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Statistical quasi Cauchyness on asymmetric spaces

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Abstract. We call a sequence (x_m) of points in an asymmetric metric space (X, d) statistically forward quasi Cauchy if $\lim_{n\to\infty} \frac{1}{n} \mid \{m \leq n : \ d(x_m, x_{m+1}) \geq \varepsilon\} \mid = 0$ for each positive ε, where $|A|$ indicates the cardinality of the set A. We prove that a subset E of X is forward totally bounded if and only if any sequence of points in E has a statistically forward quasi Cauchy subsequence. We also introduce and investigate statistically upward continuity in the sense that a function defined on X into Y is called statistically upward continuous if it preserves statistically forward quasi Cauchy sequences, i.e. $(f(x_m))$ is statistically forward quasi Cauchy whenever (x_m) is.

1. Introduction

Using the idea of sequential continuity, many kinds of continuities were introduced and investigated, not all but some of them we recall in the following: ward continuity [2, 5], statistical ward continuity [3, 4].

Metric spaces are a fundamental concept in analysis and are characterized by metric functions that define the distance between any points in space. In traditional metric spaces, this distance function is symmetric. However, in some applications or problems, situations can be encountered where the distance does not satisfy the symmetry property. Asymmetric metric spaces, which are developed to handle such situations and does not satisfy the symmetry property, have attracted great attention in mathematical modeling and applied disciplines in recent years. These spaces have become a powerful tool, especially used in fields such as optimization problems, transportation models, computer science and artificial intelligence. Such spaces are used to model situations where the distance from one point to another depends on the direction. For example, the travel time or cost between two points varies depending on the direction of travel. Asymmetric metric spaces have applications in applied mathematics and material science, such as rate-independent models for plasticity, shape memory alloys, and material failure. Another application of these spaces in abstract and applied mathematics is the study of the existence and the uniqueness of the Hamilton-Jacobi equations.

Asymmetric function is known to be first mentioned by Hausdorff in 1914. He defined the distance between two sets in a metric space using a function that does not satisfy the symmetry requirement. Then Wilson studied asymmetry by calling this concept as quasi metric [19], while Ribeiro called this notion as weak metric in 1943. Künzi and Reilly studied in quasi pseudo metrics [15, 17]. Künzi and Kočinac have a paper about selection principles on quasi uniform and quasi metric spaces [14]. Afterwards, Zimmer

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and Colins have conducted research on topics such as total boundedness, compactness and convergence on these spaces [7].

An asymmetric metric is a generalization of metric created by removing the symmetry requirement in the metric definition. The lack of symmetry requirement in asymmetric metric spaces results in two types of topology. Accordingly, basic notions such as convergence, compactness, completeness and total boundedness need to be examined in two types (see ([7, 9]).

The notion of statistical convergence was firstly given as "almost convergence" by Zygmund in 1935 [20] . Statistical convergence was formally given by Fast [12]. Although statistical convergence has been studied for approximately the last ninety years, it has been an important research topic for various authors for the last forty years [10, 13, 18].

Recall that a subset E of an asymmetric metric space (X, d) is forward totally bounded if it has a finite forward ε-net for each $\varepsilon > 0$. This is equivalent to the statement that any sequence of points in E has a forward Cauchy subsequence. This raises the question of whether the term "Cauchy" can be replaced by the term "quasi Cauchy" or "statistical quasi Cauchy". In fact we see that we can use both of them instead of the term "Cauchy".

The aim of this study is to characterize of total boundedness of a subset of an asymmetric metric space X, and to examine the relationship between statistical forward continuity and new types of continuities defined based on statistical forward quasi Cauchy sequences.

Now we present the concepts and results that may be necessary throughout the paper.

A sequence (x_n) is called *forward* (*backward*) *Cauchy* if for every $\varepsilon > 0$ there exists an $n_0 \in \mathbb{N}$ such that for $m \geq n \geq n_0$, $d(x_n, x_m) < \varepsilon$ ($d(x_m, x_n) < \varepsilon$) .

We recall the definition of forward (backward) convergence which was given in [7].

A sequence (x_n) in an asymmetric metric space X forward (backward) converges to $x \in X$ if $\lim_{n\to\infty} d(x, x_n) = 0$ ($\lim_{n\to\infty} d(x_n, x) = 0$).

Some examples are given in [7, 9] that forward convergence does not imply forward Cauchyness.

Lemma 1.1. ([8]) *Let (X, d) be an asymmetric metric space which has the property that forward convergence implies backward convergence. Then any forward convergent sequence is forward Cauchy.*

Let (X, d) be an asymmetric space and $E \subseteq X$. If every sequence taken from the set E has a subsequence that is forward (backward) convergent to an element of E, then the set E is called forward (backward) compact.

For a subset K of the set of positive integers the asymptotic density of K, denoted by $\delta(K)$, is given by

$$\delta(K) = \lim_{n\to\infty} \frac{1}{n} |\{m \leq n : m \in K\}|,$$

if this limit exists, where $|\{m \leq n : m \in K\}|$ denotes the cardinality of the set $\{m \leq n : m \in K\}$.

A sequence (x_n) of points in X is called forward quasi Cauchy if $\lim_{n\to\infty} \Delta^+ x_n = 0$ where $\Delta^+ x_n = d(x_n, x_{n+1})$ (see [8] and [1]).

A function defined on a subset E of X is called upward continuous if it preserves forward quasi Cauchy sequences, i.e. $(f(x_n))$ is a forward quasi Cauchy sequence whenever (x_n) is.

Notation 1.2. Let Y^X be the space of the functions of X into Y, then the asymmetric on Y^X is

$$\overline{\rho}(f, g) := \sup\{\overline{d}(f(w), g(w)) : w \in X\},$$

which generates the asymmetric metric space on Y^X whose topology is uniform, where $\overline{d}(x, y) = \min\{1, d_Y(x, y)\}$.

2. Statistically quasi Cauchy sequences on aasymmetric metric spaces

In this paper hereafter, $\mathbf{N}$, $\mathbf{R}$, X, and Y denote the set of positive integers, the set of real numbers, an asymmetric metric spaces with an asymmetric metric d, and d_Y respectively. .

In this section, statistical forward (backward) quasi Cauchy sequences are defined in asymmetric metric spaces, and the concept of statistical upward compactness is examined.

Definition 2.1. Let (X,d) be an asymmetric metric space and let (x_m) be a sequence in X . The sequence (x_m) is called statistical forward convergent to the point L if for each $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n} \mid \{m \le n : \ d(L,x_m) \ge \varepsilon\} \mid = 0$$

which means that $\delta(\{m : \ d(L,x_m) \ge \varepsilon\}) = 0$.
Similarly, The sequence (x_m) is called statistical backward convergent to the point L if for each $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n} \mid \{m \le n : \ d(x_m,L) \ge \varepsilon\} \mid = 0$$

which means that $\delta(\{m : \ d(x_m,L) \ge \varepsilon\}) = 0$.

Theorem 2.2. *Any forward convergent sequence in an asymmetric metric space (X,d) is statistically forward convergent.*

Proof. Let (x_m) is forward convergent with forward limit L. Take $\varepsilon > 0$. Since (x_m) forward convergent to L, there exists a positive integer m_0 which depends on ε such that

$$d(L,x_m) < \varepsilon$$

whenever $m > m_0$. Thus, for $n \ge m_0$ we get

$$\{m \le n : \ d(L,x_m) \ge \varepsilon\} \subseteq \{1,2,3,...,m_0-1,m_0\}$$

Therefore we can express it as

$$\mid \{m \le n : \ d(L,x_m) \ge \varepsilon\} \mid \le m_0$$

Thus we obtain

$$\lim_{n\to\infty} \frac{1}{n} \mid \{m \le n : \ d(L,x_m) \ge \varepsilon\} \mid \le \lim_{n\to\infty} \frac{m_0}{n} = 0$$

Thus the proof is completed. □

The converse of Theorem 2.2 is not true, i.e. a statistical forward convergent sequence does not need to be forward convergent.

Example 2.3. A sequence (x_k) defined by

$$x_k = \begin{cases} 2 & , k = m^2, (m \in \mathbb{N}) \\ 0 & , otherwise \end{cases}$$

is statistically forward convergent to zero but not forward convergent in the asymmetric metric space which is generated by the asymmetric metric $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$

$$d(x,y) = \begin{cases} y - x, & y \ge x \\ 1, & y < x \end{cases}$$

Indeed, since for each $\varepsilon > 0$

$$\mid \{k \le n : \ d(0,x_k) \ge \varepsilon\} \mid \le \mid \{k \le n : \ x_k \ne 0\} \mid \le \ \sqrt{n}$$

we obtain

$$\lim_{n\to\infty} \frac{1}{n} \mid \{k \le n : \ x_k \ge \varepsilon\} \mid \le \lim_{n\to\infty} \frac{1}{n} \mid \{k \le n : \ x_k \ne 0\} \mid \le \lim_{n\to\infty} \frac{1}{n} \sqrt{n} = 0$$

which means that (x_k) is statistically forward convergent to zero. But since $\lim_{k\to\infty} d(0,x_k) = \lim_{k\to\infty} x_k \ne 0$, (x_k) is not forward convergent to zero.

Theorem 2.4. *If forward convergence implies backward convergence in an asymmetric space (X,d), then statistical forward convergence implies statistical backward convergence.*

Proof. Let statistical forward convergence do not imply statistical backward convergence on X and let (x_m) be a sequence in X which is forward convergent to the point L. Thus, (x_m) statistically forward converges to L. Since (x_m) does not statistically backward converge to L, there exists an $\varepsilon > 0$ such that

$$\lim_{n\to\infty} \frac{1}{n} \mid \{m \leq n : \ d(x_m, L) \geq \varepsilon\} \mid > 0$$

So for infinitely many indexes of the sequence (x_m) we obtain $d(x_m, L) \geq \varepsilon$ which means (x_m) does not backward converges to L. Thus, forward convergence does not imply backward convergence on X. This contradiction completes the proof. □

Theorem 2.5. *In an asymmetric space (X, d), the limit of a statistical forward convergent sequence is unique if forward convergence on X implies backward convergence.*

Proof. Let (x_m) be a statistical forward convergent sequence. Assume that L_1 and L_2 are different statistical forward limits of (x_m). Then $d(L_1, L_2) \neq 0$.

Let $d(L_1, L_2) = \alpha$ and take $\varepsilon = \frac{\alpha}{3}$. Since (x_m) statistically forward converges to L_1 it follows that

$$\lim_{n\to\infty} \frac{1}{n} \mid \left\{m \leq n : \ d(L_1, x_m) \geq \frac{\varepsilon}{2}\right\} \mid = 0$$

Since (x_m) statistically forward converges to L_2 it also statistically backward converges. Then we obtain

$$\lim_{n\to\infty} \frac{1}{n} \mid \left\{m \leq n : \ d(x_m, L_2) \geq \frac{\varepsilon}{2}\right\} \mid = 0$$

Thus

$$\begin{aligned} 1 = \lim_{n\to\infty} \frac{1}{n} \mid \{m \leq n : \ d(L_1, L_2) \geq \varepsilon\} \mid &\leq \lim_{n\to\infty} \frac{1}{n} \mid \left\{m \leq n : \ d(L_1, x_m) \geq \frac{\varepsilon}{2}\right\} \mid \\ &+ \lim_{n\to\infty} \frac{1}{n} \mid \left\{m \leq n : \ d(x_m, L_2) \geq \frac{\varepsilon}{2}\right\} \mid = 0. \end{aligned}$$

This contradiction completes the proof. □

Definition 2.6. Let (X, d) be an asymmetric metric space and let (x_m) be a sequence in X. (x_m) is called a statistical forward Cauchy sequence if there exists a pozitive integer N such that for each $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{n} \mid \{m \leq n : \ d(x_N, x_m) \geq \varepsilon\} \mid = 0$$

Similarly, (x_m) is called a statistical backward Cauchy sequence if there exists a pozitive integer N such that for each $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{n} \mid \{m \leq n : \ d(x_m, x_N) \geq \varepsilon\} \mid = 0$$

Definition 2.7. Let (X, d) be an asymmetric space and $E \subseteq X$. If every sequence taken from the set E has a subsequence that is statistical forward (backward) convergent to an element of E, then the set E is called statistical forward (backward) compact.

Definition 2.8. Let (X,d) be an asymmetric metric space and (x_m) be a sequence in X. (x_m) is called statistically forward quasi Cauchy if for each $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{n} \left|\{m \leq n : \ \Delta x_m^+ \geq \varepsilon\}\right| = 0,$$

where $\Delta x_m^+ = d(x_m, x_{m+1})$ for each pozitive integer m. Similarly, (x_m) is called statistically backward quasi Cauchy if for each $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{n} \left|\{m \leq n : \ \Delta x_m^- \geq \varepsilon\}\right| = 0,$$

where $\Delta x_m^- = d(x_{m+1}, x_m)$ for each positive integer m.

Any statistical forward Cauchy sequence is a statistical forward quasi Cauchy sequence but the converse does not need to be true.

Example 2.9. A sequence (x_k) defined by

$$x_k = \begin{cases} 1 & , k = m^2, (m \in \mathbb{N}) \\ \sum_{i=1}^{k} \frac{1}{i} & , otherwise \end{cases}$$

is a statistical forward quasi Cauchy sequence but is not statistical forward Cauchy sequence in the asymmetric metric space which is generated by the asymmetric metric $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$

$$d(x, y) = \begin{cases} y - x, & y \geq x \\ 1, & y < x \end{cases}$$

Lemma 2.10. *Any statistical forward convergent sequence is statistically forward quasi Cauchy if forward convergence implies backward convergence on X.*

Proof. Let (x_m) be a sequence that is statistical forward convergent to a point l. Then given $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{n} \mid \left\{m \leq n : \ d(l, x_m) \geq \frac{\varepsilon}{2}\right\} \mid = 0$$

holds. Since forward convergence implies backward convergence on X statistical forward convergence implies statistical backward convergence. Thus we see that

$$\lim_{n\to\infty} \frac{1}{n} \mid \left\{m \leq n : \ d(x_m, l) \geq \frac{\varepsilon}{2}\right\} \mid = 0$$

Since

$$\lim_{n\to\infty} \frac{1}{n} \mid \left\{m \leq n : \ d(l, x_{m+1}) \geq \frac{\varepsilon}{2}\right\} \mid = 0$$

we obtain that

$$\lim_{n\to\infty} \frac{1}{n} \mid \{m \leq n : \ d(x_m, x_{m+1}) \geq \varepsilon\} \mid \leq \lim_{n\to\infty} \frac{1}{n} \mid \left\{m \leq n : \ d(x_m, l) \geq \frac{\varepsilon}{2}\right\} \mid + \lim_{n\to\infty} \frac{1}{n} \mid \left\{m \leq n : \ d(l, x_{m+1}) \geq \frac{\varepsilon}{2}\right\} \mid = 0 + 0 = 0..$$

Therefore, (x_m) is statistically forward quasi Cauchy. □

Definition 2.11. Let (X, d) be an asymmetric metric space. $E \subseteq X$ is called statistical upward (downward) compact if any sequence whose terms are in E has a statistical forward (backward) quasi Cauchy subsequence.

According to this definition a finite subset of X is statistical upward and downward compact. Any subset of statistical upward compact set is statistical upward compact, union of finitely many statistical upward compact set is statistical upward compact and intersection of any family of statistical upward compact subsets of X is statistical upward compact.

Theorem 2.12. ([8]) *A subset E of X is forward totally bounded if and only if it is upward compact.*

Theorem 2.13. *A subset E of X is forward totally bounded if and only if it is statistical upward compact.*

Proof. It is clear that forward totally boundedness of E implies statistical upward compactness of E.

To prove the converse suppose that E is not forward totally bounded. In that case, there is an $\varepsilon > 0$ such that E has not a finite forward ε-net. Let $S_\varepsilon(x)$ denotes $B^+(x, \varepsilon)$ for any $x \in E$. Now take any element of E and say x_1. Since E is not forward totally bounded, $S_\varepsilon(x_1) \neq E$. Otherwise $\{x_1\}$ would be a finite forward ε-net of E. So there is an $x_2 \in E$ such that $x_2 \notin S_\varepsilon(x_1)$, i.e. $d(x_1, x_2) \geq \varepsilon$. Since $\{x_1, x_2\}$ is not a finite forward ε-net, $S_\varepsilon(x_1) \cup S_\varepsilon(x_2) \neq E$. Now let $x_3 \notin S_\varepsilon(x_1) \cup S_\varepsilon(x_2)$. So we obtain that $d(x_1, x_3) \geq \varepsilon$ and $d(x_2, x_3) \geq \varepsilon$. Continuing in this way we can generate the sequence (x_n) such that $x_n \notin \cup_{i=1}^{n-1} S_\varepsilon(x_i)$ for $n = 2, 3, ...$ So we get $d(x_i, x_n) \geq \varepsilon$ for $i = 1, 2, ..., n-1$ and for $n = 2, 3, ...$ As a result of this, for all $n, m \in \mathbf{N}$ which satisfy $n < m$ we have $d(x_n, x_m) \geq \varepsilon$. Therefore $d(x_n, x_{n+1}) \geq \varepsilon$. Thus the sequence (x_n) constructed has not any statistical forward quasi Cauchy subsequence. This contradiction completes the proof. □

3. Statistical ward continuity of functions in asymmetric metric spaces

In this section we modify the definition of statistical forward continuity, we define statistical upward continuity and investigate the relationship between statistical upward continuity and statistical forward continuity.

Definition 3.1. A function f defined on a subset E of X to Y is called statistical forward continuous if it preserves statistical forward convergent sequences, i.e. $(f(x_n))$ is statistical forward convergent to $f(l)$ whenever (x_n) is statistical forward convergent to l.

Definition 3.2. A function f defined on a subset E of X to Y is called statistical upward continuous if it preserves statistical forward quasi Cauchy sequences, i.e. $(f(x_n))$ is statistical forward quasi Cauchy whenever (x_n) is.

Theorem 3.3. *Assume that f be a statistical upward continuous function on a subset E of X to Y and that forward convergence implies backward convergence on X , then it is statistical forward continuous.*

Proof. Take any statistical upward continuous function f on E to Y. Let (x_n) be any statistical forward convergent sequence of points in E with statistical forward limit a. Then the sequence

$$(x_1, a, x_2, a, ..., x_n, a, ...)$$

is also statistical forward convergent to a. Hence

$$(x_1, a, x_2, a, ..., x_n, a, ...)$$

is statistical forward quasi- Cauchy. As f is statistical upward continuous from E to Y, the sequence

$$(f(x_1), f(a), f(x_2), f(a), ..., f(x_n), f(a), ...)$$

is statistical forward quasi Cauchy in Y . Thus $(f(x_n))$ statistical forward converges to $f(a)$. This completes the proof of the theorem. □

Theorem 3.4. *Let f be an statistical upward continuous function from X to Y. Then statistical upward continuous image of any statistical upward compact subset of X is statistical upward compact.*

Proof. Suppose that f is statistical upward continuous, and E is a statistical upward compact subset of X. Let $\mathbf{y} = (y_n) \in f(E)$. Thus, for each $n \in \mathbf{N}$ there exists $x_n \in E$ such that $y_n = f(x_n)$. Since E is statistical upward compact, there is a statistical forward quasi Cauchy subsequence $\mathbf{t} = (t_k) = (x_{n_k})$ of $\mathbf{x}$. Statistical upward continuity of f implies that $f(\mathbf{t}) = (f(t_k))$ is statistical forward quasi Cauchy. This completes the proof of the theorem. □

Corollary 3.5. *Let f be an statistical upward continuous function from X to Y. Then statistical upward continuous image of any forward totally bounded subset of X is forward totally bounded.*

Theorem 3.6. *Let (f_n) be a sequence of statistical upward continuous functions from X to Y. Assume that forward convergence implies backward convergence on Y. If (f_n) uniformly forward converges to f, then f is statistical upward continuous.*

Proof. Let (x_m) be a statistical forward quasi Cauchy sequence of terms in X and $\varepsilon > 0$ be given. Since (f_n) is uniform forward convergent and forward convergence implies backward convergence on Y, there exists $N \in \mathbb{N}$ such that

$$d(f(x), f_n(x)) < \frac{\varepsilon}{3} \quad \text{and} \quad d(f_n(x), f(x)) < \frac{\varepsilon}{3}$$

for any $x \in X$ whenever $n \geq N$. Since f_N is statistical upward continuous, we get

$$\lim_{n\to\infty} \frac{1}{n} \mid \left\{m \le n : \ d(f_N(x_m), f_N(x_{m+1})) \ge \frac{\varepsilon}{3}\right\} \mid = 0$$

On the other hand,

$$\{m \le n : \ d(f(x_m), f(x_{m+1})) \ge \varepsilon\} \subseteq \left\{m \le n : \ d(f(x_m), f_N(x_m)) \ge \frac{\varepsilon}{3}\right\}$$
$$\cup \left\{m \le n : \ d(f_N(x_m), f_N(x_{m+1})) \ge \frac{\varepsilon}{3}\right\} \cup \left\{m \le n : \ d(f_N(x_{m+1}), f(x_{m+1})) \ge \frac{\varepsilon}{3}\right\}$$

holds. Thus we obtain

$$\lim_{n\to\infty} \frac{1}{n} \mid \{m \le n : \ d(f(x_m), f(x_{m+1})) \ge \varepsilon\} \mid$$
$$\le \lim_{n\to\infty} \frac{1}{n} \mid \left\{m \le n : \ d(f(x_m), f_N(x_m)) \ge \frac{\varepsilon}{3}\right\} \mid$$
$$+ \lim_{n\to\infty} \frac{1}{n} \mid \left\{m \le n : \ d(f_N(x_m), f_N(x_{m+1})) \ge \frac{\varepsilon}{3}\right\} \mid$$
$$+ \lim_{n\to\infty} \frac{1}{n} \mid \left\{m \le n : \ d(f_N(x_{m+1}), f(x_{m+1})) \ge \frac{\varepsilon}{3}\right\} \mid = 0$$

This completes the proof. □

Theorem 3.7. *If forward convergence implies backward convergence on Y, then the set of statistical upward continuous functions is a forward closed subset of the set of functions from X to Y in the uniform metric $\overline{\rho}$ according to d.*

Proof. Let f be an element of forward closure of the set of statistical upward continuous functions. Then there exists a sequence (f_n) whose terms are in the set of statistically upward continuous functions such that the uniform forward limit is f. To show that the function f is statistical upward continuous let take a statistical forward quasi Cauchy sequence (x_m) whose terms are in X. Let $\varepsilon > 0$ be given. Since (f_n) uniformly forward converges to f, there exists $N \in \mathbb{N}$ such that $d(f(x), f_n(x)) < \frac{\varepsilon}{3}$ for all $x \in X$ and $n \ge N$. Statistical upward continuity of f_N implies

$$\lim_{n\to\infty} \frac{1}{n} \mid \left\{m \le n : \ d(f_N(x_m), f_N(x_{m+1})) \ge \frac{\varepsilon}{3}\right\} \mid = 0$$

On the other hand,

$$\{m \le n : \ d(f(x_m), f(x_{m+1})) \ge \varepsilon\} \subseteq \left\{k \le n : \ d(f(x_m), f_N(x_m)) \ge \frac{\varepsilon}{3}\right\}$$
$$\cup \left\{m \le n : \ d(f_N(x_m), f_N(x_{m+1})) \ge \frac{\varepsilon}{3}\right\} \cup \left\{m \le n : \ d(f_N(x_{m+1}), f(x_{m+1})) \ge \frac{\varepsilon}{3}\right\}$$

holds. Thus we obtain

$$\lim_{n\to\infty} \frac{1}{n} \mid \{m \le n : \ d(f(x_m), f(x_{m+1})) \ge \varepsilon\} \mid$$
$$\le \lim_{n\to\infty} \frac{1}{n} \mid \left\{m \le n : \ d(f(x_m), f_N(x_m)) \ge \frac{\varepsilon}{3}\right\} \mid$$
$$+ \lim_{n\to\infty} \frac{1}{n} \mid \left\{m \le n : \ d(f_N(x_m), f_N(x_{m+1})) \ge \frac{\varepsilon}{3}\right\} \mid$$
$$+ \lim_{n\to\infty} \frac{1}{n} \mid \left\{m \le n : \ d(f_N(x_{m+1}), f(x_{m+1})) \ge \frac{\varepsilon}{3}\right\} \mid = 0.$$

This completes the proof. □

Corollary 3.8. *If the asymmetric metric space (Y, d) is forward compact and forward convergence implies backward convergence on Y, then the space of statistical upward continuous functions is complete in the uniform metric $\overline{\rho}$ according to d.*

Proof. It is a direct consequence of [7, Lemma 5.7] and Theorem 3.7. □

4. Conclusion

In this paper, we introduce and examine the statistical forward continuity of functions as well as new types of continuity defined based on statistical quasi Cauchy sequences in an asymmetric metric space X which is more general than the metric space. We also investigate necessary and sufficient conditions for a subset of X to be forward totally bounded. We prove that any statistical upward continuous function on a subset E of X to Y is statistical forward continuous under the condition that forward convergence implies backward convergence on X. We also prove that under suitable conditions, the space of statistical upward continuous functions is complete in the uniform metric $\overline{\rho}$ according to d. For another study, we suggest to investigate statistical forward quasi Cauchy double sequences in an asymmetric metric spaces [6, 11, 16].

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