}}{2(\varrho^p - \vartheta^p)^{\alpha}}\left[{}^{p}\mathcal{I}^{\alpha}_{(\nu^p+\varpi^p-\varrho^p)^+}\Theta(\nu^p + \varpi^p - \vartheta^p) + {}^{p}\mathcal{I}^{\alpha}_{(\nu^p+\varpi^p-\vartheta^p)^-}\Theta(\nu^p + \varpi^p - \varrho^p)\right].$$

**Lemma 2.13.** *Suppose that* $\Theta : [\nu^p, \varpi^p] \to \mathbb{R}$ *is a positive differentiable mapping on* $(\nu^p, \varpi^p)$ *with* $\varpi > \nu$ *and* $\alpha, p > 0, m \in N, \nu, \varpi \in [0, \infty)$. *Then, we have*

$$\begin{aligned}
&\frac{m^{\alpha}\Gamma(\alpha+1)}{(\varrho^p - \vartheta^p)^{\alpha} p}\Bigg[{}^{p}\mathcal{I}^{\alpha}_{((m-1)\nu^p+\varpi^p-\frac{\vartheta^p+(m-1)\varrho^p}{m})^-}\Theta((m-1)\nu^p + \varpi^p - \varrho^p) \\
&+ {}^{p}\mathcal{I}^{\alpha}_{(\nu^p+(m-1)\varpi^p-\frac{(m-1)\vartheta^p+\varrho^p}{m})^+}\Theta(\nu^p + (m-1)\varpi^p - \vartheta^p)\Bigg] - \Bigg[\Theta\left(\nu^p(m-1) + \varpi^p - \left(\frac{(m-1)\vartheta^p + \varrho^p}{m}\right)\right) \\
&+ \Theta\left(\nu^p + \varpi^p(m-1) - \left(\frac{\vartheta^p + (m-1)\varrho^p}{m}\right)\right)\Bigg] \\
&= \frac{p(\varpi^p - \nu^p)}{m}\int_0^1 \ell^{p(\alpha+1)-1}\Bigg[\Theta'\left(\nu^p(m-1) + \varpi^p - \left(\frac{m-\ell^p}{m}\vartheta^p + \frac{\ell^p}{m}\varrho^p\right)\right) \\
&- \Theta'\left(\nu^p + \varpi^p(m-1) - \left(\frac{\ell^p}{m}\vartheta^p + \frac{m-\ell^p}{m}\varrho^p\right)\right)\Bigg]d\ell
\end{aligned}$$

*holds for all* $\vartheta, \varrho \in [\nu, \varpi]$. *For convenience, where*

$$\begin{aligned}
&\Omega_1(\alpha, p, \nu, \varpi, m) \\
&= \frac{m^{\alpha}\Gamma(\alpha+1)}{(\varrho^p - \vartheta^p)^{\alpha} p}\left[{}^{p}\mathcal{I}^{\alpha}_{((m-1)\nu^p+\varpi^p-\frac{\vartheta^p+(m-1)\varrho^p}{m})^-}\Theta((m-1)\nu^p + \varpi^p - \varrho^p) + {}^{p}\mathcal{I}^{\alpha}_{(\nu^p+(m-1)\varpi^p-\frac{(m-1)\vartheta^p+\varrho^p}{m})^+}\Theta(\nu^p + (m-1)\varpi^p - \vartheta^p)\right] \\
&- \left[\Theta\left(\nu^p(m-1) + \varpi^p - \left(\frac{(m-1)\vartheta^p + \varrho^p}{m}\right)\right) + \Theta\left(\nu^p + \varpi^p(m-1) - \left(\frac{\vartheta^p + (m-1)\varrho^p}{m}\right)\right)\right].
\end{aligned}$$

## 3. Main results

### 3.1. Hermite–Hadamard–Mercer-type inequalities for $\mathfrak{h}$-convex function via Katugampola fractional integral operator

This section provides new Hermite-Hadamard-Mercer-type inequalities for $\mathfrak{h}$-convex functions by employing the Katugampola fractional integral operator.

**Theorem 3.1.** *Suppose that* $\Theta : [\nu^p, \varpi^p] \subset \mathbb{R} \to \mathbb{R}$ *is a continuous function with* $0 \le \nu < \varpi$ *and* $\alpha, p > 0$. *If* $\Theta$ *is a* $\mathfrak{h}$*-convex function on* $[\nu^p, \varpi^p]$, *then we have*

$$\begin{aligned}
\Theta\left(\nu^p + \varpi^p - \frac{\vartheta^p + \varrho^p}{2}\right) &\le \Theta(\nu^p) + \Theta(\varpi^p) - \frac{p^{\alpha}\Gamma(\alpha+1)\mathfrak{h}(1/2)}{(\varrho^p - \vartheta^p)^{\alpha}}\left[{}^{p}\mathcal{I}^{\alpha}_{\vartheta^+}\Theta(\varrho^p) + {}^{p}\mathcal{I}^{\alpha}_{\varrho^-}\Theta(\vartheta^p)\right] \\
&\le \Theta(\nu^p) + \Theta(\varpi^p) - \Theta\left(\frac{\vartheta^p + \varrho^p}{2}\right)
\end{aligned} \tag{2}$$

*and*

$$\begin{aligned}
&\frac{1}{\alpha p\mathfrak{h}(1/2)}\Theta\left(\nu^p + \varpi^p - \frac{\vartheta^p + \varrho^p}{2}\right) \\
&\le \frac{p^{\alpha-1}\Gamma(\alpha)}{(\varrho^p - \vartheta^p)^{\alpha}}\left[{}^{p}\mathcal{I}^{\alpha}_{(\nu^p+\varpi^p-\varrho^p)^+}\Theta(\nu^p + \varpi^p - \vartheta^p) + {}^{p}\mathcal{I}^{\alpha}_{(\nu^p+\varpi^p-\vartheta^p)^-}\Theta(\nu^p + \varpi^p - \varrho^p)\right] \\
&\le \left[2\left[\Theta(\nu^p) + \Theta(\varpi^p)\right] - \left[\Theta(\vartheta^p) + \Theta(\varrho^p)\right]\right]\int_0^1 \ell^{\alpha p-1}\left(\mathfrak{h}(\ell^p) + \mathfrak{h}(1-\ell^p)\right)d\ell
\end{aligned} \tag{3}$$

*for all* $\vartheta, \varrho \in [\nu, \varpi]$.

*Proof.* By using Jensen–Mercer inequality for $\mathfrak{h}$-convex function, we get

$$\Theta\left(\nu^p + \varpi^p - \frac{\vartheta_1^p + \varrho_1^p}{2}\right) \leq \Theta(\nu^p) + \Theta(\varpi^p) - \mathfrak{h}(1/2)\left[\Theta(\vartheta_1^p) + \Theta(\varrho_1^p)\right]$$

for all $\vartheta_1, \varrho_1 \in [\nu, \varpi]$. By changing of the variables $\vartheta_1^p = \ell^p\vartheta^p + (1-\ell^p)\varrho^p$ and $\varrho_1^p = (1-\ell^p)\vartheta^p + \ell^p\varrho^p$ for $\vartheta, \varrho \in [\nu, \varpi]$ and $\ell \in [0,1]$, then the above inequality leads to

$$\Theta\left(\nu^p + \varpi^p - \frac{\vartheta^p + \varrho^p}{2}\right) \leq \Theta(\nu^p) + \Theta(\varpi^p) - \mathfrak{h}(1/2)\left[\Theta(\ell^p\vartheta^p + (1-\ell^p)\varrho^p) + \Theta((1-\ell^p)\vartheta^p + \ell^p\varrho^p)\right]. \tag{4}$$

Multiplying on both sides of (4) by $\ell^{\alpha p-1}$ and then integrating the modified inequality with respect to $\ell$ over $[0,1]$, then we find that

$$\begin{aligned}
&\frac{1}{\alpha p}\Theta\left(\nu^p + \varpi^p - \frac{\vartheta^p + \varrho^p}{2}\right)\\
&\leq \frac{1}{\alpha p}\left[\Theta(\nu^p) + \Theta(\varpi^p)\right] - \mathfrak{h}(1/2)\left[\int_0^1 \ell^{\alpha p-1}\Theta(\ell^p\vartheta^p + (1-\ell^p)\varrho^p)d\ell + \int_0^1 \ell^{\alpha p-1}\Theta((1-\ell^p)\vartheta^p + \ell^p\varrho^p)d\ell\right]\\
&= \frac{1}{\alpha p}\left[\Theta(\nu^p) + \Theta(\varpi^p)\right] - \mathfrak{h}(1/2)\left[\int_\vartheta^\varrho \left(\frac{\varrho^p - \tau^p}{\varrho^p - \vartheta^p}\right)^{\alpha-1}\Theta(\tau^p)\frac{\tau^{p-1}}{\varrho^p - \vartheta^p}d\tau + \int_\vartheta^\varrho \left(\frac{\tau^p - \vartheta^p}{\varrho^p - \vartheta^p}\right)^{\alpha-1}\Theta(\tau^p)\frac{\tau^{p-1}}{\varrho^p - \vartheta^p}d\tau\right]\\
&= \frac{1}{\alpha p}\left[\Theta(\nu^p) + \Theta(\varpi^p)\right] - \frac{\Gamma(\alpha)\mathfrak{h}(1/2)}{p^{1-\alpha}(\varrho^p - \vartheta^p)^\alpha}\left[{}^p\mathcal{I}^\alpha_{\vartheta^+}\Theta(\varrho^p) + {}^p\mathcal{I}^\alpha_{\varrho^-}\Theta(\vartheta^p)\right].
\end{aligned}$$

Hence, we obtained the prove of first inequality (2). Now, suppose if $\Theta$ is an $\mathfrak{h}$-convex function and $\ell \in [0,1]$, we get

$$\begin{aligned}
\Theta\left(\frac{\vartheta^p + \varrho^p}{2}\right) &= \Theta\left(\frac{\ell^p\vartheta^p + (1-\ell^p)\varrho^p + (1-\ell^p)\vartheta^p + \ell^p\varrho^p}{2}\right)\\
&\leq \mathfrak{h}(1/2)\left[\Theta(\ell^p\vartheta^p + (1-\ell^p)\varrho^p) + \Theta((1-\ell^p)\vartheta^p + \ell^p\varrho^p)\right].
\end{aligned} \tag{5}$$

Taking product $\ell^{\alpha p-1}$ on both side of (5) and performing an integration over $[0,1]$ with respect to $\ell$, then we derive

$$\frac{1}{\alpha p}\Theta\left(\frac{\vartheta^p + \varrho^p}{2}\right) \leq \frac{\mathfrak{h}(1/2)\Gamma(\alpha)}{(\varrho^p - \vartheta^p)^\alpha p^{1-\alpha}}\left[{}^p\mathcal{I}^\alpha_{\vartheta^+}\Theta(\varrho^p) + {}^p\mathcal{I}^\alpha_{\varrho^-}\Theta(\vartheta^p)\right]$$

and then

$$-\Theta\left(\frac{\vartheta^p + \varrho^p}{2}\right) \geq -\frac{p^\alpha\Gamma(\alpha+1)\mathfrak{h}(1/2)}{(\varrho^p - \vartheta^p)^\alpha}\left[{}^p\mathcal{I}^\alpha_{\vartheta^+}\Theta(\varrho^p) + {}^p\mathcal{I}^\alpha_{\varrho^-}\Theta(\vartheta^p)\right]. \tag{6}$$

Adding $\Theta(\nu^p) + \Theta(\varpi^p)$ on both sides of the inequality (6), we obtain the second inequality of (2).

Next, we prove inequality (3). From the $\mathfrak{h}$-convexity of $\Theta$, we have

$$\begin{aligned}
\Theta\left(\nu^p + \varpi^p - \frac{\vartheta_1^p + \varrho_1^p}{2}\right) &= \Theta\left(\frac{\nu^p + \varpi^p - \vartheta_1^p + \nu^p + \varpi^p - \varrho_1^p}{2}\right)\\
&\leq \mathfrak{h}(1/2)\left(\Theta(\nu^p + \varpi^p - \vartheta_1^p) + \Theta(\nu^p + \varpi^p - \varrho_1^p)\right)
\end{aligned} \tag{7}$$

for all $\vartheta_1, \varrho_1 \in [\nu, \varpi]$. Let $\vartheta, \varrho \in [\nu, \varpi]$, $\ell \in [0,1]$, by replacing $\nu^p + \varpi^p - \vartheta_1^p = \ell^p(\nu^p + \varpi^p - \vartheta^p) + (1-\ell^p)(\nu^p + \varpi^p - \varrho^p)$ and $\nu^p + \varpi^p - \varrho_1^p = (1-\ell^p)(\nu^p + \varpi^p - \vartheta^p) + \ell^p(\nu^p + \varpi^p - \varrho^p)$ in (7), then we find

$$\Theta\left(\nu^p + \varpi^p - \frac{\vartheta^p + \varrho^p}{2}\right) \leq \mathfrak{h}(1/2)\Big[\Theta\Big(\ell^p(\nu^p + \varpi^p - \vartheta^p) + (1-\ell^p)(\nu^p + \varpi^p - \varrho^p)\Big) + \Theta\Big((1-\ell^p)(\nu^p + \varpi^p - \vartheta^p) + \ell^p(\nu^p + \varpi^p - \varrho^p)\Big)\Big]. \tag{8}$$

Multiplying on both sides of (8) by $\ell^{\alpha p-1}$ and integrating with respect to $\ell$ over $[0,1]$, we conclude

$$\begin{aligned}\frac{1}{\alpha p}\Theta\left(\nu^p + \varpi^p - \frac{\vartheta^p + \varrho^p}{2}\right) &\leq \mathfrak{h}(1/2)\Big[\int_0^1 \ell^{\alpha p-1}\Theta\left(\ell^p(\nu^p + \varpi^p - \vartheta^p) + (1-\ell^p)(\nu^p + \varpi^p - \varrho^p)\right) d\ell \\ &\quad + \int_0^1 \ell^{\alpha p-1}\Theta\left((1-\ell^p)(\nu^p + \varpi^p - \vartheta^p) + \ell^p(\nu^p + \varpi^p - \varrho^p)\right) d\ell\Big] \\ &= \frac{\mathfrak{h}(1/2)\Gamma(\alpha)}{p^{1-\alpha}(\varrho^p - \vartheta^p)^\alpha}\Big[\int_{\nu^p+\varpi^p-\varrho^p}^{\nu^p+\varpi^p-\vartheta^p} \left(\tau^p - (\nu^p + \varpi^p - \varrho^p)\right)^{\alpha-1}\tau^{p-1}\Theta(\tau^p)\, d\tau \\ &\quad + \int_{\nu^p+\varpi^p-\varrho^p}^{\nu^p+\varpi^p-\vartheta^p} \left((\nu^p + \varpi^p - \vartheta^p) - \tau^p\right)^{\alpha-1}\tau^{p-1}\Theta(\tau^p)\, d\tau\Big]\end{aligned} \tag{9}$$

so, we can be rewritten inequality (9) as

$$\frac{1}{\alpha p \mathfrak{h}(1/2)}\Theta\left(\nu^p + \varpi^p - \frac{\vartheta^p + \varrho^p}{2}\right) \leq \frac{p^{\alpha-1}\Gamma(\alpha)}{(\varrho^p - \vartheta^p)^\alpha}\left[{}^p\mathcal{I}^\alpha_{(\nu^p+\varpi^p-\varrho^p)^+}\Theta(\nu^p + \varpi^p - \vartheta^p) + {}^p\mathcal{I}^\alpha_{(\nu^p+\varpi^p-\vartheta^p)^-}\Theta(\nu^p + \varpi^p - \varrho^p)\right]. \tag{10}$$

Thus, the proof of first inequality (3) is holds. On the other hand side from the $\mathfrak{h}$ convexity of $\Theta$, we can write as

$$\Theta\left(\ell^p(\nu^p + \varpi^p - \vartheta^p) + (1-\ell^p)(\nu^p + \varpi^p - \varrho^p)\right) \leq \mathfrak{h}(\ell^p)\Theta(\nu^p + \varpi^p - \vartheta^p) + \mathfrak{h}(1-\ell^p)\Theta(\nu^p + \varpi^p - \varrho^p) \tag{11}$$

and

$$\Theta\left((1-\ell^p)(\nu^p + \varpi^p - \vartheta^p) + \ell^p(\nu^p + \varpi^p - \varrho^p)\right) \leq \mathfrak{h}(1-\ell^p)\Theta(\nu^p + \varpi^p - \vartheta^p) + \mathfrak{h}(\ell^p)\Theta(\nu^p + \varpi^p - \varrho^p). \tag{12}$$

Summing up the inequality (11) and (12) and using Jensen–Mercer inequality, we obtain

$$\begin{aligned}\Theta\left(\ell^p(\nu^p + \varpi^p - \vartheta^p)\right. &\left.+ (1-\ell^p)(\nu^p + \varpi^p - \varrho^p)\right) + \Theta\left((1-\ell^p)(\nu^p + \varpi^p - \vartheta^p) + \ell^p(\nu^p + \varpi^p - \varrho^p)\right) \\ &\leq [\mathfrak{h}(\ell^p) + \mathfrak{h}(1-\ell^p)]\left[\Theta(\nu^p + \varpi^p - \vartheta^p) + \Theta(\nu^p + \varpi^p - \varrho^p)\right] \\ &= [\mathfrak{h}(\ell^p) + \mathfrak{h}(1-\ell^p)]\left[2[\Theta(\nu^p) + \Theta(\varpi^p)] - [\Theta(\vartheta^p) + \Theta(\varrho^p)]\right].\end{aligned}$$

Taking product on both sides of above inequality by $\ell^{\alpha p-1}$ and then subsequently integrating the outcome to $\ell$ over $[0,1]$, we acquire

$$\begin{aligned}&\frac{p^{\alpha-1}\Gamma(\alpha)}{(\varrho^p - \vartheta^p)^\alpha}\left[{}^p\mathcal{I}^\alpha_{(\nu^p+\varpi^p-\varrho^p)^+}\Theta((\nu^p + \varpi^p - \vartheta^p)) + {}^p\mathcal{I}^\alpha_{(\nu^p+\varpi^p-\vartheta^p)^-}\Theta((\nu^p + \varpi^p - \varrho^p))\right] \\ &\leq \Big[2[\Theta(\nu^p) + \Theta(\varpi^p)] - [\Theta(\vartheta^p) + \Theta(\varrho^p)]\Big]_0 I_1 \ell^{\alpha p-1}[\mathfrak{h}(\ell^p) + \mathfrak{h}(1-\ell^p)]d\ell.\end{aligned}$$

Hence, the second inequality (3) is holds. This completes the proof. □

**Corollary 3.2.** *Let $\alpha = 1$ and $p = 1$, in Theorem 3.1, then we obtain the following inequalities for classical $\mathfrak{h}$-convex function*

$$\Theta\left(\nu + \varpi - \frac{\vartheta + \varrho}{2}\right) \leq \Theta(\nu) + \Theta(\varpi) - \frac{2\mathfrak{h}(1/2)}{(\varrho - \vartheta)} \int_{\vartheta}^{\varrho} \Theta(\tau)d\tau \leq \Theta(\nu) + \Theta(\varpi) - \Theta\left(\frac{\vartheta + \varrho}{2}\right)$$

*and*

$$\begin{aligned}\frac{1}{\mathfrak{h}(1/2)}\Theta\left(\nu + \varpi - \frac{\vartheta + \varrho}{2}\right) &\leq \frac{2}{(\varrho - \vartheta)} \int_{\nu+\varpi-\varrho}^{\nu+\varpi-\vartheta} \Theta(\tau)d\tau \leq [\Theta(\nu + \varpi - \vartheta) + \Theta(\nu + \varpi - \varrho)] \int_0^1 [\mathfrak{h}(\ell) + \mathfrak{h}(1-\ell)]d\ell \\ &\leq \left[2\left[\Theta(\nu) + \Theta(\varpi)\right] - \left[\Theta(\vartheta) + \Theta(\varrho)\right]\right] \int_0^1 [\mathfrak{h}(\ell) + \mathfrak{h}(1-\ell)]d\ell.\end{aligned}$$

**Remark 3.3.** By choosing $\mathfrak{h}(\ell) = \ell$, in Corollary 3.2, we derive [20, Theorem 2.1].

**Remark 3.4.** By setting $\mathfrak{h}(\ell) = \ell$, in Theorem 3.1, we arrive [11, Theorem 2.1].

**Corollary 3.5.** *If we take $\mathfrak{h}(\ell) = \ell^s, s \in (0, 1]$, in Theorem 3.1, then we find the subsequently inequalities for s-convex function*

$$\begin{aligned}\Theta\left(\nu^p + \varpi^p - \frac{\vartheta^p + \varrho^p}{2}\right) &\leq \Theta(\nu^p) + \Theta(\varpi^p) - \frac{p^\alpha \Gamma(\alpha+1)}{2^s(\varrho^p - \vartheta^p)^\alpha}\left[{}^p\mathcal{I}^\alpha_{\vartheta^+}\Theta(\varrho^p) + {}^p\mathcal{I}^\alpha_{\varrho^-}\Theta(\vartheta^p)\right] \\ &\leq \Theta(\nu^p) + \Theta(\varpi^p) - \Theta\left(\frac{\vartheta^p + \varrho^p}{2}\right)\end{aligned} \tag{13}$$

*and*

$$\begin{aligned}\frac{2^s}{\alpha}\Theta\left(\nu^p + \varpi^p - \frac{\vartheta^p + \varrho^p}{2}\right) &\leq \frac{\Gamma(\alpha)p^\alpha}{(\varrho^p - \vartheta^p)^\alpha}\left[{}^p\mathcal{I}^\alpha_{(\nu^p+\varpi^p-\varrho^p)^+}\Theta(\nu^p + \varpi^p - \vartheta^p) + {}^p\mathcal{I}^\alpha_{(\nu^p+\varpi^p-\vartheta^p)^-}\Theta(\nu^p + \varpi^p - \varrho^p)\right] \\ &\leq \left[2[\Theta(\nu^p) + \Theta(\varpi^p)] - [\Theta(\vartheta^p) + \Theta(\varrho^p)]\right]\left[\frac{1}{(\alpha+s)} + \frac{\Gamma(\alpha)\Gamma(s+1)}{\Gamma(\alpha+s+1)}\right].\end{aligned} \tag{14}$$

**Theorem 3.6.** *Suppose that $\Theta : [\nu^p, \varpi^p] \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $(\nu^p, \varpi^p)$ with $\nu < \varpi$ such that $\Theta' \in L_1[\nu^p, \varpi^p]$, then we have the following inequality*

$$\begin{aligned}&|\Omega_\Theta(\alpha, p, \nu, \varpi)| \\ &\leq p(\varpi^p - \nu^p)\left(|\Theta'(\nu^p)| + |\Theta'(\varpi^p)| - \frac{|\Theta'(\vartheta^p)| + |\Theta'(\varrho^p)|}{2}\right)\int_0^{\frac{1}{2}} [(1-\ell^p)^\alpha - (\ell^p)^\alpha]\ell^{p-1}\left(\mathfrak{h}(\ell^p) + \mathfrak{h}(1-\ell^p)\right)d\ell\end{aligned} \tag{15}$$

*holds for all $\vartheta, \varrho \in [\nu, \varpi]$ and if $|\Theta'|$ is an $\mathfrak{h}$-convex on $[\nu^p, \varpi^p]$.*

*Proof.* It follows from Lemma 2.12, $|\Theta'|$ is an $\mathfrak{h}$-convex and $\mathfrak{h}$ is a non-negative function, then

$$\begin{aligned}
&|\Omega_\Theta(\alpha,p,\nu,\varpi)| \\
&\leq \left|\frac{p(\varpi^p-\nu^p)}{2}\int_0^1[(\ell^p)^\alpha-(1-\ell^p)^\alpha]\ell^{p-1}\Theta'(\nu^p+\varpi^p-(\ell^p\vartheta^p+(1-\ell^p)\varrho^p))d\ell\right| \\
&\leq \frac{p(\varpi^p-\nu^p)}{2}\left[\int_0^1|(\ell^p)^\alpha-(1-\ell^p)^\alpha|\ell^{p-1}|\Theta'(\ell^p(\nu^p+\varpi^p-\vartheta^p)+(1-\ell^p)(\nu^p+\varpi^p-\varrho^p))|d\ell\right] \\
&= \frac{p(\varpi^p-\nu^p)}{2}\left[\int_0^{\frac{1}{2}}[(1-\ell^p)^\alpha-(\ell^p)^\alpha]\ell^{p-1}\Big(\mathfrak{h}(\ell^p)|\Theta'(\nu^p+\varpi^p-\vartheta^p)|+\mathfrak{h}(1-\ell^p)|\Theta'(\nu^p+\varpi^p-\varrho^p)|\Big)d\ell\right. \\
&\quad\left.+\int_{\frac{1}{2}}^1[(\ell^p)^\alpha-(1-\ell^p)^\alpha]\ell^{p-1}\Big(\mathfrak{h}(\ell^p)|\Theta'(\nu^p+\varpi^p-\vartheta^p)|+\mathfrak{h}(1-\ell^p)|\Theta'(\nu^p+\varpi^p-\varrho^p)|\Big)d\ell\right] \\
&= \frac{p(\varpi^p-\nu^p)}{2}\left[\int_0^{\frac{1}{2}}[(1-\ell^p)^\alpha-(\ell^p)^\alpha]\ell^{p-1}\Big(\mathfrak{h}(\ell^p)|\Theta'(\nu^p+\varpi^p-\vartheta^p)|+\mathfrak{h}(1-\ell^p)|\Theta'(\nu^p+\varpi^p-\varrho^p)|\Big)d\ell\right. \\
&\quad\left.+\int_0^{\frac{1}{2}}[(1-\ell^p)^\alpha-(\ell^p)^\alpha]\ell^{p-1}\Big(\mathfrak{h}(1-\ell^p)|\Theta'(\nu^p+\varpi^p-\vartheta^p)|+\mathfrak{h}(\ell^p)|\Theta'(\nu^p+\varpi^p-\varrho^p)|\Big)d\ell\right].
\end{aligned}$$

After simplification, we get the required result. □

**Corollary 3.7.** *Let $\alpha = 1$ and $p = 1$, then Theorem 3.6 gives the inequality for classical $\mathfrak{h}$-convex function*

$$\begin{aligned}
&\left|\frac{\Theta(\nu+\varpi-\vartheta)+\Theta(\nu+\varpi-\varrho)}{2}-\frac{1}{\varrho-\vartheta}\int_{\nu+\varpi-\varrho}^{\nu+\varpi-\vartheta}\Theta(\tau)d\tau\right| \\
&\leq(\varpi-\nu)\Big(|\Theta'(\nu)|+|\Theta'(\varpi)|-\frac{|\Theta'(\vartheta)|+|\Theta'(\varrho)|}{2}\Big)\int_0^{\frac{1}{2}}(1-2\ell)[\mathfrak{h}(\ell)+\mathfrak{h}(1-\ell)]d\ell.
\end{aligned}$$

**Remark 3.8.** If we take $\mathfrak{h}(\ell) = \ell$, in Theorem 3.6, then, we have [11, Theorem 2.3].

**Corollary 3.9.** *If we take $\mathfrak{h}(\ell) = \ell^s$, $s \in (0, 1]$ , in Theorem 3.6, then we get the inequality for s-convex function*

$$\begin{aligned}
&\left|\frac{\Theta(\nu^p+\varpi^p-\vartheta^p)+\Theta(\nu^p+\varpi^p-\varrho^p)}{2}-\frac{\Gamma(\alpha+1)p^\alpha}{2\left(\varrho^p-\vartheta^p\right)^\alpha}\Big[{}^p\mathcal{I}^\alpha_{(\nu^p+\varpi^p-\varrho^p)^+}\Theta(\nu^p+\varpi^p-\vartheta^p)\right. \\
&\quad\left.+{}^p\mathcal{I}^\alpha_{(\nu^p+\varpi^p-\vartheta^p)^-}\Theta(\nu^p+\varpi^p-\varrho^p)\Big]\right|\leq p(\varpi^p-\nu^p)\left(|\Theta'(\nu^p)|+|\Theta'(\varpi^p)|-\frac{|\Theta'(\vartheta^p)|+|\Theta'(\varrho^p)|}{2}\right) \\
&\left[\frac{{}_2F_1(-\alpha,s+1;s+2;1/2^p)}{p(s+1)2^{p(s+1)}}-\frac{{}_2F_1(1+\alpha,-s;\alpha+2;1/2^p)}{p(\alpha+1)2^{p(\alpha+1)}}+\frac{1-\left(1-\frac{1}{2^p}\right)^{\alpha+s+1}}{p(\alpha+s+1)}-\frac{1}{2^{p(\alpha+s+1)}p(\alpha+s+1)}\right].
\end{aligned}$$

### *3.2. Generalized Hermite–Hadamard–Mercer-type inequalities for $\mathfrak{h}$-convex function via Katugampola fractional integral*

This section contain a new version of Jensen–Mercer inequality for $\mathfrak{h}$-convex function by using Katugampola fractional integral operator.

**Theorem 3.10.** *Suppose that $\Theta \in SX(\mathfrak{h}, \mathcal{I}), \nu, \varpi \in \mathcal{I}$, with $\varpi > \nu$ such that $\Theta \in L_c^p[\nu^p, \varpi^p]$. Then, we have the subsequent inequality for $\mathfrak{h}$-convex function*

$$\begin{aligned}\frac{1}{\alpha p \mathfrak{h}(1/m)}\Theta\left(\nu^p+\varpi^p-\frac{\vartheta^p+\varrho^p}{2}\right) &\le \frac{m^\alpha\Gamma(\alpha)}{(\varrho^p-\vartheta^p)^\alpha p^{1-\alpha}}\Big[{}^p\mathcal{I}^\alpha_{\left((m-1)\nu^p+\varpi^p-\frac{\vartheta^p+(m-1)\varrho^p}{m}\right)^+}\Theta((m-1)\nu^p+\varpi^p-\varrho^p)\\ &\quad+{}^p\mathcal{I}^\alpha_{\left(\nu^p+(m-1)\varpi^p-\frac{(m-1)\vartheta^p+\varrho^p}{m}\right)^-}\Theta(\nu^p+(m-1)\varpi^p-\vartheta^p)\Big]\\ &\le\frac{1}{\alpha p}\Big[[\Theta(\nu^p)+\Theta(\varpi^p)]+[\Theta((m-1)\nu^p)+\Theta((m-1)\varpi^p)]\Big]\\ &\quad-(\Theta(\vartheta^p)+\Theta(\varrho^p))\int_0^1\ell^{\alpha p-1}\left[\mathfrak{h}\left(\frac{\ell^p}{m}\right)+\mathfrak{h}\left(\frac{m-\ell^p}{m}\right)\right]d\ell\end{aligned}\tag{16}$$

*holds for all $\vartheta, \varrho \in [\nu, \varpi], \alpha, p > 0,$ and $m \in N$.*

*Proof.* Considering that $\Theta \in SX(\mathfrak{h}, \mathcal{I})$, we have

$$\Theta\left(\nu^p+\varpi^p-\frac{\vartheta_1^p+\varrho_1^p}{m}\right)\le\mathfrak{h}(1/m)\left[\Theta\left(\nu^p(m-1)+\varpi^p-\vartheta_1^p\right)+\Theta\left(\nu^p+(m-1)\varpi^p-\varrho_1^p\right)\right].$$

Let $\vartheta, \varrho \in [\nu, \varpi]$, $\ell \in [0,1]$, by changing the variables $\vartheta_1^p = \frac{\ell^p}{m}\vartheta^p + \frac{m-\ell^p}{m}\varrho^p$ and $\varrho_1^p = \frac{m-\ell^p}{m}\vartheta^p + \frac{\ell^p}{m}\varrho^p$, we get

$$\begin{aligned}\frac{1}{\mathfrak{h}(1/m)}\Theta\left(\nu^p+\varpi^p-\frac{\vartheta^p+\varrho^p}{m}\right)&\le\Theta\left(\nu^p(m-1)+\varpi^p-\left(\frac{\ell^p}{m}\vartheta^p+\frac{m-\ell^p}{m}\varrho^p\right)\right)\\&\quad+\Theta\left(\nu^p+\varpi^p(m-1)-\left(\frac{m-\ell^p}{m}\vartheta^p+\frac{\ell^p}{m}\varrho^p\right)\right).\end{aligned}$$

Taking product on both sides of above inequality by $\ell^{\alpha p-1}$ and then integrating the obtained inequality with respect to $\ell$ over $[0,1]$, we have

$$\begin{aligned}&\frac{1}{\alpha p\mathfrak{h}(1/m)}\Theta\left(\nu^p+\varpi^p-\frac{\vartheta^p+\varrho^p}{m}\right)\\&\le\int_0^1\ell^{\alpha p-1}\Theta\left(\nu^p(m-1)+\varpi^p-\left(\frac{\ell^p}{m}\vartheta^p+\frac{m-\ell^p}{m}\varrho^p\right)\right)d\tau+\int_0^1\ell^{\alpha p-1}\Theta\left(\nu^p+\varpi^p(m-1)-\left(\frac{m-\ell^p}{m}\vartheta^p+\frac{\ell^p}{m}\varrho^p\right)\right)d\tau\\&=\frac{m^\alpha}{(\varrho^p-\vartheta^p)^\alpha}\left[\int_{\nu^p(m-1)+\varpi^p-\varrho^p}^{\nu^p(m-1)+\varpi^p-\frac{\vartheta^p+(m-1)\varrho^p}{m}}(\tau^p-(\nu^p(m-1)+\varpi^p-\varrho^p))^{\alpha-1}\Theta(\tau^p)\tau^{p-1}d\tau\right.\\&\quad\left.+\int_{\nu^p+\varpi^p(m-1)-\frac{(m-1)\vartheta^p+\varrho^p}{m}}^{\nu^p+\varpi^p(m-1)-\vartheta^p}((\nu^p+\varpi^p(m-1)-\vartheta^p)-\tau^p)^{\alpha-1}\Theta(\tau^p)\tau^{p-1}d\tau\right]\\&=\frac{m^\alpha\Gamma(\alpha)}{(\varrho^p-\vartheta^p)^\alpha p^{1-\alpha}}\left[{}^p\mathcal{I}^\alpha_{((m-1)\nu^p+\varpi^p-\frac{\vartheta^p+(m-1)\varrho^p}{m})^+}\Theta((m-1)\nu^p+\varpi^p-\varrho^p)+{}^p\mathcal{I}^\alpha_{(\nu^p+(m-1)\varpi^p-\frac{(m-1)\vartheta^p+\varrho^p}{m})^-}\Theta(\nu^p+(m-1)\varpi^p-\vartheta^p)\right].\end{aligned}$$

Therefore, the first inequality of Theorem 3.10 is proved. Next, by using Jensen–Mercer inequality for $\mathfrak{h}$-convex function, we obtain

$$\Theta\left(\nu^p(m-1)+\varpi^p-\left(\frac{\ell^p}{m}\vartheta^p+\frac{m-\ell^p}{m}\varrho^p\right)\right)\le\Theta(\nu^p(m-1))+\Theta(\varpi^p)-\left(\mathfrak{h}\left(\frac{\ell^p}{m}\right)\Theta(\vartheta^p)+\mathfrak{h}\left(\frac{m-\ell^p}{m}\right)\Theta(\varrho^p)\right)\tag{17}$$

and

$$\Theta\left(\nu^p+\varpi^p(m-1)-\left(\frac{m-\ell^p}{m}\vartheta^p+\frac{\ell^p}{m}\varrho^p\right)\right)\le\Theta(\nu^p)+\Theta((m-1)\varpi^p)-\left(\mathfrak{h}\left(\frac{m-\ell^p}{m}\right)\Theta(\vartheta^p)+\mathfrak{h}\left(\frac{\ell^p}{m}\Theta(\varrho^p)\right)\right).\tag{18}$$

By adding (17) and (18), we obtain the following inequality by using Jensen–Mercer inequality

$$\Theta\left(\nu^p(m-1)+\varpi^p-\left(\frac{\ell^p}{m}\vartheta^p+\left(\frac{m-\ell^p}{m}\right)\varrho^p\right)\right)+\Theta\left(\nu^p+\varpi^p(m-1)-\left(\left(\frac{m-\ell^p}{m}\right)\vartheta^p+\frac{\ell^p}{m}\varrho^p\right)\right)$$
$$\leq\Theta(\nu^p)+\Theta(\varpi^p)+[\Theta((m-1)\nu^p)+\Theta((m-1)\varpi^p)]-[\Theta(\vartheta^p)+\Theta(\varrho^p)]\left(\mathfrak{h}\left(\frac{\ell^p}{m}\right)+\mathfrak{h}\left(\frac{m-\ell^p}{m}\right)\right).$$

Multiplying the above inequality on both sides by $\ell^{\alpha p-1}$ and then integrating the obtained inequality with respect to $\ell$ over $[0,1]$, then we get

$$\frac{p^{\alpha-1}m^\alpha\Gamma(\alpha)}{(\varrho^p-\vartheta^p)^\alpha}\left[{}^p\mathcal{I}^\alpha_{((m-1)\nu^p+\varpi^p-\frac{\vartheta^p+(m-1)\varrho^p}{m})^+}\Theta((m-1)\nu^p+\varpi^p-\varrho^p)+{}^p\mathcal{I}^\alpha_{(\nu^p+(m-1)\varpi^p-\frac{(m-1)\vartheta^p+\varrho^p}{m})^-}\Theta(\nu^p+(m-1)\varpi^p-\vartheta^p)\right]$$
$$\leq\frac{1}{\alpha p}\Big[[\Theta(\nu^p)+\Theta(\varpi^p)]+[\Theta((m-1)\nu^p)+\Theta((m-1)\varpi^p)]\Big]-[\Theta(\vartheta^p)+\Theta(\varrho^p)]\int_0^1\ell^{\alpha p-1}\left[\mathfrak{h}\left(\frac{\ell^p}{m}\right)+\mathfrak{h}\left(\frac{m-\ell^p}{m}\right)\right]d\ell.$$

Hence, the second inequality (16) is proved. □

Consider $m=2$, in Theorem 3.10, then immediately we find Corollary 3.11.

**Corollary 3.11.** *Suppose that $\alpha,p>0,\nu,\varpi\in[0,\infty)$ with $\varpi>\nu$ and $\Theta:[\nu^p,\varpi^p]\to\mathbb{R}$ be a positive function such that $\Theta\in L^p_c[\nu^p,\varpi^p]$ and if $\Theta$ is an $\mathfrak{h}$-convex on $[\nu^p,\varpi^p]$. Then, the following inequality*

$$\frac{1}{\alpha p\mathfrak{h}(1/2)}\Theta\left(\nu^p+\varpi^p-\frac{\vartheta^p+\varrho^p}{2}\right)\leq\frac{p^{\alpha-1}2^\alpha\Gamma(\alpha)}{(\varrho^p-\vartheta^p)^\alpha}\left[{}^p\mathcal{I}^\alpha_{(\nu^p+\varpi^p-\frac{\vartheta^p+\varrho^p}{2})^+}\Theta(\nu^p+\varpi^p-\varrho^p)+{}^p\mathcal{I}^\alpha_{(\nu^p+\varpi^p-\frac{\vartheta^p+\varrho^p}{2})^-}\Theta(\nu^p+\varpi^p-\vartheta^p)\right]$$
$$\leq\frac{1}{\alpha p}\Big[2[\Theta(\nu^p)+\Theta(\varpi^p)]\Big]-[\Theta(\vartheta^p)+\Theta(\varrho^p)]\int_0^1\ell^{\alpha p-1}\left[\mathfrak{h}\left(\frac{\ell^p}{2}\right)+\mathfrak{h}\left(\frac{2-\ell^p}{2}\right)\right]d\ell$$

*holds for all $\vartheta,\varrho\in[\nu,\varpi]$.*

**Remark 3.12.** If we choose $\mathfrak{h}(\ell)=\ell$, in Theorem 3.11, then we find [11, Corollary 3.1].

**Corollary 3.13.** *If we choose $\mathfrak{h}(\ell)=\ell^s$, $s\in(0,1]$ , in Theorem 3.10, then we obtain the following inequalities for s-convex function*

$$\frac{m^s}{\alpha p}\Theta\left(\nu^p+\varpi^p-\frac{\vartheta^p+\varrho^p}{2}\right)\leq\frac{m^\alpha\Gamma(\alpha)}{(\varrho^p-\vartheta^p)^\alpha p^{1-\alpha}}\left[{}^p\mathcal{I}^\alpha_{((m-1)\nu^p+\varpi^p-\frac{\vartheta^p+(m-1)\varrho^p}{m})^+}\Theta((m-1)\nu^p+\varpi^p-\varrho^p)\right.$$
$$\left.+{}^p\mathcal{I}^\alpha_{(\nu^p+(m-1)\varpi^p+\frac{(m-1)\vartheta^p+\varrho^p}{m})^-}\Theta(\nu^p+(m-1)\varpi^p-\vartheta^p)\right]$$
$$\leq\frac{1}{\alpha p}\Big[\Theta(\nu^p)+\Theta(\varpi^p)+\Theta((m-1)\vartheta^p)+\Theta((m-1)\varrho^p)\Big]-(\Theta(\vartheta^p)+\Theta(\varrho^p))\left[\frac{1}{m^sp(\alpha+s)}+\frac{1}{p\alpha}{}_2F_1(-s,\alpha;1+\alpha;1/m)\right].$$

**Theorem 3.14.** *Suppose that $\Theta:[\nu^p,\varpi^p]\to\mathbb{R}$ be a differentiable mapping on $(\nu^p,\varpi^p)$ such that $\Theta'\in L_1[\nu^p,\varpi^p]$ and $\alpha,p>0,m\in N$. If $|\Theta'|$ is an $\mathfrak{h}$-convex function on $[\nu^p,\varpi^p]$. Then the following inequality*

$$|\Omega_1(\alpha,p,\nu,\varpi,m)|\leq\frac{p(\varpi^p-\nu^p)}{m}\left[\frac{1}{p(\alpha+1)}(|\Theta'(\nu^p(m-1)|+|\Theta'(\varpi^p(m-1))|+|\Theta'(\nu^p)|+|\Theta'(\varpi^p)|)\right.$$
$$\left.-(|\Theta'(\vartheta^p)|+|\Theta'(\varrho^p)|)\int_0^1\ell^{p(\alpha+1)-1}\left(\mathfrak{h}\left(\frac{\ell^p}{m}\right)+\mathfrak{h}\left(1-\frac{\ell^p}{m}\right)\right)d\ell\right] \tag{19}$$

*holds for all $\vartheta,\varrho\in[\nu,\varpi]$.*

*Proof.* From Lemma 2.13 and the Jensen's inequality, we get

$$\begin{aligned}
&|\Omega_1(\alpha,p,\nu,\varpi,m)| \\
&\le \frac{p(\varpi^p-\nu^p)}{m}\left[\int_0^1 \ell^{p(\alpha+1)-1}\left|\Theta'\left(\nu^p(m-1)+\varpi^p-\left(\frac{\ell^p}{m}\vartheta^p+\left(1-\frac{\ell^p}{m}\right)\varrho^p\right)\right)\right|d\ell\right.\\
&\left.+\int_0^1 \ell^{p(\alpha+1)-1}\left|\Theta'\left(\nu^p+\varpi^p(m-1)-\left(\left(1-\frac{\ell^p}{m}\right)\vartheta^p+\frac{\ell^p}{m}\varrho^p\right)\right)\right|d\ell\right]\\
&= \frac{p(\varpi^p-\nu^p)}{m}\left[\int_0^1 \ell^{p(\alpha+1)-1}\left(|\Theta'(\nu^p(m-1)|+|\Theta'(\varpi^p)|-\left(\mathfrak{h}\left(\frac{\ell^p}{m}\right)|\Theta'(\vartheta^p)|+\mathfrak{h}\left(1-\frac{\ell^p}{m}\right)|\Theta'(\varrho^p)|\right)d\ell\right.\right.\\
&\left.+\int_0^1 \ell^{p(\alpha+1)-1}\left(|\Theta'(\nu^p)|+|\Theta'(\varpi^p(m-1))|-\left(\mathfrak{h}\left(1-\frac{\ell^p}{m}\right)|\Theta'(\vartheta^p)|+\mathfrak{h}\left(\frac{\ell^p}{m}\right)\right)|\Theta'(\varrho^p)|\right)\right)d\ell\right]\\
&= \frac{p(\varpi^p-\nu^p)}{m}\left[\left(|\Theta'(\nu^p(m-1))|+|\Theta'(\varpi^p(m-1))|+|\Theta'(\nu^p)|+|\Theta'(\varpi^p)|\right)\int_0^1 \ell^{p(\alpha+1)-1}d\ell\right.\\
&\left.-\left[|\Theta'(\vartheta^p)|+|\Theta'(\varrho^p)|\right]\int_0^1 \ell^{p(\alpha+1)-1}\left(\mathfrak{h}\left(\frac{\ell^p}{m}\right)+\mathfrak{h}\left(1-\frac{\ell^p}{m}\right)\right)d\ell\right]
\end{aligned}$$

after simplification, we get

$$\begin{aligned}
|\Omega_1(\alpha,p,\nu,\varpi,m)| \le \frac{p(\varpi^p-\nu^p)}{m}&\left[\frac{1}{p(\alpha+1)}\left(|\Theta'(\nu^p(m-1))|+|\Theta'(\varpi^p(m-1))|+|\Theta'(\nu^p)|+|\Theta'(\varpi^p)|\right)\right.\\
&\left.-\left[|\Theta'(\vartheta^p)|+|\Theta'(\varrho^p)|\right]\int_0^1 \ell^{p(\alpha+1)-1}\left(\mathfrak{h}\left(\frac{\ell^p}{m}\right)+\mathfrak{h}\left(1-\frac{\ell^p}{m}\right)\right)d\ell\right].
\end{aligned} \tag{20}$$

Therefore, we have finished the proof. □

**Corollary 3.15.** *If we assign $m = 2$, in Theorem 3.14, then we obtain the following inequality*

$$\begin{aligned}
&\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(\varrho^p-\vartheta^p)^\alpha p}\left[{}^p\mathcal{I}^\alpha_{(\nu^p+\varpi^p-\frac{\vartheta^p+\varrho^p}{2})^-}\Theta(\nu^p+\varpi^p-\varrho^p)+{}^p\mathcal{I}^\alpha_{(\nu^p+\varpi^p-\frac{\vartheta^p+\varrho^p}{2})^+}\Theta(\nu^p+\varpi^p-\vartheta^p)\right]\right.\\
&\left.-\Theta\left(\nu^p+\varpi^p-\frac{\vartheta^p+\varrho^p}{2}\right)\right| \le \frac{(\varpi^p-\nu^p)}{2(\alpha+1)}\left[|\Theta'(\nu^p)|+|\Theta'(\varpi^p)|-\frac{|\Theta'(\vartheta^p)|+|\Theta'(\varrho^p)|}{2}\right].
\end{aligned}$$

**Remark 3.16.** If we take $\alpha = p = 1$, in Corollary 3.15, then we get

$$\left|\frac{1}{\varrho-\vartheta}\int_{\nu+\varpi-\varrho}^{\nu+\varpi-\varrho}\Theta(\tau)d\tau-\Theta\left(\nu+\varpi-\frac{\vartheta+\varrho}{2}\right)\right| \le \frac{(\varpi-\nu)}{4}\left[|\Theta'(\nu)|+|\Theta'(\varpi)|-\frac{|\Theta'(\vartheta)|+|\Theta'(\varrho)|}{2}\right]$$

which is reported in [31, Corollary 3.10].

**Remark 3.17.** If we take $\mathfrak{h}(\ell) = \ell$, one has the inequality proved in [11].

**Corollary 3.18.** *If we choose $\mathfrak{h}(\ell) = \ell^s$, $s \in (0,1]$, in Theorem 3.14, we find inequality for s-convex function*

$$\begin{aligned}
|\Omega_1(\alpha,p,\nu,\varpi,m)| \le \frac{p(\varpi^p-\nu^p)}{m}&\left[\frac{1}{p(\alpha+1)}\left[|\Theta'(\nu^p(m-1))|+|\Theta'(\varpi^p(m-1))|+|\Theta'(\nu^p)|+|\Theta'(\varpi^p)|\right]\right.\\
&\left.-\left[|\Theta'(\vartheta^p)|+|\Theta'(\varrho^p)|\right]\left(\frac{1}{m^s p(\alpha+s+1)}+\frac{1}{p(\alpha+1)}{}_2F_1(-s,1+\alpha;2+\alpha;1/m)\right)\right].
\end{aligned}$$

**Theorem 3.19.** *Suppose that $\Theta : [\nu^p, \varpi^p] \to \mathbb{R}$ is a differentiable function on $(\nu^p, \varpi^p)$ such that $\Theta' \in L_1[\nu^p, \varpi^p]$, $\alpha, p > 0$, $m \in N$ and $\nu, \varpi \in [0, \infty)$ with $\varpi > \nu$. If $|\Theta'|^q$ is $\mathfrak{h}$-convex function on $[\nu^p, \varpi^p]$, then we obtain the following inequality*

$$|\Omega_1(\alpha, p, \nu, \varpi, m)| \leq \frac{p(\varpi^p - \nu^p)}{m}\Big(\frac{1}{p[p(\alpha+1)-1]+1}\Big)^{1/p} \Big[(|\Theta'(\nu^p(m-1)|^q + |\Theta'(\varpi^p)|^q - \left(\int_0^1 \left(\mathfrak{h}\left(1-\frac{\ell^p}{m}\right)|\Theta'(\vartheta^p)|^q + \mathfrak{h}\left(\frac{\ell^p}{m}\right)|\Theta'(\varrho^p)|^q\right) d\ell\right)^{\frac{1}{q}} + (|\Theta'(\nu^p)|^q + |\Theta'(\varpi^p(m-1))|^q - \left(\int_0^1 \left(\mathfrak{h}\left(\frac{\ell^p}{m}\right)|\Theta'(\vartheta^p)|^q + \mathfrak{h}\left(1-\frac{\ell^p}{m}\right)|\Theta'(\varrho^p)|^q\right) d\ell\right)^{\frac{1}{q}}\Big] \tag{21}$$

*holds for all $\vartheta, \varrho \in [\nu, \varpi]$ and $p, q > 1$, with $\frac{1}{p} + \frac{1}{q} = 1$.*

*Proof.* Drawing from Lemma 2.13 and the combined effects of the Hölder and Jensen–Mercer inequalities, in conjunction with the $\mathfrak{h}$-convexity of $|\Theta'|^q$, we can assert that

$$\begin{aligned}
|\Omega_1(\alpha, p, \nu, \varpi, m)| &\leq \frac{p(\varpi^p - \nu^p)}{m}\left[\int_0^1 \ell^{p(\alpha+1)-1}\left|\Theta'\left(\nu^p(m-1) + \varpi^p - \left(\frac{m-\ell^p}{m}\vartheta^p + \frac{\ell^p}{m}\varrho^p\right)\right)\right| d\ell \right.\\
&\quad \left. + \int_0^1 \ell^{p(\alpha+1)-1}\left|\Theta'\left(\nu^p + \varpi^p(m-1) - \left(\frac{\ell^p}{m}\vartheta^p + \frac{m-\ell^p}{m}\varrho^p\right)\right)\right| d\ell\right]\\
&\leq \frac{p(\varpi^p - \nu^p)}{m}\left(\int_0^1 \ell^{p(p(\alpha+1)-1)}\right)^{1/p}\left[\left(\int_0^1 \left|\Theta'\left(\nu^p(m-1) + \varpi^p - \left(\mathfrak{h}\left(1-\frac{\ell^p}{m}\right)\vartheta^p + \mathfrak{h}\left(\frac{\ell^p}{m}\right)\varrho^p\right)\right)\right|^q d\ell\right)^{1/q}\right.\\
&\quad \left. + \left(\int_0^1 \left|\Theta'\left(\nu^p + \varpi^p(m-1) - \left(\mathfrak{h}\left(\frac{\ell^p}{m}\right)\vartheta^p + \mathfrak{h}\left(1-\frac{\ell^p}{m}\right)\varrho^p\right)\right)\right|^q d\ell\right)^{1/q}\right]\\
&\leq \frac{p(\varpi^p - \nu^p)}{m}\left(\frac{1}{p[p(\alpha+1)-1]+1}\right)^{1/p}\Big[(|\Theta'(\nu^p(m-1))|^q + |\Theta'(\varpi^p)|^q\\
&\quad - \int_0^1 \left(\mathfrak{h}\left(1-\frac{\ell^p}{m}\right)|\Theta'(\vartheta^p)|^q + \mathfrak{h}\left(\frac{\ell^p}{m}\right)|\Theta'(\varrho^p)|^q\right) d\ell\Big)^{1/q} + \Big(|\Theta'(\nu^p)|^q + |\Theta'(\varpi^p(m-1))|^q\\
&\quad - \int_0^1 \left(\mathfrak{h}\left(\frac{\ell^p}{m}\right)|\Theta'(\vartheta^p)|^q + \mathfrak{h}\left(1-\frac{\ell^p}{m}\right)|\Theta'(\varrho^p)|^q\right) d\ell\Big)^{1/q}\Big].
\end{aligned}$$

□

If we assume that $m = 2$, then Theorem 3.19 directly leads to Corollary 3.20.

**Corollary 3.20.** *Suppose that $\Theta : [\nu^p, \varpi^p] \to \mathbb{R}$ is a differentiable function on $(\nu^p, \varpi^p)$ such that $\Theta' \in L_1[\nu^p, \varpi^p]$ and $p, q > 1$, with $\frac{1}{p} + \frac{1}{q} = 1$, $\alpha, p > 0$, $m \in N$, with $\varpi > \nu$. If $|\Theta'|^q$ is an $\mathfrak{h}$-convex on $[\nu^p, \varpi^p]$, then we have the following inequality*

$$|\Omega_1(\alpha, p, \nu, \varpi, 2)| \leq \frac{p(\varpi^p - \nu^p)}{2}\Big(\frac{1}{p[p(\alpha+1)-1]+1}\Big)^{1/p}\Big[(|\Theta'(\nu^p|^q + |\Theta'(\varpi^p)|^q - \int_0^1 \left(\mathfrak{h}\left(1-\frac{\ell^p}{2}\right)|\Theta'(\vartheta^p)|^q + \mathfrak{h}\left(\frac{\ell^p}{2}\right)|\Theta'(\varrho^p)|^q\right) d\ell\Big)^{\frac{1}{q}} + (|\Theta'(\nu^p)|^q + |\Theta'(\varpi^p)|^q - \int_0^1 \left(\mathfrak{h}\left(\frac{\ell^p}{2}\right)|\Theta'(\vartheta^p)|^q + \mathfrak{h}\left(1-\frac{\ell^p}{2}\right)|\Theta'(\varrho^p)|^q\right) d\ell\Big)^{\frac{1}{q}}\Big] \tag{22}$$

*holds for all $\vartheta, \varrho \in [\nu^p, \varpi^p]$.*

**Corollary 3.21.** *Assuming* $\mathfrak{h}(\ell) = \ell^s$ *with* $s \in (0,1]$, *in Theorem 3.19, it leads us to the following inequality*

$$\begin{aligned}
&|\Omega_1(\alpha, p, \nu, \varpi, m)| \\
&\leq \frac{p(\varpi^p - \nu^p)}{m}\left(\frac{1}{p[p(\alpha+1)-1]+1}\right)^{1/p}\left[(|\Theta'(\nu^p(m-1))|^q + |\Theta'(\varpi^p)|^q\right. \\
&- \left({}_2F_1(1/p, -s; 1+1/p; 1/m)|\Theta'(\vartheta^p)|^q + \frac{1}{m^s(ps+1)}|\Theta'(\varrho^p)|^q\right)d\ell\Bigg)^{1/q} + \Big(|\Theta'(\nu^p)|^q + |\Theta'(\varpi^p(m-1))|^q \\
&\left. - \int_0^1\left(\frac{1}{m^s(ps+1)}|\Theta'(\vartheta^p)|^q + {}_2F_1(1/p, -s; 1+1/p; 1/m)|\Theta'(\varrho^p)|^q\right)d\ell\Big)^{1/q}\right].
\end{aligned}$$

## 4. Conclusion

The implications of this work extend to various areas of mathematics and related disciplines, including optimization, functional analysis, and fractional calculus. This study introduced the concept of $\mathfrak{h}$-convex function on Jensen–Mercer-type inequality by using Katugampola fractional integral. Additionally, results are also shown for convex, $s$-convex functions in the second sense. The findings of this research open up exciting possibilities for both theoretical and practical applications.

## References


[1] M. Abbasi, A. Morassaei, F. Mirzapour, *Jensen–Mercer-type inequalities for operator* $\mathfrak{h}$*-convex functions*, Bull. Iran. Math. Soc. (2022), 1–22.

[2] T. Abdeljawad, M. A. Ali, P. O. Mohammed, A. Kashuri, *On inequalities of Hermite–Hadamard-Mercer-type involving Riemann–Liouville fractional integrals*, AIMS Math. **5** (2020), 7316–7331.

[3] D. Ai, T. Du, *A study on Newton-type inequalities bounds for twice *differentiable functions under multiplicative Katugampola fractional integrals*, Fractals 33 (2025), 2550032.

[4] M. A. Ali, H. Budak, M. Z. Abbas, M. Z. Sarikaya, A. Kashuri, *Hermite–Hadamard-type inequalities for* $\mathfrak{h}$*-convex functions via generalized fractional integrals*, J. Math. Ext. **14** (2019), 187–234.

[5] M. W. Alomari, C. Chesneau, *On* $\mathfrak{h}$*-superquadratic functions*, Afr. Mat. **33** (2022), 41.

[6] H. Budak, M. Z. Sarikaya, *A new Hermite–Hadamard inequality for* $\mathfrak{h}$*-convex stochastic processes*, RGMIA Res. Rep. Collect. **19** (2016), 30.

[7] S. I. Butt, *Generalized Jensen-Hermite-Hadamard-Mercer type inequalities for generalized strongly convex functions on fractal sets*, Turk. J. Sci. **8** (2024), 51–63.

[8] S. I. Butt, M. Nadeem, S. Qaisar, A. O. Akdemir, *Hermite-Jensen–Mercer-type inequalities for conformable integrals and related results*, Adv. Differ. Equ. **2020** (2020), 1–24.

[9] S. I. Butt, M. Umar, S. Rashid, A. O. Akdemir, Y. M. Chu, *New Hermite-Jensen–Mercer-type inequalities via k-fractional integrals*, Adv. Differ. Equ. **2020** (2020), 1–24.

[10] S. I. Butt, S. Yousaf, M. Younas, H. Ahmad, S. W. Yao, *Fractal Hadamard–Mercer-type inequalities with applications*, Fractals **30** (2022), 2240055.

[11] H. Chu, H. Rashid, Z. Hammouch, Y. M. Chu, *New fractional estimates for Hermite–Hadamard–Mercer's-type inequalities*, Alex. Eng. J. **59** (2020), 3079–3089.

[12] M. J. V. Cortezm J. E. H. Hernández, *A variant of Jensen–Mercer Inequality for* $\mathfrak{h}$*-convex functions and operator* $\mathfrak{h}$*-convex functions*, Rev. MATUA **4** (2017), 2389–7422.

[13] S. S. Dragomir, *An inequality improving the first Hermite–Hadamard inequality for convex functions defined on linear spaces and applications for semi-inner products*, J. Inequal. Pure Appl. Math. **3** (2002), 31.

[14] S. S. Dragomir, P. Charles, *Selected topics on Hermite–Hadamard inequalities and applications*, Sci. Direct **4** (2003), S1574–0358.

[15] R. J. Dwilewicz, *A short history of Convexity*, Differ. Geom. Dyn. Syst. (2009).

[16] H. Fu, Y. Peng, T. Du, *Some inequalities for multiplicative tempered fractional integrals involving the* $\lambda$*-incomplete gamma functions*, AIMS Math. **6** (2021), 7456–7478.

[17] J. W. Green, *Recent applications of convex functions*, Amer. Math. Monthly **61**(7P1) (1954), 449–454.

[18] H. Hudzik, L. Maligranda, *Some remarks on s-convex functions*, Aequationes Math. **48** (1994), 100–111.

[19] L. Jia-Bao, S. I. Butt, J. Nasir, A. Aslam, A. Fahad, J. Soontharanon, *Jensen–Mercer variant of Hermite–Hadamard type inequalities via Atangana-Baleanu fractional operator*, AIMS Math. **7** (2022), 2123–2141.

[20] M. Kian, M. Moslehian, *Refinements of the operator Jensen–Mercer inequality*, Electron. J. Linear Algebra **26** (2013), 742–753.

[21] U. N. Katugampola, *A new approach to generalized fractional derivatives*, Bull. Math. Anal. Appl. **6** (2014), 1–15.

[22] A. Kashuri, H. Budak, R. Kapucu, and R. Liko, *New inequalities for approximately* $\mathfrak{h}$*-convex functions pertaining generalized fractional integrals*, Mat. Bilten **45**(2) (2021).

[23] H. Kara, M. A. Ali, H. Budak, *Hermite–Hadamard–Mercer-type inclusions for interval-valued functions via Riemann–Liouville fractional integrals*, Turk. J. Math. **46** (2022), 2193–2207.
[24] W. Liu, M. A. Ali, A. Kashuri, H. Budak, *Some Simpson type inequalities for $\mathfrak{h}$-convex and $(\alpha, m)$-convex functions*, arXiv preprint arXiv:1210.4062 (2012).
[25] Y. Long, T. Du, *Analysis on multiplicative k-Atangana–Baleanu fractional integrals with application to various Mercer-type inequalities*, Fractals **33** (2025), 2550003.
[26] S. Mandelbrojt, L. Schwartz, *Jacques Hadamard (1865–1963)*, Bull. Amer. Math. Soc. **71**(1) (1965), 107–129.
[27] A. M. Mercer, *A variant of Jensen's inequality*, J. Inequal. Pure Appl. Math. **4** (2003), 73.
[28] A. Matkovic, J. Pecaric, I. Peric, *A variant of Jensen's inequality of Mercer's type for operators with applications*, Linear Algebra Appl. **418** (2006), 551–564.
[29] A. Matkovic, J. Pecaric, *Refinements of Jensen–Mercer's inequality for index set functions with applications*, Rev. Anal. Numér. Théor. Approx. **35** (2006), 71–82.
[30] A. Matkovic, J. Pecaric, *A variant of Jensen's inequality for convex functions of several variables*, J. Math. Inequal. **1** (2007), 45–51.
[31] H. Ogulmus, Z. M. Sarikaya, *Hermite–Hadamard–Mercer-type inequalities for fractional integrals*, Filomat **35** (2021), 2425–2436.
[32] Y. Peng, T. S. Du, *Fractional Maclaurin-type inequalities for multiplicatively convex functions and multiplicatively P-functions*, Filomat **37**(28) (2023), 9497–9509.
[33] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.
[34] F. Robertson, E. Edmund, *Jacques Hadamard (1865–1963)*, MacTutor Hist. Math. Arch., Univ. St Andrews (1965).
[35] S. G. Samko, A. A. Kilbas, O. I. Marichev, *Fractional Integrals and Derivatives*, Gordon and Breach Sci. Publ., Yverdon, 1993.
[36] P. K. Sarpong, J. A. Prah, *Applications of convex function and concave functions*, Dama Int. J. Res. (2018).
[37] S. Varošanec, *On $\mathfrak{h}$-convexity*, J. Math. Anal. Appl. **326** (2007), 303–311.
[38] M. Vivas-Cortez, M. A. Ali, A. Kashuri, H. Budak, *Generalizations of fractional Hermite–Hadamard–Mercer like inequalities for convex functions*, AIMS Math. **6**(9) (2021), 9397–9421.
[39] P. Xu, S. I. Butt, S. Yousaf, A. Aslam, T. J. Zia, *Generalized Fractal Jensen–Mercer and Hermite-Mercer-type inequalities via $\mathfrak{h}$-convex functions involving Mittag-Leffler kernel*, Alex. Eng. J. **61** (2022), 4837–4846.
[40] J. Zhao, S. I. Butt, J. Nasir, Z. Wang, I. Tlili, *Hermite-Jensen–Mercer-type inequalities for Caputo fractional derivatives*, J. Funct. Spaces (2020), 1–11.